Some Infinite Matrices Whose Leading Principal Minors Are Well-known Sequences

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Abstract

There are scattered results in the literature showing that the leading principal minors of certain infinite integer matrices form the Fibonacci and Lucas sequences. In this article, among other results, we have obtained new families of infinite matrices such that the leading principal minors of them form a famous integer (sub)sequence, such as Fibonacci, Lucas, Pell and Jacobsthal (sub)sequences.

1 Notation, Definitions and Previous Results

Throughout this article, unless noted otherwise, we will use the following notation:

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- $\alpha = (\alpha_i)_{i\geqslant 0}$ and $\beta = (\beta_i)_{i\geqslant 0}$ are two arbitrary sequences starting with $\alpha_0 = \beta_0 = \gamma$.

- If $\alpha = (\alpha_i)_{i\geqslant 0}$, then $\tilde{\alpha} = (\tilde{\alpha}_i)_{i\geqslant 0}$ where $\tilde{\alpha}_i = (-1)^i \alpha_i$ for all $i \geqslant 0$.

- $P_{\alpha,\beta}(n)$ is the generalized Pascal triangle associated with the sequences α and β (see [1]), which is introduced as follows. In fact, $P_{\alpha,\beta}(n) = [P_{i,j}]_{0\leqslant i,j\leqslant n}$ is a square matrix of order $n+1$ whose (i,j)-entry $P_{i,j}$ obeys the following rule:

  $$P_{i,0} = \alpha_i,\ P_{0,i} = \beta_i \quad \text{for } 0 \leqslant i \leqslant n,$$

  and

  $$P_{i,j} = P_{i,j-1} + P_{i-1,j} \quad \text{for } \ 1 \leqslant i,j \leqslant n.$$

- $A_{\alpha,\beta}(n) = [A_{i,j}]_{0\leqslant i,j\leqslant n}$ is the 7-matrix associated with the sequences α and β of order $n+1$, whose entries satisfy

  $$A_{i,0} = \alpha_i,\ A_{0,i} = \beta_i \quad \text{for } 0 \leqslant i \leqslant n,$$

  and

  $$A_{i,j} = A_{i-1,j-1} + A_{i-1,j} \quad \text{for } \ 1 \leqslant i,j \leqslant n.$$

  This class of matrices was first introduced in [6]. In applications, the case when $\alpha = (1,1,1,\ldots)$ and $\beta = (1,0,0,0,\ldots)$ is more important. In fact, we put

  $$L(n) := A_{(1,1,1,\ldots),(1,0,0,0,\ldots)}(n) = [L_{i,j}]_{0\leqslant i,j\leqslant n},$$

  which is called the unipotent lower triangular matrix of order $n+1$. An explicit formula for (i,j)-entry $L_{i,j}$ is also given by the following formula

  $$L_{i,j} = \begin{cases} 0 & \text{if} \quad i < j \\ \binom{i}{j} & \text{if} \quad i \geqslant j. \end{cases}$$

  Moreover, we put $U(n) = L(n)^t$, where A^t signifies the transpose of matrix A.

- $T_{\alpha,\beta}(n) = [T_{i,j}]_{0\leqslant i,j\leqslant n}$ is the Toeplitz matrix of order $n+1$ with $T_{i,0} = \alpha_i$ and $T_{0,i} = \beta_i$, $0 \leqslant i \leqslant n$, and $T_{i,j} = T_{k,l}$ whenever $i - j = k - l$. In the case when $\alpha = (c,b,b,b,\ldots)$ and $\beta = (c,a,a,a,\ldots)$, we follow the notation in [7], and put $T_{n+1}(a,b,c) := T_{\alpha,\beta}(n)$. For example

  $$T_3(a,b,c) = \begin{bmatrix} c & a & a \\ b & c & a \\ b & b & c \end{bmatrix}.$$

- An upper Hessenberg matrix, $H(n) = [h_{i,j}]_{0\leqslant i,j\leqslant n}$, is a square matrix of order $n+1$ where $h_{i,j} = 0$ whenever $i > j+1$ and $h_{i,i-1} \neq 0$ for some i, $1 \leqslant i \leqslant n$, so we have

$$H(n) = \begin{pmatrix} h_{0,0} & h_{0,1} & h_{0,2} & \dots & h_{0,n} \\ h_{1,0} & h_{1,1} & h_{1,2} & \ddots & h_{1,n} \\ 0 & h_{2,1} & h_{2,2} & \ddots & h_{2,n} \\ \vdots & \ddots & \ddots & \ddots & h_{n-1,n} \\ 0 & 0 & 0 & h_{n,n-1} & h_{n,n} \end{pmatrix}.$$

- E_{ij} denotes the square matrix having 1 in the (i,j) position and 0 elsewhere.
- $\mathrm{R}_i(A)$ (resp. $\mathrm{C}_j(A)$) denotes the row i (resp. column j) of a matrix A.

Remark. *Notice that we index all matrices in this article beginning at the $(0,0)$-th entry.*

- $\mathcal{F}_n$ is the nth Fibonacci number (A000045 in [14]), which satisfies

$$\mathcal{F}_0 = 0,\ \mathcal{F}_1 = 1, \quad \mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2} \text{ for } n \geqslant 2.$$

- $\mathcal{L}_n$ is the nth Lucas number (A000032 in [14]), which satisfies

$$\mathcal{L}_0 = 2,\ \mathcal{L}_1 = 1, \quad \mathcal{L}_n = \mathcal{L}_{n-1} + \mathcal{L}_{n-2} \text{ for } n \geqslant 2.$$

- $\mathcal{P}_n$ is the nth Pell number (A000129 in [14]), which satisfies

$$\mathcal{P}_0 = 0,\ \mathcal{P}_1 = 1, \quad \mathcal{P}_n = 2\mathcal{P}_{n-1} + \mathcal{P}_{n-2} \text{ for } n \geqslant 2.$$

- $\mathcal{J}_n$ is the nth Jacobsthal number (A001045 in [14]), which satisfies

$$\mathcal{J}_0 = 0,\ \mathcal{J}_1 = 1, \quad \mathcal{J}_n = \mathcal{J}_{n-1} + 2\mathcal{J}_{n-2} \text{ for } n \geqslant 2.$$

Remark. Let a, b, r, s be integers with $r, s \geqslant 1$. The (a,b,r,s)-*Gibonacci sequence* (or *generalized Fibonacci sequence*), $\mathcal{G}^{(a,b,r,s)} = (\mathcal{G}_n^{(a,b,r,s)})_{n\geqslant 0}$, is recursively defined by:

$$\mathcal{G}_0^{(a,b,r,s)} = a, \quad \mathcal{G}_1^{(a,b,r,s)} = b,$$

and

$$\mathcal{G}_n^{(a,b,r,s)} = r\mathcal{G}_{n-1}^{(a,b,r,s)} + s\mathcal{G}_{n-2}^{(a,b,r,s)}, \quad \text{for } n \geqslant 2.$$

The Fibonacci sequence (resp. Lucas sequence, Pell sequence, or Jacobsthal sequence) corresponds to the case $(a,b,r,s) = (0,1,1,1)$ (resp. $(2,1,1,1)$, $(0,1,2,1)$ or $(0,1,1,2)$).

