PASCAL TRIANGLE AND RESTRICTED WORDS

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Abstract. We continue to investigate combinatorial properties of functions f_m and c_m considered in our previous papers. They depend on an initial arithmetic function f_0. In this paper, the values of f_0 are the binomial coefficients.

We first consider the case when the values of f_0 are the binomial coefficients from a row of the Pascal triangle. The values of f_0 consider next are the binomial coefficients from a diagonal of the Pascal triangle. In two final cases, the values of f_0 are the central binomial coefficients and its adjacent neighbors. In each case, we derive an explicit formula for $c_1(n, k)$ and give its interpretation in terms of restricted words. In the first two cases, we also consider the functions f_m and c_m, for $(m > 0)$.

1. Introduction

In this paper, we continue to investigate the problem of the enumeration of restricted words. Previously, the functions $f_m(n)$ and $c_m(n, k)$ were defined as follows. For an initial arithmetic function f_0, $f_m(m > 0)$ is the mth invert transform of f_0. The function $c_m(n, k)$ was defined in the following way:

$$c_m(n,k) = \sum_{i_1+i_2+\cdots+i_k=n} f_{m-1}(i_1)\cdot f_{m-1}(i_2)\cdots f_{m-1}(i_k), \tag{1}$$

where the sum is over positive $i_1, i_2, \ldots, i_k$. Its connections with the problem of the enumeration of restricted words were consider. A number of results of this kind is obtained in Janjić [4, 5, 6]. Some results have also been considered by other authors, for instance, in [1, 2, 3, 7].

In this paper, we derive results for four types of initial functions, which have different kind of the binomial coefficients as values. In the first case, the values of f_0 are the binomial coefficients from a row of the Pascal triangle. In the second case, the values are from a diagonal of the Pascal triangle. In the remaining cases, the values of f_0 are the central binomial coefficients and its adjacent neighbors.

In the first two cases, we describe the restricted words counted by f_m and c_m, and derive explicit formulas for these functions. In the last two cases, we consider the function c_1 only.

2. Rows of the Pascal triangle

For a positive integer a, we want to enumerate words over the finite alphabet $\{0, 1, \ldots, a-1, \ldots\}$, in which letters in words over $\{0, 1, \ldots, a-1\}$ are arranged into ascending order. We let $\mathcal{P}_1$ denote this property.

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We define

$$f_0(n) = \binom{a}{n-1}, (n = 1, 2, \ldots).$$

It is clear that $f_0(n)$ is the number of words of length $n-1$ over the alphabet $\{0, 1, \ldots, a-1\}$ satisfying $\mathcal{P}_1$. Since $f_0(1) = 1$, using Janjić [6, Proposition 12], we obtain the following combinatorial meaning of $c_m(n, k)$.

Corollary 1. The number of words of length $n-1$ over the alphabet $\{0, 1, \ldots, a-1, \ldots, a+m-1\}$ having $k-1$ letters equal to $a+m-1$ and satisfying $\mathcal{P}_1$ is $c_m(n, k)$.

According to [6, Corollary 2], we get

Corollary 2. The number of words of length $n-1$ over the alphabet $\{0, 1, \ldots, a-1, \ldots, a+m-1\}$ satisfying $\mathcal{P}_1$ is $f_m(n)$.

We next derive an explicit formula for $c_1(n, k)$.

Proposition 1. We have

$$c_1(n, k) = \binom{ak}{n-k}.$$

Proof. We use the induction on k. From Janjić [5, Equation (3)], we have $c_1(n, 1) = f_0(n)$, which means that the statement holds for $k = 1$. Suppose that the statement is true for $k-1, (k > 1)$. Using the recurrence [6, Equation (3)], and the induction hypothesis, we obtain

$$c_1(n, k) = \sum_{i=1}^{n-k+1} \binom{a}{i-1} \binom{ak-a}{n-i-k+1},$$

and the statement follows from the Vandermonde convolution. □

As a consequence of [6, Equation (9), Proposition 7], we obtain the following explicit formulas for $c_m(n, k)$ and f_m.

Corollary 3. We have

$$c_m(n, k) = \sum_{i=k}^{n} (m-1)^{i-k} \binom{i-1}{k-1} \binom{ia}{n-i},$$

$$f_m(n) = \sum_{i=1}^{n} m^{i-1} \binom{ia}{n-i}.$$

Corollary 4. In the case $a = 2$, the sequence $f_1(1), f_1(2), \ldots$ is seqnumA002478, which is the bisection of the Narayana's cows sequence (seqnumA000930).

