ON MINKOWSKI TYPE QUESTION MARK FUNCTIONS ASSOCIATED WITH EVEN OR ODD CONTINUED FRACTIONS

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Abstract. We study analogues of Minkowski's question mark function $?(x)$ related to continued fractions with even or odd partial quotients. We prove that these functions are Hölder continuous with precise exponents, and that they linearize the appropriate versions of the Gauss and Farey maps.

1. Introduction

Minkowski [16] introduced a homeomorphism of $[0,1]$, which he denoted $?(x)$, that gives monotonic bijections between rational and dyadic numbers in $[0,1]$, and also between quadratic irrationals in $(0,1)$ and rationals in $(0,1)$. The function $?$ is singular, yet strictly increasing, continuous, and surjective. The question mark can be defined inductively on rationals by

$$?\left(\frac{p+p'}{q+q'}\right) = \frac{1}{2}?\left(\frac{p}{q}\right) + \frac{1}{2}?\left(\frac{p'}{q'}\right),$$

whenever $\frac{p}{q}$ and $\frac{p'}{q'}$ are rational numbers in lowest terms in $[0,1]$ with $p'q-pq'=1$. It can also be explicitly expressed by Denjoy's formula [5] (see also [23])

$$?([a_1,a_2,a_3,\ldots]) = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + \frac{1}{2^{a_1+a_2+a_3-1}} - \cdots ,$$

in terms of the regular continued fraction expansion

$$x = [a_1,a_2,\ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2+\cfrac{1}{\ddots}}}.$$

It is well-known (see, e.g., [4]) that $?(x)$ linearizes the classical Gauss and Farey maps

$$G\big([a_1,a_2,a_3,\ldots]\big) = [a_2,a_3,a_4,\ldots], \quad F\big([a_1,a_2,a_3,\ldots]\big) = \begin{cases} [a_1-1,a_2,a_3,\ldots] & \text{if } a_1 \geq 2 \\ [a_2,a_3,a_4,\ldots] & \text{if } a_1 = 1, \end{cases}$$

associated with regular continued fractions, or equivalently

$$G(x) = \left\{\frac{1}{x}\right\} = \frac{1}{x} - \left[\frac{1}{x}\right], \quad F(x) = \begin{cases} \frac{x}{1-x} & \text{if } x \in \left[0,\frac{1}{2}\right] \\ \frac{1-x}{x} & \text{if } x \in \left[\frac{1}{2},1\right]. \end{cases}$$

More precisely, the map $?^{-1}G?$ is decreasing and linear on each interval $(2^{-k-1},2^{-k})$, while $(?^{-1}F?)(x) = 2\operatorname{dist}(x,\mathbb{Z})$.

Salem [23] proved that $?(x)$ is singular and Hölder continuous, with best exponent $\frac{\log 2}{2\log G} \approx 0.72021$, where $G = \frac{1+\sqrt{5}}{2}$ denotes the "big" golden ratio. Several significant results about $?(x)$ have subsequently been proved [12, 20, 2, 8, 7], culminating with the very recent solution provided by Jordan and Sahlsten [10] to the longstanding Salem open problem [23] concerning the decay of its Fourier-Stieltjes coefficients. A number of generalizations of this classical map have been considered [9, 3, 19, 25, 17]. See `http://uosis.mif.vu.lt/~alkauskas/minkowski.htm` for an extensive bibliography of research in this area until 2014.

This paper is concerned with natural analogues of Minkowski's question mark function that are related to continued fractions with odd or with even partial quotients, that we asimply going to call *even continued fractions* (in short ECF) and respectively *odd continued fractions* (in short OCF). See [24] for the definition and basic properties of these two classes of continued fractions and [21] for a detailed treatment of odd continued fractions.

In Section 2 we consider the situation of even continued fractions, defining our even question mark function Q_E and proving a formula for $Q_E(x)$ in terms of the even continued fraction expansion of x. As consequences, we prove that Q_E is singular and Hölder continuous with best exponent $\frac{\log 3}{2\log(1+\sqrt{2})} \approx 0.62324$. We also show that Q_E linearizes the even Gauss and even Farey maps. As the formula in Theorem 1 makes clear, Q_E is naturally a triadic version of the Minkowski question mark function. Northshield has introduced [17] a different triadic generalization of the question mark function. At the end of the section we establish a precise connection between our even continued fraction analogue of the Stern sequence and the sequence in $\mathbb{Z}[\sqrt{2}]$ that he considers.

In Section 3 we focus on odd continued fractions, following Zhabitskaya's work [25] and considering the odd question mark function $Q_O(x)$ that coincides with her $F^0(x)$. We prove that the function Q_O is Hölder continuous with best exponent $\frac{\log\lambda}{2\log G} \approx 0.63317$, where $\lambda \approx 1.83929$ denotes the unique real root of the equation $x^3 - x^2 - x - 1 = 0$. We also prove that the map Q_O linearizes the odd Gauss and the odd Farey maps.

2. Even Partial Quotients

2.1. **Even continued fraction generation and ordering of rational numbers.** We consider the ECF expansion in $[-1,1]$ given by

$$[(e_1,a_1),(e_2,a_2),(e_3,a_3),...] = \cfrac{e_1}{a_1 + \cfrac{e_2}{a_2 + \cfrac{e_3}{a_3+\cdots}}}, \tag{2.1}$$

where $e_i \in \{\pm 1\}$ and $a_i \in 2\mathbb{N}$. For uniqueness, we require that in a finite expansion, the last e_i must equal 1, and in this case we allow a_i to also equal 1. This convention allows all rational numbers to have a unique finite even continued fraction expansion. Note that $\frac{p}{q}$ will also have a (unique) infinite expansion iff $p+q \equiv 0 \pmod 2$ iff its finite expansion terminates in a 1.

If $x = [(1,a_1),(e_2,a_2),...,(e_n,a_n)]$, then let $[(x),(\epsilon_1,\alpha_1),(\epsilon_2,\alpha_2),...]$ denote the concatenated expansion $[(1,a_1),(e_2,a_2),...,(e_n,a_n),(\epsilon_1,\alpha_1),(\epsilon_2,\alpha_2),...]$. Define

$$\begin{aligned}
\mathcal{Y}_k :=& \big\{x = [(1,a_1),(e_2,a_2),...,(e_n,a_n)] \in \mathbb{Q}\cap[0,1] : a_1+\cdots+a_n \le 2k+1\big\},\\
\mathcal{X}_k :=& \mathcal{Y}_k \setminus \mathcal{Y}_{k-1}, \qquad \mathcal{Z}_k := \{x \in \mathcal{Y}_k : \forall i, a_i \neq 1\}.
\end{aligned}$$

Our convention is to take $0,1 \in \mathcal{Y}_k$ and $0 \in \mathcal{Z}_k$. It is plain to check that

$$\begin{aligned}
&\mathcal{Z}_0 = \left\{\frac{0}{1}\right\}, \quad \mathcal{Z}_1 = \left\{\frac{0}{1},\frac{1}{2}\right\}, \quad \mathcal{Z}_2 = \left\{\frac{0}{1},\frac{1}{4},\frac{2}{5},\frac{1}{2},\frac{2}{3}\right\},\\
&\mathcal{Z}_3 = \left\{\frac{0}{1},\frac{1}{6},\frac{2}{9},\frac{1}{4},\frac{2}{7},\frac{3}{8},\frac{2}{5},\frac{5}{12},\frac{4}{9},\frac{1}{2},\frac{4}{7},\frac{5}{8},\frac{2}{3},\frac{3}{4}\right\},\\
&\mathcal{Y}_0 = \left\{\frac{0}{1},\frac{1}{1}\right\}, \quad \mathcal{Y}_1 = \left\{\frac{0}{1},\frac{1}{3},\frac{1}{2},\frac{1}{1}\right\}, \quad \mathcal{Y}_2 = \left\{\frac{0}{1},\frac{1}{5},\frac{1}{4},\frac{1}{3},\frac{2}{5},\frac{3}{7},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{1}{1}\right\}.
\end{aligned}$$

Denote $X_k := |\mathcal{X}_k|$, $Y_k := |\mathcal{Y}_k|$ and $Z_k = |\mathcal{Z}_k|$.