- Given an arbitrary sequence $\omega = (\omega_i)_{i\geqslant 0}$, we put $\tilde{\omega} = (\tilde{\omega}_i)_{i\geqslant 0}$ where $\tilde{\omega}_i = (-1)^i\omega_i$.

- To any sequence $\alpha = (\alpha_i)_{i\geqslant 0}$, we associate other sequences $\hat{\alpha} = (\hat{\alpha}_i)_{i\geqslant 0}$ and $\check{\alpha} = (\check{\alpha}_i)_{i\geqslant 0}$ (the binomial and inverse binomial transforms of α), which are defined through the following rules:

$$\hat{\alpha}_i = \sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k}\alpha_k \quad \text{and} \quad \check{\alpha}_i = \sum_{k=0}^{i}\binom{i}{k}\alpha_k.$$

- $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio and $\Phi = \frac{1-\sqrt{5}}{2}$ is the golden ratio conjugate.

Given an infinite matrix $A = [A_{i,j}]_{i,j\geqslant 0}$, denote by d_n $(n = 0, 1, 2, \ldots)$ the nth leading principal minor of A defined as the determinant of the submatrix consisting of the entries in its first $n+1$ rows and columns, i.e., $d_n = \det[A_{i,j}]_{0\leqslant i,j\leqslant n}$. Here, we are interested in computing the sequence of leading principal minors $(d_0, d_1, d_2, \ldots)$, especially, in the case that d_n is the nth Fibonacci (Lucas, Pell or Jacobsthal) number. In [9], we introduced a family of matrices $A(n) = [A_{i,j}]_{0\leqslant i,j\leqslant n}$ as follows:

$$A_{i,j} = \begin{cases} 1 & ij = 0, \\ A_{i,j-1} + A_{i-1,j} + i - j & 1 \leqslant i, j \leqslant n. \end{cases}$$

The matrix $A(3)$ for instance is given by

$$A(3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 4 & 6 & 6 \\ 1 & 7 & 14 & 20 \end{pmatrix}.$$

The leading principal minors now generate the Fibonacci numbers $\mathcal{F}_{n+1}$ (see [9]).

In [11], we defined two other families of matrices:

$$B(n) = [B_{i,j}]_{0\leqslant i,j\leqslant n} \text{ and } C(n) = [C_{i,j}]_{0\leqslant i,j\leqslant n},$$

with

$$B_{i,j} = \begin{cases} j+2 & i = 0, j \geqslant 0, \\ 4B_{i-1,j} + i^2 - 7i - 5 & j = 0, i \geqslant 1, \\ B_{i,j-1} + B_{i-1,j} - 2(i+j) & 1 \leqslant i, j \leqslant n, \end{cases}$$

and

$$C_{i,j} = \begin{cases} 2-j & i=0, j \leqslant 1, \\ 2C_{i,j-2} - C_{i,j-1} & i=0, j \geqslant 2, \\ 3C_{i-1,j} + 5(3^{i-1} - 2i - 1)/2 & j=0, i \geqslant 1, \\ C_{i-1,j-1} + C_{i-1,j} - 2i & 1 \leqslant i, j \leqslant n. \end{cases}$$

The matrices $B(3)$ and $C(3)$, for instance, are shown below:

$$B(3) = \begin{pmatrix} 2 & 3 & 4 & 5 \\ -3 & -4 & -6 & -9 \\ -27 & -37 & -51 & -70 \\ -125 & -170 & -231 & -313 \end{pmatrix},$$

and

$$C(3) = \begin{pmatrix} 2 & 1 & 3 & -1 \\ 1 & 1 & 2 & 0 \\ -2 & -2 & -1 & -2 \\ -1 & -10 & -9 & -9 \end{pmatrix}.$$

It was proved in [11] that $\det B(n) = \det C(n) = \mathcal{L}_n$.

Recently, in [8], it is determined all sequences α and β which satisfy the second-order homogeneous linear recurrence relations and for which the leading principal minors of generalized Pascal triangle $P_{\alpha,\beta} = P_{\alpha,\beta}(\infty)$, 7-matrix $A_{\alpha,\beta} = A_{\alpha,\beta}(\infty)$ or Toeplitz matrix $T_{\alpha,\beta} = T_{\alpha,\beta}(\infty)$, form the Fibonacci, Lucas, Pell and Jacobsthal sequences.

In [15], using generating functions, the authors proved that any sequence can be presented in terms of a sequence of leading principal minors of an infinite matrix. In fact, according to their methods one can easily construct matrices whose determinants are famous sequences. For example, they constructed the following matrix

$$D(\infty) = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots \\ 4 & 1 & 2 & 3 & \dots \\ 9 & 0 & 1 & 3 & \dots \\ 16 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

in which the first column is the sequence of the square positive integers and the remaining part of the matrix is the truncated Pascal's upper triangular matrix. Let $D(n)$ denote the $n+1$ by $n+1$ upper left corner matrix of $D(\infty)$. Then $\det D(n) = (-1)^n \mathcal{T}_{n+1}$, where $\mathcal{T}_n$ is the nth triangular number, i.e., $\mathcal{T}_n = \binom{n+1}{2}, n \geqslant 0$.

In [7], in particular, it was proved that, for Toeplitz matrices

$$P_n = T_n(\phi, \Phi, 1) \quad \text{and} \quad Q_n = T_n(-\phi, -\Phi, 0),$$

$\det P_n = \mathcal{F}_{n+1}$ and $\det Q_n = \mathcal{F}_{n-1}$.

Table 1-3 list several other infinite matrices, whose leading principal minors form a Fibonacci or Lucas (sub)sequence. In these tables $\mathbf{i}$ denotes $\sqrt{-1}$.

Table 1. *Some Toeplitz matrices with determinants as Fibonacci or Lucas numbers.*

α	β	$\det T_{\alpha,\beta}(n)$	Refs.
$(2,1,1,\ldots)$	$(2,-1,0,0,\ldots)$	$\mathcal{F}_{2n+3}$	[4]
$(2,1,1,\ldots)$	$(2,1,0,0,\ldots)$	$\mathcal{F}_{n+3}$	[4]
$(1,\mathbf{i},0,0,\ldots)$	$(1,\mathbf{i},0,0,\ldots)$	$\mathcal{F}_{n+2}$	[2, 4, 5, 15]
$(1,-1,0,0,\ldots)$	$(1,1,0,0,\ldots)$	$\mathcal{F}_{n+2}$	[12]
$(3,1,0,0,\ldots)$	$(3,1,0,0,\ldots)$	$\mathcal{F}_{2n+4}$	[2, 12, 13]
$(3,-1,0,0,\ldots)$	$(3,-1,0,0,\ldots)$	$\mathcal{F}_{2n+4}$	[13]
$(2,-1,1,-1,1,\ldots)$	$(2,1,0,0,\ldots)$	$\mathcal{F}_{2n+3}$	[2]
$(2,-1,1,-1,1,\ldots)$	$(2,-1,0,0,\ldots)$	$\mathcal{F}_{n+3}$	[2]
$(1,\Phi,\Phi,\ldots)$	$(1,\phi,\phi,\ldots)$	$\mathcal{F}_{n+2}$	[7]
$(0,-\Phi,-\Phi,\ldots)$	$(0,-\phi,-\phi,\ldots)$	$\mathcal{F}_{n}$	[7]

