ENUMERATING THE SYMPLECTIC DELLAC CONFIGURATIONS

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ABSTRACT. Fang and Fourier defined the symplectic Dellac configurations in order to parametrize the torus fixed points of the symplectic degenerated flag varieties, and conjectured that their numbers are the elements of a sequence $(r_n)_{n\geq 0} = (1, 2, 10, 98, 1594, \ldots)$ which appears in the study by Randrianarivony and Zeng of the median Euler numbers. In this paper, we prove the conjecture by considering a combinatorial interpretation of the integers r_n in terms of the surjective pistols (which form a well-known combinatorial model of the Genocchi numbers), and constructing an appropriate surjection from the symplectic Dellac configurations to the surjective pistols.

NOTATIONS

For all pair of integers (n, m) such that $n < m$, the set of integers $\{n, n+1, \ldots, m\}$ is denoted by $[n, m]$. If n is a positive integer, we denote by $[n]$ the set $[1, n]$. The cardinality of a finite set S is denoted by $\#S$. If a set of integers $\{i_1, i_2, \ldots, i_m\}$ has the property $i_k < i_{k+1}$ for all $k \in [m-1]$, we denote it by $\{i_1, i_2, \ldots, i_m\}_<$.

1. INTRODUCTION

Let n be a positive integer. Recall that a Dellac configuration of size n [2] is a tableau D, made of n columns and $2n$ rows, that contains $2n$ dots such that :

— every row contains exactly one dot;

— every column contains exactly two dots;

— if there is a dot in the box (j, i) of D (*i.e.*, in the intersection of its j-th column from left to right and its i-th row from bottom to top), then $j \leq i \leq j + n$.

The set of the Dellac configurations of size n is denoted by DC_n. For example, in Figure 1 are depicted the 7 elements of DC_3.

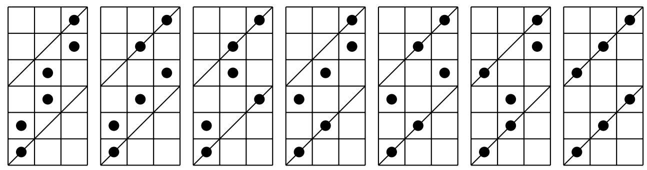


FIGURE 1. The $h_3 = 7$ elements of DC_3.

It is well-known [5] that the cardinality of DC_n is h_n where $(h_n)_{n\geq 0} = (1, 1, 2, 7, 38, 295, \ldots)$ is the sequence of the normalized median Genocchi numbers [7]. Feigin [5, 1] proved that the Poincaré polynomial of the degenerate flag variety $\mathcal{F}_n^a$ has a combinatorial interpretation in terms of the Dellac configurations of size n, in particular its Euler characteristic equals $\#\,\mathrm{DC}_n = h_n$. Afterwards, following computer experiments, Cerulli Irelli and Feigin conjectured that in the case of the symplectic degenerate flag varieties $\mathrm{Sp}\mathcal{F}_{2n}^a$ [6], the role of the sequence $(h_n)_{n\geq 0}$ is played by the sequence of positive integers $(r_n)_{n\geq 0} = (1, 2, 10, 98, 1594, \ldots)$ [8] defined by Randrianarivony and Zeng [10] following $r_n = D_n(1)/2^n$ where $D_0(x) = 1$ and

$$D_{n+1}(x) = (x + 1)(x + 2)D_n(x + 2) - x(x + 1)D_n(x).$$

Now, Fang and Fourier [4] have defined a combinatorial model of the Euler characteristic $\chi(\mathrm{Sp}\mathcal{F}_{2n}^a)$ of the symplectic degenerate flag variety $\mathrm{Sp}\mathcal{F}_{2n}^a$, through the set SpDC_{2n} of the symplectic Dellac configurations of size $2n$.

Definition 1 (Fang and Fourier [4])**.** A symplectic Dellac configuration of size $2n$ is an element S of DC_{2n} such that, for all $i \in [4n]$ and $j \in [2n]$, there is a dot in the box (j, i) of S if and only if there is a dot in its box $(2n+1-j, 4n+1-i)$ (in other words, there exists a central reflection of S with respect to the center of S). The set of the symplectic Dellac configurations of size $2n$ is denoted by SpDC_{2n}.

For example, in Figure 2 are depicted the 10 elements of SpDC_4.

Proposition 2 (Fang and Fourier [4])**.** *For all $n \geq 1$, the Euler characteristic of $Sp\mathcal{F}_{2n}^a$ is the cardinality of $SpDC_{2n}$.*

Conjecture 3 (Cerulli Irelli and Feigin, Fang and Fourier [4])**.** *The cardinality of $SpDC_{2n}$ equals r_n for all $n \geq 1$.*

The aim of this paper is to prove the above conjecture. To do so, we use a combinatorial interpretation of the integers r_n in terms of the surjective pistols. Recall that, for a given $n \geq 1$, a surjective pistol $f \in \mathcal{P}_n$ is a surjective map $f : [2n] \twoheadrightarrow \{2, 4, \ldots, 2n\}$ such that

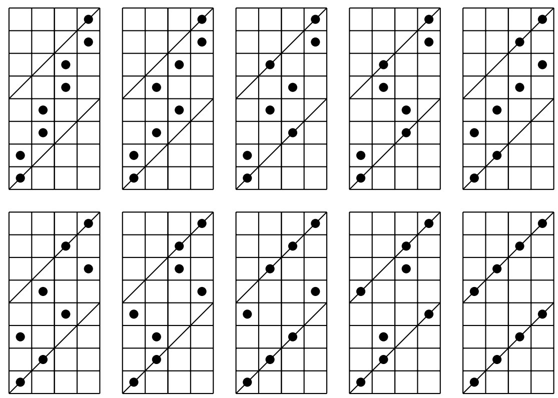
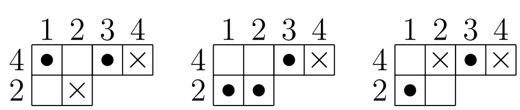


FIGURE 2. The 10 elements of SpDC_4.

$f(j) \geq j$ for all $j \in [2n]$. For a given element $f \in \mathcal{P}_n$, an integer $j \in [2n-2]$ is said to be a doubled fixed point if there exists $j' < j$ such that $f(j') = f(j) = j$ (in particular j is even). Let $\mathrm{ndf}(f)$ be the number of elements of $\{2, 4, \dots, 2n\}$ that are not doubled fixed points of f (by definition $\mathrm{ndf}(f) \geq 1$ because $2n$ is never considered as a doubled fixed point, even though $f(2n-1) = f(2n) = 2n$ for all f). From now on, we assimilate every surjective pistol $f \in \mathcal{P}_n$ into the sequence $(f(1), f(2), \dots, f(2n))$, in which the images of the even integers that are doubled fixed points (respectively not doubled fixed points) are underlined (respectively written in bold characters). Also, we represent $f \in \mathcal{P}_n$ by a tableau made of n left-justified rows of length $2, 4, 6, \dots, 2n$ (from bottom to top) by plotting a dot inside the $(f(j)/2 - j)$-th box (from bottom to top) of the j-th column of the tableau for all $j \in [2n]$; with precision, if j is an even integer that is not a doubled fixed point of f, we plot a symbol $\times$ instead of a dot. For example, we represent in Figure 3 the 3 elements of $\mathcal{P}_2$, whose numbers of non doubled fixed points are respectively 2, 1 and 2.



$f_1 = (4, \mathbf{2}, 4, \mathbf{4})$ $\quad f_2 = (2, \underline{2}, 4, \mathbf{4})$ $\quad f_3 = (2, \mathbf{4}, 4, \mathbf{4})$

FIGURE 3. The $G_6 = 3$ elements of $\mathcal{P}_2$.

Randrianarivony and Zeng [10] proved the following Formula for all $n \geq 1$:

$$r_n = \sum_{f \in \mathcal{P}_n} 2^{\mathrm{ndf}(f)}. \tag{1}$$

For example, in the case $n = 2$, we do obtain $r_2 = 2^2 + 2 + 2^2$ as seen in Figure 3. We know from Dumont [3] that the surjective pistols form a combinatorial interpretation of the sequence of the Genocchi numbers $(G_{2k})_{k \geq 1} = (1, 1, 3, 17, 155, 2073, \dots)$ [9] : for all $n \geq 1$, the cardinality of $\mathcal{P}_n$ equals G_{2n+2}.

Now, we are going to obtain (in Proposition 5) an analogous formula for the cardinality SpDC_{2n}, in terms of the combinatorial objects defined as follows.

Definition 4. Let $\mathcal{T}_n$ be the set of tableaux T made of n columns and $2n$ rows, that contain $2n$ dots such that :

— every row contains exactly one dot;

— every column contains exactly two dots;

— if there is a dot in the box (j, i) of T, then $j \leq i$.

(This is in fact the Definition of DC_n, minus the condition that each box (j, i) that contains a dot implies $i \leq j + n$.)

If a dot of T is located in a box (j, i) such that $i \geq 2n + 1 - j$, we say that it is *free* and we represent it by a star instead of a dot. Let $\mathrm{fr}(T)$ be the number of free dots of T.

For example, in Figure 4 are depicted the 3 elements of $\mathcal{T}_2$, and their numbers of free dots are respectively 2, 1 and 2.

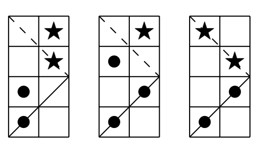


FIGURE 4. The 3 elements of $\mathcal{T}_2$.

By considering the $\binom{n+1}{2} = (n+1)n/2$ possible locations of the two dots of the last column of any $T \in \mathcal{T}_n$ (from left to right), it is easy to obtain the induction formula $\# \mathcal{T}_n = \binom{n+1}{2} \# \mathcal{T}_{n-1}$ for all $n \geq 2$, and finally to compute

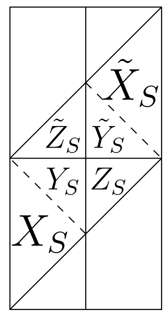
$$\# \mathcal{T}_n = (n+1)! n! / 2^n.$$

Proposition 5. *For all $n \geq 1$, we have*

$$\# \, SpDC_{2n} = \sum_{T \in \mathcal{T}_n} 2^{fr(T)}.$$

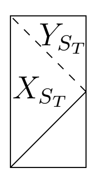
Proof. Any $S \in \mathrm{SpDC}_{2n}$ can be partitioned as follows,

$S =$



where there are no dots in the blank areas, and where the area $\tilde{X}_S$ (respectively $\tilde{Y}_S$ and $\tilde{Z}_S$) is symmetrical to the area X_S (respectively Y_S and Z_S) following the center of S. Now, for any dot of Y_S (respectively Z_S), say, located in the box (j, i) with $2n \geq i \geq 2n+1-j$ (respectively $2n \geq i \geq j$), we can define a new configuration $s_i(S) \in \mathrm{SpDC}_{2n}$ by relocating this dot in the box $(2n+1-j, i)$ of Z_S (respectively Y_S), and relocating the dot located in the box $(2n+1-j, 4n+1-j)$ of $\tilde{Y}_S$ (respectively $\tilde{Z}_S$), in the box $(j, 4n+1-i)$ of $\tilde{Z}_S$ (respectively $\tilde{Y}_S$). Thus, if E_i is defined as the set of the configurations $S \in \mathrm{SpDC}_{2n}$ whose i-th row (from bottom to top) doesn't contain its dot in X_S, then it is clear that s_i is an involution of E_i. Also, if $S \in E_{i_1} \cap E_{i_2}$ for some $(i_1, i_2) \in [n+1, 2n]^2$, obviously $s_{i_1} \circ s_{i_2}(S) = s_{i_2} \circ s_{i_1}(S)$. Consequently, for all $S \in \mathrm{SpDC}_{2n}$, there exists one unique $T \in \mathcal{T}_n$ such that S is obtained by applying a finite number of these involutions on the configuration $S_T \in \mathrm{SpDC}_{2n}$ defined by Z_{S_T} and $\tilde{Z}_{S_T}$ being empty, and

$T =$



so that S_T generates a total amount of $2^{\# Y_{S_T}}$ elements of SpDC_{2n}, where $\# Y_{S_T}$ is the number of dots located in Y_{S_T} (in other words $\# Y_{S_T} = \mathrm{fr}(T)$). $\square$

For example, we depict in Figure 5 how the 3 elements of $\mathcal{T}_2$ generate the $10 = 2^2 + 2 + 2^2$ elements of SpDC_4.

Now, Conjecture 3 is a corollary of the following Theorem in view of Formula (1) and Proposition 5.

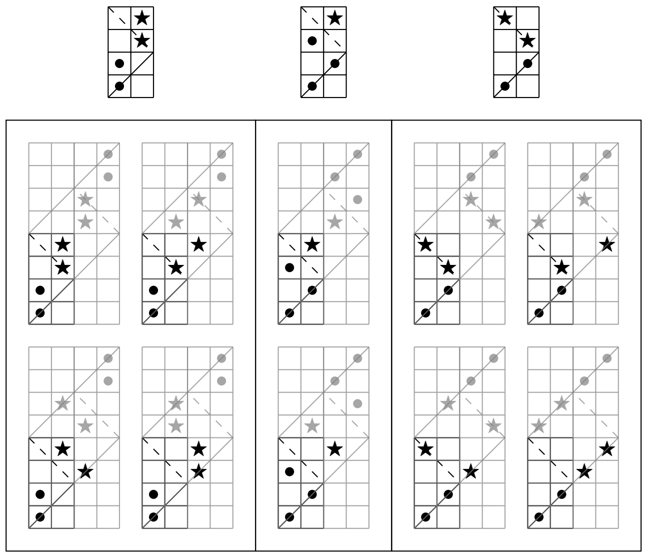


FIGURE 5. Generation of the 2^2+2+2^2 elements of SpDC_4 from the 3 elements of $\mathcal{T}_2$.

Theorem 6. *There exists a surjective map $\varphi : \mathcal{T}_n \twoheadrightarrow \mathcal{P}_n$ such that*

$$\sum_{T\in\varphi^{-1}(f)} 2^{fr(T)} = 2^{ndf(f)} \tag{2}$$

for all $f \in \mathcal{P}_n$.

The rest of this paper aims at proving Theorem 6, and is organized as follows. In Section 2, we introduce the j-tableaux (a generalization of the tableaux $T \in \mathcal{T}_n$), on which we define a family of paths, namely, the T-paths. In Section 3, we use these paths to define the pistol labeling of a tableau (in Algorithm 13), which produces (in Definition 16) the Definition of φ. In Section 4, we define the notion of (f,j)-insertion of a dot into a j-tableau, which allows to formulate Algorithm 25 and produces the Definition of a map $\phi : \mathcal{P}_n \to \mathcal{T}_n$. In Section 5, we first prove that $\varphi \circ \phi$ is the identity map of $\mathcal{P}_n$ (hence $\phi : \mathcal{P}_n \to \mathcal{T}_n$ is injective and $\varphi : \mathcal{T}_n \to \mathcal{P}_n$ is surjective), then we make the image $\phi(\mathcal{P}_n) \subset \mathcal{T}_n$ of ϕ explicit, and we prove that $\phi \circ \varphi_{|\phi(\mathcal{P}_n)}$ is the identity map of this set. Finally, in Section 6, we finish the proof of Theorem 6, *i.e.*, we show that Formula (2) is true for all $f \in \mathcal{P}_n$. To do so, we make $\varphi^{-1}(f)$ explicit by defining Algorithm 42 and Algorithm 44, which allow to construct every element of $\varphi^{-1}(f)$ from one given element of it (like $\phi(f)$).

2. j-TABLEAUX AND T-PATHS

Definition 7. Let $j \in [n]$, a *j-tableau* $T \in \mathcal{T}_n^j$ is a tableau made of n columns (denoted by $C_1^T, C_2^T, \ldots, C_n^T$ from left to right) and $2n$ rows (denoted by $R_1^T, R_2^T, R_n^T, R_{2n-1}^T, R_{2n-2}^T, \ldots, R_{n+1}^T, R_{2n}^T$ from bottom to top), that contain between $2j$ and $2n$ dots above the line $y = x$ (for all $i \in [2n]$, if the row R_i^T contains a dot, it is denoted by d_i^T) such that :

— each column $C_1^T, C_2^T, \ldots, C_{j-1}^T$ contains exactly two dots and the other columns contain at most two dots;

— each row $R_1^T, R_2^T, \ldots, R_{j-1}^T$ contains exactly one dot and the other rows contain at most one dot.