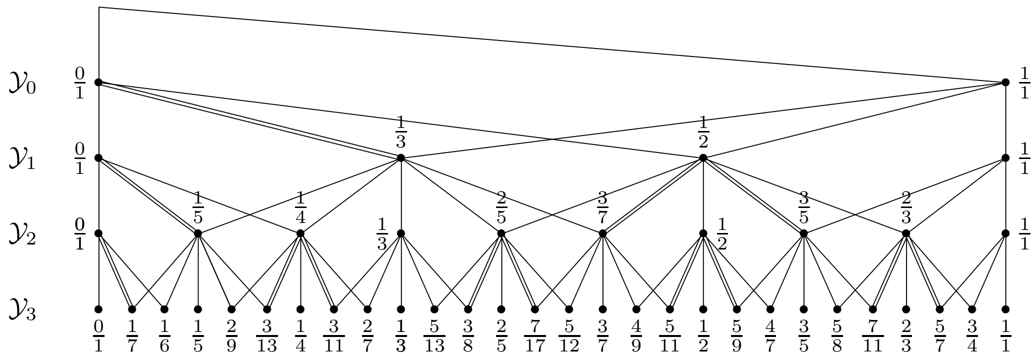

FIGURE 1. The ternary ECF diagram $\mathcal{D}_E$

Observe that $\mathcal{Y}_k = \cup_{x\in\mathcal{Z}_k}\{x, [(x),(1,1)]\}$, and hence $Y_k = 2Z_k$. For $x \in \mathcal{Z}_k$, consider $c_k(x) := 2k+2-\sum a_i$. Then

$$\mathcal{Z}_{k+1} = \left(\bigcup_{x\in\mathcal{Z}_k, x\neq 0} \{x, [(x),(1,c_k(x))], [(x),(-1,c_k(x))]\} \right) \cup \{0, [(1,c_0)]\}.$$

Hence $Z_{k+1} = 3Z_k - 1$. Since $Z_0 = 1$, we have $Z_k = \frac{3^k+1}{2}$ and we conclude that $Y_k = 3^k + 1$.

Note that if $x, y \in \mathcal{Z}_k$, and $x < y$, then $[(x),(e,a)] < [(y),(\epsilon,\alpha)]$ for $e, \epsilon \in \{-1,1\}$ and $a, \alpha \in \{1\} \cup 2\mathbb{Z}$. Inductively, this holds for any two continued fractions with initial expansions equal to those of x and y, respectively.

Hence we may obtain the ordered set $\mathcal{Y}_{k+1}$ from $\mathcal{Y}_k$ by replacing 0 with 0, $[(1,c_k(0)),(1,1)]$, $[(1,c_k(0))]$ and each of the nonzero elements $x = [(e_1,a_1),(e_2,a_2),...(e_n,a_n)] \in \mathcal{Z}_k$ with the following five elements:

$$[(x),(1,c_k(x))],\ [(x),(1,c_k(x)),(1,1)],\ x,\ [(x),(-1,c_k(x)),(1,1)],\ [(x),(-1,c_k(x))].$$

These numbers are in this order if $\Pi_{i=1}^n(-e_i) = -1$ and are in the reverse order if $\Pi_{i=1}^n(-e_i) = 1$. By induction, we have that for any $x \in \mathcal{Z}_k$, the neighbors of x in $\mathcal{Y}_k$ are $[(x),(1,c_{k-1}(x)),(1,1)]$ and $[(x),(-1,c_{k-1}(x)),(1,1)]$, with the understanding that if $x \in \mathcal{X}_k$ and $c_{k-1}(x) = 0$, we have $[(x),(1,0),(1,1)] = [(x),(1,1)]$ and $[(x),(-1,0),(1,1)] = [(x),(-1,1)]$. In any case, we observe that $[(x),(1,c_k(x))]$ is the mediant of x and its neighbor $[(x),(1,c_{k-1}(x)),(1,1)]$ in $\mathcal{Y}_k$, and $[(x),(1,c_k(x)),(1,1)]$ is the mediant of x and $[(x),(1,c_k(x))]$. Likewise, $[(x),(-1,c_k(x))]$ is the mediant of x and its neighbor $[(x),(-1,c_{k-1}(x)),(1,1)]$ and $[(x),(-1,c_k(x)),(1,1)]$ is the mediant of x and $[(x),(-1,c_k(x))]$. In other words, we obtain $\mathcal{Y}_{k+1}$ from $\mathcal{Y}_k$ by inserting between each pair of elements (say, $\frac{p}{q} \in \mathcal{Z}_k$ and $\frac{r}{s} \in \mathcal{Y}_k \setminus \mathcal{Z}_k$) the successive mediants $\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}$ and $\frac{p}{q} \oplus \frac{p+r}{q+s} = \frac{2p+r}{2q+s}$. This gives an ECF analogue $\mathcal{D}_E$ of the Stern-Brocot tree, which is illustrated in Figure 1, in which the ternary tree structure of the $\mathcal{Z}_k$ is also apparent.

2.2. **The even Farey type map F_E and the even Gauss map T_E.** The sets $\mathcal{Y}_k$ and $\mathcal{Z}_k$ arise naturally in connection with the map $F_E : [0,1] \to [0,1]$ defined by

$$F_E(x) = \begin{cases} \frac{x}{1-2x} & \text{if } 0 \le x < \frac{1}{3} \\ \frac{1}{x} - 2 & \text{if } \frac{1}{3} \le x < \frac{1}{2} \\ 2 - \frac{1}{x} & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

This map has an infinite invariant measure $d\nu_E(x) = \frac{dx}{x(1-x)}$ and was considered in different contexts in [1] and [22].

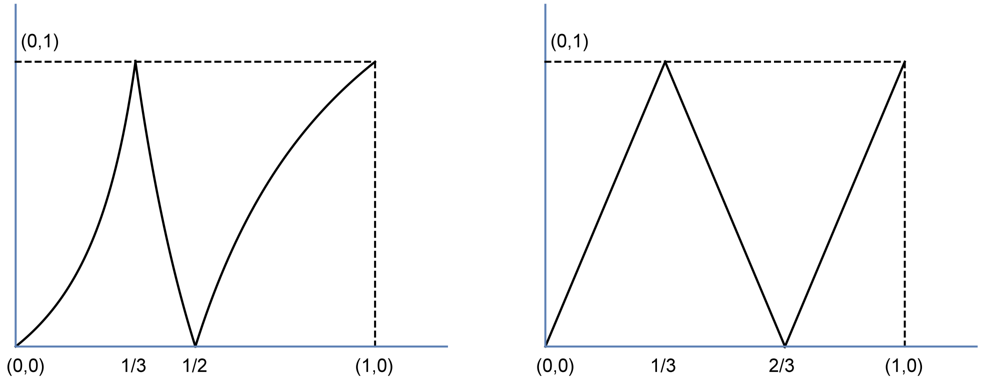

FIGURE 2. The even Farey map F_E and its linearization $\overline{F}_E$

Symbolically, F_E acts on the ECF representation by subtracting 2 from the leading digit a_1 of x when $a_1 \geq 4$ (which corresponds to x between 0 and $\frac{1}{3}$), and by simply removing (a_1, e_2) when $a_1 = 2$ (which corresponds to x between $\frac{1}{3}$ and 1), i.e.

$$F_E([(1,a_1),(e_2,a_2),(e_3,a_3),\ldots]) = \begin{cases} [(1,a_1-2),(e_2,a_2),(e_3,a_3),\ldots] & \text{if } a_1 \geq 4 \\ [(1,a_2),(e_3,a_3),(e_4,a_4),\ldots] & \text{if } a_1 = 2. \end{cases}$$

Then the sets $\mathcal{Y}_k$ and $\mathcal{Z}_k$ can be described as

$$\mathcal{Y}_k = F_E^{-k}(\{0,1\}) \quad \text{and} \quad \mathcal{Z}_k = F_E^{-k}(\{0\}), \quad k \geq 0.$$

The induced transformation R_E of F_E on $[\frac{1}{3},1]$ acts on the ECF expansion in $[\frac{1}{3},1]$ as:

$$R_E\big([(1,2),(e_2,a_2),(e_3,a_3),\ldots]\big) = [(1,2),(e_3,a_3),(e_4,a_4),\ldots].$$

Recall that the even Gauss map T_E acts on $[0,1]$ by

$$T_E(x) = \left|\frac{1}{x} - 2k\right| \quad \text{if } x \in \left[\frac{1}{2k+1},\frac{1}{2k-1}\right],$$

and it acts on ECF expansions (2.1) restricted to $[0,1]$ by

$$T_E\big([(1,a_1),(e_2,a_2),(e_3,a_3),\ldots]\big) = [(1,a_2),(e_3,a_3),(e_4,a_4),\ldots].$$

Furthermore, $d\mu_E(x) = (\frac{1}{1+x} - \frac{1}{1-x})dx$ is a T_E-invariant measure [24].

We will also consider the map $\widetilde{T}_E : [-1,1] \to [-1,1]$ acting on the ECF expansion (2.1) as

$$\widetilde{T}_E\big([(e_1,a_1),(e_2,a_2),(e_3,a_3),\ldots]\big) = [(e_2,a_2),(e_3,a_3),(e_4,a_4),\ldots].$$

Equivalently $\widetilde{T}_E(x) = e_2 T_E(x)$ if $x > 0$ and $\widetilde{T}_E$ is an even function, that is

$$\widetilde{T}_E(x) = \begin{cases} \frac{1}{x} - 2k & \text{if } x \in [\frac{1}{2k+1},\frac{1}{2k-1}] \\ \widetilde{T}_E(-x) & \text{if } x \in (-1,0), \end{cases}$$

and the push forward $d\widetilde{\mu}_E(x) = \frac{dx}{1+x}$ of $\nu_E|_{[1/3,1]}$ under φ is a $\widetilde{T}_E$-invariant measure. It is plain to check that R_E and $\widetilde{T}_E$ are conjugated, or more precisely $\widetilde{T}_E = \varphi R_E \varphi^{-1}$, where $\varphi : [\frac{1}{3},1] \to [-1,1]$, $\varphi(x) = \frac{1}{x} - 2$, with $\phi^{-1}(y) = \frac{1}{2+y}$. It is also plain to see that $\widetilde{T}_E$ is an extension of T_E. More precisely we have $\pi\widetilde{T}_E = T_E\pi$, where $\pi : [-1,1] \to [0,1]$, $\pi(x) = |x|$. The push forward of $\widetilde{\mu}_E$ under π is the T_E-invariant measure μ_E.