Table 2. *Some matrices with determinants as Fibonacci or Lucas numbers.*

α	β	Determinant	Ref.
$(1,\mathbf{i},0,\ldots)$	$(1,\mathbf{i},0,\ldots)$	$\det\big(T_{\alpha,\beta}(n)+E_{11}\big)=\mathcal{L}_{n+1}$	[4]
$(1,1,1,\ldots)$	$(1,-1,0,\ldots)$	$\det\big(T_{\alpha,\beta}(n)+\sum_{i=1}^{n}E_{ii}\big)=\mathcal{F}_{2n+2}$	[4]
$(1,\mathbf{i},0,\ldots)$	$(1,\mathbf{i},0,\ldots)$	$\det\big(T_{\alpha,\beta}(n)+2E_{00}\big)=\mathcal{L}_{n+2}$	[3]

Table 3. *Some Pascal and 7-matrices with determinants as Fibonacci or Lucas numbers.*

γ	α	β	Determinant	Ref.
1	$\alpha_i=\alpha_{i-1}+c$	$\beta_i=\beta_{i-1}-c^{-1}$	$\det P_{\alpha,\beta}(n)=\mathcal{F}_{n+2}$	[10]
1	$\alpha_i=\alpha_{i-1}+1$	$\beta_1=0,\ \beta_i=\beta_{i+2}$	$\det A_{\alpha,\beta}(n)=\mathcal{F}_{n+1}$	[10]
1	$\alpha_i=\alpha_{i-1}-1$	$\beta_1=0,\ \beta_i=\beta_{i+2}$	$\det A_{\alpha,\beta}(n)=\mathcal{F}_{n+1}$	[10]
1	$\alpha_i=\alpha_{i-1}+\mathbf{i}$	$\beta_1=2\mathbf{i},\ \beta_i=\beta_{i+2}$	$\det A_{\alpha,\beta}(n)=\mathcal{L}_{n+1}$	[10]
1	$\alpha_i=\alpha_{i-1}-\mathbf{i}$	$\beta_1=-2\mathbf{i},\ \beta_i=\beta_{i+2}$	$\det A_{\alpha,\beta}(n)=\mathcal{L}_{n+1}$	[10]

The purpose of this article is to find some *new infinite families of matrices* whose leading principal minors form a subsequence of the sequences $\mathcal{F}$, $\mathcal{L}$, $\mathcal{P}$ and $\mathcal{J}$.

2 Preliminaries

In this section, we prove several auxiliary results to be used later. First of all, we introduce the notion of equimodular matrices (see [16]).

Definition 1 *Infinite matrices* $A = [A_{i,j}]_{i,j\geqslant 0}$ *and* $B = [B_{i,j}]_{i,j\geqslant 0}$ *are equimodular if their leading principal minors are the same, that is*

$$\det[A_{i,j}]_{0\leqslant i,j\leqslant n} = \det[B_{i,j}]_{0\leqslant i,j\leqslant n} \quad \textit{for all} \quad n \geqslant 0.$$

Lemma 1 (Corollary 2.6, [8]) *Let* α *and* β *be two sequences starting with a common first term. Then, the following matrices are equimodular:*

$$T_{\alpha,\beta}(\infty), \quad T_{\beta,\alpha}(\infty), \quad P_{\check{\alpha},\check{\beta}}(\infty), \quad P_{\check{\beta},\check{\alpha}}(\infty), \quad A_{\check{\alpha},\beta}(\infty) \quad \text{and} \quad A_{\check{\beta},\alpha}(\infty).$$

The following lemma, which is taken from [7], determines recursive and explicit formula for the leading principal minors for special classes of Toeplitz matrices.

Lemma 2 *([7]) Let* $a, b, c \in \mathbb{C}$. *For each positive integer* n, *let* $T_n = T_n(a, b, c)$. *Then* $\det T_1 = c$, $\det T_2 = c^2 - ab$ *and for* $n \geqslant 3$,

$$\det T_n = (2c - a - b) \det T_{n-1} - (c - a)(c - b) \det T_{n-2}.$$

Moreover, an explicit formula for the case when $n \geqslant 3$, *is as follows:*

$$\det T_n = \begin{cases} 0 & \textit{if} \quad a = b = c, \\ [c + a(n-1)](c-a)^{n-1} & \textit{if} \quad a = b \neq c, \\ \frac{b}{b-a}(c-a)^n - \frac{a}{b-a}(c-b)^n & \textit{if} \quad a \neq b. \end{cases}$$

Lemma 3 *Let* $a, c \in \mathbb{C}$. *Let* $\alpha = (\alpha_i)_{i\geqslant 0}$ *with* $\alpha_0 = c$ *and for each* $i \geqslant 1$, $\alpha_i = a$. *Then there hold:*

(1) *for each* $i \geqslant 0$, $\check{\alpha}_i = c + a(2^i - 1)$.

(2) $\hat{\alpha}_0 = c$ *and for each* $i \geqslant 1$, $\hat{\alpha}_i = \begin{cases} a - c & i \textit{ is odd}, \\ c - a & i \textit{ is even}. \end{cases}$

Proof. The following easy computations

$$\check{\alpha}_i = \sum_{k=0}^{i} \binom{i}{k} \alpha_k = c + a \sum_{k=1}^{i} \binom{i}{k} = c + a(2^i - 1),$$

and

$$\begin{aligned}\hat{\alpha}_i &= \textstyle\sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k}\alpha_k = (-1)^i c + a\sum_{k=1}^{i}(-1)^{i+k}\binom{i}{k} \\ &= (-1)^i(c-a) = \begin{cases} a-c & i \text{ is odd,} \\ c-a & i \text{ is even,}\end{cases}\end{aligned}$$

conclude the results. □

Unless noted otherwise, we use the following notation in this section:

- $\mathcal{G} = \mathcal{G}^{(a,b,r,1)}$, where a, b and r are integers with $r \geqslant 1$,
- $\bar{\mathcal{G}} = \mathcal{G}^{(0,1,r,1)}$, where $r \geqslant 1$ is an integer. Moreover, we assume that $\bar{\mathcal{G}}_{-1} = 1$.

