ENUMERATION OF MEANDERS AND MASUR–VEECH VOLUMES

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Abstract. A *meander* is a topological configuration of a line and a simple closed curve in the plane (or a pair of simple closed curves on the 2-sphere) intersecting transversally. In physics, meanders provide a model of polymer *folding*, and their enumeration is directly related to the entropy of the associated dynamical systems.

We combine recent results on Masur–Veech volumes of the moduli spaces of meromorphic quadratic differentials in genus zero and our previous result that horizontal and vertical separatrix diagrams of integer quadratic differentials are asymptotically uncorrelated to derive two applications to asymptotic enumeration of meanders.

First, we get simple asymptotic formulae for the number of pairs of transverse simple closed curves on a sphere and for the number of closed meanders of fixed combinatorial type when the number of crossings $2N$ goes to infinity.

Second, we compute the asymptotic probability of getting a simple closed curve on a sphere by identifying the endpoints of two arc systems (one on each of the two hemispheres) along the common equator. Here the total number of minimal arcs of the two arc systems is considered as a fixed parameter while the number of all arcs (same for each of the two hemispheres) grows.

The number of all meanders with $2N$ crossings grows exponentially when N grows. However, the additional combinatorial constraints we impose in this article yield polynomial asymptotics.

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1. Introduction and statements of main results

In the seminal paper [Mi] M. Mirzakhani has computed the asymptotics for the number of simple closed hyperbolic geodesics on a hyperbolic surface of constant negative curvature and frequencies of simple closed hyperbolic geodesics of fixed combinatorial type.

We count the asymptotics for the number of *pairs* of transverse simple closed curves of a fixed combinatorial type on a sphere when the number of intersections tends to infinity. The similar enumeration problems in higher genera will be considered in the sequel.

M. Mirzakhani establishes a relations between counting of simple closed curves and Weil–Petersson volumes of the moduli spaces of bordered hyperbolic surfaces. Counting pairs of transverse simple closed curves leads naturally to Masur–Veech volumes of the moduli spaces of meromorphic quadratic differentials with at most simple poles.

In this section we state our results on meander enumeration. The link with quadratic differentials and Masur-Veech volumes will be explained in Section 2.

1.1. Counting meanders with given number of minimal arcs.

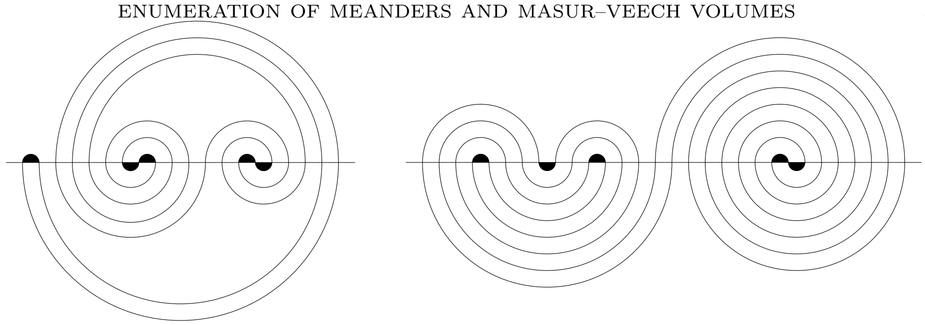
A closed *plane meander* is a smooth closed curve in the plane transversally intersecting the horizontal line as in Figure 1. According to the paper [LZv] of S. Lando and A. Zvonkine (serving as a reference paper in the literature on meanders) the notion "meander" was suggested by V. I. Arnold in [Ar] though meanders were studied already by H. Poincaré [Po]. Meanders appear in various contexts, in particular in physics, see [DiGG1]. Counting meanders has a reputation of a notoriously difficult problem. The number of meanders with $2n$ crossings is conjecturally asymptotic to $R^{2n}n^{-\alpha}$ where R and α are some constants. A conjectural value of the critical exponent α is given in [DiGG2].

We say that a closed meander has a *maximal arc* ("*rainbow*" in terminology of [ACPRS]) if it has an arc joining the leftmost and the rightmost crossings with the horizontal line. Otherwise meander *does not a have maximal arc*. Meander on the left of Figure 1 has maximal arc, while the one on the right – does not.

By *minimal arc* ("*pimple*" in terminology of [ACPRS], or "*internal arch*" in terminology of [DiGG1]) we call an arc which does not have any crossings inside. The areas between the horizontal line and the minimal arcs of meanders are colored in black in Figure 1; each of the two meanders has $p = 5$ minimal arcs.

By convention, in this paper we do not consider the trivial closed meander represented by a circle. All other closed meanders satisfy $p \geq 3$ when they have a maximal arc and $p \geq 4$ when they do not.

Let $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ be the numbers of closed meanders respectively with and without maximal arc ("rainbow") and having at most $2N$ crossings with the



Contributes to $\mathcal{M}_5^+(N)$ Contributes to $\mathcal{M}_5^-(N)$

FIGURE 1. Meander with a maximal arc ("rainbow") on the left and without one on the right. Both meanders have 5 minimal arcs ("pimples").

horizontal line and exactly p minimal arcs ("pimples"). We consider p as a parameter and we study the leading terms of the asymptotics of $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ as $N \to +\infty$.

Theorem 1. *For any fixed p the numbers $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ of closed meanders with p minimal arcs (pimples) and with at most $2N$ crossings have the following asymptotics as $N \to +\infty$:*

$$\mathcal{M}_p^+(N) = 2(p+1) \cdot \frac{cyl_{1,1}\big(\mathcal{Q}(1^{p-3}, -1^{p+1})\big)}{(p+1)!\,(p-3)!} \cdot \frac{N^{2p-4}}{4p-8} \;+\; o(N^{2p-4}) = \tag{1.1}$$

$$= \frac{2}{p!\,(p-3)!}\left(\frac{2}{\pi^2}\right)^{p-2} \cdot \binom{2p-2}{p-1}^2 \cdot \frac{N^{2p-4}}{4p-8} \;+\; o(N^{2p-4})\,.$$

$$\mathcal{M}_p^-(N) = \frac{2\,cyl_{1,1}\big(\mathcal{Q}(1^{p-4}, 0, -1^{p})\big)}{p!\,(p-4)!} \cdot \frac{N^{2p-5}}{4p-10} \;+\; o(N^{2p-5}) = \tag{1.2}$$

$$= \frac{4}{p!\,(p-4)!}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2}^2 \cdot \frac{N^{2p-5}}{4p-10} \;+\; o(N^{2p-5})\,.$$

The quantities $cyl_{1,1}\big(\mathcal{Q}(1^{p-3}, -1^{p+1})\big)$ and $cyl_{1,1}\big(\mathcal{Q}(1^{p-4}, 0, -1^{p})\big)$ in the above formulae are related to Masur–Veech volumes of the moduli space of meromorphic quadratic differentials. Their definition and role would be discussed in section 2. Theorem 1 is proved in section 3.5 with exception for the explicit expressions for these two quantities evaluated in Corollary 4.1 in section 4.