3. Diagonals of Pascal triangle

The following problem which we investigate is: For a positive integer a, find the numbers of words over the alphabet $\{0, 1, \ldots, a-1, \ldots\}$ such that subwords from $\{0, 1, \ldots, a-1\}$ have no rises. We let $\mathcal{P}_2$ denote this property. We show that for this problem, the values of the initial function are figurate numbers, that is, the numbers on a diagonal of the Pascal triangle. We let $d(n-1, a)$ denote the number words length $n-1$ satisfying $\mathcal{P}_2$.

Proposition 2. The following equation holds:

$$d(n-1,a)=\binom{n+a-2}{a-1}. \tag{2}$$

Proof. We first have $d(0,a)=1$, since the empty word has no a rise. Assume that $n>1$. The following recurrence holds:

$$d(n,a+1)=d(n,a)+d(n-1,a+1),(n>0).$$

Namely, each word of length $n-1$ over $\{0,1,\ldots,a-1,a\}$ that has no rises may begin with a letter from $\{0,1,\ldots,a-1\}$. The letter a does not appear in a word, which yields that there are $d(n-1,a)$ such words. There remains the words of length $n-1$ beginning with a. Obviously, there are $d(n-2,a+1)$ such words. It follows that

$$\begin{array}{rcl} d(n-1,a+1)-d(n-2,a+1) & = & d(n-1,a), \\ d(n-2,a+1)-d(n-3,a+1) & = & d(n-2,a), \\ & \vdots & \\ d(1,a+1)-d(0,a+1) & = & d(1,a). \end{array}$$

Adding expressions on the left-hand sides and the right-hand sides, we obtain the following recurrence:

$$d(n-1,a+1)=\sum_{i=1}^{n} d(i-1,a). \tag{3}$$

To prove (2), we use induction on a. If $a=1$, then $d(n-1,1)=1$, since the alphabet consists of the empty word. Assume that the statement holds for $a\geq 1$. Then (3) takes the form

$$d(n-1,a+1)=\sum_{i=1}^{n}\binom{i+a-2}{a-1},$$

and the statement holds according to the horizontal recurrence for the binomial coefficients. □

Therefore, in the case $f_0(n)=\binom{n+a-2}{a-1},(n=1,2,\ldots)$, since $f_0(1)=1$, using Janjić [6, Proposition 12], we obtain

Corollary 5. The number $c_m(n,k),(m\geq 1)$ is the number of words of length $n-1$ over the alphabet $\{0,1,\ldots,a-1,\ldots,a+m-1\}$ having $k-1$ letters equal to $a+m-1$ and satisfying $\mathcal{P}_2$.

According to [6, Corollary 2], we get

Corollary 6. The number $f_m(n)$ equals the number of words of length $n-1$ over the alphabet $\{0,1,\ldots,a-1,\ldots,a+m-1\}$ satisfying $\mathcal{P}_2$.

To derive an explicit formula for $c_1(n,k)$, we need the following lemma.

Lemma 1. Let $u\geq v\geq w\geq 1$ be integers. Then

$$\binom{u}{v}=\sum_{i=w}^{u-v+w}\binom{i-1}{w-1}\binom{u-i}{v-w}. \tag{4}$$

Proof. We know that $\binom{u}{v}$ is the number of binary words of length u with v zeros. We let i denote the position of the wth zero in a word. It is clear that $w \le i \le u-v+w$. For a fixed i, the number of words is $\binom{i-1}{w-1}\binom{u-i}{v-w}$. Summing over all i, we obtain (4). □

Note 1. The identity (4) generalizes the horizontal recurrence for the binomial coefficients, which we obtain for either $w = 1$ or $w = v$.

Next, we derive a formula for $c_1(n,k)$.

Proposition 3. We have

$$c_1(n,k) = \binom{n+ak-k-1}{ak-1}.$$

Proof. We use induction on k. From [6, Equation (3)], we have

$$c_1(n,1) = f_0(n) = \binom{n+a-2}{a-1},$$

which means that the statement is true for $k = 1$ Using induction, we conclude that the statement is equivalent to the following binomial identity:

$$\binom{n+ak-k-1}{ak-1} = \sum_{i=1}^{n-k+1} \binom{i+a-2}{a-1}\binom{n+ak-k-a-i}{ak-a-1}.$$

We prove that this identity follows from Identity 4. Namely, taking $w+1$ instead of w in Identity 4, and then replacing $i-1$ by j yields

$$\binom{u}{v} = \sum_{j=w}^{u-v+w} \binom{j}{w}\binom{u-j-1}{v-w-1}.$$

Taking, in particular, $w = a-1$ and replacing j by $i+a-2$ implies

$$\binom{u}{v} = \sum_{i=1}^{u-v+1} \binom{i+a-2}{a-1}\binom{u-1-i}{v-a}.$$

Finally, taking $u = n+ak-k-1, v = ak-1$, we obtain the desired result. □

Corollary 7. The following formulas holds:

(1)
$$c_m(n,k) = \sum_{i=k}^{n} (m-1)^{i-k}\binom{i-1}{k-1}\binom{n+ai-i-1}{ai-1}.$$

(2)
$$f_m(n) = \sum_{k=1}^{n} m^{i-1}\binom{n+ai-i-1}{ai-1}.$$

Proof. The proof follows from [6, Equation (9), Proposition 7]. □

We state some particular cases. Note that the case $a = 2$ was considered in [6, Example 31].