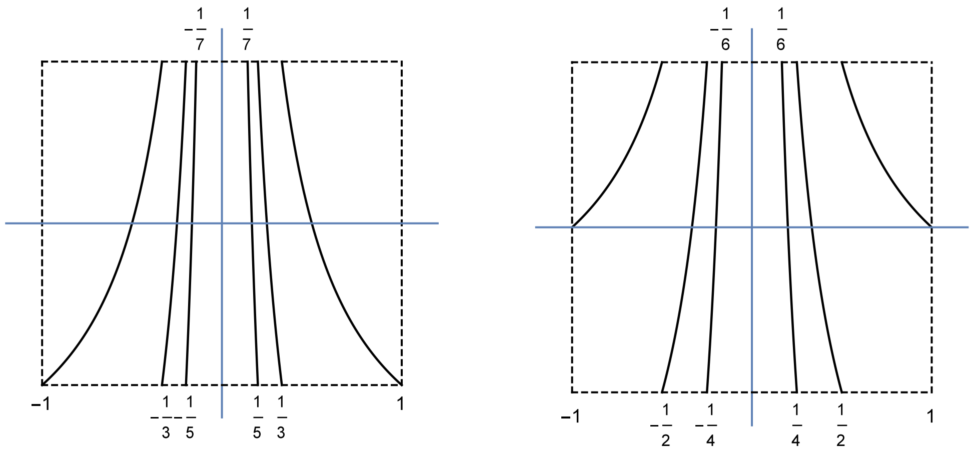

FIGURE 3. The maps $\widetilde{T}_E$ and $\widetilde{T}_O$

2.3. **The even Minkowski type question map Q_E.** We are now ready to define the ECF analogue of Minkowski's question mark function, and prove an explicit formula for it in terms of the ECF expansion.

Definition 1. *For $x \in \mathcal{Y}_k$, define*

$$Q_E(x) := \frac{|\{y \in \mathcal{Y}_k : y < x\}|}{3^k}.$$

Proposition 2. *This does not depend on the choice of k and hence $Q_E(x)$ is well-defined for $\mathbb{Q} \cap [0, 1]$.*

Proof. Case 1. Suppose $x \in \mathcal{Z}_k$. Then

$$|\{y \in \mathcal{Y}_k : y < x\}| = 2|\{z \in \mathcal{Z}_k : z < x\}|$$

and

$$|\{z \in \mathcal{Z}_{k+1} : z < x\}| = 3|\{z \in \mathcal{Z}_k : z < x\}|$$

since if $x, z \in \mathcal{Z}_k$ and $0 < z < x$, then $[(z), (\pm 1, c_k(z))] < x$, and exactly one of $[(x), (\pm 1, c_k(x))]$ is less than x. We therefore have $|\{y \in \mathcal{Y}_{k+1} : y < x\}| = 3|\{y \in \mathcal{Y}_k : y < x\}|$, so by induction, $\frac{|\{y \in \mathcal{Y}_k : y < x\}|}{3^k} = \frac{|\{y \in \mathcal{Y}_{k+j} : y < x\}|}{3^{k+j}}$ for any $j \in \mathbb{N}$, and $Q_E(x)$ is well-defined.

Case 2. Suppose $x \notin \mathcal{Z}_k$. Then

$$|\{y \in \mathcal{Y}_k : y < x\}| = 2|\{z \in \mathcal{Z}_k : z < x\}| - 1$$

and

$$|\{z \in \mathcal{Z}_{k+1} : z < x\}| = 3|\{z \in \mathcal{Z}_k : z < x\}| - 1,$$

so

$$\begin{aligned} |\{y \in \mathcal{Y}_{k+1} : y < x\}| &= 2|\{z \in \mathcal{Z}_{k+1} : z < x\}| - 1 \\ &= 6|\{z \in \mathcal{Z}_k : z < x\}| - 3 = 3|\{y \in \mathcal{Y}_k : y < x\}|, \end{aligned}$$

and as in the previous case, we conclude by induction that $Q_E(x)$ is well-defined. □

Theorem 1. *Let $x = [0; (e_1, a_1), (e_2, a_2), ...(e_n, a_n)]$. Then*

$$Q_E(x) = \sum_{k=1}^{n} \frac{-w_k \Pi_{i=1}^{k}(-e_i)}{3^{\sum_{i=1}^{k} \lfloor \frac{a_i}{2} \rfloor}}$$

where $w_k = 2$ if $a_k \in 2\mathbb{N}$ and $w_k = 1$ if $a_k = 1$.

Proof. Let $y = [(x), (e_{n+1}, a_{n+1})]$.

Case 1. Suppose $a_{n+1} = 1$. In this case, $e_{n+1} = 1$ as well, so $y \in \mathcal{X}_k$. In the ordered $\mathcal{Y}_k$, y is adjacent to x. If $\Pi_{i=1}^{n}(-e_i) = 1$ then $y > x$, and if $\Pi_{i=1}^{n}(-e_i) = -1$ then $y < x$. Hence

$$Q_E(y) = Q_E(x) - \frac{\Pi_{i=1}^{n+1}(-e_i)}{3^k}.$$

Case 2. Suppose $a_{n+1} = 2j$. In this case, $y \in \mathcal{X}_{k+j}$. At this level, the neighbors of x are $[(x), (1, 2j)]$, $[(x), (1, 2j), (1, 1)]$, x, $[(x), (-1, 2j), (1, 1)]$, $[(x), (-1, 2j)]$ in this order if $\Pi_{i=1}^{n}(-e_i) = -1$, and in the opposite order if $\Pi_{i=1}^{n}(-e_i) = 1$. Hence

$$Q_E(y) = Q_E(x) - \frac{2\Pi_{i=1}^{n+1}(-e_i)}{3^{j+k}}.$$

Working backwards from the tail of the continued fraction, repeated application of these relations yields the formula stated above. □

We will see that by continuity, the formula also holds for infinite even continued fraction expansions, with the finite sum replaced by an infinite one. For rationals which have both an infinite and a finite even continued fraction expansion, the infinite expansion is obtained from the finite one by replacing the last term $[...(1, 1)]$ with $...(1, 2), (-1, 2), (-1, 2), (-1, 2)...]$. Using the fact that $\sum_{k=1}^{\infty} \frac{2}{3^k} = 1$, it is straightforward to check that the two sums coincide.

Theorem 2. *$Q_E(x)$ is Hölder continuous, with best exponent $\frac{\log 3}{2\log(1+\sqrt{2})}$.*

Before proving this, we need a fact about the growth of the continuants associated to even continued fraction expansions.

Proposition 3. *Let $\frac{p_n}{q_n} = [(1, a_1), (e_2, a_2), ..., (e_n, a_n)]$. Then*

$$q_n < (1 + \sqrt{2})^{\sum_{i=1}^{n} \frac{a_i}{2}}.$$

Proof. Let $\theta = 1 + \sqrt{2}$. Observe that $q_1 = a_1 < \theta^{a_1/2}$ holds for all $a_i \in \mathbb{R}$, and $q_0 = 1 = \theta^0$. We have the relation $q_k = a_k q_{k-1} + e_k q_{k-2}$. Assuming the claim holds for $n = k-2, k-1$, then

$$a_k q_{k-1} + e_k q_{k-2} \le a_k \theta^{\sum_{i=1}^{k-1} \frac{a_i}{2}} + \theta^{\sum_{i=1}^{k-2} \frac{a_i}{2}}$$

so it is sufficient to show that

$$a_k \theta^{\frac{a_{k-1}}{2}} + 1 \le \theta^{\frac{a_k + a_{k-1}}{2}}$$

or equivalently, that

$$a_k + \theta^{\frac{-a_{k-1}}{2}} \le \theta^{\frac{a_k}{2}}.$$

Since $a_k \neq 0$, we must have $a_{k-1} \ge 2$, so it is sufficient to prove

$$a_k + \frac{1}{\theta} \le \theta^{a_k/2}$$

which is always true: we verify that

$$1 + \frac{1}{\theta} < \theta^{1/2} \text{ and } 2 + \frac{1}{\theta} = \theta.$$

For $a_i > 2$, it is sufficient to observe that $\theta^{x/2} - x$ is increasing for $x \ge 2$, with derivative $\frac{1}{2}(\theta)^{x/2}\log(\theta) - 1 > 0$. □

Remark. The exponent in the proposition is the best possible, and it is attained by the convergents of $\sqrt{2} - 1 = [(1, 2), (1, 2), (1, 2), ...]$.