Note that the number $\mathcal{M}_p^+(N)$ grows as N^{2p-4} while $\mathcal{M}_p^-(N)$ grows as N^{2p-5}. This means that for large N all but negligible fraction of meanders having any given number p of minimal arcs (pimples) do have a maximal arc (rainbow) as the left one in Figure 1.

As the reader could observe in the statement of Theorem 1, our approach to counting meanders differs from the traditional one: we fix the combinatorics of the meander and then count the asymptotic number of meanders of chosen combinatorial type as the number of intersections N tends to infinity. Our settings can be seen as a zero temperature limit in the thermodynamical sense, where the complexity of a meander is measured in terms of the number of minimal arcs.

Applying Stirling's formula we get the following asymptotics for the coefficients in formulae (1.1) and (1.2) for large values of parameter p:

$$\frac{2}{p!\,(p-3)!}\left(\frac{2}{\pi^2}\right)^{p-2}\cdot\binom{2p-2}{p-1}^2\cdot\frac{1}{4p-8}\ \sim\ \frac{\pi^2}{256}\cdot\left(\frac{32e^2}{\pi^2p^2}\right)^p \qquad \text{for } p\gg 1\,.$$

$$\frac{4}{p!\,(p-4)!}\left(\frac{2}{\pi^2}\right)^{p-3}\cdot\binom{2p-4}{p-2}^2\cdot\frac{1}{4p-10}\ \sim\ \frac{\pi^2e^2}{128p}\cdot\left(\frac{32e^2}{\pi^2p^2}\right)^{p-1} \qquad \text{for } p\gg 1\,.$$

(we again recall that in our setting we always assume that $N \gg p$).

In section 3.5 we provide an analogous statement, Theorem 5, which counts meanders in the setting where the combinatorial type is specified in a more detailed way.

1.2. **Counting meanders with given reduced arc systems.** Extending the horizontal segment of a plane meander to the infinite line and passing to a one-point compactification of the plane we get a meander on the 2-sphere. A meander on the sphere is a pair of transversally intersecting labeled simple closed curves. It will be always clear from the context whether we consider meanders in the plane or on the sphere. Essentially, we follow the following dichotomy: enumerating meanders, as in the previous section, we work with meanders in the plane, while considering frequencies of pairs of simple closed curves among more complicated pairs of multicurves, as in the current section, we work with meanders on the sphere.

Each meander defines a pair of arc systems in discs as in Figure 2. An arc system on the disc (also known as a "chord diagram") can be encoded by the dual tree, see the trees in dashed lines on the right pictures in Figure 2. Namely, the vertices of the tree correspond to the faces in which the arc system cuts the disc; two vertices are joined by an edge if and only if the corresponding faces have common arc. It is convenient to simplify the dual tree by forgetting all vertices of valence two. We call the resulting tree the *reduced* dual tree.

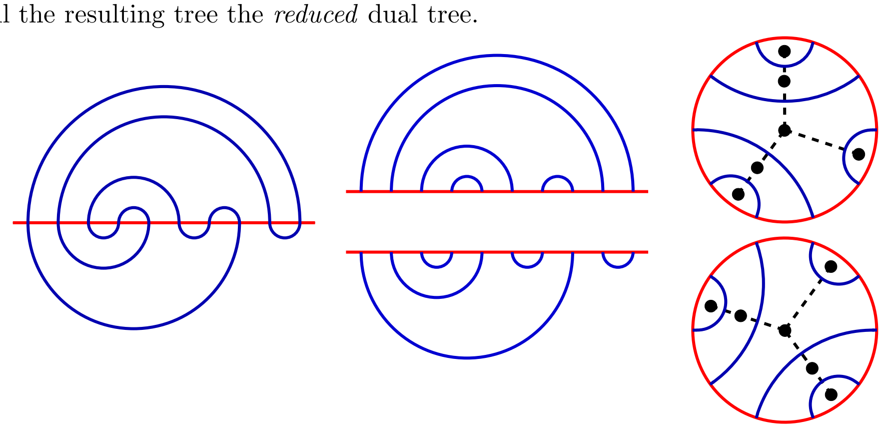


FIGURE 2. A closed meander on the left. The associated pair of arc systems in the middle. The same arc systems on the discs and the associated dual trees on the right.

It is much easier to count arc systems (for example, arc systems sharing the same reduced dual tree). However, this does not simplify counting meanders since identifying a pair of arc systems with the same number of arcs by the common equator, we sometimes get a meander and sometimes — a curve with several connected components, see Figure 3.

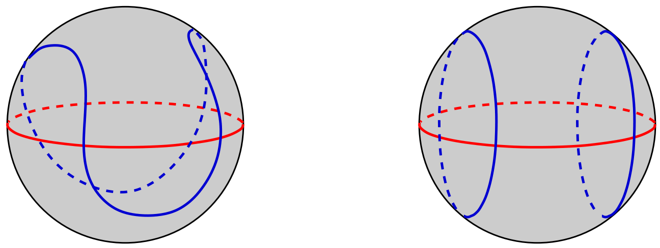


FIGURE 3. Gluing two hemispheres with arc systems along the common equator we may get either a single simple closed curve (as on the left picture) or a multicurve with several connected components (as on the right picture).