Example 1. In the case $a = 3$, we have $f_0(n) = \binom{n+1}{2}, (n \ge 1)$. Hence, $f_0(n)$ is the nth triangular number. We thus obtain

(1) The number $\binom{n+2k-1}{3k-1}$ is the number of ternary words of length $n-1$ having $k-1$ letters equal to 2, and avoiding $01, 02, 12$.

(2) The number

$$\sum_{i=k}^{n}(m-1)^{i-k}\binom{i-1}{k-1}\binom{n+2i-1}{3i-1}$$

is the number of words of length $n-1$ over the alphabet $\{0,1,\ldots,m+2\}$ having $k-1$ letters equal to $m+2$ and avoiding $01,02,12$.

(3) The number

$$\sum_{i=1}^{n} m^{i-1}\binom{n+2i-1}{3i-1}$$

is the number of words of length $n-1$ over the alphabet $\{0,1,\ldots,m+2\}$ avoiding $01,02,12$.

This case is also related with the enumeration of some Dyck paths. Namely, the sequence $f_0(1), f_0(2),\ldots$ makes the second column of the Narayana triangle. It follows that $f_0(n)$ is the number of Dyck paths of semilength $n+1$ having exactly two peaks. In the formula

$$c_1(n,k)=\sum_{i_1+i_2+\cdots+i_k=n} f_0(i_1)\cdot f_0(i_2)\cdots f_0(i_k),$$

the product $f_0(i_1)\cdot f_0(i_2)\cdots f_0(i_k)$ equals the number of Dyck paths of semilength $n+k$ obtained by the concatenation of k Dyck paths with exactly two peaks. We thus obtain the following Euler-type identity.

Identity 1. The following sets have the same number of elements

(1) The set of quaternary words of length $n-1$ in which $k-1$ letters equal 3 and which avoid $01,02,12$.

(2) The set Dyck paths of semilength $n+k$ obtained by the concatenation of k Dyck paths with exactly two peaks.

4. Central binomial coefficients ant its adjacent neighbors

In two concluding examples, we give combinatorial properties for $c_1(n,k)$ only.

We start with a slight generalization of [6, Proposition 12]. Let W be a set of words with a property $\mathcal{P}$, over a finite alphabet α. Assume that the empty word has the property $\mathcal{P}$. For a positive integer i, we denote by W_{i-1} the set of words of length $l(i-1)$. In particular, $W_0=\emptyset$ yields $l(0)=0$. We let $f_0(i)$ denote the number of words from W_{i-1}. In particular, we have $f_0(1)=1$. Consider the equation $i_1+i_2+\cdots+i_k=n, (i_t>0, t=1,2,\ldots,k)$. For $x\notin\alpha$, we want to count words from the alphabet $\alpha\cup\{x\}$ of the form:

$$w_{i_1-1}, x, w_{i_2-1}, x, \ldots w_{i_{k-1}-1}, x, w_{i_k-1},$$

where $w_{i_t-1}\in W_{i_t-1}, (t=1,2,\ldots,k)$. We let N denote the number of such words. For fixed $i_1,i_2,\ldots,i_k$, the word has length

$$l(i_1-1)+\cdots+l(i_k-1)+k-1,$$

and its $k-1$ letters equal to x. Choosing suitable $i_1,i_2,\ldots,i_k$, each of $k-1$ letters x may be put at any prescribed place in a word. Summing over all $i_1,i_2,\ldots,i_k$, implies

Proposition 4. We have

$$N=\sum_{i_1+i_2+\cdots+i_k=n} f_0(i_1)\cdots f_0(i_k).$$

Remark 1. Taking, in particular, $l(i-1)=i-1$ for all i, we obtain [6, Proposition 12].