Proof. Notice that since each $a_i = 2$, the denominators satisfy the recurrence relation $q_k = 2q_{k-1} + q_{k-2}$, and hence are given by the sequence $1, 1, 3, 7, 17, \ldots$, which has the closed form

$$q_k = \frac{(1+\sqrt{2})^k + (1-\sqrt{2})^k}{2}.$$

Asymptotically, $q_k \sim \frac{1}{2}\theta^k$, so the bound $q_k \leq \theta^{\sum_{i=1}^k \frac{a_i}{2}} = \theta^k$ cannot be improved. □

Proof of Theorem 2. Let $x < x' \in \mathbb{Q} \cap [0,1]$, and let $y = Q_E(x)$, $y' = Q_E(x')$. Consider $\mathcal{Y}_k$ for the first k such that we have $x \leq r < r' \leq x'$ for some $r, r' \in \mathcal{Y}_k$. From the bound on the denominators proved in proposition 3 we must have $x' - x \geq r' - r \geq \frac{1}{\theta^{2k+2}}$ and since there can be at most 5 elements of $\mathcal{Y}_k$ between x and x', we have $y' - y \leq \frac{6}{3^k}$. We have $(2k+2)\log\theta \geq \log\frac{1}{x'-x}$ and $(k - \log_3 6)\log 3 \leq \log\frac{1}{y'-y}$ which together give

$$y' - y < C(x' - x)^{\frac{\log 3}{2\log\theta}}.$$

To see that this is the best possible exponent, consider $x = \sqrt{2} - 1 = [(1,2),(1,2),(1,2),\ldots]$. Let $\frac{p_k}{q_k}$ be the kth convergent, $\underbrace{[(1,2),(1,2),\ldots,(1,2)]}_{k \text{ times}}$. We have $Q_E(x) = \sum_{j=1}^{\infty}\frac{2(-1)^{j+1}}{3^j}$, and $Q_E(\frac{p_k}{q_k}) = \sum_{j=1}^{k}\frac{2(-1)^{j+1}}{3^j}$, so $|Q_E(x) - Q_E(\frac{p_k}{q_k})|$ is of order $\frac{1}{3^k}$. Observe that $|x - \frac{p_k}{q_k}| < |\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}| < \frac{1}{q_k^2}$. We know that q_k is of the same order as θ^k, so we have

$$\left|x - \frac{p_k}{q_k}\right|^{\frac{\log 3}{2\log\theta}} \lesssim \theta^{-2k\cdot\frac{\log 3}{2\log\theta}} = \frac{1}{3^k}.$$

Hence the exponent $\frac{\log 3}{2\log\theta} = \frac{\log 3}{2\log(1+\sqrt{2})}$ is best possible. □

Theorem 3. *$Q_E(x)$ is singular.*

Proof. Let $x = [(e_1,a_1),(e_2,a_2),\ldots]$ with ECF convergents $\frac{p_n}{q_n} = [(e_1,a_1),\ldots(e_n,a_n)]$, and let $Q_E(x) = y$. Let $t_n := [(e_{n+2},a_{n+2}),(e_{n+3},a_{n+3}),\ldots]$. We have

$$x = \frac{(a_{n+1}+t_n)p_n + e_{n+1}p_{n-1}}{(a_{n+1}+t_n)q_n + e_{n+1}q_{n-1}}$$

and (see [13])

$$\left|x - \frac{p_n}{q_n}\right| = \left|\frac{e_{n+1}(q_n p_{n-1} - p_n q_{n-1})}{q_n((a_{n+1}+t_n)q_n + e_{n+1}q_{n-1})}\right| = \frac{1}{q_n((a_{n+1}+t_n)q_n + e_{n+1}q_{n-1})}.$$

Since $|t_n| \leq 1$, we have (in the case where $a_{n+1} > 2$)

$$\frac{1}{q_n^2(a_{n+1}+2)} < \left|x - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2(a_{n+1}-2)}.$$

Applying the formula for $Q_E(x)$, we have $y - Q_E(\frac{p_n}{q_n}) = \sum_{k=n}^{\infty}\frac{-2\Pi_{i=1}^k(-e_i)}{3^{\sum_{i=1}^k \frac{a_i}{2}}}$ so

$$\frac{1}{3^{(\sum_{i=1}^n \frac{a_i}{2})+1}} < \left|y - Q_E\left(\frac{p_n}{q_n}\right)\right| < \frac{2}{3^{(\sum_{i=1}^n \frac{a_i}{2})-1}}.$$

Let $r_n = \left|\frac{y-Q_E(p_n/q_n)}{x-p_n/q_n}\right|$. Then

$$\begin{aligned}
\frac{q_n^2(a_{n+1}-2)}{3^{(\sum_{i=1}^n \frac{a_i}{2})+1}} < r_n < \frac{2q_n^2(a_{n+1}+2)}{3^{(\sum_{i=1}^n \frac{a_i}{2})-1}}, \quad \text{and} \\
\frac{r_n}{r_{n-1}} &< \frac{2q_n^2(a_{n+1}+2)}{3^{(\sum_{i=1}^n \frac{a_i}{2})-1}} \cdot \frac{3^{(\sum_{i=1}^{n-1} \frac{a_i}{2})+1}}{q_{n-1}^2(a_n-2)} \\
&= \frac{18(a_{n+1}+2)}{3^{\frac{a_n}{2}}(a_n-2)} \cdot \left(\frac{q_n}{q_{n-1}}\right)^2 < \frac{18(a_{n+1}+2)}{3^{\frac{a_n}{2}}(a_n-2)} \cdot (a_n+1)^2 \\
&< \frac{18(a_{n+1}+2)(a_n+1)}{3^{\frac{a_n}{2}}}.
\end{aligned}$$

If for some x, the a_i are unbounded, then we may take a subsequence a_{i_k} such that $2 < a_{i_k} < a_{i_{k+1}}$. The above will imply that $\lim \frac{r_{i_k}}{r_{i_{k-1}}} = 0$, which implies that if the derivative of $?(x)$ exists and is finite, it must be equal to 0. As we will see in the next proposition, the a_i are in fact unbounded for almost every x. Since $Q_E(x)$ is monotone, the derivative must in fact exist almost everywhere, and hence $Q_E(x)$ is singular. □

Proposition 4. *The set of x with bounded even partial quotients has measure* 0.

Proof. It is well known that almost every number is normal with respect to the regular continued fraction. (The results of [14] can perhaps be extended to show that this in fact implies being normal with respect to the even continued fraction, although we only need a much weaker result.)

For $k > 0$ a number which is normal with respect to the regular continued fraction expansion will have at some point in its expansion three consecutive $a_i, a_{i+1}, a_{i+2} > k$, so after applying the singularization and insertion algorithm (see [15] for example) to obtain the even continued fraction expansion we must have at least one even partial quotient $a_j > k$. Hence almost every number has unbounded even partial quotients. □

2.4. **The linearization of the map F_E.** The formula proved in Theorem 1 and the continuity of Q_E provide the formula

$$Q_E\big([(1,2k_1),(e_1,2k_2),(e_2,2k_3),\dots]\big) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}e_1\cdots e_n}{3^{k_1+\cdots+k_n}}. \tag{2.2}$$

Consider the continuous piecewise linear maps $\overline{F}_E, \overline{T}_E : [0,1] \to [0,1]$ defined by

$$\overline{F}_E(y) = \begin{cases} 3y & \text{if } y \in [0,\frac{1}{3}] \\ 2-3y & \text{if } y \in [\frac{1}{3},\frac{1}{2}] \\ 3y-2 & \text{if } y \in [\frac{1}{2},1] \end{cases} \quad \text{and} \quad \overline{T}_E(y) = \begin{cases} 2-3^k y & \text{if } y \in [3^{-k}, 2\cdot 3^{-k}] \\ 3^k y - 2 & \text{if } y \in [2\cdot 3^{-k}, 3^{-k+1}]. \end{cases}$$

Lemma 5. *The homeomorphism Q_E of* $[0,1]$ *linearizes the maps F_E and G_E as follows:*

(i) $Q_E^{-1}F_E Q_E = \overline{F}_E$.
(ii) $Q_E^{-1}T_E Q_E = \overline{T}_E$.

Proof. Let $x = [(1,2k_1),(e_1,2k_2),(e_2,2k_3),\dots]$.