Lemma 4 *The sequences $\check{\mathcal{G}} = (\check{\mathcal{G}}_i)_{i\geqslant 0}$ and $\check{\check{\mathcal{G}}} = (\check{\check{\mathcal{G}}}_i)_{i\geqslant 0}$ satisfy linear recursions of order 2. More precisely, we have*

$$\begin{cases} \check{\mathcal{G}}_0 = \mathcal{G}_0 = a, \quad \check{\mathcal{G}}_1 = \mathcal{G}_0 + \mathcal{G}_1 = a+b, \\ \check{\mathcal{G}}_i = (r+2)\check{\mathcal{G}}_{i-1} - r\check{\mathcal{G}}_{i-2}, \quad i \geqslant 2, \end{cases} \tag{1}$$

and

$$\begin{cases} \check{\check{\mathcal{G}}}_0 = \mathcal{G}_0 = a, \quad \check{\check{\mathcal{G}}}_1 = \mathcal{G}_0 - \mathcal{G}_1 = a-b, \\ \check{\check{\mathcal{G}}}_i = (2-r)\check{\check{\mathcal{G}}}_{i-1} + r\check{\check{\mathcal{G}}}_{i-2}, \quad i \geqslant 2. \end{cases} \tag{2}$$

Proof. By the definition, we have the following recurrence relation

$$\mathcal{G}_n = r\mathcal{G}_{n-1} + \mathcal{G}_{n-2}, \tag{3}$$

which can be written as

$$\mathcal{G}_n - r\mathcal{G}_{n-1} - \mathcal{G}_{n-2} = 0.$$

Its characteristic equation is

$$x^2 - rx - 1 = 0,$$

and we denote by t_1 and t_2 the characteristic roots. Now, the general solution of Eq. (3) is given by

$$\mathcal{G}_i = C_1 t_1^i + C_2 t_2^i, \tag{4}$$

where C_1 and C_2 are constants. In order to verify Eq. (1), we first notice that, for every $i \geqslant 0$:

$$\begin{aligned}\check{\mathcal{G}}_i = \textstyle\sum_{k=0}^{i}\binom{i}{k}\mathcal{G}_k &= \textstyle\sum_{k=0}^{i}\binom{i}{k}[C_1 t_1^k + C_2 t_2^k] \qquad \text{(by Eq. (4))} \\ &= C_1(t_1+1)^i + C_2(t_2+1)^i.\end{aligned}$$

Therefore, we verify Eq. (1) by a direct calculation. Clearly $\check{\mathcal{G}}_0 = \mathcal{G}_0 = a$ and $\check{\mathcal{G}}_1 = \mathcal{G}_0 + \mathcal{G}_1 = a + b$. Let us for the moment assume that $i \geqslant 2$. Then, we have

$$\begin{aligned}
&(r+2)\check{\mathcal{G}}_{i-1} - r\check{\mathcal{G}}_{i-2}\\
&= (r+2)[C_1(t_1+1)^{i-1} + C_2(t_2+1)^{i-1}] - r[C_1(t_1+1)^{i-2} + C_2(t_2+1)^{i-2}]\\
&= C_1(t_1+1)^{i-2}[(r+2)(t_1+1) - r] + C_2(t_2+1)^{i-2}[(r+2)(t_2+1) - r]\\
&= C_1(t_1+1)^{i-2}[rt_1 + 2t_1 + 2] + C_2(t_2+1)^{i-2}[rt_2 + 2t_2 + 2]\\
&= C_1(t_1+1)^{i-2}[(rt_1+1) + 2t_1 + 1] + C_2(t_2+1)^{i-2}[(rt_2+1) + 2t_2 + 1]\\
&= C_1(t_1+1)^{i} + C_2(t_2+1)^{i} \quad \text{(note that } rt_1 + 1 = t_1^2 \text{ and } rt_2 + 1 = t_2^2.\text{)}\\
&= \check{\mathcal{G}}_i,
\end{aligned}$$

as required.

The second part (i.e., the equalities in Eq. (2)) is similar and we leave it to the reader.□

Lemma 5 *If $n \geqslant 1$, then $\mathcal{G}_n = a\bar{\mathcal{G}}_{n-1} + b\bar{\mathcal{G}}_n$.*

Proof. We will proceed by induction with respect to n. The case $n = 1$ is trivial:

$$a\bar{\mathcal{G}}_0 + b\bar{\mathcal{G}}_1 = a \cdot 0 + b \cdot 1 = b = \mathcal{G}_1.$$

Assume the identity is true when $1 \leqslant n < k$, that is,

$$\mathcal{G}_n = a\bar{\mathcal{G}}_{n-1} + b\bar{\mathcal{G}}_n, \quad n = 1, 2, 3, \ldots, k-1. \tag{5}$$

We now prove that it is also true for $n = k$. Easy calculations show that

$$\begin{aligned}
\mathcal{G}_k = r\mathcal{G}_{k-1} + \mathcal{G}_{k-2} \quad &= \quad r(a\bar{\mathcal{G}}_{k-2} + b\bar{\mathcal{G}}_{k-1}) + (a\bar{\mathcal{G}}_{k-3} + b\bar{\mathcal{G}}_{k-2})\\
&\qquad \text{(by the induction hypothesis, Eq. (5))}\\
&= \quad a(r\bar{\mathcal{G}}_{k-2} + \bar{\mathcal{G}}_{k-3}) + b(r\bar{\mathcal{G}}_{k-1} + \bar{\mathcal{G}}_{k-2})\\
&= \quad a\bar{\mathcal{G}}_{k-1} + b\bar{\mathcal{G}}_k,
\end{aligned}$$

which completes the proof. □

Lemma 6 *If $m, n \geqslant 0$, then*

$$\bar{\mathcal{G}}_{n+m} = \bar{\mathcal{G}}_n\bar{\mathcal{G}}_{m-1} + \bar{\mathcal{G}}_{n+1}\bar{\mathcal{G}}_m, \tag{6}$$

and

$$\bar{\mathcal{G}}_{m+1}\bar{\mathcal{G}}_{m-1} - \bar{\mathcal{G}}_m^2 = (-1)^m. \tag{7}$$

Proof. If $m = 0$ or $n = 0$, then the proof is straightforward. Therefore, we may assume that $m, n \geqslant 1$. Put

$$A := \begin{bmatrix} \bar{\mathcal{G}}_2 & \bar{\mathcal{G}}_1 \\ \bar{\mathcal{G}}_1 & \bar{\mathcal{G}}_0 \end{bmatrix} = \begin{bmatrix} r & 1 \\ 1 & 0 \end{bmatrix}.$$

We claim that

$$A^m = \begin{bmatrix} \bar{\mathcal{G}}_{m+1} & \bar{\mathcal{G}}_m \\ \bar{\mathcal{G}}_m & \bar{\mathcal{G}}_{m-1} \end{bmatrix}, \qquad m \geqslant 1.$$

To prove this we will proceed by induction with respect to m. It is clear for $m = 1$. Suppose that the claim is true for $m - 1$. Our task is to show that it is also true for m. Again, we verify the result by a direct calculation:

$$\begin{aligned} A^m = A \cdot A^{m-1} &= \begin{bmatrix} r & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \bar{\mathcal{G}}_m & \bar{\mathcal{G}}_{m-1} \\ \bar{\mathcal{G}}_{m-1} & \bar{\mathcal{G}}_{m-2} \end{bmatrix} \\ &\quad \text{(by the induction hypothesis)} \\ &= \begin{bmatrix} r\bar{\mathcal{G}}_m + \bar{\mathcal{G}}_{m-1} & r\bar{\mathcal{G}}_{m-1} + \bar{\mathcal{G}}_{m-2} \\ \bar{\mathcal{G}}_m & \bar{\mathcal{G}}_{m-1} \end{bmatrix} \\ &= \begin{bmatrix} \bar{\mathcal{G}}_{m+1} & \bar{\mathcal{G}}_m \\ \bar{\mathcal{G}}_m & \bar{\mathcal{G}}_{m-1} \end{bmatrix}, \end{aligned}$$

as claimed.