In particular, a tableau $T \in \mathcal{T}_n$ is also a j-tableau for all $j \in [n]$.

Definition 8. Let $j \in [n]$, $T \in \mathcal{T}_n^j$ and $i \in [j, 2n]$ such that the intersection of the row R_i^T with the columns $C_1^T, C_2^T, \ldots, C_{j-1}^T$ is empty. The *T-path from the box $C_j^T \cap R_i^T$* is the sequence $(i_0, i_1, \ldots) \in [2n]^{\mathbb{N}}$ defined by $i_0 = i$, and, for all $k \in \mathbb{N}$, by the following rules.

(1) If $i_k \in [j, n] \sqcup [n + j, 2n]$, then $i_{k+1} = i_k$.

(2) If i_k is of the kind $n + j_k$ with $j_k \in [j - 1]$, then $d_{i_{k+1}}^T$ is defined as the upper dot of the column $C_{j_k}^T$.

(3) Otherwise $i_k \in [j - 1]$, and $d_{i_{k+1}}^T$ is defined as the lower dot of the column $C_{i_k}^T$.

Remark 9. In the context of Definition 8, the T-path from the box $C_j^T \cap R_i^T$ becomes stationary if and only if $i_k \in [j, n] \sqcup [n + j, 2n]$ for k big enough.

Proposition-Definition 10. *With the notations of Definition 8, the T-path $(i_0, i_1, \ldots)$ from the box $C_j^T \cap R_i^T$ becomes stationary, i.e., the integer i_k belongs to the set $[j, n] \sqcup [n + j, 2n]$ for k big enough. This integer is said to be the arrival of this T-path. Also, let π_j^T be the map that maps every integer $i \in [j, 2n]$ such that the first $j - 1$ boxes of R_i^T are empty, to the arrival of the T-path from the box $C_j^T \cap R_i^T$. Then π_j^T is bijective.*

Proof. Suppose that $i_k \notin [j, n] \sqcup [n + j, 2n]$ for all $k \geq 0$. Since $[2n]$ is a finite set, and because $(i_k)_{k \geq 0}$ is defined by induction, there exists $0 \leq k_1 < k_2$ such that $i_{k_1} = i_{k_2}$. Now, Rule (1) of Definition 8 is never applied, so the sequence $(i_k)_{k \geq 0}$ is reversible : for all $k > 0$, let $j_{k-1} \in [n]$ such that $d_{i_k}^T \in C_{j_{k-1}}^T$; if $d_{i_k}^T$ is the upper dot of $C_{j_k}^T$, then $i_{k-1} = n + j_{k-1}$, otherwise $i_{k-1} = j_{k-1}$. Consequently, the equality $i_{k_1} = i_{k_2}$ implies $i = i_0 = i_{k_2 - k_1}$. Since $k_2 - k_1 > 0$ and, for all $k > 0$, the dot $d_{i_k}^T$ belongs to a column $C_{j_{k-1}}^T$ for some $j_{k-1} \in [j-1]$, then d_i^T cannot belong to C_j^T, which is absurd. So $i_k \in [j,n] \sqcup [n+j,2n]$ for k big enough.

Let $k_{\min}$ be the smallest integer $k \geq 0$ such that i_k is the arrival of the T-path. As stated before, the sequence $(i_0, i_1, \ldots, i_{k_{\min}})$ is reversible because it never involves Rule (1) of Definition 8, so the application π_j^T is injective. Finally, the number of integers $i \in [j, 2n]$ such that the first $j-1$ boxes of R_i^T are empty, is exactly $2(n-j+1) = \#([j,n] \sqcup [n+j,2n])$: by definition of $\mathcal{T}_j^n$, the first $j-1$ rows of T contain exactly $j-1$ dots, and the first $j-1$ columns of T contain exactly $2(j-1)$ dots, so, among the $2n-j+1$ rows $R_j^T, R_{j+1}^T, \ldots, R_{2n}^T$, exactly $j-1$ of them contain their dot in one of their $j-1$ first boxes. So π_j^T is bijective. □

Remark 11. The fixed points of π_j^T are the integers $i \in [j,n] \sqcup [j+n, 2n]$ such that the first $j-1$ boxes of R_i^T are empty.

For example (in this case $n = 7$ and $j = 4$), consider the 4-tableau $T_0 \in \mathcal{T}_7^4$ which appears in Figure 6. In this example the columns $C_4^{T_0}, C_5^{T_0}, C_6^{T_0}, C_7^{T_0}$ are empty. The set of integers $i \in [j,n] \sqcup [n+j, 2n] = [4,7] \sqcup [11,14]$ such that the $j-1 = 3$ first boxes of $R_i^{T_0}$ are empty, is $\{4,5,7,8,9,11,12,13\}$. For all $i \in \{4,5,7,11,12,13\}$, we obtain $\pi_4^{T_0}(i) = i$ because $i \in [j,n] \sqcup [n+j,2n]$. In Figure 6, we show that the T_0-path from the box $C_4^{T_0} \cap R_8^{T_0}$ (respectively $C_4^{T_0} \cap R_9^{T_0}$) is the sequence $(i_k)_{k \geq 0} = (8,2,10,6,6,6,\ldots)$, which becomes stationary at $i_3 = 6$, element of $[j,n] \sqcup [n+j,2n]$, hence $\pi_4^{T_0}(8) = 6$ (respectively the sequence $(i'_k)_{k\geq 0} = (9,14,14,14,\ldots)$, which becomes stationary at $i'_1 = 14 \in [j,n] \sqcup [n+j,2n]$, hence $\pi_4^{T_0}(9) = 14$). As a summary, we obtain

$$\pi_4^{T_0} = \begin{pmatrix} 4 & 5 & 7 & 8 & 9 & 11 & 12 & 13 \\ 4 & 5 & 7 & 6 & 14 & 11 & 12 & 13 \end{pmatrix}.$$

3. From the tableaux to the surjective pistols

3.1. **Pistol labeling of a tableau.**

Let $T \in \mathcal{T}_n$. We consider a vectorial version of the statistic of free dots $\mathrm{fr} : \mathcal{T}_n \to [n]$, through $\overrightarrow{\mathrm{fr}} : \mathcal{T}_n \to \{0,1\}^n$ defined by

$$\overrightarrow{\mathrm{fr}}(T) = [\mathrm{fr}_1(T), \mathrm{fr}_2(T), \ldots, \mathrm{fr}_n(T)]$$

where $\mathrm{fr}_i(T) = 1$ if and only if the dot d_{n+i}^T is free.

We are going to give (in Algorithm 13) three labels to every dot of T :

— a *digital* label, *i.e.*, an element of $[0, n-1]$;

— a *type* label, *i.e.*, either the letter α or β;

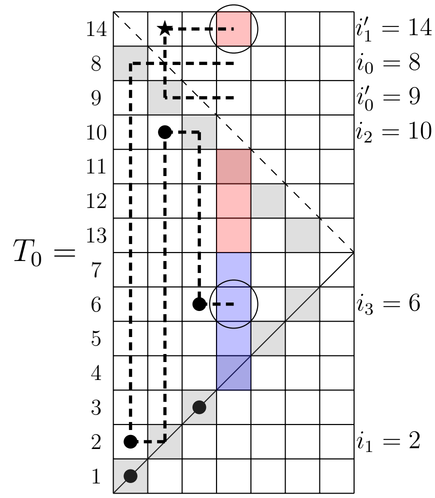


FIGURE 6. T_0-paths $(8, 2, 10, 6, 6, 6, \dots)$ and $(9, 14, 14, 14, \dots)$ from the respective boxes $C_4^{T_0} \cap R_8^{T_0}$ and $C_4^{T_0} \cap R_9^{T_0}$.

— a *parity* label, *i.e.*, either the letter o (for *odd*) or e (for *even*).

If a dot d is labeled with the type label $t \in \{\alpha, \beta\}$, the digital label $h \in [0, n-1]$ and the parity label $p \in \{o, e\}$, we denote the data of these three labels by t_h^p, and we name it the *pistol label* of d. Sometimes, we will also write that d is labeled with t_h if we know its digital label h and its type label t but not its parity label.

Definition 12. Let $T \in \mathcal{T}_n$ and $i \in [n]$. The dots d_i^T and d_{n+i}^T are said to be *twin dots*. Let j_1 and j_2 such that $d_i^T \in C_{j_1}^T$ and $d_{n+i}^T \in C_{j_2}^T$. The dot $d_{i,\min}^T$ is defined as d_i^T if $j_1 \leq j_2$, as d_{n+i}^T otherwise.

Algorithm 13 (pistol labeling of a tableau)**.** For j from n down to 1, assume that each of the $2(n-j)$ dots of the columns $C_{j+1}^T, \dots, C_n^T$ have already received its pistol label. At this step, in the parts I., II. and III., we give every dot of C_j^T its digital, type and parity label respectively.

I. The digital labels. For all $i \in [j, 2n]$, if the dot d_i^T belongs to C_j^T, let $i' = \pi_j^T(i) \in [j, n] \sqcup [n+j, 2n]$. We define the digital label of d_i^T as $i' - j$ if $i' \in [j, n]$, as $i' - n - j$ if $i' \in [n+j, 2n]$.

II. The type labels. For all $i \in [j, 2n]$, if the dot d_i^T belongs to C_j^T, let $h \in [0, n-j]$ be its digital label. We consider $j' = j + h \in [j, n]$, and $i' = \pi_j^T(i) \in \{j', n+j'\}$.

1 - Assume first that $j' > j$. By hypothesis, the two dots of $C_{j'}^T$ have already received their pistol labels. If they have

different type labels, we define $(\gamma, \bar{\gamma})$ as (α, β), otherwise we define $(\gamma, \bar{\gamma})$ as (β, α).

a) If one of the dots of $C^T_{j'}$ is labeled with β^e_0, then we define the type label of d^T_i as α if $i' = j'$, as β if $i' = n + j'$.

b) Otherwise, we define the type label of d^T_i as γ if $d^T_{i'} = d^T_{j',\min}$, as $\bar{\gamma}$ otherwise.

2 - If $j' = j$, let $d \neq d^T_i$ be the other dot of C^T_j.

a) If the digital label of d is 0, then we define the type label of d^T_i as α if $i' = j$, as β if $i' = n + j$.

b) Otherwise, the type label t of d has already been defined by Rule II.1- of this algorithm.

i. If $t = \alpha$, we define the type label of d^T_i as α if $i \neq i'$ and $i' = j$, as β otherwise.

ii. If $t = \beta$, we define the type label of d^T_i as α if $d^T_{i'} = d^T_{j,\min}$, as β otherwise.

III. The parity labels. Let $h_1 \leq h_2$ be the digital labels of the dots of C^T_j.

1 - If the type labels of the dots of C^T_j are different, we label with o the dot whose type label is α, and with e the dot whose type label is β.

2 - Otherwise, it is necessary that $h_1 \neq h_2$ (if $h_1 = h_2$, then the type labels of the dots of C^T_j are defined by Rule II.1- or Rule II.2-a) of this algorithm, and in both case the type labels are different).

a) If the type label of the dots of C^T_j is α, we label with e the dot whose digital label is h_1, and with o the other dot.

b) If they both have the type label β, we label with o the dot whose digital label is h_1, and with e the other dot.

For example, we depict in Figure 7 the pistol labeling of a tableau $T_1 \in \mathcal{T}_7$. On the left of this figure appears the tableau T_1 per se (and we specified on the left the indices of its rows, and on the right its vector statistic $\overrightarrow{\mathrm{fr}}(T_1) = [1,1,0,0,1,1,1]$) ; on the right appears its pistol-labeled version. The details of this pistol labeling are given in Appendix A.

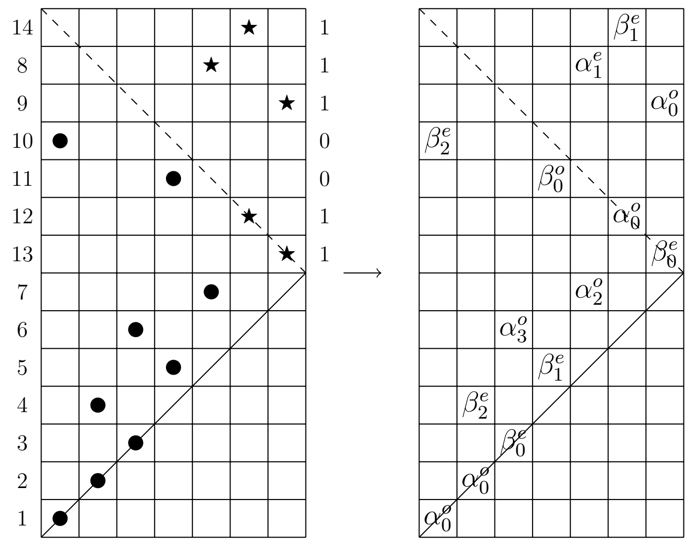


FIGURE 7. Tableau $T_1 \in \mathcal{T}_7$ (on the left) mapped to its pistol labeling (on the right).

Remark 14. We enumerate here a few facts about the pistol labeling of $T \in \mathcal{T}_n$.

(a) For all $j \in [n]$ and $i \in [j, 2n]$, if the dot $d_i^T \in C_j^T$, then its digital label belongs to the set $[0, n-j]$.

(b) If a dot d_i^T in a column C_j^T is labeled with α_0^e, then by Rule III.2-a) of Algorithm 13 the other dot of C_j^T has the pistol label α_h^o for some $h \in [n-j]$. Also, the type label α of d_i^T has necessarily been defined by Rule II.2-b)i., and in particular $i \in [n+1, n+j-1]$.

(c) Every column of T contains exactly one dot whose parity label is o (respectively e).

(d) By Rule II.2-a) and Rule III.1- of Algorithm 13, the pistol labels of the two dots of C_n^T are α_0^o and β_0^e. Consequently, the type label of d_n^T (respectively d_{2n}^T) is defined either by Rule II.1-a) or Rule II.2-a), and in either case it is α (respectively β). Also, whether its parity label is defined by Rule III.1- or Rule III.2-a) (respectively Rule III.1- or Rule III.2-b)), it equals o (respectively e), and its pistol label is α_{n-j}^o (respectively β_{n-j}^e) where C_j^T is the column that contains d_n^T (respectively d_{2n}^T).

(e) For all $i \in [n]$, if one of the dots of C_i^T has the pistol label β_0^e, then the other dot of C_i^T has the type label α (otherwise the parity labels of the two dots of C_i^T would have been defined by

Rule III.2-b) of Algorithm 13, following which the digital label of the dot labeled with e cannot be 0).

(f) For all $j \in [n]$ and $i \in [j, 2n]$, if $d_i^T \in C_j^T$, let $h \in [0, n-j]$ be its digital label and $j' = j + h$, then
$$i \in \left\{ \left(\pi_j^T\right)^{-1}(j'), \left(\pi_j^T\right)^{-1}(n+j') \right\}.$$
Consequently, if the dots of C_j^T are $d^T_{\left(\pi_j^T\right)^{-1}(j')}$ and $d^T_{\left(\pi_j^T\right)^{-1}(n+j')}$, then the twin dots $d_{j'}^T$ and $d_{n+j'}^T$ are located in the columns $C_1^T, C_2^T, \ldots, C_j^T$.

(g) For all $i \in [n]$, if no dot of C_i^T is labeled with β_0^e, then the type label of $d_{i,\min}^T$ is α if the dots of C_i^T have different type labels, otherwise it is β.

Definition 15. Following Remark 13.(c), for all $T \in \mathcal{T}_n$ and $j \in [n]$, we define the *odd* dot (respectively *even* dot) of C_j^T as the dot whose parity label is o (respectively e).