We now consider the more specialized setting where we fix a pair of plane trees and count meanders whose corresponding pair of arc systems have these given dual trees. Let us mention that everywhere in this paper we consider only *plane trees*, that is trees embedded into the plane.

Let $(\mathcal{T}_{top}, \mathcal{T}_{bottom})$ be a pair of plane trees with no vertices of valence 2. We consider arc system with the same number of arcs $n \leq N$ on a labeled pair of oriented discs having $\mathcal{T}_{top}$ and $\mathcal{T}_{bottom}$ as reduced dual trees. We draw the arc system corresponding to $\mathcal{T}_{top}$ on the northern hemisphere, and the arc system corresponding to $\mathcal{T}_{bottom}$ on the southern hemisphere. To simplify gluing of the two hemispheres, we assume that all segments of the boundary circle between adjacent endpoints of the arcs have equal length and that the arcs are orthogonal to the boundary circle. There are $2n$ ways to isometrically identify the boundaries of two hemispheres into the sphere in such way that the endpoints of the arcs match. We consider all possible triples

$$(n\text{-arc system of type } \mathcal{T}_{top};\ n\text{-arc system of type } \mathcal{T}_{bottom};\ \text{identification})$$

as described above for all $n \leq N$. Define

$$\mathrm{P}_{connected}(\mathcal{T}_{top}, \mathcal{T}_{bottom}; N) := \frac{\text{number of triples giving rise to meanders}}{\text{total number of different triples}}\,. \tag{1.3}$$

Theorem 2. *For any pair of trivalent plane trees $\mathcal{T}_{bottom}, \mathcal{T}_{top}$, having the total number p of leaves (vertices of valence one) the following limit exists:*

$$\lim_{N\to+\infty} \mathrm{P}_{connected}(\mathcal{T}_{bottom}, \mathcal{T}_{top}; N) \;=\; \mathrm{P}_1(\mathcal{Q}(1^{p-4}, -1^p)) \;= \tag{1.4}$$

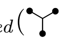
$$=\; \frac{cyl_1(\mathcal{Q}(1^{p-4}, -1^p))}{\operatorname{Vol}\mathcal{Q}(1^{p-4}, -1^p)} \;=\; \frac{1}{2}\left(\frac{2}{\pi^2}\right)^{p-3}\cdot\binom{2p-4}{p-2}\,.$$

The quantity $cyl_1\big(\mathcal{Q}(1^{p-4}, -1^p)\big)$ in the above formula is related to Masur–Veech volume of the moduli space of meromorphic quadratic differentials. Its definition and role would be discussed in section 2.

The quantity in this Theorem should be interpreted as the asymptotic probability $\mathrm{P}_{connected}(\mathcal{T}_{bottom}, \mathcal{T}_{top})$ with which a random choice of twist identifying a pair of random arc systems of fixed combinatorial types of the same cardinality defines a meander. To be more accurate, one should rather speak of asymptotic *frequency* of meanders among resulting multicurves.

Theorem 2 is proved at the end of section 3.4.We will actually state and prove a more general statement, Theorem 4, where not only trivalent trees are considered.

The fact that this asymptotic frequency is nonzero is already somehow unexpected. For example, for the pair of trees as on the right side of Figure 2 the corresponding asymptotic frequency is equal to

$$\mathrm{P}_{connected}($$  ,  $$) = \frac{280}{\pi^6} \approx 0.291245\,,$$

which is not even close to 0.

Stirling's formula gives the following asymptotics for $\mathrm{P}_1(\mathcal{Q}(1^{p-4}, -1^p))$ for large values of parameter p:

$$\mathrm{P}_1(\mathcal{Q}(1^{p-4}, -1^p)) = \frac{1}{2}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2} \sim \frac{2}{\sqrt{\pi p}} \cdot \left(\frac{8}{\pi^2}\right)^{p-3} \quad \text{for } p \gg 1$$

(we recall that in our setting we always assume that $N \gg p$).

Another unexpected fact that follows from Theorem 2 is that the way the leaves (univalent vertices) are distributed between the two trees is irrelevant: the answer depends only on the total number p of leaves. This observation suggests an alternative (and much less restrictive) way to fix combinatorics of the meanders. Namely, we can fix only the total number p of leaves (vertices of valence one) of the two trees together, where $p \geq 4$.

Theorem 3. *Let $p \geq 4$. The frequency $\mathrm{P}_{connected}(p; N)$ of meanders obtained by all possible identifications of all arc systems with at most N arcs represented by all possible pairs of (not necessarily trivalent) plane trees having the total number p of leaves (vertices of valence one) has the same limit $\mathrm{P}_1(\mathcal{Q}(1^{p-4}, -1^p))$ as the frequency $\mathrm{P}_{connected}(\mathcal{T}_{bottom}, \mathcal{T}_{top}; N)$ of meanders represented by any individual pair of trivalent plane trees with the total number p of leaves:*

$$(1.5)\qquad \lim_{N\to+\infty} \mathrm{P}_{connected}(p; N) \;=\; \mathrm{P}_1(\mathcal{Q}(1^{p-4}, -1^p)) \;= \;= \;\frac{cyl_1(\mathcal{Q}(1^{p-4}, -1^p))}{\operatorname{Vol}\mathcal{Q}(1^{p-4}, -1^p)} \;=\; \frac{1}{2}\left(\frac{2}{\pi^2}\right)^{p-3} \cdot \binom{2p-4}{p-2}.$$

Theorem 3 is proved at the end of section 3. The proof is based on the fact that the contribution of any pair of trees where at least one of the trees has a vertex of valency 4 or higher is negligible in comparison with the contribution of any pair of trivalent trees.

Acknowledgements. The vague idea that counting results concerning linear involutions (see Appendix B) might have applications to meanders was discussed by the authors in independent conversations with M. Kontsevich and with M. Mirzakhani on the early stage of the project [DGZZ]. We are grateful to M. Kontsevich and to M. Mirzakhani for these discussions and for their insights in enumerative geometry which were very inspiring for us.

We thank MPIM in Bonn, where part of this work was performed, for providing us with friendly and stimulating environment.

2. Idea of proof

2.1. **Pairs of transverse multicurves on the sphere as pillowcase covers.**

A *multicurve* on the sphere, is a collection of pairwise nonintersecting smooth simple closed curves in the sphere.