Now, we have $A^{n+m} = A^n \cdot A^m$, that is

$$\begin{bmatrix} \bar{\mathcal{G}}_{n+m+1} & \bar{\mathcal{G}}_{n+m} \\ \bar{\mathcal{G}}_{n+m} & \bar{\mathcal{G}}_{n+m-1} \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{G}}_{n+1} & \bar{\mathcal{G}}_n \\ \bar{\mathcal{G}}_n & \bar{\mathcal{G}}_{n-1} \end{bmatrix} \cdot \begin{bmatrix} \bar{\mathcal{G}}_{m+1} & \bar{\mathcal{G}}_m \\ \bar{\mathcal{G}}_m & \bar{\mathcal{G}}_{m-1} \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} \bar{\mathcal{G}}_{n+m+1} & \bar{\mathcal{G}}_{n+m} \\ \bar{\mathcal{G}}_{n+m} & \bar{\mathcal{G}}_{n+m-1} \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{G}}_{n+1}\bar{\mathcal{G}}_{m+1} + \bar{\mathcal{G}}_n\bar{\mathcal{G}}_m & \bar{\mathcal{G}}_{n+1}\bar{\mathcal{G}}_m + \bar{\mathcal{G}}_n\bar{\mathcal{G}}_{m-1} \\ \bar{\mathcal{G}}_n\bar{\mathcal{G}}_{m+1} + \bar{\mathcal{G}}_{n-1}\bar{\mathcal{G}}_m & \bar{\mathcal{G}}_n\bar{\mathcal{G}}_m + \bar{\mathcal{G}}_{n-1}\bar{\mathcal{G}}_{m-1} \end{bmatrix}.$$

Comparing $(0, 1)$-entries on both sides of this equation yields

$$\bar{\mathcal{G}}_{n+m} = \bar{\mathcal{G}}_{n+1}\bar{\mathcal{G}}_m + \bar{\mathcal{G}}_n\bar{\mathcal{G}}_{m-1},$$

which is the Eq. (6).

Moreover, since $\det A = -1$, $\det A^m = (\det A)^m = (-1)^m$, or equivalently

$$\det \begin{bmatrix} \bar{\mathcal{G}}_{m+1} & \bar{\mathcal{G}}_m \\ \bar{\mathcal{G}}_m & \bar{\mathcal{G}}_{m-1} \end{bmatrix} = (-1)^m,$$

and this immediately implies the following:

$$\bar{\mathcal{G}}_{m+1}\bar{\mathcal{G}}_{m-1} - \bar{\mathcal{G}}_m^2 = (-1)^m,$$

which is the Eq. (7). This completes the proof of the lemma. □

An immediate consequence of Lemma 6 is the following, which contains a list of well-known identities.

Corollary 1 *If $m, n \geqslant 0$, then the following identities hold:*

(i) $\mathcal{F}_n\mathcal{F}_{m-1} + \mathcal{F}_{n+1}\mathcal{F}_m = \mathcal{F}_{m+n}$,

(ii) $\mathcal{F}_{m+1}\mathcal{F}_{m-1} - \mathcal{F}_m^2 = (-1)^m$, *(Cassini's identity)*

(iii) $\mathcal{P}_n\mathcal{P}_{m-1} + \mathcal{P}_{n+1}\mathcal{P}_m = \mathcal{P}_{m+n}$,

(iv) $\mathcal{P}_{m+1}\mathcal{P}_{m-1} - \mathcal{P}_m^2 = (-1)^m$. *(Simpson's identity)*

3 Main Results

As we mentioned in the Introduction, we are going to obtain some new matrices whose leading principal minors form the Fibonacci, Lucas, Pell and Jacobsthal sequences. We start with the following theorem.

Theorem 1 *If $\mathcal{G} = \mathcal{G}^{(a,b,r,1)}$, where a, b and r are integers with $r \geqslant 1$, then for any nonnegative integer n there holds*

$$\det T_{\tilde{\mathcal{G}},\mathcal{G}}(n) = \begin{cases} a & n = 0, \\ (2b - ar)^{n-1}(a\mathcal{G}_{n-1} + b\mathcal{G}_n) & n \geqslant 1. \end{cases}$$

Proof. For $n = 0$, we have nothing to do. Therefore, we may assume that $n \geqslant 1$. We use the matrix factorization method. Let $T(n)$ denote the matrix $T_{\tilde{\mathcal{G}},\mathcal{G}}(n)$. We claim that

$$T(n) = L(n) \cdot H(n), \tag{8}$$

where $L(n) = (L_{i,j})_{0\leqslant i,j\leqslant n}$ with

$$L_{i,j} = \begin{cases} (-1)^{i+j}\bar{\mathcal{G}}_{i+j-1} & j = 0, 1,\ i \geqslant 0, \\ 0 & i = 0,\ j \geqslant 2, \\ L_{i-1,j-1} & i \geqslant 1,\ j \geqslant 2, \end{cases} \tag{9}$$

(we recall that $\bar{\mathcal{G}}_{-1} = 1$) and $H(n) = (H_{i,j})_{0\leqslant i,j\leqslant n}$ with

$$H_{i,j} = \begin{cases} (-1)^i \mathcal{G}_i & i = 0,1,\ j = 0, \\ \mathcal{G}_{j-i} & i = 0,1,\ j \geqslant 1, \\ r\mathcal{G}_0 - 2\mathcal{G}_1 & i - j = 1,\ i \geqslant 2, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The matrix $L(n)$ is a lower triangular matrix with 1's on the diagonal, whereas $H(n)$ is an upper Hessenberg matrix. The matrices $L(4)$ and $H(4)$ for instance are given by