3.2. A map from the tableaux to the surjective pistols.

Definition 16 (Map $\varphi : \mathcal{T}_n \to \{2, 4, \ldots, 2n\}^{[2n]}$)**.** Let $T \in \mathcal{T}_n$, we define a map $\varphi(T) : [2n] \to \{2, 4, \ldots, 2n\}$ as follows : for all $j \in [n]$, following Remark 14.(a) and 13.(c), let $t_o \in \{\alpha, \beta\}$ and $h_o \in [0, n-j]$ (respectively $t_e \in \{\alpha, \beta\}$ and $h_e \in [0, n-j]$) be the type and digital labels of the odd dot (respectively even dot) of C_j^T. We first define $\varphi(T)(2j-1)$ as $2(j + h_o)$. Afterwards,

— if $t_e = \alpha$ and $h_e = 0$, we also define $\varphi(T)(2j)$ as $2(j + h_0)$;

— otherwise, we define $\varphi(T)(2j)$ as $2(j + h_e)$.

Lemma 17. *Let $T \in \mathcal{T}_n$, $f = \varphi(T)$ and $i \in [n]$. If the dot $d_{i,\min}^T$ is located in the column C_j^T, then there exists $k \in \{2j-1, 2j\}$ such that $f(k) = 2i$, and j is the integer $j_{\min} = \lceil k_{min}/2 \rceil$ where*
$$k_{\min} = \min\{k \in [2i] : f(k) = 2i\}.$$

Proof. Since $d_{i,\min}^T = d_i^T$ or d_{n+i}^T, by Part I. of Algorithm 13 its digital label is $i - j$. Consequently, either $d_{i,\min}^T$ is the odd dot of C_j^T, in which case $\varphi(T)(2j-1) = 2i$, or it is the even dot and $\varphi(T)(2j) = 2i$ because $d_{i,\min}^T$ cannot be labeled with α_0^e in view of Remark 14.(b). In either case there exists $k \in \{2j-1, 2j\}$ such that $f(k) = 2i$, so $j \geq j_{\min}$. Reciprocally, since $f(k_{\min}) = 2i$, by Definition 16 one dot of $C_{j_{\min}}^T$ has the digital label $i - j_{\min}$. By Definition 8, this implies that there exists $j' \leq j_{\min}$ such that $d_i^T \in C_{j'}^T$ or $d_{n+i}^T \in C_{j'}^T$, hence $j \leq j' \leq j_{\min}$, and $j = j_{\min}$. $\square$

Corollary 18. *In particular, for all $T \in \mathcal{T}_n$, the map $\varphi(T)$ is surjective, thus belongs to $\mathcal{P}_n$.*

For example, the tableau $T_1 \in \mathcal{T}_7$ depicted in Figure 7 provides the surjective pistol $f_1 = (2, \mathbf{6}, 4, \mathbf{8}, 12, \underline{6}, 8, \mathbf{10}, 14, \mathbf{12}, 12, \mathbf{14}, 14, \mathbf{14}) \in \mathcal{P}_7$ (whose vector statistic is $\overrightarrow{\mathrm{ndf}}(f_1) = [1,1,0,1,1,1,1]$) depicted in Figure 8.

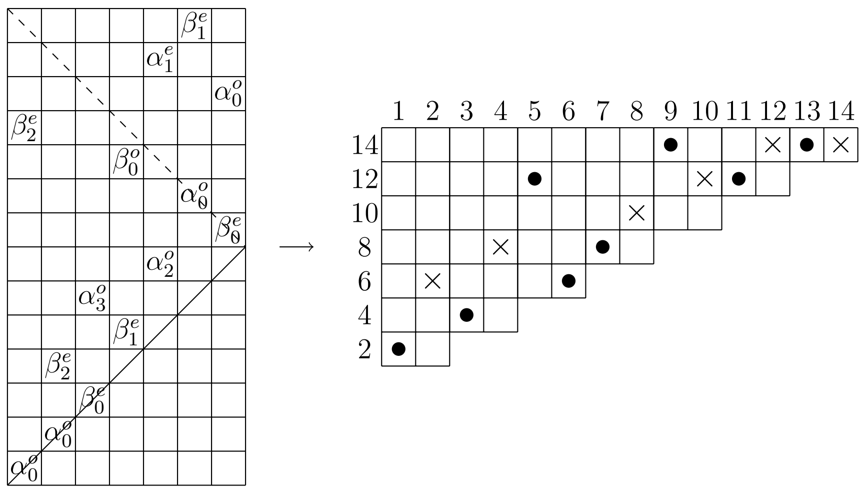


FIGURE 8. The pistol-labeled version of $T_1 \in \mathcal{T}_7$ (on the left) is mapped by φ to the surjective pistol $f_1 = (2, \mathbf{6}, 4, \mathbf{8}, 12, \underline{6}, 8, \mathbf{10}, 14, \mathbf{12}, 12, \mathbf{14}, 14, \mathbf{14}) \in \mathcal{P}_7$ (on the right).

We now introduce a vectorial version of the statistic of non-doubled fixed points $\mathrm{ndf} : \mathrm{DC}_n \to [n]$ through $\overrightarrow{\mathrm{ndf}} : \mathrm{DC}_n \to \{0,1\}^n$ defined by

$$\overrightarrow{\mathrm{ndf}}(f) = [\mathrm{ndf}_1(f), \mathrm{ndf}_2(f), \ldots, \mathrm{ndf}_n(f)]$$

where $\mathrm{ndf}_i(f) = 1$ if and only if $2i$ is not a doubled fixed point of $f \in \mathcal{P}_n$.

In the example of $T_1 \in \mathcal{T}_7$ and $f_1 = \varphi(T_1) \in \mathcal{P}_7$, note that

$$\overrightarrow{\mathrm{ndf}}(f_1) = [1,1,0,1,1,1,1] \neq [1,1,0,0,1,1,1] = \overrightarrow{\mathrm{fr}}(T_1).$$

In order to define a statistic on tableaux that would be preserved by φ, we introduce the notion of grounded dots hereafter.

Definition 19. Let $T \in \mathcal{T}_n$ and $i \in [n]$. We say that the dot d_{n+i}^T is *grounded* if it is not free and if one of the dots of the column C_i^T has the pistol label β_0^e. Let $\overrightarrow{\mathrm{ng}}(T) = [\mathrm{ng}_1(T), \mathrm{ng}_2(T), \ldots, \mathrm{ng}_n(T)]$ where $\mathrm{ng}_i(T) \in \{0,1\}$ equals 0 if and only if d_{n+i}^T is grounded for all $i \in [n]$.

For example, consider the tableau $T_1 \in \mathcal{T}_7$ depicted in Figure 7, we depict in Figure 9 the pistol labeling of T_1 in which every non-grounded dot has been encircled, which gives

$$\overrightarrow{\mathrm{ng}}(T_1) = [1, 1, 0, \mathbf{1}, 1, 1, 1] = \overrightarrow{\mathrm{ndf}}(f_1).$$

Note that in general the dot d^T_{2n}, always being free, is never grounded, even though the column C^T_n always has a dot labeled with β^e_0 (which is similar to $2n$ never being considered as a doubled fixed point of $f \in \mathcal{P}_n$ even though $f(2n-1) = f(2n) = 2n$).

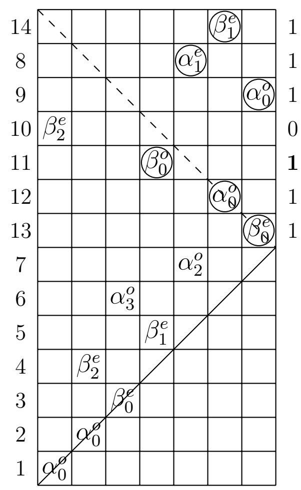


FIGURE 9. Pistol labeling of the tableau $T_1 \in \mathcal{T}_7$.

Lemma 20. *Let $T \in \mathcal{T}_n$ and $f = \varphi(T) \in \mathcal{P}_n$. For all $i \in [n]$, the integer $2i$ is a fixed point of f if and only if C^T_i has a dot labeled with β^e_0.*

Proof. If C^T_i has a dot labeled with β^e_0, by Definition 16 we have $f(2i) = 2i$. Reciprocally, suppose that no dot of C^T_i has the pistol label β^e_0, and that $f(2i) = 2i$. If the digital label of the even dot of C^T_i was $h_e > 0$, we would have $f(2i) = 2(i + h_e) > 2i$, so its digital label is necessarily $h_e = 0$, and since its pistol label is not β^e_0 by hypothesis, then it must be α^e_0. In view of Remark 14(b), this implies that the other dot of C^T_i has the pistol label $\alpha^o_{h_o}$ for some $h_o > 0$. We then have $f(2i) = 2(i + h_o) > 2i$, which is absurd. □

Proposition 21. *Let $f \in \mathcal{P}_n$ and $T \in \varphi^{-1}(f) \subset \mathcal{T}_n$. We have $\overrightarrow{ng}(T) = \overrightarrow{ndf}(f)$.*

Proof. Let $i \in [n]$. If d^T_{n+i} is a grounded dot, then in particular the even dot of C^T_i has the pistol label β^e_0, so $f(2i) = 2i$ by Lemma 20.

Also, the dot d^T_{n+i} is not free, *i.e.*, it is located in a column C^T_j with $j \leq i$. Let also $j' \leq j$ such that $d^T_{i,\min} \in C^T_{j'}$. By Lemma 17, there exists $k \in \{2j'-1, 2j'\}$ such that $f(2k) = 2i$. Consequently, if $j' < i$, then $2i$ is a doubled fixed point of f. Otherwise, we have $j' = j = i$, so the two dots of C^T_i are d^T_i and d^T_{n+i}, in which case it is straightforward from Algorithm 13 that their pistol labels are respectively α^o_0 and β^e_0, thus $f(2i-1) = f(2i) = 2i$, and $2i$ is still a doubled fixed point of f.

Reciprocally, if d^T_{n+i} is not a grounded dot, then either it is free, or the even dot of C^T_i is not labeled with β^e_0. If the even dot of C^T_i is not labeled with β^e_0, then $2i$ is not a fixed point of f by Lemma 20, in particular it is not a doubled fixed point. Assume now that the even dot of C^T_i is labeled with β^e_0 but that d^T_{n+i} is free (which implies that $d^T_{i,\min} = d^T_i$). We have $f(2i) = 2i$ by Lemma 20. By Remark 14(f), since d^T_{n+i} is free, the dot of C^T_i labeled with β^e_0 is $d^T_{\left(\pi^T_i\right)^{-1}(i)}$. This forces its label β to have been defined by Rule II.2-b) of Algorithm 13, and in view of Remark 14(e), it has been defined with precision by Rule II.2-b)i., which implies in this situation that $d^T_i \in C^T_i$. As a summary, the two dots of C^T_i are $d^T_{i,\min} = d^T_i$ (whose pistol label is β^e_0), and another dot whose pistol label is α^o_k with $k \neq 0$. By Definition 16, we then have $f(2i) = 2i$ and $f(2i-1) = 2(i+k) > 2i$. Also, since $d^T_{i,\min} = d^T_i$, by Lemma 17 we know that $\min\{k \in [2i] : f(k) = 2i\}$ belongs to $\{2i-1, 2i\}$, so it is $2i$, which is consequently a fixed point of f but not a doubled fixed point. □

4. From the surjective pistols to the tableaux

4.1. Insertion labels and (f,j)-insertions.

Definition 22 (Insertion of a dot into a j-tableau)**.** Let $f \in \mathcal{P}_n$, $j \in [n]$ and $T \in \mathcal{T}^j_n$. We consider $i \in [j,n] \sqcup [n+j, 2n]$, and $i' = \left(\pi^T_j\right)^{-1}(i) \in [j, 2n]$. Following Proposition-Definition 10, the $j-1$ first boxes of $R^T_{i'}$ are empty. Now, if the box $C^T_j \cap R^T_{i'}$ is also empty, we define a new j-tableau by plotting a dot in this box. This operation is called the *insertion of a dot into the box $C^T_j \cap R^T_i$.*

For example (in this case $n = 7$ and $j = 4$), the insertion of a dot into the box $C^{T_0}_4 \cap R^{T_0}_6$ (respectively the box $C^{T_0}_4 \cap R^{T_0}_{14}$) of the 4-tableau $T_0 \in \mathcal{T}^4_7$ depicted in Figure 7, leads to plotting a dot in the box $C^{T_0}_4 \cap R^{T_0}_8$ (respectively the box $C^{T_0}_4 \cap R^{T_0}_9$).

Definition 23 (labeled j-tableaux)**.** Let $j \in [n]$, we denote by $\mathfrak{T}^j_n$ the set of tableaux $T \in \mathcal{T}^j_n$ whose dots are labeled with the letter a or b, whose columns $C^T_{j+1}, \dots, C^T_n$ are empty, and whose column C^T_j contains at most one dot.

Definition 24 ((f,j)-insertion of labeled dots into a labeled j-tableau)**.** Let $j \in [n]$ and $T \in \mathfrak{T}^j_n$. We consider $l \in \{a,b\}$ and $h \in [0, n-j]$. The *(f,j)-insertion in T of a dot labeled with l at the height h* consists of the following. Let $i = j+h \in [j,n]$.

1. Suppose that $i = j$.
   (a) If R^T_j is empty, we insert a dot labeled with l in the box $C^T_j \cap R^T_j$.
   (b) Otherwise, we insert a dot labeled with l in the box $C^T_j \cap R^T_j$ (respectively $C^T_j \cap R^T_{n+j}$) if $l = a$ (respectively $l = b$).
2. Suppose that $i > j$.
   (a) If R^T_i is empty,
      i. if $l = b$ and $f(2i) = 2i$, then we insert a dot labeled with l in the box $C^T_j \cap R^T_{n+i}$;
      ii. otherwise, we insert a dot labeled with l in the box $C^T_j \cap R^T_i$.
   (b) Otherwise, let $l' \in \{a,b\}$ be the label of the dot of R^T_i. If $l = l'$ (respectively $l \neq l'$), then we insert a dot labeled with l in the box $C^T_j \cap R^T_i$ (respectively $C^T_j \cap R^T_{n+i}$).

4.2. **A map from the surjective pistols to the tableaux.** Let $f \in \mathcal{P}_n$, and $T^1 \in \mathfrak{T}^1_n$ be the empty labeled 1-tableau. For j from 1 to n, we are going to define (in Algorithm 25) a labeled $(j+1)$-tableau $T^{j+1} \in \mathfrak{T}^{j+1}_n$ by filling $C^{T^j}_j$ with two dots located above the line $y = x$, and labeled with the letter a or b.

Algorithm 25. For j from 1 to n, we consider the induction hypothesis $H(j)$ defined as follows.

(A) $T^j \in \mathfrak{T}^j_n$.
(B) If the row $R^{T^j}_j$ is empty and $f(2j) > 2j$, then $f(2k) \neq 2j$ for all $k \in [2j-2]$ (hence $f(2j-1) = 2j$ because f is surjective).

Hypothesis $H(1)$ is obviously true and we initiate the following algorithm for $j = 1$. Let $(\delta_o, \delta_e) = (f(2j-1)/2 - j, f(2j)/2 - j) \in [0, n-j]^2$.

I. We define first two labels l_o and l_e as follows.
   1 - If the row $R^{T^j}_j$ is empty, let $(l_o, l_e) = (a, b)$.
   2 - Otherwise, let d be the dot of $R^{T^j}_j$.

a) If d is labeled with a, let $(l_o, l_e) = (a, b)$.
b) Else,
   i. if $\delta_o < \delta_e$, let $(l_o, l_e) = (b, b)$;
   ii. if $\delta_o \geq \delta_e$, let $(l_o, l_e) = (a, a)$.

II. Then, we define two heights $(h_o, h_e) \in [0, n-j]^2$ as follows. The height h_o is defined as δ_o. Afterwards,

1 - if $l_e = a$ and $\delta_o = \delta_e$, we define h_e as 0;

2 - otherwise, we define h_e as δ_e.

We finally define T^{j+1} as the tableau obtained first by (f, j)-inserting in T^j a dot labeled with l_o at the height h_o, then by (f, j)-inserting in the resulted tableau a dot labeled with l_e at the level h_e. We prove now that Hypothesis $H(j+1)$ is true.

(A) Following the condition (A) of Hypothesis $H(j)$, since T^{j+1} is obtained by plotting two dots in $C_j^{T^j}$ and in two empty rows of T^j, we only need to prove that $R_j^{T^{j+1}}$ contains a dot. Either $R_j^{T^j}$ contains a dot, in which case $R_j^{T^{j+1}}$ too, or, following the condition (B) of Hypothesis $H(j)$, we have $\delta_o = 0$ or $\delta_e = 0$, hence $h_o = 0$ or $h_e = 0$, which implies that the box $C_j^{T^{j+1}} \cap R_j^{T^{j+1}}$ contains a dot by Rule 1.(a) of Definition 24. So $T^{j+1} \in \mathfrak{T}_n^{j+1}$.