Definition 2.1. We say that two multicurves on the sphere form a *transverse connected pair* if any intersection between any connected component of the first curve and any connected component of the second curve is transverse and if in addition the union of two multicurves is connected.

Having a transverse connected pair of multicurves we always assume that the pair is ordered. By convention, the first multicurve is called "horizontal" and the second one — "vertical". We consider natural equivalence classes of transverse connected pairs of multicurves up to diffeomorphisms preserving the orientation of the sphere and respecting horizontal and vertical labelling.

Let $\mathcal{G}$ be the graph defined by a transverse connected pair of multicurves. The vertices of $\mathcal{G}$ are intersections of the multicurves, so all vertices of $\mathcal{G}$ have valence 4. Hence, all faces of the dual graph $\mathcal{G}^*$ are 4-gons. The edges of $\mathcal{G}^*$ dual to horizontal edges of $\mathcal{G}$ will be called vertical, and those dual to the vertical edges of $\mathcal{G}$ will be called horizontal. By construction, any two non-adjacent edges of any face of $\mathcal{G}^*$ are either both horizontal or both vertical.

Let us choose metric squares with side $\frac{1}{2}$ as faces of $\mathcal{G}^*$. By the Two Color Theorem every plane graph whose faces all have even numbers of sides is bipartite (see e. g. [So], pp. 136–137), so $\mathcal{G}$ is bipartite. This means that the squares of $\mathcal{G}^*$ can be colored in the chessboard manner. A square-tiled surface which admits a chessboard coloring is called *pillowcase cover*. It is a ramified cover over the standard "pillow" (obtained by isometric identification of the boundaries of two squares with side $\frac{1}{2}$) ramified only over the corners of the pillow, see Figure 4. By convention we memorize which sides of the pillow are horizontal, and which ones are vertical. Conversely, every pillowcase cover defines a transverse connected pair of multicurves: collections of closed horizontal and vertical curves passing through the centers of the squares of the tiling.

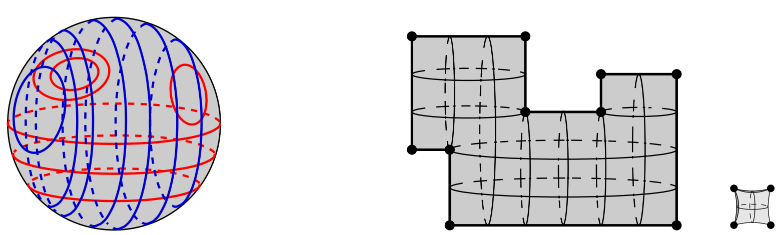


Figure 4. Graph dual to a transverse connected pair of multicurves on a sphere defines a pillowcase cover.

We have proved the following statement.

Proposition 1. *There is a natural one-to-one correspondence between transverse connected pairs of multicurves on the sphere and pillowcase covers of genus 0, where the square tiling is given by the graph dual to the graph formed by the union of two multicurves.*

The "pillow" as above defines a unique quadratic differential q_0 on $\mathbb{C}\mathrm{P}^1$ having simple poles at the four corners of the pillow and no other singularities. A pillowcase cover is thus endowed with the meromorphic quadratic differential $q = \pi^* q_0$ that is a pullback of the pillow differential q_0 by the covering map π. Simple poles of q correspond to bigons of $\mathcal{G}$; zeroes of order $j \in \mathbb{N}$ correspond to $(2j+4)$-gons.

Remark. "Pillowcase covers" are defined differently by different authors. In the original paper [EO2], A. Eskin and A. Okounkov define pillowcase covers as ramified covers π of degree $2d$ over the sphere having the following ramification type. They fix a partition μ with entries $\mu_i \le 2d$ and a partition ν of an even number bounded from above by $2d$ into odd parts. The cover π has the profile $[\nu, 2^{d-|\nu|/2}]$ over one corner of the pillow and the profile $[2^d]$ over three other corners. Additionally, π has profile $[\mu_i, 1^{2d-\mu_i}]$ over $\ell(\mu)$ distinct non-corner points of the pillow.

Lemma B.2 in [AEZ2] relates the asymptotic numbers of pillowcase covers of degree at most $2N$ defined in these two different ways in the strata of genus 0 moduli spaces as $N \to +\infty$.

2.2. **Couting pillowcase covers.** The moduli space of meromorphic quadratic differentials on $\mathbb{C}\mathrm{P}^1$ with exactly p simple poles is naturally stratified by the strata $\mathcal{Q}(\nu, -1^{|\nu|+4})$ of quadratic differentials with prescribed orders of zeroes ν (ν_i zeroes of order i) and with $p = |\nu| + 4$ simple poles (see e.g. [Zor1] for references). Here

$$|\nu| = 1 \cdot \nu_1 + 2 \cdot \nu_2 + 3 \cdot \nu_3 + \dots . \tag{2.1}$$

Under the above interpretation, transverse connected pairs of multicurves having fixed number of bigonal faces correspond to pillowcase covers with fixed number of simple poles. The transverse connected pairs of multicurves having fixed number ν_1 of hexagonal faces, fixed number ν_2 of octagonal faces, fixed number ν_j of $2(j+2)$-gonal faces for $j \in \mathbb{N}$ correspond to pillowcase covers in the fixed stratum $\mathcal{Q}(\nu, -1^{|\nu|+4})$.

In particular, the number of bigonal faces equals $|\nu| + 4$. The number of squares is the total number of crossings between the two multicurves. Generally speaking, pillowcase covers play the role of integer points in strata of moduli spaces of quadratic differentials.

Gluing two hemispheres with arc systems along the common equator, we get a transverse connected pair of multicurves. The horizontal multicurve has a single connected component, it is just a simple closed curve represented by the equator, whereas the vertical multicurve may have several connected components. Such transverse connected pairs of multicurves correspond to pillowcase covers having a single horizontal cylinder of height $\frac{1}{2}$.

Labeled connected pairs of transverse simple closed curves correspond to pillowcase covers having a single horizontal cylinder of height $\frac{1}{2}$ and a single vertical cylinder of height $\frac{1}{2}$. Closed meanders in the plane correspond to pillowcase covers as above with a marked vertical side of one of the squares of the tiling.