$$\begin{aligned} L(4) &= \begin{pmatrix} \bar{\mathcal{G}}_{-1} & -\bar{\mathcal{G}}_0 & 0 & 0 & 0 \\ -\bar{\mathcal{G}}_0 & \bar{\mathcal{G}}_1 & -\bar{\mathcal{G}}_0 & 0 & 0 \\ \bar{\mathcal{G}}_1 & -\bar{\mathcal{G}}_2 & \bar{\mathcal{G}}_1 & -\bar{\mathcal{G}}_0 & 0 \\ -\bar{\mathcal{G}}_2 & \bar{\mathcal{G}}_3 & -\bar{\mathcal{G}}_2 & \bar{\mathcal{G}}_1 & -\bar{\mathcal{G}}_0 \\ \bar{\mathcal{G}}_3 & -\bar{\mathcal{G}}_4 & \bar{\mathcal{G}}_3 & -\bar{\mathcal{G}}_2 & \bar{\mathcal{G}}_1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\bar{\mathcal{G}}_0 & 1 & 0 & 0 & 0 \\ \bar{\mathcal{G}}_1 & -\bar{\mathcal{G}}_2 & 1 & 0 & 0 \\ -\bar{\mathcal{G}}_2 & \bar{\mathcal{G}}_3 & -\bar{\mathcal{G}}_2 & 1 & 0 \\ \bar{\mathcal{G}}_3 & -\bar{\mathcal{G}}_4 & \bar{\mathcal{G}}_3 & -\bar{\mathcal{G}}_2 & 1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} H(4) &= \begin{pmatrix} \mathcal{G}_0 & \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 & \mathcal{G}_4 \\ -\mathcal{G}_1 & \mathcal{G}_0 & \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 \\ 0 & r\mathcal{G}_0 - 2\mathcal{G}_1 & 0 & 0 & 0 \\ 0 & 0 & r\mathcal{G}_0 - 2\mathcal{G}_1 & 0 & 0 \\ 0 & 0 & 0 & r\mathcal{G}_0 - 2\mathcal{G}_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a & b & \mathcal{G}_2 & \mathcal{G}_3 & \mathcal{G}_4 \\ -b & a & \mathcal{G}_1 & \mathcal{G}_2 & \mathcal{G}_3 \\ 0 & ra - 2b & 0 & 0 & 0 \\ 0 & 0 & ra - 2b & 0 & 0 \\ 0 & 0 & 0 & ra - 2b & 0 \end{pmatrix}. \end{aligned}$$

In what follows, for convenience, we will let $T = T(n)$, $L = L(n)$ and $H = H(n)$. For the proof of the claimed factorization Eq. (8) we compute the (i, j)-entry of $L \cdot H$, that is

$$(L \cdot H)_{i,j} = \sum_{k=0}^{n} L_{i,k} H_{k,j}.$$

As a matter of fact, we should establish

$$\begin{aligned} \mathrm{R}_0(L \cdot H) &= \mathrm{R}_0(T) = & (\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_n), \\ \mathrm{C}_0(L \cdot H) &= \mathrm{C}_0(T) = & (\mathcal{G}_0, -\mathcal{G}_1, \ldots, (-1)^n \mathcal{G}_n), \end{aligned}$$

and

$$(L \cdot H)_{i,j} = (L \cdot H)_{i-1,j-1} \quad \text{for} \quad 1 \leqslant i, j \leqslant n. \tag{11}$$

Let us do the required calculations. First, suppose that $i = 0$. Then

$$(L \cdot H)_{0,j} = \sum_{k=0}^{n} L_{0,k} H_{k,j} = L_{0,0} H_{0,j} = H_{0,j} = \mathcal{G}_j,$$

which shows that $\mathrm{R}_0(L \cdot H) = \mathrm{R}_0(T)$.

Next, assume that $j = 0$. In this case, using Lemma 5, we obtain

$$\begin{aligned} (L \cdot H)_{i,0} &= \textstyle\sum_{k=0}^{n} L_{i,k} H_{k,0} = L_{i,0} H_{0,0} + L_{i,1} H_{1,0} \\ &= (-1)^i [a \bar{\mathcal{G}}_{i-1} + b \bar{\mathcal{G}}_i] = (-1)^i \mathcal{G}_i, \end{aligned}$$

and hence we have $\mathrm{C}_0(L \cdot H) = \mathrm{C}_0(T)$.

Finally, we must establish Eq. (11). To do this, we assume that $1 \leqslant i, j \leqslant n$. We distinguish two cases separately: $j = 1$ and $j \geqslant 2$.

Case 1. $j = 1$. We calculate the sum in question:

$$\begin{aligned} (L \cdot H)_{i,1} &= \sum_{k=0}^{n} L_{i,k} H_{k,1} = L_{i,0} H_{0,1} + L_{i,1} H_{1,1} + L_{i,2} H_{2,1} \\ &= L_{i,0} H_{0,1} + L_{i,1} H_{1,1} + L_{i-1,1} H_{2,1} \\ &= (-1)^i \bar{\mathcal{G}}_{i-1} b + (-1)^{i+1} \bar{\mathcal{G}}_i a + (-1)^i \bar{\mathcal{G}}_{i-1} (ra - 2b) \\ &= (-1)^i [r \bar{\mathcal{G}}_{i-1} - \bar{\mathcal{G}}_i] a + (-1)^{i+1} \bar{\mathcal{G}}_{i-1} b \\ &= (-1)^{i+1} \bar{\mathcal{G}}_{i-2} a + (-1)^{i+1} \bar{\mathcal{G}}_{i-1} b \\ &= (-1)^{i+1} [a \bar{\mathcal{G}}_{i-2} + b \bar{\mathcal{G}}_{i-1}] \\ &= (-1)^{i+1} \mathcal{G}_{i-1} \quad \text{(by Lemma 5)} \\ &= (L \cdot H)_{i-1,0}, \end{aligned}$$

as required.

CASE 2. $j \geqslant 2$. We proceed analogously. In this case, we have

$$
\begin{aligned}
& (L \cdot H)_{i,j} \\
= \ & \sum_{k=0}^{n} L_{i,k}H_{k,j} = L_{i,0}H_{0,j} + L_{i,1}H_{1,j} + L_{i,2}H_{2,j} + \sum_{k=3}^{n} L_{i,k}H_{k,j} \\
= \ & (-1)^i \bar{\mathcal{G}}_{i-1}\mathcal{G}_j + (-1)^{i+1}\bar{\mathcal{G}}_i\mathcal{G}_{j-1} + \sum_{k=3}^{n} L_{i-1,k-1}H_{k-1,j-1} \\
& (\text{Note that } H_{2,j} = 0 \text{ for } j \geqslant 2) \\
= \ & (-1)^i \bar{\mathcal{G}}_{i-1}\mathcal{G}_j + (-1)^{i+1}[\bar{\mathcal{G}}_{i-2} + r\bar{\mathcal{G}}_{i-1}]\mathcal{G}_{j-1} + \sum_{k=2}^{n} L_{i-1,k}H_{k,j-1} \\
& (\text{Note that } L_{i-1,n} = 0) \\
= \ & (-1)^i \bar{\mathcal{G}}_{i-1}[\mathcal{G}_j - r\mathcal{G}_{j-1}] + (-1)^{i+1}\bar{\mathcal{G}}_{i-2}\mathcal{G}_{j-1} + \sum_{k=2}^{n} L_{i-1,k}H_{k,j-1} \\
= \ & (-1)^i \bar{\mathcal{G}}_{i-1}\mathcal{G}_{j-2} + (-1)^{i-1}\bar{\mathcal{G}}_{i-2}\mathcal{G}_{j-1} + \sum_{k=2}^{n} L_{i-1,k}H_{k,j-1} \\
= \ & \sum_{k=0}^{n} L_{i-1,k}H_{k,j-1} = (L \cdot H)_{i-1,j-1},
\end{aligned}
$$

as required.