(i) There are three cases to be considered:

Case 1. $x \in (\frac{1}{2}, 1)$, where $k_1 = 1$ and $e_1 = -1$. The we successively infer:

$$\begin{aligned} Q_E(x) &= Q_E\big([(1,2),(-1,2k_2),(e_2,2k_3),\ldots]\big) = 2\bigg(\frac{1}{3} + \frac{1}{3^{k_2+1}} - \frac{e_2}{3^{k_2+k_3}} + \cdots\bigg), \\ (\overline{F}_E Q_E)(x) &= 3Q_E(x) - 2 = 2\bigg(\frac{1}{3^{k_2}} - \frac{e_2}{3^{k_2+k_3}} + \frac{e_2 e_3}{3^{k_2+k_3+k_4}} - \cdots\bigg), \\ (Q_E F_E)(x) &= Q_E\big([(1,2k_2),(e_2,2k_3),(e_3,2k_4),\ldots]\big) \\ &= 2\bigg(\frac{1}{3^{k_2}} - \frac{e_2}{3^{k_2+k_3}} + \frac{e_2 e_3}{3^{k_2+k_3+k_4}} - \cdots\bigg) = (\overline{F}_E Q_E)(x). \end{aligned}$$

Case 2. $x \in (\frac{1}{3}, \frac{1}{2})$, where $k_1 = 1$, $e_1 = 1$, and we have

$$\begin{aligned} Q_E(x) &= Q_E\big([(1,2),(1,2k_2),(e_2,2k_3),\ldots]\big) = 2\bigg(\frac{1}{3} - \frac{1}{3^{k_2+1}} + \frac{e_2}{3^{k_2+k_3+1}} - \cdots\bigg), \\ (\overline{F}_E Q_E)(x) &= 2 - 3Q_E(x) = 2\bigg(\frac{1}{3^{k_2}} - \frac{e_2}{3^{k_2+k_3}} + \frac{e_2 e_3}{3^{k_2+k_3+k_4}} - \cdots\bigg) = (Q_E F_E)(x). \end{aligned}$$

Case 3. $x \in (0, \frac{1}{3})$, where $k_1 \geq 2$ and we have:

$$\begin{aligned} Q_E(x) &= 2\bigg(\frac{1}{3^{k_1}} - \frac{e_1}{3^{k_1+k_2}} + \frac{e_1 e_2}{3^{k_1+k_2+k_3}} - \cdots\bigg), \\ (\overline{F}_E Q_E)(x) &= 3Q_E(x) = 2\bigg(\frac{1}{3^{k_1-1}} - \frac{e_1}{3^{k_1-1+k_2}} + \frac{e_1 e_2}{3^{k_1-1+k_2+k_3}} - \cdots\bigg). \\ (Q_E F_E)(x) &= Q_E\big([(1,2k_1-2),(e_1,2k_2),(e_2,2k_3),\ldots]\big) \\ &= 2\bigg(\frac{1}{3^{k_1-1}} - \frac{e_1}{3^{k_1-1+k_2}} + \frac{e_1 e_2}{3^{k_1-1+k_2+k_3}} - \cdots\bigg) = (\overline{F}_E Q_E)(x). \end{aligned}$$

(ii) We always have

$$(Q_E T_E)(x) = 2\bigg(\frac{1}{3^{k_2}} - \frac{e_2}{3^{k_2+k_3}} + \frac{e_2 e_3}{3^{k_2+k_3+k_4}} - \cdots\bigg),$$

and consider separately the following two possible situations:

Case 1. $x \in (\frac{1}{2k+1}, \frac{1}{2k})$, where $k_1 = k$, $e_1 = 1$, $y := Q_E(x) \in [3^{-k}, 2 \cdot 3^{-k}]$, and

$$\begin{aligned} (\overline{T}_E Q_E)(x) &= \overline{T}_E(y) = \overline{T}_E\bigg(2\Big(\frac{1}{3^k} - \frac{1}{3^{k+k_2}} + \frac{e_2}{3^{k+k_2+k_3}} - \frac{e_2 e_3}{3^{k+k_1+k_2+k_3}} + \cdots\Big)\bigg) \\ &= 2 - 3^k y = 2\bigg(\frac{1}{3^{k_2}} - \frac{e_2}{3^{k_2+k_3}} + \frac{e_2 e_3}{3^{k_2+k_3+k_4}} - \cdots\bigg) = (Q_E T_E)(x). \end{aligned}$$

Case 2. $x \in (\frac{1}{2k}, \frac{1}{2k-1})$, where $k_1 = k$, $e_1 = -1$, $y := Q_E(x) \in [2 \cdot 3^{-k}, 3^{-k+1}]$, and

$$\begin{aligned} (\overline{T}_E Q_E)(x) &= \overline{T}_E(y) = \overline{T}_E\bigg(2\Big(\frac{1}{3^k} + \frac{1}{3^{k+k_2}} - \frac{e_2}{3^{k+k_2+k_3}} + \frac{e_2 e_3}{3^{k+k_1+k_2+k_3}} - \cdots\Big)\bigg) \\ &= 3^k y - 2 = 2\bigg(\frac{1}{3^{k_2}} - \frac{e_2}{3^{k_2+k_3}} + \frac{e_2 e_3}{3^{k_2+k_3+k_4}} - \cdots\bigg) = (Q_E T_E)(x). \qquad \square \end{aligned}$$

2.5. **The ECF Stern Sequence and Stern Polynomials.** We now consider the integer sequence of denominators of the fractions in our analogue $\mathcal{D}_E$ of the Stern-Brocot tree, giving an ECF version of the Stern sequence. As we will see, this ends up being closely related to a triadic version of the Stern sequence that has been constructed by Northshield in [17]. It is convenient to work on $[-1,1)$, since $|\{x \in [-1,1) : \sum a_i \leq k\}| = 2 \cdot 3^k$ so $n \mapsto 3n$ corresponds to moving down a level in the extension of the diagram $\mathcal{D}_E$ to $[-1,1)$. Let $\{b_n\}$ be the sequence of the denominators of the fractions in the extension of $\mathcal{D}_E$ to $[-1,1)$. We have the relations

$$\begin{aligned} b_{3n} &= b_n, \\ b_{3n+1} &= w(n)b_n + b_{n+1}, \\ b_{3n+2} &= b_n + w(n+1)b_{n+1}, \end{aligned}$$

where $w(n)=2$ if n is even and 1 if n is odd. We let $b_0=0$. From these relations, we derive

$$\begin{aligned} B_o(x) = \sum_{n \text{ odd}} b_n x^n &= \sum_{n \text{ odd}} b_{3n}x^{3n} + \sum_{n \text{ even}} b_{3n+1}x^{3n+1} + \sum_{n \text{ odd}} b_{3n+2}x^{3n+2} \\ &= (x^{-2}+1+x^2)\sum_{n \text{ odd}} b_n x^{3n} + 2(x^{-1}+x)\sum_{n \text{ even}} b_n x^{3n} \end{aligned}$$

and

$$\begin{aligned} B_e(x) = \sum_{n \text{ even}} b_n x^n &= \sum_{n \text{ even}} b_{3n}x^{3n} + \sum_{n \text{ odd}} b_{3n+1}x^{3n+1} + \sum_{n \text{ even}} b_{3n+2}x^{3n+2} \\ &= (x^{-2}+1+x^2)\sum_{n \text{ even}} b_n x^{3n} + (x^{-1}+x)\sum_{n \text{ odd}} b_n x^{3n}. \end{aligned}$$

Although we do not immediately obtain an infinite product form for the generating function (as in the case of the Stern sequence), we will see that this is possible for a slight modification of our sequence. Rewriting the above in matrix form, we have

$$\begin{pmatrix} B_o(x) \\ B_e(x) \end{pmatrix} = \begin{pmatrix} x^{-2}+1+x^2 & 2(x^{-1}+x) \\ (x^{-1}+x) & x^{-2}+1+x^2 \end{pmatrix} \begin{pmatrix} B_o(x^3) \\ B_e(x^3) \end{pmatrix}.$$

The matrix $\begin{pmatrix} x^{-2}+1+x^2 & 2(x^{-1}+x) \\ (x^{-1}+x) & x^{-2}+1+x^2 \end{pmatrix}$ has an eigenvector $\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$ with eigenvalue $(x^{-2}+\sqrt{2}x^{-1}+1+\sqrt{2}x+x^2)$, so we obtain the relation

$$\sqrt{2}B_o(x)+B_e(x) = (x^{-2}+\sqrt{2}x^{-1}+1+\sqrt{2}x+x^2)(\sqrt{2}B_o(x^3)+B_e(x^3)),$$

from which we obtain the infinite product representation

$$\sqrt{2}B_o(x)+B_e(x) = \prod (x^{-6n}+\sqrt{2}x^{-3n}+1+\sqrt{2}x^{3n}+x^{6n}).$$

The sequence obtained from $\{b_n\}$ by multiplying the odd terms by $\sqrt{2}$ is what Northshield denotes $\{b_n\}$ in [17], in which many properties of the sequence are proved, including an infinite product representation in Section 4. Our $\{b_n\}$ are A277750 in [18], which appear as the denominators of Northshield's R_n.