Having translated our counting problems into the language of pillowcase covers, we are ready to present our approach in detail.

Pillowcase covers of fixed combinatorial type and Masur–Vech volumes. For any (generalized) partition $\nu = [0^{\nu_0} 1^{\nu_1} 2^{\nu_2} \dots]$ denote by $\operatorname{Vol} \mathcal{Q}(\nu, -1^{|\nu|+4})$ the Masur–Veech volume of the stratum $\mathcal{Q}(\nu, -1^{|\nu|+4})$ of genus 0 meromorphic quadratic differentials with at most simple poles (for the precise definition of the

Masur–Veech volume see e. g. [AEZ2]). Then the following formula holds:

$$\operatorname{Vol}\mathcal{Q}(\nu,-1^{|\nu|+4}) := 2\pi^2\cdot\big(f(0)\big)^{\nu_0}\big(f(1)\big)^{\nu_1}\big(f(2)\big)^{\nu_2}\cdots\,, \tag{2.2}$$

where $|\nu| = 1\cdot\nu_1 + 2\cdot\nu_2 + \dots$ and

$$f(j) = \frac{j\,!!}{(j+1)!!}\cdot\pi^j\cdot\begin{cases}\pi & \text{if } j \text{ is odd}\\ 2 & \text{if } j \text{ is even}\,.\end{cases}$$

(here we use the notation

$$j\,!! := \begin{cases}1\cdot 3\cdot 5\cdot\dots\cdot j, & \text{when } j \text{ is odd},\\ 2\cdot 4\cdot 6\cdot\dots\cdot j, & \text{when } j \text{ is even}\,.\end{cases}$$

and the common convention $0!! := 1$). This formula was originally conjectured by M. Kontsevich and recently proved in [AEZ2]. In this setting zeroes and poles of quadratic differentials are *labeled*.

As it follows from the definition of the Masur–Veech volume, the number of pillowcase covers in the stratum $\mathcal{Q}(\nu,-1^{|\nu|+4})$ in the moduli space of meromorphic quadratic differentials with labeled zeroes and poles tiled with at most $2N$ squares has asymptotics

$$\operatorname{Vol}\mathcal{Q}(\nu,-1^{|\nu|+4})\cdot\frac{N^d}{2d} + o(N^d) \quad\text{as}\quad N\to+\infty\,, \tag{2.3}$$

where

$$d = \dim_{\mathbb{C}}\mathcal{Q}(\nu,-1^{|\nu|+4}) = \ell(\nu) + |\nu| + 2 \tag{2.4}$$

and

$$\ell(\nu) := \nu_0 + \nu_1 + \dots\,. \tag{2.5}$$

As "combinatorial type" of a pillowcase cover one can use the number p of bigons, as in Theorems 1 and 3. In this setting formulae (2.3) and (2.4) imply that all but negligible part of transverse connected pairs of multicurves having large number N of intersections would have only bigons, squares, and hexagons as faces and would correspond to pillowcase covers in the principal stratum $\mathcal{Q}(1^{p-4},-1^p)$.

As an alternative choice of "combinatorial type" of a pillowcase cover one can specify the number of hexagons, octagons, etc, separately, thus fixing the stratum $\mathcal{Q}(\nu,-1^{|\nu|+4})$. This corresponds to the setting of Theorems 4 and 5 below. Under either choice we have a simple asymptotic formula for the number of transverse connected pairs of multicurves of fixed combinatorial type with at most $2N$ intersections.

Remark 2.1 (Labeled versus non-labeled zeroes and poles). When we introduced pillowcase covers in section 2.1 and identified them with transverse connected pairs of multicurves in Proposition 1, we did not label zeroes and poles of the corresponding quadratic differential, which was quite natural in this setting. Traditionally, one labels zeroes and poles of a pillowcase cover in the contex of Masur–Veech volumes. So we do label zeroes and poles in the current section. We remind the setting every time when there may be any ambiguity.

Pillowcase covers with a single horizontal cylinder. Enumeration of pillowcase covers with a single horizontal cylinder was performed in [DGZZ]. In Section 4 we reproduce the relevant computations in the case of the sphere. The number of pillowcase covers tiled with at most $2N$ squares lying in the stratum $\mathcal{Q}(\nu,-1^{|\nu|+4})$ and having a single horizontal cylinder of minimal possible height $\frac{1}{2}$ has asymptotics

$$cyl_1(\mathcal{Q}(\nu,-1^{|\nu|+4}))\cdot\frac{N^d}{2d}+o(N^d)\quad\text{as}\quad N\to+\infty\,, \tag{2.6}$$

where the coefficient $cyl_1(\mathcal{Q}(\nu,-1^{|\nu|+4}))$ is positive and is given by the explicit formula (see (4.2)) that is particularly simple for the principal stratum, see (4.3). (Here we assume that zeroes and poles of the corresponding quadratic differentials are labeled.)

Pillowcase covers with a single horizontal and a single vertical cylinder. We are particularly interested in counting pillowcase covers having a single horizontal cylinder of height $\frac{1}{2}$ and a single vertical cylinder of width $\frac{1}{2}$.

The number $\mathcal{P}_\nu^{labeled}(N)$ of genus 0 pillowcase covers in the stratum $\mathcal{Q}(\nu,-1^{|\nu|+4})$ with at most $2N$ squares having a single horizontal cylinder of height $1/2$ and a single vertical cylinder of width $1/2$ has asymptotics

$$\mathcal{P}_\nu^{labeled}(N)=cyl_{1,1}\left(\mathcal{Q}(\nu,-1^{|\nu|+4})\right)\cdot\frac{N^d}{2d}+o(N^d)\text{ as }N\to+\infty\,, \tag{2.7}$$

where the constant $cyl_{1,1}\left(\mathcal{Q}(\nu,-1^{|\nu|+4}\right)$ satisfies the relation

$$cyl_{1,1}(\mathcal{Q}(\nu,-1^{|\nu|+4}))=\frac{\left(cyl_1(\mathcal{Q}(\nu,-1^{|\nu|+4}))\right)^2}{\operatorname{Vol}\mathcal{Q}(\nu,-1^{|\nu|+4})}\,. \tag{2.8}$$

The both statements are formulated more precisely in Theorem 6 in Section 4 and follow from the results of [DGZZ].