In the following two cases, we restricted our investigation to find explicit formulas for $c_1(n,k)$ and its combinatorial interpretations. We first assume that the values of f_0 are the central binomial coefficients, that is

$$f_0(n)=\binom{2n-2}{n-1}.$$

It is clear that $f_0(n)$ is the number of binary words of length $2n-2$, in which the number of zeros equals the number of ones. We let $\mathcal{P}_3$ denote this condition. In this case, we take $l(n-1)=2n-2$. Since the empty word satisfies $\mathcal{P}_3$, we have $l(0)=0$. Hence, Proposition 4 may be applied to obtain

Corollary 8. The number $c_1(n,k)$ is the number of ternary words of length $2n-k-1$ having $k-1$ letters equal 2, and all binary subwords satisfy $\mathcal{P}_3$.

We derive an explicit formula for $c_1(n,k)$.

Proposition 5. We have

$$c_1(n,n)=1,$$
$$c_1(n,k)=\frac{2^{n-k}k(k+2)\cdots[k+2(n-k-1)]}{(n-k)!},(k<n).$$

Proof. It is well-known that the generating function $g(x)$, for the sequence

$$\left\{\binom{2n-2}{n-1}:n=1,2,\dots\right\}, \tag{5}$$

is $g(x)=\frac{1}{\sqrt{1-4x}}$. We know that $c_1(n,k)$ is the coefficient of x^n in the expansion of $[xg(x)]^k$ into powers of x. The formula follows by the use of Taylor expansion for the binomial series. □

The fact that $c_1(n,1)=f_0(n)$ leads to the following identity:

Identity 2. For each $n\geq 1$, we have

$$\prod_{i=1}^{n}(n+i)=2^n(2n-1)!!.$$

We also consider the particular case $k=2$. Then $c_1(n,2)=4^{n-2},(n>1)$. Hence, powers of 4 have the following property.

Corollary 9. For $n\geq 2$, the number 4^{n-2} is the number of ternary words of length $2n-3$ in which one letter is 2 and in each binary subword, the number of ones and zeros are equal.

The number $c_1(n,k)$ may be interpreted in terms of lattice paths. Namely, It is a well-known fact that $f_0(n)$ is the number of lattice paths from $(0,0)$ to $(n-1,n-1)$ using the steps $(0,1)$ and $(1,0)$. We may consider the symbol x in Proposition 4 as the $(1,1)$-step possible only on the main diagonal. We thus obtain the following Euler-type identity.

Identity 3. The following sets have the same number of elements.

(1) The set of ternary words having $k-1$ letters equal 2, in which each binary subword has the same number of zeros and ones.
(2) The set of the lattice paths from $(0,0)$ to $(n+k-2, n+k-2)$ using steps $(0,1),(1,0)$ and $k-1$ steps $(1,1)$ possible only on the main diagonal.

Our final example is the case $f_0(n) = \binom{2n-1}{n}, (n = 1,2,\ldots)$. Combinatorially, $f_0(n)$ is the number of binary words of length $2n-1$ in which the number of ones is by 1 greater than the number of zeros. We let $\mathcal{P}_4$ denote this property. Obviously, the empty word does not satisfy this condition, so that, Proposition 4 can not be applied.

We count the number of words of the form $w_{i_1}, 2, w_{i_2}, 2, \ldots w_{i_{k-1}}, 2, w_{i_k}$ when $i_1 + i_2 + \cdots + i_k = n$, and $l(w_i) = 2i-1$, for $i \in \{i_1, i_2, \ldots, i_k\}$. It is easy to see that the following result holds:

Corollary 10. The number $c_1(n,k)$ is number of ternary words of length $n+k-1$ having the following properties:

(1) No word either begins or ends with 2.
(2) No two 2 can be adjacent.
(3) Each binary subword satisfies $\mathcal{P}_4$.

It is known that a generating function for the sequence $f_0(1), f_0(2), \ldots$ is

$$g(x) = \frac{1}{2x\sqrt{1-4x}} - \frac{1}{2x}.$$

To obtain an explicit formula for $c_1(n,k)$, we have to expand $[xg(x)]^k$ into powers of x. Using the binomial formula and the expansion of the binomial series, we obtain

$$[xg(x)]^k = \frac{1}{2^k}\sum_{j=0}^{\infty}\left(\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} i(i+2)\cdots(i+2j-2)\right)\frac{2^j}{j!}x^j. \tag{6}$$

Since the least power of x on the left-hand side is k, we obtain the following:

Identity 4.

$$\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} i(i+2)\cdots(i+2j-2), (j<k).$$

Remark 2. The identity may easily be proved directly.

We thus obtain

Proposition 6. The following formula holds:

$$c_1(n,k) = \frac{2^{n-k}}{n!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\cdot\prod_{t=0}^{n-1}(i+2t).$$

From equation (6), we obtain

Identity 5. The following formula holds:

$$\prod_{i=1}^{n}(n+i-1) = 2^{n-1}(2n-1)!!.$$

Remark 3. We note that a number of other combinatorial interpretations of our results may be found in Sloane's [8].

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