Dilcher and Stolarsky have considered a polynomial version of the Stern sequence in [6]. The ECF Stern sequence can be similarly generalized, by setting $b(0,x)=0$, $b(1,x)=1$, $b(2,x)=1$, and

$$\begin{aligned} b(3n,x) &= b(n,x^4), \\ b(3n+1,x) &= \begin{cases} (1+x)b(n,x^4)+x^3b(n+1,x^4) & \text{if } n \text{ is even} \\ b(n,x^4)+x^2b(n+1,x^4) & \text{if } n \text{ is odd,} \end{cases} \\ a(3n+2,x) &= \begin{cases} b(n,x^4)+x^2b(n+1,x^4) & \text{if } n \text{ is even} \\ b(n,x^4)+(x^2+x)b(n+1,x^4) & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

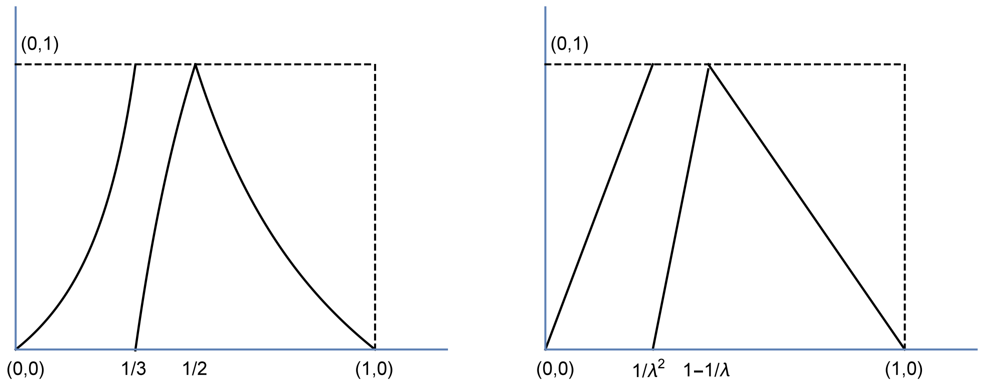

FIGURE 4. The odd Farey map F_O and its linearization $\overline{F}_O$

The above relations are derived from replacing the mediant construction with the polynomial version used by Dilcher and Stolarsky. It is immediate from the definition that $b(n,1)$ recovers the ECF Stern seqence, and that is a $b(n,x)$ has coefficients in $\{0,1\}$. It would be interesting to find a combinatorial interpretation of the ECF Stern sequence or its polynomial generalization.

3. Odd Partial Quotients

3.1. **Odd continued fraction generation and ordering of rational numbers.** In this section, we consider the OCF in $[-1,1]$ given by

$$[(e_1,a_1),(e_2,a_2),(e_3,a_3),...] = \cfrac{e_1}{a_1+\cfrac{e_2}{a_2+\cfrac{e_3}{a_3+\cdots}}}, \tag{3.1}$$

where $e_i \in \{\pm 1\}$, a_i is odd, $e_1 = 1$, and $a_i + e_{i+1} > 0$. For uniqueness of representations, we require that in a finite expansion, if the last $a_j = 1$, then $e_j = 1$.

3.2. **The odd Farey type map F_O and the odd Gauss map T_O.** We consider the Farey type map $F_O : [0,1] \to [0,1]$ associated to OCF expansions and defined by

$$F_O(x) = \begin{cases} \frac{x}{1-2x} & \text{if } 0 \le x < \frac{1}{3} \\ 3 - \frac{1}{x} & \text{if } \frac{1}{3} < x \le \frac{1}{2} \\ \frac{1}{x} - 1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases} \tag{3.2}$$

Symbolically, F_O acts on the OCF representation by subtracting 2 from the leading digit a_1 of x when $(a_1,e_2) \neq (3,-1)$ and $(a_1,e_1) \neq (1,1)$ (which correspond to x between 0 and $\frac{1}{3}$), and by simply removing (a_1,e_2) when $(a_1,e_2) \in \{(3,-1),(1,1)\}$ (which corresponds to x between $\frac{1}{3}$ and 1), i.e.

$$F_O([(1,a_1),(e_2,a_2),(e_3,a_3),\ldots]) = \begin{cases} [(1,a_1-2),(e_2,a_2),(e_3,a_3)] & \text{if } (a_1,e_2) \notin \{(3,-1),(1,1)\} \\ [(1,a_2),(e_3,a_3),(e_4,a_4),\ldots] & \text{if } (a_1,e_2) \in \{(3,-1),(1,1)\}. \end{cases}$$

The following result follows from direct verification:

Lemma 6. *The infinite measure $d\nu_O(x) = \frac{1}{x} + \frac{1}{G+1-x}$ is F_O-invariant.*

The induced transformation R_O of F_O on $[\frac{1}{3},1]$ acts on the OCF expansion in $[\frac{1}{3},1]$ as:

$$R_O\big([(1,a_1),(e_2,a_2),(e_3,a_3),\ldots]\big) = [(1,a_1),(e_3,a_3),(e_4,a_4),\ldots],$$

where $(a_1, e_2) \in \{(3,-1),(1,1)\}$. Recall that the odd Gauss map T_O acts on $[0,1]$ by

$$T_O(x) = \begin{cases} 2k+1-\frac{1}{x} & \text{if } x \in [\frac{1}{2k+1}, \frac{1}{2k}] \\ \frac{1}{x}-(2k-1) & \text{if } x \in [\frac{1}{2k}, \frac{1}{2k-1}], \end{cases}$$

and it acts on OCF expansions (3.1) restricted to $[0,1]$ by

$$T_O\big([(1,a_1),(e_2,a_2),(e_3,a_3),\ldots]\big) = [(1,a_2),(e_3,a_3),(e_4,a_4),\ldots].$$

Recall also that $d\mu_O(x) = (\frac{1}{G-1+x} + \frac{1}{G+1-x})dx$ is a finite T_O-invariant measure [24].

We will consider instead the map $\widetilde{T}_O : [-1,1] \to [-1,1]$ acting on the OCF expansion (3.1) in $[-1,1]$ as:

$$\widetilde{T}_O\big([(e_1,a_1),(e_2,a_2),(e_3,a_3),\ldots]\big) = [(e_2,a_2),(e_3,a_3),(e_4,a_4),\ldots].$$

Equivalently $\widetilde{T}_O(x) = e_2 T_O(x)$ if $x>0$ and $\widetilde{T}_O$ is an even function, that is

$$\widetilde{T}_O(x) = \begin{cases} \frac{1}{x}-1 & \text{if } y \in (\frac{1}{2},1) \\ \frac{1}{x}-(2k-1) & \text{if } x \in (\frac{1}{2k}, \frac{1}{2k-2}],\ k \geq 2 \\ \widetilde{T}_O(-x) & \text{if } x \in (-1,0). \end{cases}$$

It is plain to check that R_O and $\widetilde{T}_O$ are conjugated, or more precisely $\widetilde{T}_O = \psi R_O \psi^{-1}$, where

$$\psi(x) = \begin{cases} \frac{1}{x}-1 & \text{if } x \in (\frac{1}{2},1) \\ \frac{1}{x}-3 & \text{if } x \in (\frac{1}{3},\frac{1}{2}) \end{cases} \quad \text{and} \quad \psi^{-1}(y) = \begin{cases} \frac{1}{3+y} & \text{if } -1<y<0 \\ \frac{1}{1+y} & \text{and } 0<y<1. \end{cases}$$

The push-forward $\widetilde{\mu}_O$ of the measure $\nu_O|_{[1/3,1]}$ by ψ yields a $\widetilde{T}_O$-invariant measure given by

$$\begin{aligned} \int_{1/3}^{1} f\big(\psi(x)\big)\,\frac{dx}{x(G+1-x)} &= \int_{1/3}^{1/2} f\left(\frac{1}{x}-3\right)\frac{dx}{x(G+1-x)} + \int_{1/2}^{1} f\left(\frac{1}{x}-1\right)\frac{dx}{x(G+1-x)} \\ &= \int_{-1}^{1} f(y)\,d\widetilde{\mu}_O(y), \end{aligned}$$

that is

$$d\widetilde{\mu}_O(y) = \frac{1}{G^2}\cdot\frac{dy}{y+G+1}\,\chi_{[-1,0]} + \frac{1}{G^2}\cdot\frac{dy}{y+G-1}\,\chi_{[0,1]}.$$

Again, $\widetilde{T}_O$ is an extension of T_O with $\pi\widetilde{T}_O = G_O\pi$, where $\pi : [-1,1]\to[0,1]$, $\pi(x)=|x|$. The push forward of $\widetilde{\mu}_O$ under π is the T_O-invariant measure μ_O.

The map $\widetilde{T}_O$ coincides with the map T introduced and investigated by Rieger in Chapters 2 and 3 of [21]. Note also that $G d\widetilde{\mu}_O(y) = d\rho(y)$, where ρ is the T-invariant measure considered in [21, Theorem 6.1].

3.3. **The odd Minkowski type question mark function Q_O.** Let λ be the unique real root of $x^3-x^2-x-1=0$. Following [25], we define the map

$$Q_O(x) = \sum_{k=1}^{\infty} \frac{-\Pi_{i=1}^{k}(-e_i)}{\lambda^{\sum_{i=1}^{k} a_i}}, \tag{3.3}$$

which coincides with Zhabitskaya's $F^0(x)$.