The relation 2.8 can be viewed as a statement about independence of horizontal and vertical decompositions of pillowcase covers: the asymptotic fraction of pillowcase covers having a single horizontal cylinder of height $\frac{1}{2}$ among all pillowcase covers in $\mathcal{Q}(\nu,-1^{|\nu|+4})$ tiled with at most $2N$ squares is the same as the asymptotic fraction of pillowcase covers having a single horizontal cylinder of height $\frac{1}{2}$ and a single vertical cylinder of width $\frac{1}{2}$ among all pillowcase covers having a single vertical cylinder of width $\frac{1}{2}$ and tiled with at most $2N$ squares.

Forgetting the labeling of zeroes and poles we get the asymptotics of the number of connected pairs of transverse simple closed curves of fixed combinatorial type with at most $2N$ crossings.

Further remarks. It is worth mentioning that all the above quantities have combinatorial nature, but were computed by alternative methods. The Masur–Veech volumes in genus zero are closely related to Hurwitz numbers counting covers of the sphere of some very special ramification type. However, all attempts to compute these volumes by purely combinatorial methods have (up to now) failed even for covers of the simplest ramification type, see e. g. [AEZ1]. The proof in [AEZ2] of the formula for the Masur–Veech volumes implicitly uses the analytic Riemann–Roch theorem in addition to combinatorics.

The result about pillowcase covers with a single horizontal and a single vertical cylinder is proved in [DGZZ] using ergodicity of the Teichmüller geodesic flow with respect to the Masur–Veech measure and Moore's ergodicity theorem. The proof was inspired by the approach of M. Mirzakhani to counting simple closed geodesics on hyperbolic surfaces.

3. From arc systems and meanders to pillowcase covers

In this section we give precise bijections between meanders and pillowcase covers with a single maximal cylinder in both horizontal and vertical directions. We consider meanders in the plane in sections 3.1–3.3 and meanders on the sphere in section 3.4.

3.1. **Orientation, marking and weight.** We have seen in Proposition 1 from Section 2.1 that transverse connected pairs of multicurves on the sphere are in bijection with pillowcase covers of genus 0. A pillowcase cover arising from a pair of arc systems has a single horizontal cylinder of height $\frac{1}{2}$. In particular, a pillowcase cover arising from a meander has a single horizontal and a single vertical cylinder of height (respectively width) $\frac{1}{2}$.

However, pairs of arc systems and meanders (both in the plane and on the sphere) carry an extra marking. Namely, a pair of arc systems comes with a given choice of a top and bottom sides. Furthermore, the pillowcase cover corresponding to a *plane* meander has a special square corresponding to the leftmost intersection. Summarizing, we get the following result:

Lemma 3.1. *There is a natural bijection between meanders in the plane and pillowcase covers with a marked oriented vertical side of one of the squares that have a single horizontal and a single vertical cylinder of height (width)* $\frac{1}{2}$.

In order to provide exact counting of meanders we present the conventions for counting pillowcase covers and see how these quantities are related to arc systems and meander counting. The pillowcase covers considered in Lemma 3.1 are not well suited for counting. We will consider pillowcase covers with a marked *vertex* of the square tiling.

Convention 3.1. By convention, the marked vertex is located at the end of the marked oriented vertical edge on the top boundary component of the single horizontal cylinder.

Note that the two boundary components of the single horizontal cylinder do not intersect. Thus, the marked vertex uniquely defines the top boundary component and, hence, provides us with the canonical orientation of the waist curve of the single horizontal cylinder.

Let us reconstruct the labeled pair of arc systems in the plane from a pillowcase cover of genus zero tiled with a single horizontal band of squares and having a marked vertex. If the marked vertex of the square tiling is a simple pole of the quadratic differential, there is a single vertical side of the square tiling adjacent to it, and the choice of the vertical side is canonical. If the marked vertex of the square tiling is a regular point of the quadratic differential, there are two adjacent vertical sides, so there are two ways to chose a distinguished vertical side which, generally, lead to two different arc systems. We say "generally" because it might happen that the pillowcase cover is particularly symmetric (like pillowcase covers associated to arc systems from Figure 3) and the resulting two arc systems are isomorphic.

As soon as we are interested only in the asymptotic counting we can simply neglect this issue: the pillowcase covers with extra symmetries occur too rarely to affect the asymptotics. To perform exact count we establish the following standard Convention.

Convention 3.2. We always count a marked or non-marked pillowcase cover with a weight reciprocal to the order $|\operatorname{Aut}|$ of the automorphism group of the cover. In the current context we keep track of which sides of the pillowcase cover are horizontal and which ones are vertical, but we do not label either the sides or the vertices of the pillowcase cover. By definition, the automorphism group Aut acts by flat isometries sending horizontal (respectively vertical) sides of the tiling to horizontal (respectively vertical) sides and keeping the marked point (if any) fixed.

In particular, if we have a marked point at a regular vertex of a pillowcase cover, the automorphism group is either trivial or $\mathbb{Z}/2\mathbb{Z}$. If we have a marked point at a zero of degree j of a pillowcase cover, the automorphism group is a (usually trivial) subgroup of the cyclic group $\mathbb{Z}/(j+2)\mathbb{Z}$.

3.2. **Meanders with a given number of minimal arcs and pillowcase covers.** In this section and in the next one we continue to work with *plane* meanders. Under Conventions 3.1 and 3.2, any collection of weighted pillowcase covers on the sphere with a single band of horizontal squares and with a marked regular point defines twice as much arc systems; the weighted collection of pillowcase covers as above with a marked zero of degree j defines $(j+2)$ times more arc systems for any $j \in \mathbb{N}$.