(B) If $2j+2$ is not a fixed point of f and if there exists $k \in [2j]$ that is mapped to $2j+2$ by f, suppose that k is the smallest integer to have that property and let $j' = \lceil k/2 \rceil \leq j$; at the j'-th step of the algorithm, a dot is (f, j')-inserted in $T^{j'}$ at the level $h = j + 1 - j'$. Since k is minimal, the row $R_{j+1}^{T^{j'}}$ is empty, so the box $C_{j'}^{T^{j'+1}} \cap R_{j+1}^{T^{j'+1}}$ contains a dot by Rule 2.(a)ii. of Definition 24.

So the above algorithm is well-defined and, following Hypothesis $H(n+1)$, produces a tableau $T^{n+1} \in \mathfrak{T}_n^{n+1}$, in other words, a tableau $T \in \mathcal{T}_n$ whose dots are labeled with the letter a or b. We define $\Phi(f)$ as this tableau $T \in \mathcal{T}_n$.

For example, consider the surjective pistol

$$f_1 = (2, \mathbf{6}, 4, \mathbf{8}, 12, \underline{6}, 8, \mathbf{10}, 14, \mathbf{12}, 12, \mathbf{14}, 14, \mathbf{14}) \in \mathcal{P}_7,$$

whose graphical representation is depicted in Figure 8. We depict in Figure 10 the insertion-labeled version of the tableau $\Phi(f_1) \in \mathcal{T}_7$, which is in fact the tableau $T_1 \in \mathcal{T}_7$ depicted in Figure 7, mapped to f_1 by φ (see Figure 8). The details of this computation are given in Appendix B.

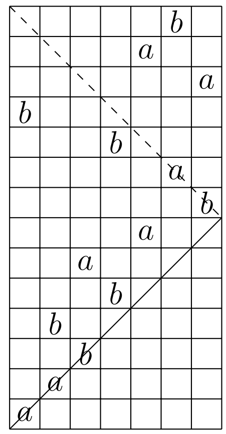


FIGURE 10. The insertion labeling of the tableau $\Phi(f_1) \in \mathcal{T}_7$.

5. CONNECTION BETWEEN φ AND ϕ

Lemma 26. *Let $f \in \mathcal{P}_n$, $T = \Phi(f) \in \mathcal{T}_n$ and $i \in [n]$. If $d^T_{i,\min} = d^T_{n+i}$, then $f(2i) = 2i$ and the two dots of C^T_i have different insertion labels.*

Proof. In general, the dot $d^T_{i,\min}$ is, by its definition, always plotted by Rule 1.(a) or Rule 2.(a) of Definition 24. Now, if $d^T_{i,\min} = d^T_{n+i}$, then with precision it must be plotted by Rule 2.(a)i., following which $f(2i) = 2i$.

Afterwards, if the insertion labels of the dots of C^T_i are defined by Rule I.1- of Algorithm 25, then they are different by definition. If they are defined by Rule I.2-, then d^T_i belongs to a column C^T_j with $j < i$, which implies that it was plotted by Rule 2.(a)ii. of Definition 24. Since $f(2i) = 2i$, this implies that its insertion label is a, hence the insertion labels of the dots of C^T_i are different by Rule I.2-a) of Algorithm 25. $\square$

Lemma 27. *Let $f \in \mathcal{P}_n$ and $T = \Phi(f) \in \mathcal{T}_n$. The type label of a dot of T is α if and only if its insertion label is a.*

Proof. Assume that the Lemma is true for the dots of the columns $C^T_{j+1}, C^T_{j+2}, \ldots, C^T_n$ for some $j \in [n]$. First of all, we prove that for all $k \in [j+1, n]$, if $d^T_{k,\min} = d^T_{n+k}$, then the even dot of the column C^T_k is labeled with β^e_0 : if so, then by Lemma 26, the two dots of C^T_k have different insertion labels, hence different type labels by hypothesis (because $k > j$). With precision, the dot whose type label is β has been (f, j)-inserted in T^k with the label $l_e = b$ at the height $h_e = f(2k)/2 - k = 0$, so the pistol label of this dot is β^e_0 in view of Rule III.1- of Algorithm 13.

Now, let $d_i^T \in C_j^T$, $j' = j + h$ where $h \in [0, n-j]$ is the digital label of d_i^T, and $i' = \pi_j^T(i) \in \{j', n+j'\}$.

— If the type label of d_i^T is defined by Rule II.1-a) of Algorithm 13, then $f(2j') = 2j'$: indeed, by Remark 14(e), the type label of the odd dot of $C_{j'}^T$ is α, so by hypothesis the insertion labels of the dots of $C_{j'}^T$ are a and b, which implies that $(l_o, l_e) = (a, b)$ at the j'-th step of Algorithm 25, hence $h_e = \delta_e = f(2j)/2 - j$ is the digital label 0 of the even dot of $C_{j'}^T$. Since $j' > j$, the dot $d_{i'}^T$ has been plotted by Rule 2.(a) of Definition 24, and since $f(2j') = 2j'$, its insertion label equals a (respectively b) if and only if $i' = j'$ (respectively $i' = n + j'$) following Rule 2.(a)ii. (respectively Rule 2.(a)i.) of Definition 24, hence if and only if its type label is α (respectively β) in this context.

— If the type label of d_i^T is defined by Rule II.1-b) of Algorithm 13, let $(c, \bar{c})$ be defined as (a, b) if the two dots of $C_{j'}^T$ have different type labels, as (b, a) otherwise. The aim of this part is to prove that the insertion label of d_i^T is c (respectively $\bar{c}$) if its type label is γ (respectively $\bar{\gamma}$), *i.e.*, if $d_{i'}^T = d_{j',\min}^T$ (respectively if $d_{i'}^T \neq d_{j',\min}^T$). Now, if $d_{i'}^T = d_{j',\min}^T$ (respectively $d_{i'}^T \neq d_{j',\min}^T$), then in this context we know that $i' = j'$ (respectively $i' = n + j'$), because we showed at the beginning of the proof that if $d_{j',\min}^T = d_{n+j'}^T$ then one of the dots of $C_{j'}^T$ is labeled with β_0^e, which is not true by hypothesis. So the insertion label of $d_{i'}^T$ is c (respectively $\bar{c}$) following Rule I.2- of Algorithm 25 (respectively following that very same rule and the fact that $d_{n+j'}^T$ has been plotted by Rule 2.(b) of Definition 24).

— If the type label of d_i^T is defined by Rule II.2-a) of Algorithm 13, it is straightforward that the two dots of C_j^T are $d_{i_o}^T$ (labeled with α_0^o) and $d_{i_e}^T$ (labeled with β_0^e) where $i_o = \left(\pi_j^T\right)^{-1}(j)$ and $i_e = \left(\pi_j^T\right)^{-1}(n+j)$. Now, at the j-th step of Algorithm 25, we prove that $(l_o, l_e) = (a, b)$ and that the insertion labels of $d_{i_o}^T$ and $d_{i_e}^T$ are respectively a and b.

  * If (l_o, l_e) has been defined by Rule I.1- of Algorithm 25, it is straightforward that it is (a, b), and by Rule 1.(a) (respectively Rule 1.(b)) of Definition 24, the dot $d_{i_o}^T = d_j^T$ belongs to C_j^T and is labeled with a (respectively the dot $d_{i_e}^T$ was plotted by inserting a dot labeled with b in the box $C_j^T \cap R_{n+j}^T$).

  * Else, let d be the dot of R_j^T. It has been plotted by Rule 2.(a)ii. of Definition 24, so, since $f(2j-1) = f(2j) = 2j$ by hypothesis, it implies that its insertion label is a, hence $(l_o, l_e) = (a, b)$ by Rule I.2-a) of Algorithm 25. Consequently, by Rule 1.(b) of Definition 24, the dot $d_{i_o}^T$ (respectively $d_{i_e}^T$) was plotted by inserting a dot labeled with a (respectively b) in the box $C_j^T \cap R_j^T$ (respectively $C_j^T \cap R_{n+j}^T$).

— If the type label of d_i^T is defined by Rule II.2-b) of Algorithm 13, the insertion label of the other dot d of C_j^T being a if and only if its type label is $t = \alpha$ has already been proved above.

  * If $t = \alpha$, suppose first that the type label of d_i^T is α, *i.e.*, that $i \neq i' = j$. The equality $i \neq i'$ implies that d_i^T has been plotting by Rule 1.(b) of Definition 24, and the equality $i' = j$ then implies that its insertion label is a. Afterwards, if the type label of d_i^T is β, then either $i = i'$ or $i' = n + j$. If $i' = n + j$, then d_i^T has been plotted by Rule 1.(b) of Definition 24 with the insertion label b. Assume finally that $i = i' = j$. Then, at the j-th step of Algorithm 25, the pair of labels (l_o, l_e) has been defined by Rule I.1-, *i.e.*, it equals (a, b). Since the insertion label of d is a by hypothesis, then the insertion label of d_i^T is b.
  * If $t = \beta$, since the digital label of d is not 0 by hypothesis, by Rule III. of Algorithm 13 no dot of C_j^T is labeled with β_0^e. Following the beginning of this proof, this implies that $d_{j,\min}^T = d_j^T$. In view of this, the type label of d_i^T is α if and only if $i' = j$. If $i' = j$, either the pair of labels (l_o, l_e) is defined by Rule I.1- of Algorithm 25 hence is (a, b), and the insertion label of d_i^T is a because that of d is b by hypothesis, or d_i^T has been plotted by Rule 1.(b) of Definition 24, so its insertion label is a because $i' = j$. Assume finally that $i' \neq j$, *i.e.*, that $i' = n+j$ and the type label of d_i^T is β. It has necessarily been plotted by Rule 1.(b) of Definition 24, and its insertion label is b because $i' = n + j$.

So the Lemma is true by induction. □

Proposition 28. *The composition $\varphi \circ \Phi$ is the identity map of $\mathcal{P}_n$.*

Proof. Let $f \in \mathcal{P}_n$, $T = \Phi(f) \in \mathcal{T}_n$ and $g = \varphi(T) \in \mathcal{P}_n$. We want to prove that $g = f$. Let $j \in [n]$. By Part II. of Algorithm 25 and Definition 24, we know that one of the dots d_o of C_j^T has the digital label $h_o = \delta_o = f(2j-1)/2 - j$, and that the other dot d_e of C_j^T has the digital label h_e that has the following property :

— if $\delta_o = \delta_e$ and $l_e = a$, then $h_e = 0$;

— otherwise $h_e = \delta_e = f(2j)/2 - j$.

Also, by Lemma 27, the type label of d_o (respectively d_e) is α if and only if $l_o = a$ (respectively $l_e = a$).

We prove now that the parity labels of d_o and d_e are o and e respectively, and, at the same time, that $g_{|\{2j-1,2j\}} = f_{|\{2j-1,2j\}}$.

1 - If their type labels are different, we know that their insertion labels are different, and by Part I. of Algorithm 25 this implies that $l_o = a$ and $l_e = b$, hence the type labels of d_o and d_e are α and β respectively. As a result, by Part III.1- of Algorithm 13, the parity labels of d_o and d_e are o and e. Also, since $l_e \neq a$, the digital labels of d_o and d_e are δ_o and δ_e respectively, so, by Definition 16, we have $g(2j-1) = 2(j+\delta_o) = f(2j-1)$ and $g(2j) = 2(j+\delta_e) = f(2j)$.

2 - Otherwise, the insertion labels of d_o and d_e are the same, so they have been defined by Rule I.2-(b) of Algorithm 25.

1) If the type label of d_o and d_e is β, their insertion label is b, so it has been defined by Rule I.2-(b)i. of Algorithm 25. In particular $\delta_o < \delta_e$, and since in that case δ_o and δ_e are the digital labels of d_o and d_e respectively, by Rule III.2-b) of Algorithm 13 the parity labels of d_o and d_e are o and e respectively, and by Definition 16, we have $g(2j-1) = 2(j+\delta_o) = f(2j-1)$ and $g(2j) = 2(j+\delta_e) = f(2j)$.

2) If their type labels are α, their insertion label is a, which has been defined by Rule I.2-(b)ii. of Algorithm 25. In particular $\delta_o \geq \delta_e$. Let h_o and h_e be the digital labels of d_o and d_e respectively. Since we are in the context III.2- of Algorithm 13, we know that $h_o \neq h_e$. If $\delta_o = \delta_e$ then $(h_o, h_e) = (\delta_o, 0)$, otherwise $(h_o, h_e) = (\delta_o, \delta_e)$, so in any case $h_o > h_e$ and by Rule III.2-a) of Algorithm 13 the parity labels of d_o and d_e are o and e respectively. Afterwards,

— If $\delta_o = \delta_e$, hence $(h_o, h_e) = (\delta_o, 0)$, then by Definition 16 we have $g(2j-1) = 2(j+\delta_o) = f(2j-1)$ and $g(2j) = 2(j+\delta_o) = 2(j+\delta_e) = f(2j)$;

— otherwise $(h_o, h_e) = (\delta_o, \delta_e)$ and, by Definition 16, we have $g(2j-1) = 2(j+\delta_o) = f(2j-1)$ and $g(2j) = 2(j+\delta_e) = f(2j)$.

So $g_{|\{2j-1,2j\}} = f_{|\{2j-1,2j\}}$ for all $j \in [n]$. □

Proposition 28 implies that the maps $\phi : \mathcal{P}_n \to \mathcal{T}_n$ and $\varphi : \mathcal{T}_n \to \mathcal{P}_n$ are respectively injective and surjective. We intend now to make the image of Φ explicit.

Definition 29. Let $T \in \mathcal{T}_n$, we define $\mathcal{S}(T) \subset [n]$ as the set of integers $i \in [n]$ such that :

— the dot d_{n+i}^T is not free;

— the twin dots d_i^T and d_{n+i}^T are not in the same column;

— no dot of C_i^T has the pistol label β_0^e.

For all such i, we define $\mu_T(i)$ as 1 if $d_{i,\min}^T = d_i^T$, as -1 otherwise.

Afterwards, we define $\mathcal{C}(T) \subset [n-1]$ as the set of integers $j \in [n]$ such that C_j^T contains twin dots, say, the dots d_i^T and d_{n+i}^T for some $i \in [n]$, such that no dot of C_i^T is labeled with β_0^e. For all such j, we define $t_T(j)$ as the type label of d_i^T.

Remark 30. For all $T \in \mathcal{T}_n$, by Proposition 21, we have the formula

$$\mathrm{fr}(T) + \#\mathcal{S}(T) + \#\mathcal{C}(T) = \mathrm{ng}(T) = \mathrm{ndf}(f)$$

where $f = \varphi(T)$.

Remark 31. In the proof of Lemma 27, we showed that for all T of the kind $\phi(f)$ for some $f \in \mathcal{P}_n$, and for all $i \in [n]$, if $d_{i,\min}^T = d_{n+i}^T$, then one of the dots of C_i^T is labeled with β_0^e, hence $i \notin \mathcal{S}(T)$.

Definition 32. Let $\tilde{\mathcal{T}}_n$ be the subset of $\mathcal{T}_n$ made of the tableaux T that have the following properties : for all $i \in [n]$,

(a) if $i \in \mathcal{S}(T)$, then $d_{i,\min}^T = d_i^T$;

(b) if $j \in \mathcal{C}(T)$, then $t_T(j) = \alpha$.

Lemma 33. *The image of* $\Phi : \mathcal{P}_n \to \mathcal{T}_n$ *is a subset of* $\tilde{\mathcal{T}}_n$.