Theorem 4. *Q_O is is Hölder continuous, with best exponent $\frac{\log\lambda}{2\log G}$.*

In preparation for this result, we need two preliminary facts about the (ordered) set

$$\mathcal{Y}_n := \big\{x \in \mathbb{Q}\cap[0,1] : x = [(e_1,a_1),(e_2,a_2),\ldots,(e_k,a_k)] \text{ and } a_1+\cdots+a_k \leq n+1\big\}.$$

Note that in this section we use the same notation $\mathcal{Y}_n$ and $\mathcal{X}_n$ as in Section 2, but now they denote odd continued fraction analogues. The facts that we need follow from the structure of the analogue of the Stern-Brocot tree for odd continued fractions, which we denote $\mathcal{D}$, as in [25].

Proposition 7. *For a reduced fraction* $\frac{p}{q} \in \mathcal{Y}_n$,

$$q \leq G^{n+2}.$$

Proof. In fact, the largest denominator in $\mathcal{Y}_n$ is given by the $n+2$th Fibonacci number. This can be directly verified for the first few n, and follows inductively from the fact that every element of $\mathcal{X}_{n+1} := \mathcal{Y}_{n+1} \setminus \mathcal{Y}_n$ is the mediant of two adjacent elements of $\mathcal{Y}_n$. Since no two elements of $\mathcal{X}_{n+1}$ are adjacent in $\mathcal{Y}_{n+1}$, the largest denominator in Y_{n+2} is at most the sum of the largest denominator in Y_{n+1} and the largest denominator in Y_n. Since this recurrence relation is in fact satisfied by the convergents of $[(1,1),(1,1),(1,1),...]$, we obtain the stated (sharp) upper bound for the denominators. □

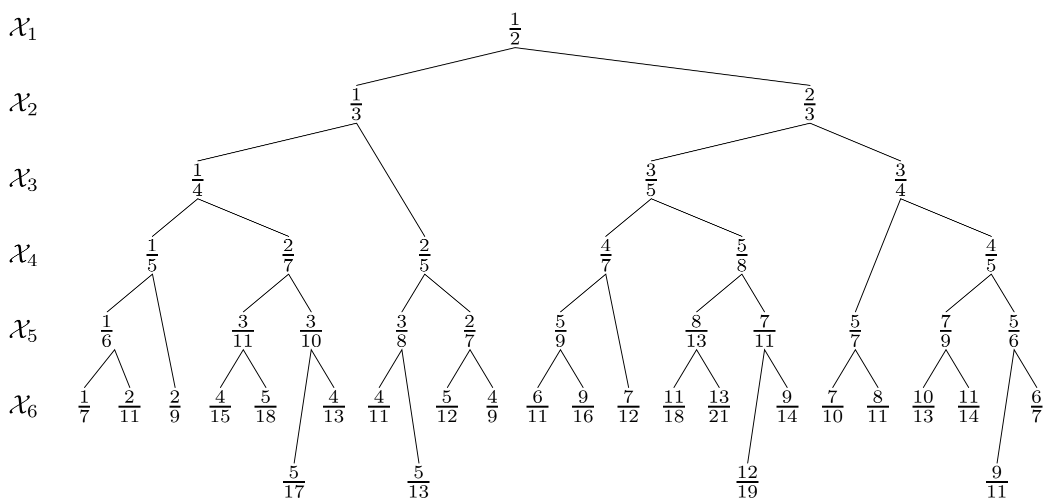

FIGURE 5. Zhabitskaya's odd Farey tree

Proposition 8. *There exists an absolute constant C such that if x and y are adjacent elements of $\mathcal{Y}_n$, then*

$$|Q_O(x) - Q_O(y)| \leq C\lambda^{-n}.$$

Proof. First, suppose that $y \in \mathcal{X}_n$. We have already noted that no two elements of $\mathcal{X}_n$ are adjacent in $\mathcal{Y}_n$, so it must be the case that $y \in \mathcal{X}_n$ is a descendant of x, in the sense that it is obtained from x by (perhaps repeatedly) taking mediants. Suppose $x = [(e_1,a_1),(e_2,a_2)...,(e_j,a_j)]$. There are three possible "moves" in the tree $\mathcal{D}$, each corresponding to a possible relationship between an element $x \in \mathcal{X}_k$ and its descendant in $\mathcal{X}_{k+1}$ or $\mathcal{X}_{k+2}$. The first type of move is appending $(1,1)$ to the tail of the continued fraction of x. The second (possible only when $a_j > 1$) is appending $(-1,1),(1,1)$ to the tail, and the third (possible only when $a_j = 1$) is to remove $(e_j,a_j) = (1,1)$ and replace (e_{j-1},a_{j-1}) with $(e_{j-1},a_{j-1}+2)$. Suppose we call a move (of any of the three types) a left move if the result is less than the input, and a right move if the result is greater than the input. Not only is y obtained from x by a series of these moves, but since y is adjacent to x, it must be obtained either by a right move followed by only left moves, or a left move followed by only right moves. Note that moves of the first type will be right moves iff $\Pi_{i=1}^{j}(-e_i) = 1$, and hence moves of the second or third types are left moves in this case. Note also that each move has the end result of switching the sign of the product of the $-e_i$. We now consider three cases:

Case 1. If the first move is of the first type, then the second move must be as well, in order to switch direction. Subsequent moves must all have the same direction as the second, so they must alternate between type three moves (since the type one moves leave $(1,1)$ as the last term) and type one moves. In this case, the continued fraction of y is of the form $[(e_1,a_1),(e_2,a_2)...,(e_j,a_j),(1,1+2k)]$ or $[(e_1,a_1),(e_2,a_2)...,(e_j,a_j),(1,1+2k),(1,1)]$.

Case 2. If the first move is of second type, then the second move must be of third type, after which it must alternate between first type and third type. Hence the continued fraction of y is of the form $[(e_1,a_1),(e_2,a_2)...,(e_j,a_j),(-1,1+2k)]$ or $[(e_1,a_1),(e_2,a_2)...,(e_j,a_j),(-1,1+2k),(1,1)]$.

Case 3. If the first move is of third type, then the second move must be of second type, after which it must alternate between first type and third type. Hence the contined fraction of y is of the form $[(e_1,a_1),(e_2,a_2)...,(e_{j-1},a_{j-1}+2),(-1,1),(1,1+2k)]$ or $[(e_1,a_1),(e_2,a_2)...,(e_{j-1},a_{j-1}+2),(-1,1),(1,1+2k),(1,1)]$.

In any case, what we need is that fact that $|Q_O(x)-Q_O(y)|\leq C\lambda^{-\sum_{i=1}^{j}a_i}\lambda^{-2k-1}$, and the consequence that since $y\in\mathcal{X}_n$, then

$$|Q_O(x)-Q_O(y)|\leq C\lambda^{-n}.$$

For the first two cases this is an immediate consequence of the forumla for Q_O and the possible continued fractions for y. In the third case, we note that

$$|Q_O(x)-Q_O(y)|=\lambda^{-\sum_{i=1}^{j-1}a_i}|(1-\lambda^{-1})-(\lambda^{-2}+\lambda^{-3}-\lambda^{-4-2k}+\lambda^{-4-2k-1})|$$

From the definition of λ, $1-\lambda^{-1}-\lambda^{-2}-\lambda^{-3}=0$, so

$$|Q_O(x)-Q_O(y)|\leq 2\lambda^{-2k-2}\lambda^{-\sum_{i=1}^{j-1}a_i}\leq 2\lambda^{-\sum_{i=1}^{j}a_i}\lambda^{-2k-1}.$$

Essentially, what we have used in the third case is that $Q_O(x)$ does not depend on the representation of x. Although we have adopted a convention that if the last $a_j=1$ then we require $e_j=1$, the formula for Q_O gives the same results for $[(e_1,a_1),(e_2,a_2)...,(e_{j-1},a_{j-1}),(1,1)]$ and the equivalent $[(e_1,a_1),(e_2,a_2)...,(e_{j-1},a_{j-1}+2),(-1,1)]$, as a consequence of the definition of λ.

Finally, by increasing the constant C by a factor of λ, we may remove our initial assumption that $y\in\mathcal{X}_n$, since given any two adjacent elements of $\mathcal{Y}_n$, at least one of them must be in $\mathcal{X}_n$ or $\mathcal{X}_{n-1}$. ☐

We are now ready to prove Theorem 4, in much the same manner as Theorem 2.