Returning back to the Eq. (8), we conclude that

$$\det T = \det(L \cdot H) = \det L \cdot \det H = \det H.$$

Now, expanding the determinant of H along the rows $2, 3, \ldots, n$, one can easily get

$$\det H = (2b - ra)^{n-1}(a\mathcal{G}_{n-1} + b\mathcal{G}_n).$$

This completes the proof of the theorem. $\square$

Although we have restricted ourselves to integral values in Theorem 1, it holds for arbitrary values of all parameters a, b and r. In the sequel, we will consider some applications of Theorem 1 in some simple, but interesting, cases.

Corollary 2 *Let $m \geqslant 0$ be an integer and $\alpha = (\mathcal{F}_{m+i})_{i \geqslant 0}$. Then there holds*

$$\det T_{\tilde{\alpha},\alpha}(n) = \mathcal{L}_m^{n-1}\mathcal{F}_{2m+n} = (\mathcal{F}_{m-1} + \mathcal{F}_{m+1})^{n-1}\mathcal{F}_{2m+n}, \quad n \geqslant 0. \qquad (12)$$

In particular, we have

(1) *If* $m=0$*, then* $\det T_{\tilde{\mathcal{F}},\mathcal{F}}(n)=2^{n-1}\mathcal{F}_n$.

(2) *If* $m=1$*, then* $\det T_{\tilde{\alpha},\alpha}(n)=\mathcal{F}_{n+2}$.

Proof. Let $d_n=\det T_{\tilde{\alpha},\alpha}(n)$. Using Theorem 1 and Corollary 1 (i), we have

$$\begin{aligned}
d_n &= \begin{cases} \mathcal{F}_m & n=0, \\ (2\mathcal{F}_{m+1}-\mathcal{F}_m)^{n-1}(\mathcal{F}_m\mathcal{F}_{m+n-1}+\mathcal{F}_{m+1}\mathcal{F}_{m+n}) & n\geqslant 1, \end{cases} \\
&= \begin{cases} \mathcal{F}_m & n=0, \\ (\mathcal{F}_{m-1}+\mathcal{F}_{m+1})^{n-1}\mathcal{F}_{2m+n} & n\geqslant 1, \end{cases} \\
&= \begin{cases} (\mathcal{F}_{m-1}+\mathcal{F}_{m+1})^{-1}\mathcal{F}_{2m} & n=0, \\ (\mathcal{F}_{m-1}+\mathcal{F}_{m+1})^{n-1}\mathcal{F}_{2m+n} & n\geqslant 1, \end{cases} \\
&= (\mathcal{F}_{m-1}+\mathcal{F}_{m+1})^{n-1}\mathcal{F}_{2m+n}, \quad n\geqslant 0, \\
&= \mathcal{L}_m^{n-1}\mathcal{F}_{2m+n}, \quad n\geqslant 0.
\end{aligned}$$

The rest follows immediately. □

Corollary 3 *Let* $m\geqslant 0$ *be an integer and* $\alpha=(\mathcal{P}_{m+i})_{i\geqslant 0}$*. Then for every* $n\geqslant 0$ *there holds*

$$\det T_{\tilde{\alpha},\alpha}(n)=(\mathcal{P}_{m-1}+\mathcal{P}_{m+1})^{n-1}\mathcal{P}_{2m+n}.$$

In particular, for $m=0$ *we conclude that* $\det T_{\tilde{\mathcal{P}},\mathcal{P}}(n)=2^{n-1}\mathcal{P}_n$.

Proof. Let $d_n=\det T_{\tilde{\alpha},\alpha}(n)$. Again, using Theorem 1 and Corollary 1 (iii), we have

$$\begin{aligned}
d_n &= \begin{cases} \mathcal{P}_m & n=0, \\ (2\mathcal{P}_{m+1}-2\mathcal{P}_m)^{n-1}(\mathcal{P}_m\mathcal{P}_{m+n-1}+\mathcal{P}_{m+1}\mathcal{P}_{m+n}) & n\geqslant 1, \end{cases} \\
&= \begin{cases} \mathcal{P}_m & n=0, \\ (\mathcal{P}_{m-1}+\mathcal{P}_{m+1})^{n-1}\mathcal{P}_{2m+n} & n\geqslant 1, \end{cases} \\
&= \begin{cases} (\mathcal{P}_{m-1}+\mathcal{P}_{m+1})^{-1}\mathcal{P}_{2m} & n=0, \\ (\mathcal{P}_{m-1}+\mathcal{P}_{m+1})^{n-1}\mathcal{P}_{2m+n} & n\geqslant 1, \end{cases} \\
&= (\mathcal{P}_{m-1}+\mathcal{P}_{m+1})^{n-1}\mathcal{P}_{2m+n}, \quad n\geqslant 0.
\end{aligned}$$

The rest follows immediately. □

Notice that, using Lemma 4, if $\alpha = (\mathcal{F}_{i+1})_{i\geqslant 0} = (1,1,2,3,5,8,\ldots)$, then

$$\begin{cases} \breve{\alpha} = (\mathcal{F}_{2i+1})_{i\geqslant 0} = (1,2,5,13,34,\ldots) & \text{(A001519 in [14])} \\ \check{\breve{\alpha}} = (\mathcal{F}_{i-1})_{i\geqslant 0} = (1,0,1,1,2,3,5,\ldots) & \text{(A212804 in [14])} \end{cases}$$

Corollary 4 *If $\alpha = (\mathcal{F}_{i+1})_{i\geqslant 0}$, then the following infinite integer matrices:*

$$T_{\tilde{\alpha},\alpha} = \begin{pmatrix} 1 & 1 & 2 & 3 & \cdot \\ -1 & 1 & 1 & 2 & \cdot \\ 2 & -1 & 1 & 1 & \cdot \\ -3 & 2 & -1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad T_{\alpha,\tilde{\alpha}} = \begin{pmatrix} 1 & -1 & 2 & -3 & \cdot \\ 1 & 1 & -1 & 2 & \cdot \\ 2 & 1 & 1 & -1 & \cdot \\ 3 & 2 & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$P_{\check{\breve{\alpha}},\breve{\alpha}} = \begin{pmatrix} 1 & 2 & 5 & 13 & \cdot \\ 0 & 2 & 7 & 20 & \cdot \\ 1 & 3 & 10 & 30 & \cdot \\ 1 & 4 & 14 & 44 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \qquad P_{\breve{\alpha},\check{\breve{\alpha}}} = \begin{pmatrix} 1 & 0 & 1 & 1 & \cdot \\ 2 & 2 & 3 & 4 & \cdot \\ 5 & 7 & 10 & 14 & \cdot \\ 13 & 20 & 30 & 44 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$A_{\check{\breve{\alpha}},\alpha} = \begin{pmatrix} 1 & 1 & 2 & 3 & \cdot \\ 0 & 2 & 3 & 5 & \cdot \\ 1 & 2 & 5 & 8 & \cdot \\ 1 & 3 & 7 & 13 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \qquad A_{\breve{\alpha},\tilde{\alpha}} = \begin{pmatrix} 1 & -1 & 2 & -3 & \cdot \\ 2 & 0 & 1 & -1 & \cdot \\ 5 & 2 & 1 & 0 & \cdot \\ 13 & 7 & 3 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