Proof. Let $f \in \mathcal{P}_n$ and $T = \Phi(f) \in \mathcal{T}_n$. The tableau T having the property (a) of Definition 32 comes from Remark 31. Now, let $j \in \mathcal{C}(T)$ and $i \geq j$ such that C_j^T contains the twin dots d_i^T and d_{n+i}^T. They both have the same digital label $i-j$ (which implies that $h_o = h_e = i-j$ at the j-th step of Algorithm 25), so by Part III. of Algorithm 13 their type labels are different. In view of Lemma 27, this implies that their insertion labels are $l_o = a$ and $l_e = b$. Now, at the beginning of the j-th step of Algorithm 25, the row $R_i^{T^j}$ is empty, so the (f,j)-insertion in T^j of a dot labeled with $l_o = a$ at the height $h_o = i-j$ leads to plotting a dot labeled with a in the box $C_j^{T^j} \cap R_i^{T^j}$ following Rule 1.(a) or Rule

2.(a)ii. of Definition 24. So d_i^T is the dot of C_j^T whose insertion label is a, and its type label is α by Lemma 27, hence T has the property (b) of Definition 32. $\square$

Definition 34. For all $T \in \mathcal{T}_n$ and $j \in [n]$, we define ϵ_j^T as the set of the pistol labels of the dots of C_j^T.

Lemma 35. *Let $f \in \mathcal{P}_n$ and $(T,T') \in \varphi^{-1}(f)^2$. Let $j \in [n]$ such that $\epsilon_j^T \neq \epsilon_j^{T'}$. There exists $k \in \mathcal{C}(T) \cap \mathcal{C}(T') \cap [j-1]$ such that $t_T(k) \neq t_{T'}(k)$.*

Proof. Suppose first that the dots of C_j^T (respectively $C_j^{T'}$) have the same type label. If, with precision, the four dots of C_j^T and $C_j^{T'}$ have the same type label, since $\varphi(T) = \varphi(T') = f$, in view of Definition 16 the set of the digital labels of the dots of C_j^T equals the set of the digital labels of the dots of $C_j^{T'}$, and by Part III.2- of Algorithm 13 we have in fact $\epsilon_j^T = \epsilon_j^{T'}$, which is false by hypothesis. So, should T and T' be transposed, we can suppose that the dots of C_j^T (respectively $C_j^{T'}$) have the type label α (respectively β). By Part III.2- of Algorithm 13, we have $\epsilon_j^T = \left\{\alpha_{h_o}^o, \alpha_{h_e}^e\right\}$ with $h_o > h_e$, and $\epsilon_j^{T'} = \left\{\beta_{h'_o}^o, \beta_{h'_e}^e\right\}$ with $h'_o < h'_e$. Following Definition 16, this implies that $f(2j-1) \geq f(2j)$ and $f(2j-1) < f(2j)$, which is absurd.

So, should T and T' be transposed, we can suppose that the dots of C_j^T have different type labels, and by Part III.1- of Algorithm 13, we have $\epsilon_j^T = \left\{\alpha_{h_o}^o, \beta_{h_e}^e\right\}$ for some $(h_o, h_e) \in [0, n-j]^2$, which, following Definition 16, implies that $f(2j-1) = 2(j+h_o)$ and $f(2j) = 2(j+h_e)$. Now, if the dots of $C_j^{T'}$ had different type labels, then by Part III.1- of Algorithm 13 and Definition 16 we would have $\epsilon_j^T = \epsilon_j^{T'}$, so it is necessary that the dots of $C_j^{T'}$ have the same type label. This implies several things, enumerated hereafter.

(A) Since no dot of $C_j^{T'}$ is labeled with β_0^e in view of Remark 14(e), then by Lemma 20 the integer $2j$ is not a fixed point of f, hence no dot of C_j^T is labeled with β_0^e (*i.e.*, we have $h_e > 0$).

(B) Suppose that $d_{j,\min}^{T'} \in C_j^{T'}$ (hence $d_{j,\min}^{T'} = d_j^{T'}$). Then its type label is defined by Rule II.2- of Algorithm 13. Since no dot of $C_j^{T'}$ has the pistol label β_0^e, then it is with precision defined by Rule II.2-b). Since it is $d_{j,\min}^{T'}$, whether it is defined by Rule II.2-b)i. or Rule II.2-b)ii., the type labels of the two dots of $C_j^{T'}$ are different, which is absurd. So $d_{j,\min}^{T'}$ belongs to a column $C_{j'}^{T'}$ with $j' < j$.

(C) Consequently, by Lemma 17, we know that $d_{j,\min}^T \in C_{j'}^T$.

(D) Following Remark 14(e), the type label of $d^T_{j,\min}$ is α, whereas the type label of $d^{T'}_{j,\min}$ is β.

Now, if $j' \in \mathcal{C}(T)$, then with precision $d^T_{j,\min} = d^T_j$ because the other dot of $C^T_{j'}$ is d^T_{n+j} (in particular $t_{j'}(T) = \alpha$ following (D)), also $\epsilon^T_{j'} = \left\{\alpha^o_{j-j'}, \beta^e_{j-j'}\right\}$ in view of Rule II.1-b) and Rule III.1- of Algorithm 13, and by Definition 16 we have $f(2j'-1) = f(2j') = 2j$. Still by Definition 16, since one of the dots of $C^{T'}_{j'}$ (the dot $d^{T'}_{j,\min}$ by (D)) doesn't have the type label α, then the two dots of $C^{T'}_{j'}$ have the same digital label $j - j'$ (incidentally, by Part III.2- of Algorithm 13 this implies that they have different type labels, *i.e.*, that the dot of $C^{T'}_{j'}$ that is not $d^{T'}_{j,\min}$ has the type label α). Let d be the twin dot of $d^{T'}_{j,\min}$, *i.e.*, the dot defined as $d^{T'}_j$ if $d^{T'}_{j,\min} = d^{T'}_{n+j}$, as $d^{T'}_{n+j}$ if $d^{T'}_{j,\min} = d^{T'}_j$. Since $d^{T'}_{j,\min} \in C^{T'}_{j'}$ and the other dot of $C^{T'}_{j'}$ has the digital label $j - j'$, by Definition 8 this implies that d is located in a column $C^{T'}_{j''}$ with $j'' \leq j'$. By Definition of $d^{T'}_{j,\min}$, we have $j'' \geq j'$ hence $j'' = j'$. In other words, the integer j' belongs to $\mathcal{C}(T')$ and $d^{T'}_{j,\min} = d^{T'}_j$ has the type label β in view of (D), hence $t_{j'}(T') = \beta \neq t_{j'}(T)$, which is exactly the statement of the lemma.

Otherwise (if $j' \notin \mathcal{C}(T)$), suppose that $\epsilon^T_{j'} = \epsilon^{T'}_{j'}$. Since the type labels of $d^T_{j,\min} \in C^T_{j'}$ and $d^{T'}_{j,\min} \in C^{T'}_{j'}$ are respectively α and β and their digital label $j-j'$, in view of Part III. of Algorithm 13 this implies that $\epsilon^T_{j'} = \epsilon^{T'}_{j'} = \left\{\alpha^o_{j-j'}, \beta^e_{j-j'}\right\}$. In particular, the two dots of C^T_j have the same digital label $j - j'$. By Definition 8, this means that both the twin dots d^T_j and d^T_{n+j} are located in columns $C^T_{j''}$ with $j'' \leq j'$. By Definition of $d^T_{j,\min} \in C^T_{j'}$, this forces those two dots to be the dots of $C^T_{j'}$, which contradicts $j' \notin \mathcal{C}(T)$ since no dot of C^T_j is labeled with β^e_0 in view of (A). So, necessarily $\epsilon^T_{j'} \neq \epsilon^{T'}_{j'}$ and we are in the situation of the beginning of the proof with j being replaced by j'. This produces some integer $j^{(2)} \in [j'-1]$ such that $d^T_{j',\min} \in C^T_{j^{(2)}}$ and $d^{T'}_{j',\min} \in C^{T'}_{j^{(2)}}$ do not have the same type label. If the statement of the Lemma is false, then it in fact produces a strictly decreasing sequence of integers $(j^{(2)}, j^{(3)}, \ldots) \in [n]^{\mathbb{N}}$, which is absurd. So the Lemma is true. $\square$

Proposition 36. *The map $\varphi_{|\tilde{T}_n}$ is injective.*

Proof. Let $(T_1, T_2) \in \left(\tilde{T}_n\right)^2$ such that $\varphi(T_1) = \varphi(T_2) =: f \in \mathcal{P}_n$. By Lemma 35 and the property (b) of Definition 32, we know that $\epsilon^{T_1}_j = \epsilon^{T_2}_j$ for all $j \in [n]$. Let $\mathcal{H}(j)$ be the induction hypothesis that for all $k \in [j-1]$ and $i \in [k, 2n]$, the dot $d^{T_1}_i$ belongs to $C^{T_1}_k$ only if $d^{T_2}_i$ belongs to $C^{T_2}_k$. Hypothesis $\mathcal{H}(1)$ is obviously true. Suppose that Hypothesis

$\mathcal{H}(j)$ is true for some $j \in [n-1]$. The fact that the dots of T_1 and T_2 are located at the same levels in their $j-1$ first columns implies that $\pi_j^{T_1} = \pi_j^{T_2} =: \pi_j$. Let $i_1 \in [j,n]$ such that $d^1 = d_{i_1}^{T_1} \in C_j^{T_1}$, we consider the digital label $h \in [0, n-j]$ of d^1, $j' = j+h$, and $i_1' = \pi_j(i_1) \in \{j', n+j'\}$. We denote by $d^2 = d_{i_2}^{T_2}$ the dot of $C_j^{T_2}$ that has the same pistol label as d^1. We intend to prove that $d^2 = d_i^{T_2}$, *i.e.*, that $i_1' = i_2'$ because π_j is injective. Since $\epsilon_j^{T_1} = \epsilon_j^{T_2}$ and the column $C_j^{T_1}$ has a dot labeled with β_0^e if and only if the column $C_j^{T_2}$ has a dot labeled with β_0^e (in view of Lemma 20), then the type label of d^1 and d^2 is defined by the same rule among Rules II.1-a),II.1-b),II.2-a),II.2-b)i. and II.2-b)ii. of Algorithm 13. The equality $i_1' = i_2'$ is then straightforward in each case in view of $d_{j',\min}^{T_1} = d_{j'}^{T_1}$ and $d_{j',\min}^{T_2} = d_{j'}^{T_2}$ following the condition (a) of Definition 32. So Hypothesis $\mathcal{H}(j)$ is true. By induction, Hypothesis $\mathcal{H}(n)$ is true, thence $T_1 = T_2$. □

Corollary 37. *The map $\Phi \circ \varphi_{|\tilde{T}_n}$ is the identity map of $\tilde{T}_n$. (In view of Lemma 33, it implies that the image of $\Phi : \mathcal{P}_n \to \mathcal{T}_n$ is exactly $\tilde{\mathcal{T}}_n$.)*

Proof. Let $T \in \tilde{T}_n$, $f = \varphi(T) \in \mathcal{P}_n$ and $T' = \phi(f) \in \tilde{T}_n$. By Proposition 28, we know that $\varphi(T') = f$, so $T = T'$ in view of Proposition 36. □

6. Proof of Theorem 6

We now know that the injection $\phi : \mathcal{P}_n \hookrightarrow \mathcal{T}_n$ induces a bijection from $\mathcal{P}_n$ to $\tilde{\mathcal{T}}_n \subset \mathcal{T}_n$, whose inverse map is $\varphi_{|\tilde{\mathcal{T}}_n}$, and which maps the statistic $\overrightarrow{\mathrm{ndf}}$ to the statistic $\overrightarrow{\mathrm{ng}}$ in view of Proposition 21. To finish the proof of Theorem 6, it remains to show Formula (2) for all $f \in \mathcal{P}_n$, which we do in this section with the help of Algorithm 42 and Algorithm 44, which compute $\varphi^{-1}(f)$.

Definition 38. Let $f \in \mathcal{P}_n$ and $j \in [n]$. We define $\mathcal{T}_f(j)$ as the set of the tableaux $T \in \varphi^{-1}(f)$ such that $j \in \mathcal{C}(T)$. Let $T_0 \in \mathcal{T}_f(j)$. We define $\mathcal{T}(T_0, j)$ as the set of tableaux $T \in \varphi^{-1}(f)$ such that $\mathcal{C}(T) \cap [j-1] = \mathcal{C}(T_0) \cap [j-1]$ and $t_T(k) = t_{T_0}(k)$ for all $k \in \mathcal{C}(T) \cap [j-1]$ (this set is not empty because it contains T_0).

Lemma 39. *Using the notations of Definition 38, if $T \in \mathcal{T}(T_0, j)$, then the dots of C_j^T have the same levels as the dots of $C_j^{T_0}$, and $j \in \mathcal{C}(T)$.*

Proof. Let $i \in [j,n]$ such that the dots of $C_j^{T_0}$ are the twin dots $d_i^{T_0}$ and $d_{n+i}^{T_0}$. Since $j \in \mathcal{C}(T_0)$, we have $i > j$ (otherwise the dot $d_{n+i}^{T_0}$ would have the pistol label β_0^e). Consequently, the type labels of the twin dots of $C_j^{T_0}$ are both defined by Rule II.1-b) of Algorithm 13, so they are different. Let $j' \leq j$ such that $d_{j,\min}^{T_0} \in C_{j'}^{T_0}$. With precision, we have $j' < j$: otherwise, we would have $d_{j,\min}^{T_0} = d_j^{T_0} \in C_j^{T_0}$ and i would equal j, which is false. The type label of $d_{j,\min}^{T_0}$ is α following Rule II.1-b) of Algorithm 13. Now, if $T \in \mathcal{T}(T_0, j)$, suppose that the type label of $d_{j,\min}^{T}$ is β. By Lemma 17, we know that $d_{j,\min}^{T} \in C_{j'}^{T}$. Also, by hypothesis and Lemma 35, it is necessary that $\epsilon_j^{T_0} = \epsilon_j^{T}$. Since $d_{j,\min}^{T_0}$ is labeled with $\alpha_{j-j'}$ and $d_{j,\min}^{T}$ with $\beta_{j-j'}$, it is necessary that $\epsilon_j^{T_0} = \epsilon_j^{T} = \left\{\alpha_{j-j'}^{o}, \beta_{j-j'}^{e}\right\}$. In particular, by Remark 14(f), the dots of $C_{j'}^{T_0}$ are $d^{T_0}_{\left(\pi_{j'}^{T_0}\right)^{-1}(j)}$ and $d^{T_0}_{\left(\pi_{j'}^{T_0}\right)^{-1}(n+j)}$, and the dots of $C_{j'}^{T}$ are $d^{T}_{\left(\pi_{j'}^{T}\right)^{-1}(j)}$ and $d^{T}_{\left(\pi_{j'}^{T}\right)^{-1}(n+j)}$. Now, by Definition 12 of $d_{j,\min}^{T_0} \in C_{j'}^{T_0}$ and $d_{j,\min}^{T} \in C_{j'}^{T}$, these two dots are $d_j^{T_0}$ and $d_{n+j}^{T_0}$ (respectively d_j^{T} and d_{n+j}^{T}). In other words, since no dot of $C_j^{T_0}$ is labeled with β_0^e (hence no dot of C_j^T is labeled with β_0^e in view of Lemma 20), we have $j' \in \mathcal{C}(T_0) \cap \mathcal{C}(T)$ and $t_{T_0}(j') = \alpha \neq \beta = t_T(j')$, which is absurd by hypothesis. So the type label of $d_{j,\min}^{T}$ is α, hence the two dots of C_j^T have different type labels. Afterwards, since $\varphi(T_0) = \varphi(T) = f$ and the type labels of the dots of $C_j^{T_0}$ and C_j^T are not both α, by Definition 16 the digital label of the odd dot (respectively even dot) of C_j^T is the same as the digital label of the odd dot (respectively even dot) of $C_j^{T_0}$. Since the dots of $C_j^{T_0}$ are twins, they have the same digital label $i-j$, and so do the dots of C_j^T. Consequently, by Remark 14(f), the dots of C_j^T are $d^{T}_{\left(\pi_{j}^{T}\right)^{-1}(i)} = d_i^T$ and $d^{T}_{\left(\pi_{j}^{T}\right)^{-1}(n+i)} = d_{n+i}^T$.