Lemma 3.2. *Let the initial closed meander in the plane have p minimal arcs, where $p \geq 3$. The associated pillowcase cover has $p+1$ simple poles if the initial meander has a maximal arc and p simple poles if it does not.*

Proof. A maximal arc becomes indistinguishable from a minimal arc after passing to a labeled pair of transverse simple closed curves on the sphere. Minimal and maximal arcs are in bijective correspondence with bigons in the partition of the sphere by the union of these transverse simple closed curves. Bigons, in turn, are in bijective correspondence with simple poles of the associated pillowcase cover. □

Recall that $\mathcal{M}_p^+(N)$ and $\mathcal{M}_p^-(N)$ denote the number of meanders with p minimal arcs and respectively with and without a maximal arc. Denote $\mathcal{P}_p(N)$ the number of pillowcase covers of genus zero tiled with at most $2N$ squares, having exactly p simple poles, a single horizontal cylinder of height $\frac{1}{2}$ and a single vertical cylinder of width $\frac{1}{2}$. Denote by $\mathcal{P}_{p,j}(N)$, where $j = 0, 1, 2, \ldots$, the number of pillowcase covers as above having in addition a marked point at a regular vertex when $j = 0$ and at a zero of order j when $j > 0$.

Note that a pillowcase cover of genus 0 with p simple poles cannot have zeroes of order greater than $p-4$.

Lemma 3.3. *Under Convention 3.2 on the weighted count of pillowcase covers the following equalities hold:*

$$\mathcal{M}_p^+(N) = 2(p+1) \cdot \mathcal{P}_{p+1}(N) \tag{3.1}$$

$$\mathcal{M}_p^-(N) = \sum_{j=0}^{p-4} (j+2) \cdot \mathcal{P}_{p,j}(N) \, - \, \frac{1}{2}\, \mathcal{M}_{p-1}^+(N)\,. \tag{3.2}$$

Proof. If the meander has $2n$ intersections, then the associated pillowcase cover is tiled with $2n$ squares with side $\frac{1}{2}$.

To every closed meander with a maximal arc and with p minimal arcs we associated a canonical pillowcase cover of genus zero with $p+1$ simple poles, a single horizontal cylinder of height $\frac{1}{2}$ and a single vertical cylinder of width $\frac{1}{2}$, see Proposition 1. Conversely, to every such pillowcase cover we can associate $2(p+1)$ meanders with one maximal arc and p minimal arcs. Indeed, choose any of the $(p+1)$ simple poles and choose independently one of the two possible orientations of the waist curve of the horizontal cylinder. Cutting this waist curve at the intersection with the single vertical edge of the square tiling adjacent to the selected pole we get a closed meander in the plane with a maximal arc.

It might happen that some of the resulting $2(p+1)$ meanders are pairwise isomorphic. However, this implies that the automorphism group of the pillowcase cover is nontrivial, and Convention 3.2 provides the exact count. This completes the proof of equality (3.1).

Similarly, to every closed meander without a maximal arc and with p minimal arcs we assigned a canonical pillowcase cover of genus zero having p simple poles, a single horizontal cylinder of height $\frac{1}{2}$, a single vertical cylinder of width $\frac{1}{2}$, and a marked vertex following Convention 3.1. The assumption that the initial meander does not have any maximal arc excludes coincidence of the marked point with a simple pole on the top side. In order to exclude a maximal arc on the bottom side, one needs to subtract a half of $\mathcal{M}^+_{p-1}(N)$ (that is, $p \cdot \mathcal{P}_{p,-1}(N)$). At the end of section 3.1 we have seen that under Convention 3.2 on the weight with which we count pillowcase covers with a marked vertex, any collection of weighted pillowcase covers on the sphere with a single horizontal cylinder of height $\frac{1}{2}$ and with a marked regular point defines twice as much closed meanders in the plane; the weighted collection of pillowcase covers as above with a marked zero of degree j defines $(j+2)$ times more closed meanders in the plane for any $j \in \mathbb{N}$. As before, if some of the resulting meanders are isomorphic we do not count them several times since by definition of the automorphism group Aut of the corresponding pillowcase cover, the resulting multiplicity coincides with the order $|\operatorname{Aut}|$ of the automorphism group. This completes the proof of equality (3.2). □

3.3. **Meanders and pilowcase covers in a given stratum.** We now introduce finer counting with respect to a fixed stratum. For a partition $\nu = [1^{\nu_1} 2^{\nu_2} \dots]$ denote by $\mathcal{M}^+_\nu(N)$ and $\mathcal{M}^-_\nu(N)$ the number of meanders leading to pillowcase covers in the stratum $\mathcal{Q}(\nu, -1^{|\nu|+4})$ of meromorphic quadratic differentials respectively with a maximal arc and without maximal arcs. We say that such meanders are *of type* ν. Similarly, let $\mathcal{P}_\nu(N)$ be the number of pillowcase covers in the stratum $\mathcal{Q}(\nu, -1^{|\nu|+4})$ of genus zero tiled with at most $2N$ squares, with a single horizontal cylinder of height $\frac{1}{2}$ and a single vertical cylinder of width $\frac{1}{2}$. Denote by $\mathcal{P}_{\nu,j}(N)$, $j = 0, 1, 2, \dots$, the number of pillowcase covers as above having in addition a marked point at a regular vertex when $j = 0$ and at a zero of order j when $j > 0$. By definition, we let $\mathcal{P}_{\nu,j}(N) = 0$ for any N when $\nu_j = 0$. Recall that by Convention 3.2 we count pillowcase covers with weights reciprocal to the orders of their automorphism groups.

Lemma 3.4. *Under Convention 3.2 on weights with which we count pillowcase covers the following equalities hold*

$$\mathcal{M}^+_\nu(N) = 2(|\nu| + 4) \cdot \mathcal{P}_\nu(N) \tag{3.3}$$

$$\mathcal{M}^-_\nu(N) = \sum_{j=0}^{|\nu|} (j+2) \cdot \mathcal{P}_{\nu,j}(N) \, - \, \frac{1}{2}\, \mathcal{M}^+_\nu(N)\,. \tag{3.4}$$

Proof. The proof is completely analogous to the proof of Lemma 3.3. □

3.4. **Asymptotic frequency of meanders: general setting.** In this section we return to meanders on the sphere. Let $\mathcal{T}$ be a plane tree. We associate to $\mathcal{T}$ a generalized integer partition $\nu = \nu(\mathcal{T}) = [0^{\nu_0}1^{\nu_1}2^{\nu_2}\dots]$ where ν_j denotes the number of internal vertices of valence $j+2$ for $j \in \mathbb{N}$. The number of leaves, or equivalently of vertices of valence 1, is then expressed in terms of the (generalized) partition ν as $2+|\nu|$ where $|\nu|$ is the sum of the partition (see (2.1)).