are equimodular. In precisely, we have

$$\begin{aligned} \det T_{\tilde{\alpha},\alpha}(n) &= \det T_{\alpha,\tilde{\alpha}}(n) = \det P_{\check{\breve{\alpha}},\breve{\alpha}}(n) = \det P_{\breve{\alpha},\check{\breve{\alpha}}}(n) \\ &= \det A_{\check{\breve{\alpha}},\alpha}(n) = \det A_{\breve{\alpha},\tilde{\alpha}}(n) = \mathcal{F}_{n+2}, \end{aligned}$$

for all $n \geqslant 0$.

Proof. Follows directly from Corollary 2 (2) and Lemmas 1 and 4. □

Corollary 5 *Let a, b and c be arbitrary complex numbers, and for each positive integer n, let $T_n = T_n(a,b,c)$. Then* $\det T_1 = c$, $\det T_2 = c+1$ *and for $n \geqslant 3$,*

$$\det T_n = \det T_{n-1} + \det T_{n-2}, \tag{13}$$

if and only if

$$(a,b,c) \in \Big\{ \big(c-\phi, c-\Phi, c\big), \big(c-\Phi, c-\phi, c\big) \Big\}. \tag{14}$$

Furthermore, if Eq. (14) *holds, then we have* $\det T_n = c\mathcal{F}_n + \mathcal{F}_{n-1}$. *In particular, there hold*

(1) *If* $c = 0$, *then* $\det T_n = \mathcal{F}_{n-1}$,
(2) *If* $c = 1$, *then* $\det T_n = \mathcal{F}_{n+1}$,
(3) *If* $c = -1$, *then* $\det T_n = -\mathcal{F}_{n-2}$,
(4) *If* $c = 2$, *then* $\det T_n = \mathcal{F}_{n+2}$,
(5) *If* $c = -2$, *then* $\det T_n = -\mathcal{L}_{n-1}$,
(6) *If* $c = 3$, *then* $\det T_n = \mathcal{L}_{n+1}$.

Proof. The sufficiency is clear. To prove the necessity, using Lemma 2 we have

$$\det T_n = (2c-a-b)\det T_{n-1} - (c-a)(c-b)\det T_{n-2},$$

and by comparing this with Eq. (13), we then obtain the system of equalities

$$\begin{cases} 2c-a-b = 1 \\ (c-a)(c-b) = -1 \end{cases}$$

which has two solutions of the forms

$$a = \frac{(2c-1) \pm \sqrt{5}}{2} \quad \text{and} \quad b = \frac{(2c-1) \mp \sqrt{5}}{2}.$$

The rest of the proof is obvious. □

Corollary 6 *Let* a, b *and* c *be arbitrary complex numbers, and for each positive integer* n, *let* $T_n = T_n(a,b,c)$. *Then* $\det T_1 = c$, $\det T_2 = 2c+1$ *and for* $n \geqslant 3$,

$$\det T_n = 2\det T_{n-1} + \det T_{n-2}, \tag{15}$$

if and only if

$$(a,b,c) \in \Big\{ \big(c-1+\sqrt{2}, c-1-\sqrt{2}, c\big), \big(c-1-\sqrt{2}, c-1+\sqrt{2}, c\big) \Big\}. \tag{16}$$

Furthermore, if Eq. (16) *holds, then we have* $\det T_n = c\mathcal{P}_n + \mathcal{P}_{n-1}$. *In particular, there hold*

(1) *If* $c = 0$, *then* $\det T_n = \mathcal{P}_{n-1}$,
(2) *If* $c = 2$, *then* $\det T_n = \mathcal{P}_{n+1}$.

Proof. The sufficiency is clear. To prove the necessity, using Lemma 2 we have

$$\det T_n = (2c - a - b)\det T_{n-1} - (c-a)(c-b)\det T_{n-2},$$

and by comparing this with Eq. (15), we then obtain the system of equalities

$$\begin{cases} 2c - a - b = 2 \\ (c-a)(c-b) = -1 \end{cases}$$

which has two solutions of the forms

$$a = c - 1 \pm \sqrt{2} \quad \text{and} \quad b = c - 1 \mp \sqrt{2}.$$

The rest of the proof is obvious. $\square$

Corollary 7 *Let a, b and c be arbitrary complex numbers, and for each positive integer n, let $T_n = T_n(a, b, c)$. Then $\det T_1 = c$, $\det T_2 = c + 2$ and for $n \geqslant 3$,*

$$\det T_n = \det T_{n-1} + 2\det T_{n-2}, \tag{17}$$

if and only if

$$(a, b, c) \in \Big\{\big(c+1, c-2, c\big), \big(c-2, c+1, c\big)\Big\}. \tag{18}$$

Furthermore, if Eq. (18) *holds, then we have $\det T_n = c\mathcal{J}_n + 2\mathcal{J}_{n-1}$. In particular, we have*

(1) *If $c = 1$, then $\det T_n = \mathcal{J}_{n+1}$.*
(2) *If $c = 3$, then $\det T_n = \mathcal{J}_{n+2}$.*

Proof. The sufficiency is clear. To prove the necessity, using Lemma 2 we have

$$\det T_n = (2c - a - b)\det T_{n-1} - (c-a)(c-b)\det T_{n-2},$$

and by comparing this with Eq. (17), we then obtain the system of equalities

$$\begin{cases} 2c - a - b = 1 \\ (c-a)(c-b) = -2 \end{cases}$$

which has two solutions of the forms

$$(a, b) \in \big\{(c+1, c-2), (c-2, c+1)\big\}.$$

The rest of the proof is obvious. $\square$

Corollary 8 *Let* $\alpha = (\alpha_i)_{i\geqslant 0}$ *and* $\beta = (\beta_i)_{i\geqslant 0}$ *be two sequences starting with a common first term. Then there hold:*

(1) *If* $\alpha_i = 2^{i+1} - 1$ *and* $\beta_i = 2(1 - 2^{i-1})$, *then* $\det P_{\alpha,\beta}(n) = \mathcal{J}_{n+2}$,

(2) *If* $\alpha_i = 2^{i+2} - 1$ *and* $\beta_i = 2(2^{i-1} + 1)$, *then* $\det P_{\alpha,\beta}(n) = \mathcal{J}_{n+3}$.

Proof. Both statements follow directly from Lemma 1, Lemma 3 and Corollary 7. □

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