Finally, let $f = \varphi(T_0)$. By hypothesis $j \in \mathcal{C}(T_0)$, so $f(2j) > 2j$ in view of Lemma 20. Since $f = \varphi(T)$, then Lemma 20 also implies that $j \in \mathcal{C}(T)$. □

Definition 40. Using the notations of Definition 38, by Lemma 39 we can decompose $\mathcal{T}(T_0, j)$ into the disjoint union $\mathcal{T}(T_0, j, \alpha) \sqcup \mathcal{T}(T_0, j, \beta)$ where, for all $\gamma \in \{\alpha, \beta\}$, the subset $\mathcal{T}(T_0, j, \gamma)$ is the set of the tableaux $T \in \mathcal{T}(T_0, j)$ such that $t_T(j) = \gamma$.

6.1. **An operation on $\mathcal{S}(T)$.**

Definition 41. Let $T \in \mathcal{T}_n$ and $\{i_1, i_2, \ldots, i_m\}_< = \mathcal{S}(T)$. For all $k \in [m]$, we define μ_k^T as 1 if $d_{i_k,\min}^T = d_{i_k}^T$, as -1 otherwise.

Algorithm 42. Let $T \in \mathcal{T}_n$ and $\{i_1, i_2, \ldots, i_m\}_< = \mathcal{S}(T)$. We consider $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \{-1, 1\}^m$, and we define a tableau $S_\mu(T)$ as follows. Let T'_1 be the empty 1-tableau. For j from 1 to n, suppose that T'_j is a j-tableau (which is true for $j=1$). In particular the map $\pi_j^{T'_j}$ is defined. Let $(r_1,r_2) \in [j,2n]^2$ such that $d^T_{r_1}$ and $d^T_{r_2}$ are the two dots of C^T_j. For all $p \in \{1,2\}$, we consider the integer $r'_p = \pi^T_j(r_p) \in [j,n] \sqcup [n+j,2n]$.

1 - If $r'_p \in \{i_k, n+i_k\}$ for some $k \in [m]$, let $(r_\gamma, r_{\bar\gamma})$ be the pair $(i_k, n+i_k)$ if $\mu_k = 1$, or the pair $(n+i_k, i_k)$ if $\mu_k = -1$. We define the integer r''_p as $\left(\pi_j^{T'_j}\right)^{-1}(r_\gamma)$ if the type label of $d^T_{r_p}$ is the same as that of $d^T_{i_k,\min}$, as $\left(\pi_j^{T'_j}\right)^{-1}(r_{\bar\gamma})$ otherwise.

2 - Otherwise, we define the integer r''_p as $\left(\pi_j^{T'_j}\right)^{-1}(r'_p)$.

Since $\pi_j^{T'_j}$ is bijective, the integers r''_1 and r''_2 are different, and by definition the rows $R^{T'_j}_{r''_1}$ and $R^{T'_j}_{r''_2}$ are empty. We then define the $(j+1)$-tableau T'_{j+1} by plotting two dots in the boxes $C_j^{T'_j} \cap R^{T'_j}_{r''_1}$ and $C_j^{T'_j} \cap R^{T'_j}_{r''_2}$. This algorithm produces a $(n+1)$-tableau T'_n which we denote by $S_\mu(T)$, and which belongs to $\mathcal{T}_n$ as a $(n+1)$-tableau.

For example, in Figure 11, we consider a tableau $T \in \mathcal{T}_7$ such that $\mathcal{S}(T) = \{3,5\}$ and $(\mu^T_1, \mu^T_2) = (-1,1)$. In this figure, the tableau T is depicted with its pistol labeling.

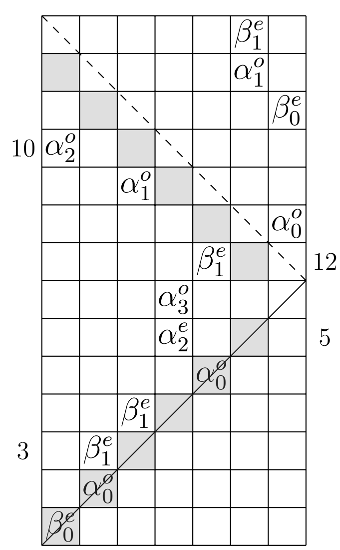


FIGURE 11. Tableau $T \in \mathcal{T}_7$ such that $\mathcal{S}(T) = \{3,5\}$.

In Figure 12, we represent the pistol-labeled versions of the tableaux $S_\mu(T)$ for all $\mu \in \{-1,1\}^2$.

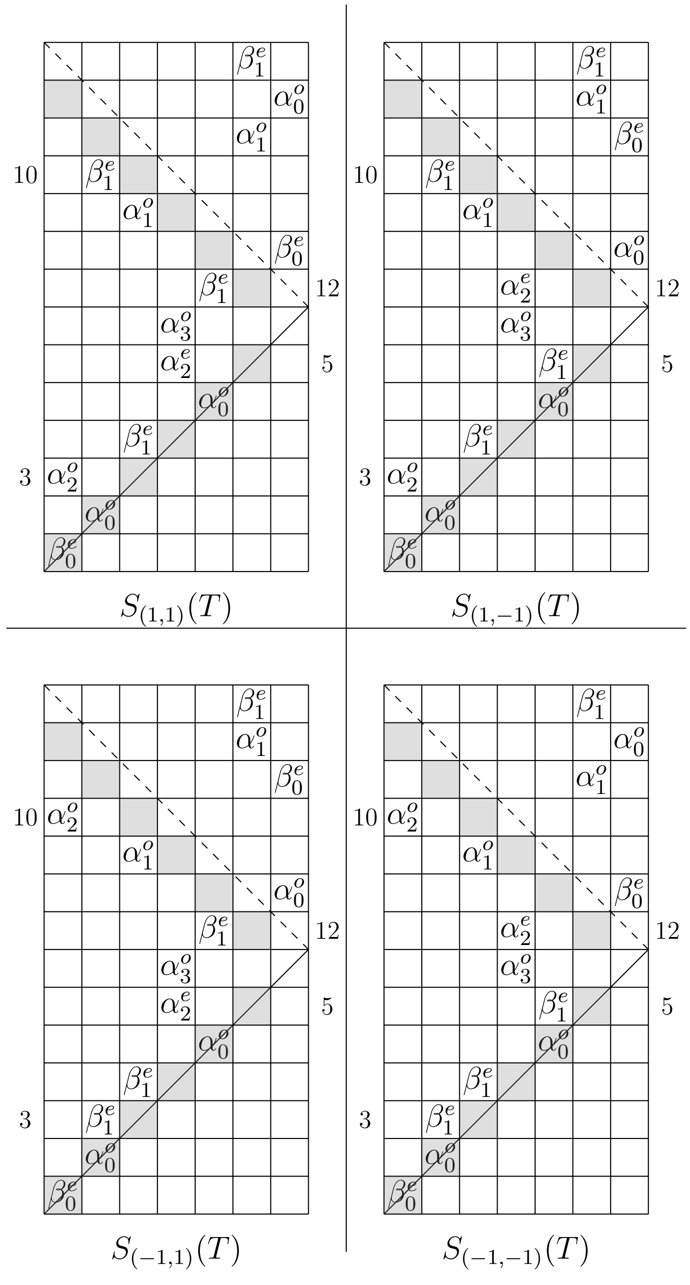


FIGURE 12. The tableaux $S_\mu(T)$ for all $\mu \in \{-1,1\}^2$.

Note that if $\mu_0 = (\mu_1^T, \mu_2^T)$ $(= (-1,1))$, then $S_{\mu_0}(T) = T$ (in the bottom left-hand corner in Figure 12). Afterwards, for all $\mu \in \{-1,1\}^2$

and $j \in [7]$, we have $\epsilon_j^{S_\mu(T)} = \epsilon_j^T$, consequently $\varphi(S_\mu(T)) = \varphi(T)$; also, we have $\mathcal{C}(S_\mu(T)) = \mathcal{C}(T) = \{3\}$, and $t_{S_\mu(T)}(3) = \beta = t_T(3)$. All these remarks are generalized in the easy following result.

Proposition 43. *Let $T \in \mathcal{T}_n$, $f = \varphi(T) \in \mathcal{P}_n$ and $\{i_1, i_2, \dots, i_m\}_< = \mathcal{S}(T)$. For all $\mu = (\mu_1, \mu_2, \dots, \mu_m) \in \{-1,1\}^m$, the tableau $S_\mu(T)$ is the unique tableau $T' \in \varphi^{-1}(f)$ such that :*

— *$\mathcal{S}(T') = \mathcal{S}(T)$ and for all $k \in [m]$, we have $\mu_k^{T'} = \mu_k$;*

— *$\mathcal{C}(T') = \mathcal{C}(T)$ and for all $j \in \mathcal{C}(T)$, we have $t_T(j) = t_{t'}(j)$.*

6.2. **An operation on $\mathcal{C}(T)$.**

Algorithm 44. Let $T \in \varphi^{-1}(f)$ for some given $f \in \mathcal{P}_n$, $j_0 \in \mathcal{C}(T)$ and $\gamma \in \{\alpha, \beta\}$. We define a tableau $M_{j_0,\gamma}(T)$ as follows. First of all, let $\{i_1, i_2, \dots, i_m\}_< = \mathcal{S}(T)$, we define $\tilde{T} \in \varphi^{-1}(f)$ as $S_\mu(T)$ where μ is the sequence $(1, 1, \dots, 1) \in \{-1,1\}^m$. Afterward, we define $T^{j_0+1} \in \mathfrak{T}_n^{j_0+1}$ as follows. Let $(c, \bar{c})$ be defined as (a, b) if $\gamma = \alpha$, as (b, a) otherwise.

— For all $j < j_0$ and $i \in [j, 2n]$, if the column $C_j^{\tilde{T}}$ contains the dot $d_i^{\tilde{T}}$ whose type label is α (respectively β), then the column $C_j^{T^{j_0+1}}$ contains the dot $d_i^{T^{j_0+1}}$ labeled with the letter a (respectively b).

— If d_i^T and d_{n+i}^T are the twin dots of $C_{j_0}^{\tilde{T}}$, then the column $C_{j_0}^{T^{j_0+1}}$ contains the twin dots $d_i^{T^{j_0+1}}$ and $d_{n+i}^{T^{j_0+1}}$ labeled with the letters c and $\bar{c}$ respectively.

Afterwards, we define $M_{j_0,\gamma}(T) \in \mathcal{T}_n$ as the tableau produced by the restriction of Algorithm 25 from step $j_0 + 1$ (using T^{j_0+1}) to step n.

Remark 45. With the notations of Algorithm 44, for all $\mu \in \{-1,1\}^m$, we have the equality $M_{j_0,\gamma}(T) = M_{j_0,\gamma}(S_\mu(T))$.

For example, consider the tableau $T \in \mathcal{T}_7$ of Figure 11, with $\mathcal{S}(T) = \{3, 5\}$ and $\mathcal{C}(T) = \{3\}$. To compute $M_{3,\alpha}(T)$ and $M_{3,\beta}(T)$, we first need to make $f = \varphi(T)$ explicit, which can be read from the pistol labeling of T in Figure 11 (or from any pistol labeling of the tableaux depicted in Figure 12 for that matter) :

$$f = (6, \mathbf{2}, 4, \mathbf{6}, 8, \mathbf{8}, 14, \mathbf{12}, 10, \mathbf{12}, 14, \mathbf{14}, 14, \mathbf{14}) \in \mathcal{P}_7,$$

whose graphical representation is depicted in Figure 13.

Following the notations of Algorithm 44, we have $\tilde{T} = S_{(1,1)}(T) \in \mathcal{T}_7$, which is represented at the top left-hand corner of Figure 12. We then use $\tilde{T}$ to compute the insertion labeled versions of $M_{3,\alpha}(T)$ and $M_{3,\beta}(T)$ in Figure 14.

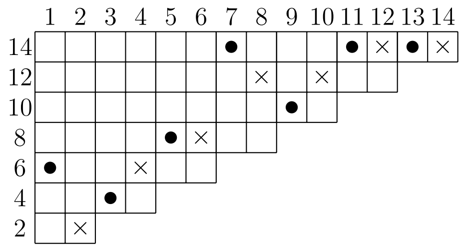


FIGURE 13. The surjective pistol $f = \varphi(T) \in \mathcal{P}_7$.

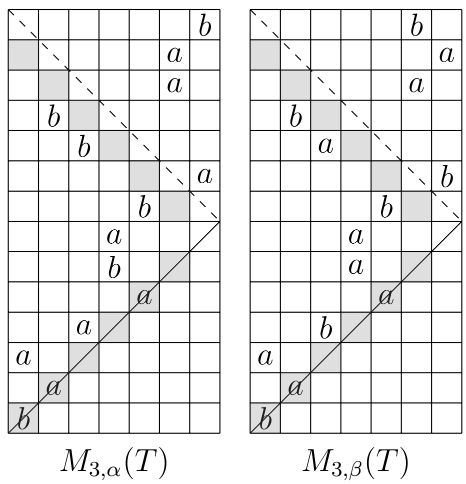


FIGURE 14. The insertion-labeled versions of the tableaux $M_{3,\alpha}(T)$ and $M_{3,\beta}(T)$.

Lemma 46. *With the notations of Algorithm 44, let $T' = M_{j_0,\gamma}(T)$ and $i \in [n]$. If $d^{T'}_{i,\min} = d^{T'}_{n+i}$, then $f(2i) = 2i$ and the two dots of $C^{T'}_i$ have different insertion labels.*

Proof. Let $j \in [n]$ such that $C^{T'}_j$ contains $d^{T'}_{n+i}$. Since $C^{T'}_{j_0}$ contains twin dots, we know that $j \neq j_0$. If $j < j_0$, we also have $d^{\tilde{T}}_{n+i} = d^{\tilde{T}}_{i,\min}$. By Definition of $\tilde{T}$, this implies that $i \notin \mathcal{S}(\tilde{T})$. Since $d^{\tilde{T}}_{n+i}$ is not free, it is then necessary that $C^{\tilde{T}}_i$ contains a dot labeled with β^e_0, hence $f(2i) = 2i$ by Lemma 20. If $j > j_0$, the proof of $f(2i) = 2i$ is the same as in the proof of Lemma 26.

Afterwards, since $f(2i) = 2i$ and $f(2j_0) \neq 2j_0$ because $j_0 \in \mathcal{C}(\tilde{T})$, we have $i \neq j_0$. If $i < j_0$, then by Lemma 20 and Remark 14(e) the type labels of the dots of $C^{\tilde{T}}_i$ are different, hence the insertion labels of the dots of T' are different by definition. If $i > j_0$, the proof of the insertion labels of the dots of T' being different is the same as in the proof of Lemma 26. $\square$

Lemma 47. *With the notations of Algorithm 42, the type label of a dot of the tableau $T' = M_{j_0,\gamma_0}(T)$ is α if and only if its insertion label is a.*

Proof. The proof of the Lemma for the dots of $C_{j_0+1}^{T'}, C_{j_0+2}^{T'}, \ldots, C_n^{T'}$ is the same as that of Lemma 27 where Lemma 46 plays the role of Lemma 26.

Now, let $i \in [j_0, 2n]$ such that $C_{j_0}^{T'}$ contains the twin dots $d_i^{T'}$ and $d_{n+i}^{T'}$. Since $j_0 \in \mathcal{C}(\tilde{T})$, we know that $i > j_0$. Let $(c, \bar{c})$ and $\bar{\gamma}_0$ be defined as (a, b) and β respectively if $\gamma_0 = \alpha$, as (b, a) and α otherwise. By Definition the insertion labels of $d_i^{T'}$ and $d_{n+i}^{T'}$ are c and $\bar{c}$ respectively. Afterwards, suppose that $C_i^{T'}$ contains a dot d_e labeled with β_0^e. By Remark 14(e), the other dot d_o of $C_i^{T'}$ has the type label α. Since $i > j_0$, we know that the insertion labels of d_o and d_e are a and b respectively, so these insertion labels have been defined following Rule I.2-a) of Algorithm 25. Since the digital label of d_e is $0 = h_e = f(2i)/2 - i$ in this situation, by Lemma 20 the column $C_i^{\tilde{T}}$ contains a dot labeled with β_0^e, which contradicts $j_0 \in \mathcal{C}(\tilde{T})$. So no dot of $C_i^{T'}$ is labeled with β_0^e, and the type labels of $d_i^{T'}$ and $d_{n+i}^{T'}$ are defined by Rule II.1-b) of Algorithm 13. Since the insertion labels of the dots of $C_i^{T'}$ are defined by Rule I.2- of Algorithm 25, and since they are different labels if and only if these two dots of different type labels because $i > j_0$, then by Rule I.1-b) of Algorithm 13, the type labels of $d_i^{T'}$ and $d_{n+i}^{T'}$ are respectively α and β if $c_0 = a$ (*i.e.*, if $\gamma_0 = \alpha$), as β and α otherwise (if $\gamma_0 = \beta$), in other words their type labels are respectively γ_0 and $\bar{\gamma}_0$, and the Lemma is true for these two dots.