Proof of Theorem 4. Suppose $x<x'\in[0,1]$. Let $y=Q_O(x)$ and $y'=Q_O(x')$. Let k be the least integer such that we have $x\leq r\leq r'\leq x'$ for some $r,r'\in\mathcal{Y}_k$. The bound from proposition 7 gives $x'-x\geq r'-r\geq G^{-2k}$. Since we have taken k to be the least possible, there are at most 3 elements of $\mathcal{Y}_k$ in the interval $[x,x']$, so $y'-y\leq 5C\lambda^{-k}$. We have $2k\log G\geq\log\frac{1}{x'-x}$ and $(k-\log_\lambda 3C)\log\lambda\leq\log\frac{1}{y'-y}$, which together give

$$y'-y<C'(x'-x)^{\frac{\log\lambda}{2\log G}}.$$

To see that this is best possible, consider $x=G-1=[(1,1),(1,1),(1,1)...]$ and its convergents. If $x_n=[\underbrace{(1,1),(1,1),...,(1,1)}_{n\text{ times}}]$ then $|x-x_n|$ is of the order G^{-2n}. On the other hand,

$$|Q_O(x)-Q_O(x_n)|=\sum_{k=n}^{\infty}\frac{-\Pi_{i=1}^{k}(-e_i)}{\lambda^{\sum_{i=1}^{k}1}}$$

is of order λ^{-n}. Since $|x-x_n|^{\frac{\log\lambda}{2\log G}}$ is of order $(G^{-2n})^{\frac{\log\lambda}{2\log G}}=\lambda^{-n}$, we conclude that this is the best possible exponent. □

3.4. **The linearization of the map F_O.** With notation as in (3.1), formula (3.3) provides

$$Q_O(x)=\frac{1}{\lambda^{a_1-1}}-\frac{e_1}{\lambda^{a_1+a_2-1}}+\frac{e_1e_2}{\lambda^{a_1+a_2+a_3-1}}-\cdots \tag{3.4}$$

Consider the piecewise linear maps $\overline{F}_O,\overline{T}_O:[0,1]\to[0,1]$ defined by

$$\overline{F}_O(y)=\begin{cases}\lambda^2 y & \text{if } y\in[0,\lambda^2]\\ \lambda(\lambda^2 y-1) & \text{if } y\in[\lambda^2,1-1\frac{1}{\lambda}]\\ \lambda(1-y) & \text{if } y\in[1-\frac{1}{\lambda},1],\end{cases}$$

$$\overline{T}_O(y)=\begin{cases}\lambda-\lambda^{2k-1}y & \text{if } y\in(\frac{\lambda-1}{\lambda^{2k-1}},\frac{1}{\lambda^{2k-2}}),\ k\geq 1\\ \lambda^{2k-1}y-\lambda & \text{if } y\in(\frac{1}{\lambda^{2k-2}},\frac{\lambda-1}{\lambda^{2k+1}}),\ k\geq 2.\end{cases}$$

Lemma 9. *The homeomorphism Q_O of $[0,1]$ linearizes the maps F_O and T_O as follows:*
(i) $Q_O^{-1}F_OQ_O=\overline{F}_O$.
(ii) $Q_O^{-1}T_OQ_O=\overline{T}_O$.

Proof. Let $x=[(1,a_1),(e_1,a_2),(e_2,a_3),\ldots]\in[0,1]$ be as in (3.1).

(i) There are three cases that have to be considered:

Case 1. $x\in(0,\frac{1}{3})$, where $(a_1,e_1)\notin\{(3,-1),(1,1)\}$. The we successively infer:

$$\begin{aligned}Q_O(x)&=\frac{1}{\lambda^{a_1-1}}-\frac{e_1}{\lambda^{a_1+a_2-1}}+\frac{e_1e_2}{\lambda^{a_1+a_2+a_3-1}}-\cdots\in\left(0,\frac{1}{\lambda^2}\right),\\ (Q_OF_O)(x)&=Q_O\big([(1,a_1-2),(e_1,a_2),(e_2,a_3),\ldots]\big)\\ &=\frac{1}{\lambda^{a_1-3}}-\frac{e_1}{\lambda^{a_1+a_2-3}}+\frac{e_1e_2}{\lambda^{a_1+a_2+a_3-3}}-\cdots=\lambda^2Q_O(x)=(\overline{F}_OQ_O)(x).\end{aligned}$$

Case 2. $x\in(\frac{1}{3},\frac{1}{2})$, so $a_1=3$, $e_1=-1$ and we have:

$$\begin{aligned}Q_O(x)&=\lambda\left(\frac{1}{\lambda^{a_1}}-\frac{e_1}{\lambda^{a_1+a_2}}+\frac{e_1e_2}{\lambda^{a_1+a_2+a_3}}-\cdots\right)\\ &=\lambda\left(\frac{1}{\lambda^3}+\frac{1}{\lambda^{a_2+3}}-\frac{e_2}{\lambda^{a_2+a_3+3}}+\frac{e_2e_3}{\lambda^{a_2+a_3+a_4+3}}-\cdots\right)\\ &=\frac{1}{\lambda^2}+\frac{1}{\lambda^{a_2+2}}-\frac{e_2}{\lambda^{a_2+a_3+2}}+\frac{e_2e_3}{\lambda^{a_2+a_3+a_4+2}}-\cdots\in\left(\frac{1}{\lambda^2},1-\frac{1}{\lambda}\right),\\ (\overline{F}_OQ_O)(x)&=\lambda(\lambda^2Q_O(x)-1)=\lambda\left(\frac{1}{\lambda^{a_2}}-\frac{e_2}{\lambda^{a_2+a_3}}+\frac{e_2e_3}{\lambda^{a_2+a_3+a_4}}-\cdots\right)\\ &=Q_Q\big([(1,a_2),(e_2,a_3),(e_3,a_4),\ldots]\big)=(Q_OF_O)(x).\end{aligned}$$

Case 3. $x\in(\frac{1}{2},1)$, so $a_1=e_1=1$ and we have:

$$\begin{aligned}Q_O(x)&=1-\frac{1}{\lambda^{a_2}}+\frac{e_2}{\lambda^{a_2+a_3}}-\frac{e_2e_3}{\lambda^{a_2+a_3+a_4}}-\cdots,\\ (\overline{F}_OQ_O)(x)&=\lambda\big(1-Q_O(x)\big)=\frac{1}{\lambda^{a_2-1}}-\frac{e_2}{\lambda^{a_2+a_3-1}}+\frac{e_2e_3}{\lambda^{a_2+a_3+a_4-1}}-\cdots\\ &=(Q_OF_O)(x).\end{aligned}$$

(ii) Note first that

$$\begin{aligned} Q_O(x) &= \frac{1}{\lambda^{a_1-1}} - \frac{1}{\lambda^{a_1+a_2-1}} + \frac{e_2}{\lambda^{a_1+a_2+a_3-1}} - \frac{e_2e_3}{\lambda^{a_1+a_2+a_3+a_4-1}} + \cdots , \\ (Q_OT_O)(x) &= \frac{1}{\lambda^{a_2-1}} - \frac{e_2}{\lambda^{a_2+a_3-1}} + \frac{e_2e_3}{\lambda^{a_2+a_3+a_4-1}} - \cdots , \end{aligned}$$

and for every $k \in \mathbb{N}$ we have

$$Q_O\left(\frac{1}{2k-1}\right) = \frac{1}{\lambda^{2k-2}} \quad \text{and} \quad Q_O\left(\frac{1}{2k}\right) = \frac{1}{\lambda^{2k-2}} - \frac{1}{\lambda^{2k-1}} = \frac{\lambda-1}{\lambda^{2k-1}}.$$

Two situations can occur:
Case 1. $x \in (\frac{1}{2k}, \frac{1}{2k-1})$, $k \geq 1$, so $a_1 = 2k-1$, $e_1 = 1$ and we have

$$\begin{aligned} Q_O\left(\frac{1}{2k}\right) &= \frac{\lambda-1}{\lambda^{2k-1}} < Q_O(x) < Q_O\left(\frac{1}{2k-1}\right) = \frac{1}{\lambda^{2k-2}}, \\ (\overline{T}_OQ_O)(x) &= \lambda - \lambda^{2k-1}Q_O(x) \\ &= \lambda - \lambda^{2k-1}\left(\frac{1}{\lambda^{2k-2}} - \frac{1}{\lambda^{2k-2+a_2}} + \frac{e_2}{\lambda^{2k-2+a_2+a_3}} - \cdots\right) = (Q_OT_O)(x). \end{aligned}$$

Case 2. $x \in (\frac{1}{2k-1}, \frac{1}{2k-2})$, $k \geq 2$, so $a_1 = 2k-1$, $e_1 = -1$ and we have

$$\begin{aligned} Q_O\left(\frac{1}{2k-1}\right) &= \frac{1}{\lambda^{2k-2}} < Q_O(x) < Q_O\left(\frac{1}{2k-2}\right) = \frac{\lambda-1}{\lambda^{2k-3}} = \frac{\lambda+1}{\lambda^{2k-1}}, \\ (\overline{T}_OQ_O)(x) &= \lambda^{2k-1}Q_O(x) - \lambda \\ &= \lambda^{2k-1}\left(\frac{1}{\lambda^{2k-2}} + \frac{1}{\lambda^{2k-2+a_2}} - \frac{e_2}{\lambda^{2k-2+a_2+a_3}} + \cdots\right) - \lambda = (Q_OT_O)(x). \qquad \square \end{aligned}$$

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