Finally, a thorough analysis of the rules of Algorithm 13 following which the type labels of the dots of $C_1^{T'}, C_2^{T'}, \ldots, C_{j_0-1}^{T'}$ are defined show that these type labels are the same as in $\tilde{T}$, hence that the type label of each dot is α if and only if its insertion label is a by definition. $\square$

Proposition 48. *With the notations of Algorithm 42, the tableau $T' = M_{j_0,\gamma}(T)$ is an element of $\mathcal{T}(T, j_0, \gamma)$.*

Proof. Let $g = \varphi(T')$. The proof of $g_{|[2j_0+1,2n]} = f_{|[2j_0+1,2n]}$ is the same as in the proof of Proposition 28 where Lemma 47 plays the role of Lemma 20. Also, in the proof of Lemma 47, we show with precision that $\epsilon_j^{T'} = \epsilon_j^{\tilde{T}}$ for all $j \leq j_0$, so $g = f$.

Afterwards, for all $(i, j) \in [j_0]^2$, the twin dots $d_i^{T'}$ and $d_{n+i}^{T'}$ are by definition the two dots of $C_j^{T'}$ if and only if $d_i^{\tilde{T}}$ and $d_{n+i}^{\tilde{T}}$ are the two dots of $C_j^{\tilde{T}}$. In that case, the integer j belongs to $\mathcal{C}(T')$ if and only if no dot of $C_i^{T'}$ is labeled with β_0^e, which, in view of Lemma 20, is equivalent with $f(2i) > 2i$ and no dot of $C_i^{\tilde{T}}$ being labeled with β_0^e, hence with $j \in \mathcal{C}(\tilde{T})$. So $\mathcal{C}(T') \cap [j_0] = \mathcal{C}(\tilde{T}) \cap [j_0] = \mathcal{C}(T) \cap [j_0]$.

Finally, if $j < j_0$, the type label of $d_i^{T'}$ being α is equivalent with its insertion label being a (by Lemma 47), hence with the type label of $d_i^{\tilde{T}}$ being α by definition. By Proposition 43, this is also equivalent with the type label of d_i^T being α. In other words $t_{T'}(j) = t_T(j)$, and $T' \in \mathcal{T}(T, j_0)$. With precision, by Part I.2- of Algorithm 44, the insertion label of the lower dot of $C_{j_0}^{T'}$ is c defined as a if $\gamma_0 = \alpha$, as b otherwise. So its type label is γ_0 by Lemma 47, and $T' \in \mathcal{T}(T, j_0, \gamma)$. $\square$

Remark 49. Proposition 48 implies that for all $T \in \mathcal{T}_n$, $j \in \mathcal{C}(T)$ and $\gamma \in \{\alpha, \beta\}$, the set $\mathcal{T}(T, j, \gamma)$ is not empty.

Remark 50. For all $f \in \mathcal{P}_n$, we can now construct every element of $\varphi^{-1}(f)$. Indeed, every two elements T and T' of $\varphi^{-1}(f)$ are linked by a finite numbers of applications of the kind S_μ and M_{j_0,γ_0}. To prove it, it is enough to show that we can obtain $\phi(f) \in \varphi^{-1}(f)$ by applying a finite number of these applications to any element $T \in \varphi^{-1}(f)$. Recall that $\phi(f)$ is the unique element of $\tilde{T}_n$ in $\varphi^{-1}(f)$ because $\varphi_{|\tilde{T}_n}$ is injective by Proposition 36, so we only need to show that T is mapped to an element of $\tilde{T}_n$ by a finite number of these applications. We do that as follows.

If $\mathcal{C}(T)$ is not empty, let j_0 be its minimal element. We define T_1 as $M_{j_0,\alpha}(T)$. Afterwards, if $\mathcal{C}(T_1) \cap [j_0 + 1, n]$ is not empty, we set j_1 as its minimal element, and we define T_2 as $M_{j_1,\alpha}(T_1)$. Clearly, by induction, we define a finite sequence $(T = T_0, T_1, T_2, \dots, T_k)$ (for some $k \geq 0$, where the case $k = 0$ corresponds to $\mathcal{C}(T)$ being empty) such that $t_{T_k}(j) = \alpha$ for all $j \in \mathcal{C}(T_k)$ in view of Proposition 48. Finally, let $m = \#\mathcal{S}(T_k) \in [0, n]$. If $m = 0$, then obviously $T_k \in \tilde{T}_n$. Otherwise, let $\mu = (1, 1, \dots, 1) \in \{-1, 1\}^m$, then $S_\mu(T_k) \in \tilde{T}_n$ in view of Proposition 43.

6.3. **Proof of Formula (2).**

Lemma 51. *Let $f \in \mathcal{P}_n$, $T_0 \in \varphi^{-1}(f)$ and $k \in \{0\} \sqcup \mathcal{C}(T_0)$. We consider $T \in \mathcal{T}(T_0, k)$ (where $\mathcal{T}(T_0, 0)$ is defined as $\varphi^{-1}(f)$). If $\mathcal{C}(T_0) \cap [k+1, n] = \emptyset$, then $\mathcal{C}(T) \cap [k+1, n] = \emptyset$. Otherwise, we have*

$$\min \mathcal{C}(T_0) \cap [k+1, n] = \min \mathcal{C}(T) \cap [k+1, n].$$

Proof. Let j_0 (respectively j) be defined as $n+1$ if $\mathcal{C}(T_0) \cap [k+1, n] = \emptyset$ (respectively $\mathcal{C}(T) \cap [k+1, n] = \emptyset$), as $\min \mathcal{C}(T_0) \cap [k+1, n]$ (respectively

$\min \mathcal{C}(T) \cap [k+1,n])$ otherwise. The proof of the Lemma consists in proving the equality $j_0 = j$. Assume that $j_0 \neq j$. Since $\mathcal{T}(T_0, j) = \mathcal{T}(T, j)$, should (T_0, T) be replaced with (T, T_0), we can suppose that $j_0 > j$ (which implies that $\mathcal{C}(T) \neq \emptyset$ and $j \in [n]$). Since $f = \varphi(T)$, by Definition 16 we know that $f(2j-1) = f(2j) = 2i$ where the dots of C_j^T are the twin dots d_i^T and d_{n+i}^T. Also, by Part III. of Algorithm 13, since d_i^T and d_{n+i}^T have the same digital label $i-j$, then they have different type labels, and we obtain $\epsilon_j^T = \left\{\alpha_{i-j}^o, \beta_{i-j}^e\right\}$. Also, in this situation $d_{i,\min}^T = d_i^T \in C_j^T$. But f is also $\varphi(T_0)$, so by Lemma 35 it is necessary that $\epsilon_j^{T_0} = \epsilon_j^T = \left\{\alpha_{i-j}^o, \beta_{i-j}^e\right\}$. Now, since $j \notin \mathcal{C}(T_0)$ and $C_i^{T_0}$ has no dot labeled with β_0^e (otherwise, by Lemma 20 it would imply that $f(2i) = 2i$ and that C_i^T also contains a dot labeled with β_0^e, which would contradict $j \in \mathcal{C}(T)$), this implies that the dots of $C_j^{T_0}$ are not twin dots. Still, since $\epsilon_j^{T_0} = \left\{\alpha_{i-j}^o, \beta_{i-j}^e\right\}$, by Remark 14(f) the dots of $C_j^{T_0}$ are $d^{T_0}_{\left(\pi_j^{T_0}\right)^{-1}(i)}$ and $d^{T_0}_{\left(\pi_j^{T_0}\right)^{-1}(n+i)}$; since they are not the twin dots $d_i^{T_0}$ and $d_{n+i}^{T_0}$, this implies that either $d_i^{T_0}$ or $d_{n+i}^{T_0}$ belongs to a column $C_{j'}^{T_0}$ with $j' < j'$, hence $d_{i,\min}^{T_0} \notin C_j^{T_0}$, which is absurd in view of Lemma 17 and the fact that $d_{i,\min}^T \in C_j^T$. So $j_0 = j$. $\square$

Lemma 52. *Let $f \in \mathcal{P}_n$ and $(T, T') \in \varphi^{-1}(f)^2$. Let $i \in [n]$ such that d_{n+i}^T is not free and $d_{n+i}^{T'}$ is free. Let also $(j_1, j_2) \in [i]^2$ such that $d_{i,\min}^T \in C_{j_1}^T$ and $d \in C_{j_2}^T$ where d is the twin dot of $d_{i,\min}^T$. Then, there exists $k \in \mathcal{C}(T) \cap \mathcal{C}(T') \in [j_2 - 1]$ such that $t_T(k) \neq t_{T'}(k)$.*

Proof. By Lemma 35, it suffices that show that $\epsilon_{j_2}^T \neq \epsilon_{j_2}^{T'}$. Since $d_{n+i}^{T'}$ is free, and in view of Lemma 17, we know that $d_{i,\min}^{T'} = d_i^{T'} \in C_{j_1}^{T'}$. Also, since $d_{n+i}^{T'}$ is in particular non grounded, Proposition 21 and Lemma 20 imply that no dot of C_i^T or $C_i^{T'}$ is labeled with β_0^e. Now, by Rule I. of Algorithm 13, the digital label of d is $i - j_2$. By Definition 16, this implies that either $f(2j_2 - 1) = 2i$ or $f(2j_2) = 2i$, hence at least one of the dots of $C_{j_2}^{T'}$ has the digital label $i - j_2$. In fact, since d_{n+i}^T is free, by Remark 14(f) there exists exactly one such dot : the dot $d' = d^{T'}_{\left(\pi_j^{T'}\right)^{-1}(i)}$. Now, the type labels of d and d' are defined by the same rule of Algorithm 13, and this rule is either Rule II.1-b) or Rule II.2-.

Suppose that the type labels of d and d' are defined by Rule II.1-b). Since $d \in \left\{d_i^T, d_{n+i}^T\right\} \setminus \left\{d_{i,\min}^T\right\}$ but $d' = d^{T'}_{\left(\pi_j^{T'}\right)^{-1}(i)}$ where $d_i^{T'} = d_{i,\min}^{T'}$, the type labels of d and d' are different. Assume now that $\epsilon_{j_2}^T = \epsilon_{j_2}^{T'}$. Then these two sets equal $\left\{\alpha_{i-j_2}^o, \beta_{i-j_2}^e\right\}$, which contradicts d' being the only dot of $C_{j_2}^{T'}$ that has the digital label $i - j_2$. So $\epsilon_{j_2}^T \neq \epsilon_{j_2}^{T'}$.

Suppose finally that the type labels of d and d' are defined by Rule II.2-. In this situation, since $i = j_2$, we know that $d = d^T_{n+i}$. Whether its type label is defined by Rule II.2-a), Rule II.2-b)i. or Rule II.2-b)ii., it equals β. Afterwards, since the dots of $C^{T'}_{j_2}$ (among which is d') have different digital labels, the type label of d' is defined by Rule II.2-b), and whether it follows Rule II.2-b)i. or Rule II.2-b)ii., the dots of $C^{T'}_{j_2}$ have the same type label. Consequently, if we suppose that $\epsilon^T_{j_2} = \epsilon^{T'}_{j_2}$, then they must have the type label β following Rule II.2-b)ii., which is absurd because it implies that $d^{T'}_i \neq d^{T'}_{i,\min}$. So $\epsilon^T_{j_2} \neq \epsilon^{T'}_{j_2}$. ☐

Proposition 53. *For all $f \in \mathcal{P}_n$, we have*

$$\sum_{T \in \varphi^{-1}(f)} 2^{fr(T)} = 2^{ndf(f)}. \tag{3}$$

Proof. Let $T_0 \in \varphi^{-1}(f)$. If $\mathcal{C}(T_0) = \emptyset$, then $\mathcal{C}(T) = \emptyset$ for all $T \in \varphi^{-1}(f)$ by Lemma 51, and $\mathrm{fr}(T) = \mathrm{fr}(T_0)$ by Corollary 52. Consequently, we obtain

$$\sum_{T \in \varphi^{-1}(f)} 2^{\mathrm{fr}(T)} = 2^{\mathrm{fr}(T_0)} \times \#\varphi^{-1}(f). \tag{4}$$

Now, by Proposition 43, we know that $\#\varphi^{-1}(f) = 2^{\#\mathcal{S}(T_0)}$. Also, by Remark 30, the integer $\mathrm{fr}(T_0) + \mathcal{S}(T_0)$ equals $\mathrm{ndf}(f)$ because $\mathcal{C}(T_0) = 0$ by hypothesis, hence $2^{\mathrm{fr}(T_0)} \times \#\varphi^{-1}(f) = 2^{\mathrm{ndf}(f)}$, and Formula (4) becomes Formula (3).

It remains to prove Formula (3) if there exists $T \in \varphi^{-1}(f)$ such that $\mathcal{C}(T)$ is not empty, *i.e.*, if there exists $j \in [n]$ such that $\mathcal{T}_f(j) \neq \emptyset$. Under that hypothesis, let $\{j_1, j_2, \ldots, j_p\}_< = \{j \in [n] : \mathcal{T}_f(j) \neq \emptyset\}$. Let T_p be any element of $\mathcal{T}_f(j_p)$, and $\gamma \in \{\alpha, \beta\}$. We consider $\tilde{T}_p \in \mathcal{T}(T_p, j_p, \gamma)$ (which is not empty in view of Remark 49). For all $T \in \mathcal{T}(T_p, j_p, \gamma)$, we have $\mathrm{fr}(T) = \mathrm{fr}(\tilde{T}_p)$ by Corollary 52. Consequently, we obtain

$$\sum_{T \in \mathcal{T}(T_p, j_p, \gamma)} 2^{\mathrm{fr}(T)} = 2^{\mathrm{fr}(\tilde{T}_p)} \times \#\mathcal{T}(T_p, j_p, \gamma). \tag{5}$$

Now, in view of Proposition 43, the cardinality of $\mathcal{T}(T_p, j_p, \gamma)$ equals $2^{\#\mathcal{S}(\tilde{T}_p)}$, so Formula (5) becomes

$$\sum_{T \in \mathcal{T}(T_p, j_p, \gamma)} 2^{\mathrm{fr}(T)} = 2^{\mathrm{fr}(\tilde{T}_p) + \#\mathcal{S}(\tilde{T}_p)}. \tag{6}$$

By Remark 30, we know that $\mathrm{fr}(\tilde{T}_p)+\#\mathcal{S}(\tilde{T}_p)=\mathrm{ndf}(f)-\mathcal{C}(\tilde{T}_p)$, and by hypothesis $\tilde{T}_p\in\mathcal{T}(T_p,j_p)$, so $\mathcal{C}(\tilde{T}_p)=\mathcal{C}(T_p)$, and Formula (6) becomes

$$\sum_{T\in\mathcal{T}(T_p,j_p,\gamma)} 2^{\mathrm{fr}(T)}=2^{\mathrm{ndf}(f)-\#\mathcal{C}(T_p)}. \tag{7}$$

Since Formula (7) is true for all $\gamma\in\{\alpha,\beta\}$, and in view of the equality $\mathcal{T}(T_p,j_p)=\mathcal{T}(T_p,j_p,\alpha)\sqcup\mathcal{T}(T_p,j_p,\beta)$, we obtain

$$\sum_{T\in\mathcal{T}(T_p,j_p)} 2^{\mathrm{fr}(T)}=2^{\mathrm{ndf}(f)-\#\mathcal{C}(T_p)+1}. \tag{8}$$

Suppose now that for some $q\in[2,p]$, and for all $T_q\in\mathcal{T}_f(j_q)$, we have the Formula

$$\sum_{T\in\mathcal{T}(T_q,j_q)} 2^{\mathrm{fr}(T)}=2^{\mathrm{ndf}(f)-\#(\mathcal{C}(T_q)\cap[j_q-1])} \tag{9}$$

(it is true for $q=p$ in view of Formula 8). Let $T_{q-1}\in\mathcal{T}_f(j_{q-1})$, $\gamma\in\{\alpha,\beta\}$ and $\tilde{T}_{q-1}\in\mathcal{T}(T_{q-1},j_{q-1},\gamma)$ (which is not empty in view of Remark 49). We first intend to prove the following Formula :

$$\sum_{T\in\mathcal{T}(T_{q-1},j_{q-1},\gamma)} 2^{\mathrm{fr}(T)}=2^{\mathrm{ndf}(f)-\#(\mathcal{C}(T_{q-1})\cap[j_{q-1}])}. \tag{10}$$

— If $[j_{q-1}+1,n]\cap\mathcal{C}(\tilde{T}_{q-1})=\emptyset$, by Lemma 51, it is necessary that $[j_{q-1}+1,n]\cap\mathcal{C}(T)=\emptyset$ for all $T\in\mathcal{T}(T_{q-1},j_{q-1},\gamma)$, and $\mathrm{fr}(\tilde{T}_{q-1})=\mathrm{fr}(T)$ by Corollary 52, hence

$$\sum_{T\in\mathcal{T}(T_{q-1},j_{q-1},\gamma)} 2^{\mathrm{fr}(T)}=2^{\mathrm{fr}(\tilde{T}_{q-1})}\times\#\mathcal{T}(T_{q-1},j_{q-1},\gamma). \tag{11}$$

By Proposition 43, we have $\#\mathcal{T}(T_{q-1},j_{q-1},\gamma)=2^{\#\mathcal{S}(\tilde{T}_{q-1})}$. By Remark 30, the integer $\mathrm{fr}(\tilde{T}_{q-1})+\mathcal{S}(\tilde{T}_{q-1})$ equals the integer $\mathrm{ndf}(f)-\mathcal{C}(\tilde{T}_{q-1})=\mathrm{ndf}(f)-\#(\mathcal{C}(T_{q-1})\cap[j_{q-1}])$ because $\mathcal{C}(\tilde{T}_{q-1})\cap[j_{q-1}+1,n]=\emptyset$ by hypothesis, hence Formula (11) becomes Formula (10).

— Otherwise, let $j=\min[j_{q-1}+1,n]\cap\mathcal{C}(\tilde{T}_{q-1})$. By Lemma 51, it is necessary that j is also $\min[j_{q-1}+1,n]\cap\mathcal{C}(T)$ for all $T\in\mathcal{T}(T_{q-1},j_{q-1},\gamma)$. In other words, the set $\mathcal{T}(T_{q-1},j_{q-1},\gamma)$ is in fact $\mathcal{T}(\tilde{T}_{q-1},j)$. Let $q'>q-1$ such that $j=j_{q'}$. By hypothesis, we know that

$$\sum_{T\in\mathcal{T}(\tilde{T}_{q-1},j_{q'})} 2^{\mathrm{fr}(T)}=2^{\mathrm{ndf}(f)-\#(\mathcal{C}(\tilde{T}_{q-1})\cap[j_{q'}-1])}. \tag{12}$$

Since $j_{q'} = \min \mathcal{C}(\tilde{T}_{q-1}) \cap [j_{q-1}+1, n]$ and $\tilde{T}_{q-1} \in \mathcal{T}(T_{q-1}, j_{q-1})$, we have $\#(\mathcal{C}(\tilde{T}_{q-1}) \cap [j_{q'}-1]) = \#(\mathcal{C}(T_q) \cap [j_{q-1}])$, hence Formula (12) becomes Formula (10) in view of $\mathcal{T}(T_{q-1}, j_{q-1}, \gamma) = \mathcal{T}(T_{q-1}, j_{q-1}, \gamma)$.

So Formula (10) is true for all $\gamma \in \{\alpha, \beta\}$, and in view of

$$\mathcal{T}(T_{q-1}, j_{q-1}) = \mathcal{T}(T_{q-1}, j_{q-1}, \alpha) \sqcup \mathcal{T}(T_{q-1}, j_{q-1}, \beta),$$

we obtain

$$\begin{aligned} \sum_{T \in \mathcal{T}(T_{q-1}, j_{q-1})} 2^{\mathrm{fr}(T)} &= 2^{\mathrm{ndf}(f) - \#(\mathcal{C}(T_{q-1}) \cap [j_{q-1}]) + 1} \\ &= 2^{\mathrm{ndf}(f) - \#(\mathcal{C}(T_{q-1}) \cap [j_{q-1}-1])}. \end{aligned}$$

So Formula (9) is true for all $q \in [p]$ by induction. In particular, for $q = 1$, we obtain Formula (3). □

This ends the proof of Theorem 6.

Acknowledgements

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Appendices

A. Pistol labeling of the tableau $T_1 \in \mathcal{T}_7$

We give in Figure 15 the details of the pistol labeling of the tableau $T_1 \in \mathcal{T}_7$ depicted in Figure 7. From j from 7 down to 1, we show how the two dots of the column $C_j^{T_1}$ receive their pistol labels. In the following, we specify which rule of Part II. of Algorithm 13 is applied, for j from 7 down to 1.

— $j = 7$: Rule II.2-a) for both dots $d_8^{T_1}$ and $d_{12}^{T_1}$.

— $j = 6$: Rule II.1-a) for $d_{14}^{T_1}$, then Rule II.2-b)ii. for $d_9^{T_1}$.

— $j = 5$: Rule II.1-a) for $d_7^{T_1}$ and Rule II.1-b) for $d_{13}^{T_1}$.

— $j = 4$: Rule II.1-b) for $d_5^{T_1}$, then Rule II.2-b)ii. for $d_{10}^{T_1}$.

— $j = 3$: Rule II.1-b) for $d_6^{T_1}$, then Rule II.2-b)i. for $d_3^{T_1}$.

— $j = 2$: Rule II.1-b) for $d_4^{T_1}$, then Rule II.2-b)ii. for $d_2^{T_1}$.

— $j = 1$: Rule II.1-a) for $d_{11}^{T_1}$, then Rule II.2-b)i. for $d_1^{T_1}$.

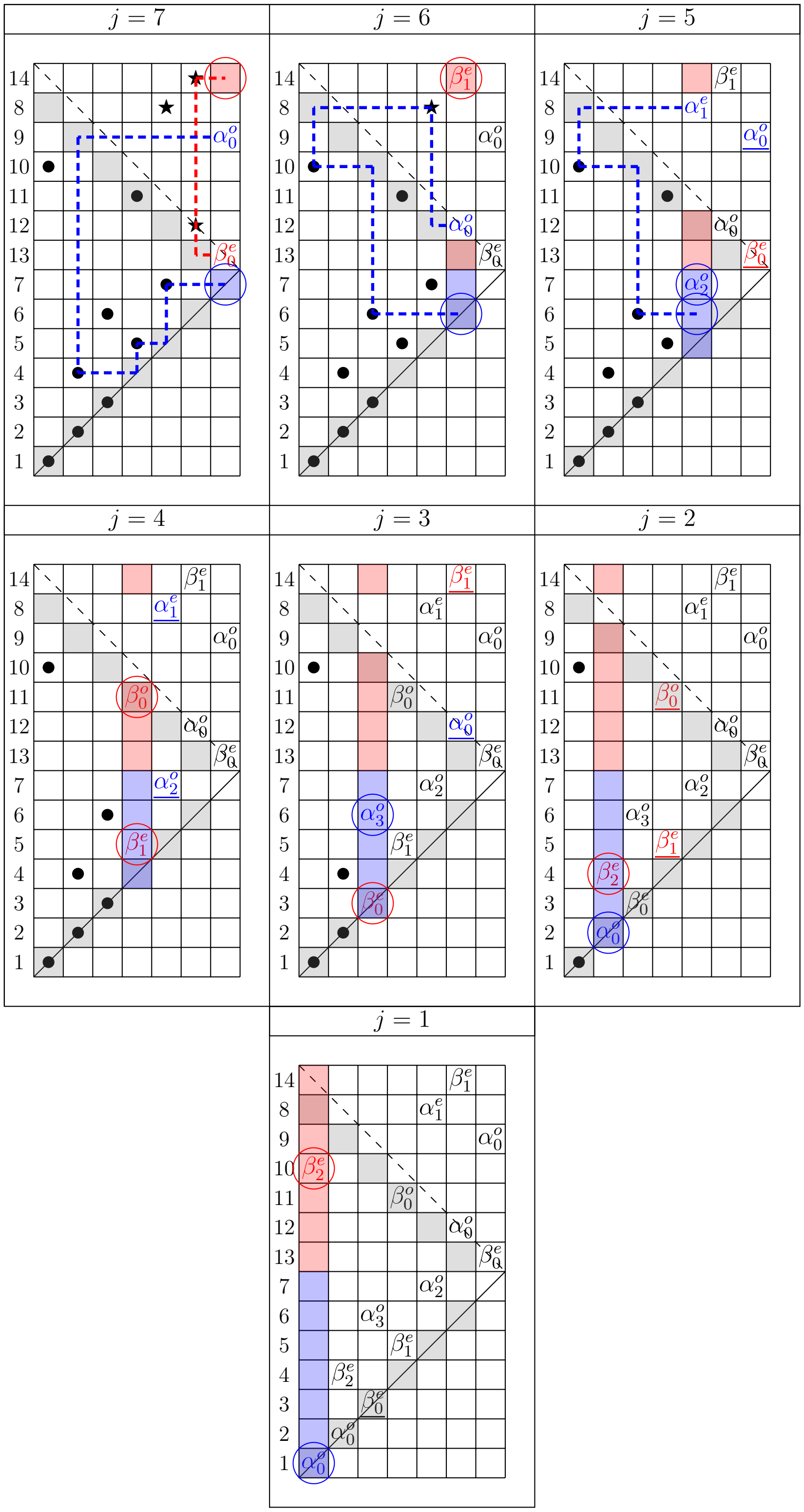

Figure 15. Pistol labeling of $T_1 \in \mathcal{T}_7$.

B. Computation of $\Phi(f_1)$

We give in Figure 16 the details of the computation of $\Phi(f_1) \in \mathcal{T}_7$ where $f_1 \in \mathcal{P}_7$ is the surjective pistol depicted in Figure 8. From j from 1 to 7, we show how the two labeled dots of $C_j^{\Phi(f_1)}$ are inserted. At each step j, on the left of every suitable row, we specify the integer $\delta \in [0, 7-j]$ it corresponds with (in blue, for dots labeled with a, and in red for dots labeled with b). In the following table, we make explicit every rule of Algorithm 25 and Definition 24 that leads to the plotting of the dots of $C_j^{\Phi(f_1)}$.

j	Rule of Algorithm 25	Rules of Definition 24
1	I.1-	1.(a) and 2.(a)i.
2	I.1-	1.(a) and 2.(a)ii.
3	I.1-	1.(a) and 2.(a)ii.
4	I.2-b)i.	1.(b) and 2.(a)ii.
5	I.2-b)ii.	2.(b) and 2.(a)ii.
6	I.2-a)	2.(b) and 2.(a)i.
7	I.1-	1.(b) and 1.(b)

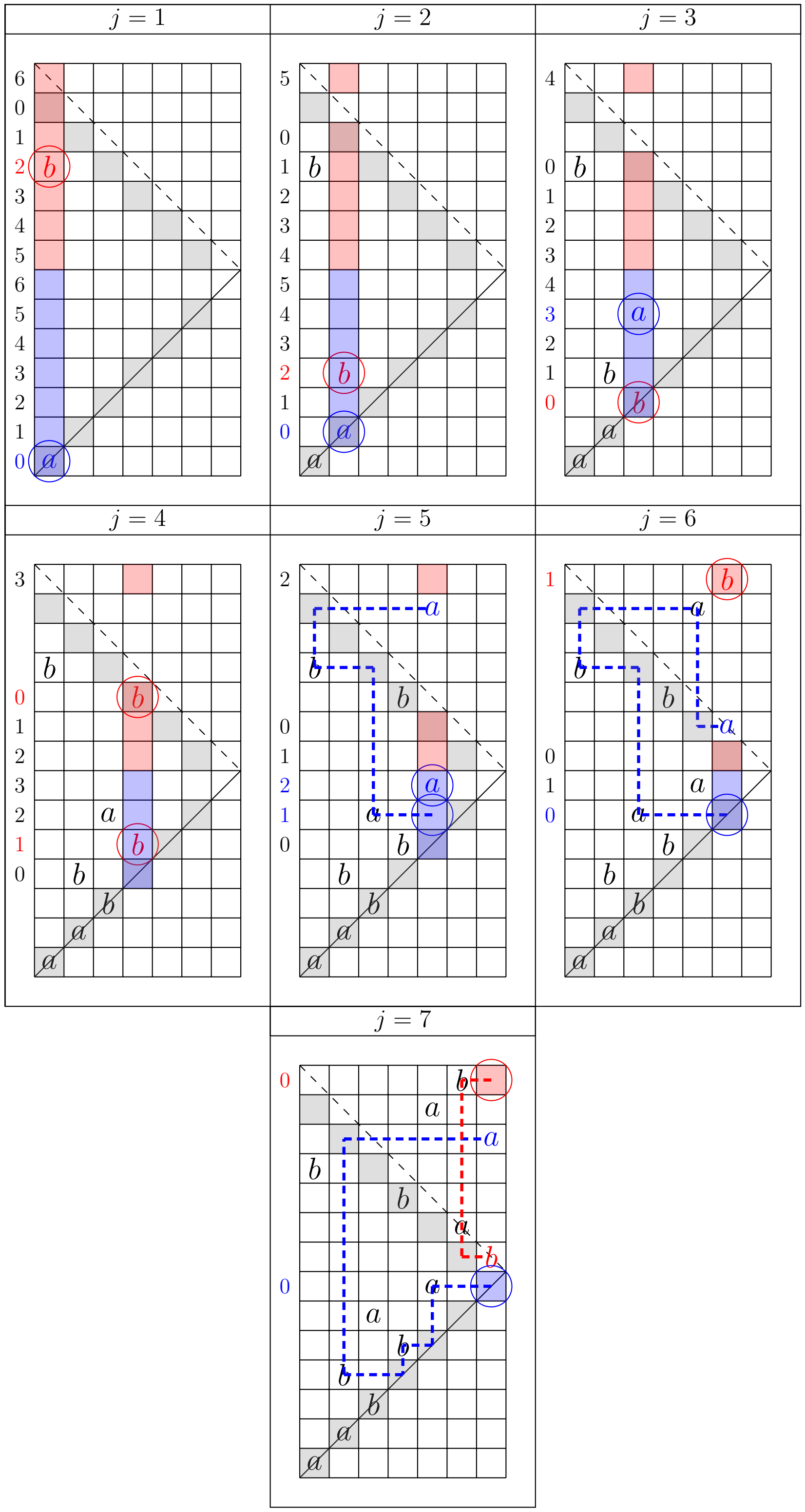

FIGURE 16. Computation of $\Phi(f_1) \in \mathcal{T}_7$.

References

[1] G. Cerulli Irelli, E.Feigin, M. Reineke, *Quiver Grassmannians and degenerate flag varieties*, Algebra & Number Theory 6 (2012), no. 1, 165–194.

[2] H. Dellac, *Problem 1735*, L'Intermédiaire des Mathématiciens, 7 (1900), 9–10.

[3] D. Dumont, *Interprétations combinatoires des nombres de Genocchi*, Duke Math. J. 41 (1974), 305–318.

[4] X. Fang, G. Fourier, *Torus fixed points in Schubert varieties and normalized median Genocchi numbers*, Sém. Lothar. Combin. 75 (2015), Art. B75f, 12 pp.

[5] E. Feigin, *Degenerate flag varieties and the median Genocchi numbers*, Math. Res. Lett. 18 (2011), no. 6, 1163–1178.

[6] E. Feigin, M. Finkelberg, P. Littelmann, *Symplectic degenerate flag varieties*, Canad. J. Math. 66 (2014), no. 6, 1250–1286.

[7] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A000366` (2011).

[8] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A098279` (2011).

[9] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A110501` (2011).

[10] A. Randrianarivony, J. Zeng, *Une famille de polynômes qui interpole plusieurs suites classiques de nombres*, Adv. in Appl. Math. 17 (1996), no. 1, 1–26.