Given two generalized partitions $\iota = [0^{\iota_0}1^{\iota_1}2^{\iota_2}\dots]$ and $\kappa = [0^{\kappa_0}1^{\kappa_1}2^{\kappa_2}\dots]$ we define their sum as $\nu = \iota + \kappa = [0^{\iota_0+\kappa_0}1^{\iota_1+\kappa_1}2^{\iota_2+\kappa_2}\dots]$. We say that ι is a *subpartition* of ν and denote it as $\iota \subset \nu$. For a subpartition $\iota \subset \nu$ we define the difference $\kappa = \nu - \iota$.

The following Lemma recalls what graphs of horizontal saddle connections have horizontally one-cylinder pillowcase covers in a given stratum of meromorphic quadratic differentials in genus zero.

Lemma 3.5. *A ribbon graph $\mathcal{D}$ represents the graph of horizontal saddle connections of some pillowcase cover in a stratum $\mathcal{Q}(\nu,-1^{|\nu|+4})$ having a single horizontal cylinder if and only if it is represented by a pair of plane trees with associated partitions ν_{top} and ν_{bottom} such that the sum $\nu_{top}+\nu_{bottom}=\nu$.*

Proof. The Lemma was proved in section 2.2. □

We formulate and prove the following generalization of Theorem 2 giving a formula for the limit of the fraction (1.3) of meanders which we get identifying arc systems of types $\mathcal{T}_{top}$ and $\mathcal{T}_{bottom}$ with the same number of arcs, see Figure 3.

Though we agreed in section 1.2 to consider reduced trees, suppressing the vertices of valence 2, it is often convenient to keep several marked points, so we state the Theorem below in this slightly more general setting. Note that since $f(0)=2$, the number ν_0 of zeroes in the partition ν affects the value of the function $\operatorname{Vol}\mathcal{Q}(\nu,-1^{|\nu|+4})$. Adding an extra marked point we double the Masur–Veech volume of the corresponding stratum.

Theorem 4. *For any pair of plane trees $\mathcal{T}_{top},\mathcal{T}_{bottom}$ with associated generalized partitions ν_{top} and ν_{bottom} the following limit exists and is positive:*

$$\lim_{N\to+\infty}\mathrm{P}_{connected}(\mathcal{T}_{top},\mathcal{T}_{bottom};N)=\mathrm{P}_1(\mathcal{Q}(\nu,-1^{|\nu|+4}))>0$$

where $\nu=\nu_{top}+\nu_{bottom}$ and $\mathrm{P}_1(\mathcal{Q}(\nu,-1^{|\nu|+4}))$ is defined by

$$\mathrm{P}_1\left(\mathcal{Q}(\nu,-1^{|\nu|+4})\right)=\frac{cyl_1(\mathcal{Q}(\nu,-1^{|\nu|+4}))}{\operatorname{Vol}\mathcal{Q}(\nu,-1^{|\nu|+4})}\,. \tag{3.5}$$

Here $\operatorname{Vol}\mathcal{Q}(\nu,-1^{|\nu|+4})$ is given by formula (2.2), and $cyl_1(\mathcal{Q}(\nu,-1^{|\nu|+4}))$ takes the value

$$cyl_1(\mathcal{Q}(\nu,-1^{|\nu|+4}))=2\sum_{\mu\subset\nu}\binom{|\nu|+4}{|\mu|+2}\binom{\nu_0}{\mu_0}\binom{\nu_1}{\mu_1}\binom{\nu_2}{\mu_2}\cdots\,. \tag{3.6}$$

Proof. The trees $\mathcal{T}_{top}$ and $\mathcal{T}_{bottom}$ represent the trees formed by the horizontal saddle connections of the pillowcase cover. Vertices of valence one are in bijective correspondence with simple poles. Vertices of valence two represent marked points (if any). Vertices of valence $j+2$ are in bijective correspondence with zeroes of degree j for $j\in\mathbb{N}$. Recall that the type $\nu=[1^{\nu_1}2^{\nu_2}3^{\nu_3}\dots]$ of the graph $\mathcal{T}_{bottom}\sqcup\mathcal{T}_{top}$

encodes the total number ν_j of vertices of valence $j+2$ in $\mathcal{T}_{bottom} \sqcup \mathcal{T}_{top}$ for $j \in \mathbb{N}$. We conclude that a pair of arc systems having $\mathcal{T}_{bottom}$ and $\mathcal{T}_{top}$ as dual trees defines a pillowcase cover in the stratum $\mathcal{Q}(\nu, -1^{|\nu|+4})$ of meromorphic quadratic differentials.

We are ready to express the numerator and the denominator of (1.3) in terms of pillowcase covers. First, note that arc systems are defined on a pair of labeled oriented discs (called top and bottom, or northern and southern hemispheres).

When $\mathcal{T}_{bottom}$ and $\mathcal{T}_{top}$ are not isomorphic as ribbon graphs, the "total number of different triples" in the denominator of (1.3) is equal to the weighted number of pillowcase covers tiled with at most $2N$ squares that form a single horizontal band and having the non-labeled ribbon graph $\mathcal{D} := \mathcal{T}_{bottom} \sqcup \mathcal{T}_{top}$ as the diagram of horizontal saddle connections. Here we identify triples leading to isomorphic pairs of labeled multicurves. We do not label either of components, vertices, or edges of the graph $\mathcal{D}$, but we consider the plane trees $\mathcal{T}_{bottom}$ and $\mathcal{T}_{top}$ as ribbon graphs, so the corresponding topological discs are oriented.

When $\mathcal{T}_{bottom}$ and $\mathcal{T}_{top}$ are isomorphic as ribbon graphs, the "total number of different triples" in the denominator of (1.3) is equal to twice the weighted number of pillowcase covers as above.

When $\mathcal{T}_{bottom}$ and $\mathcal{T}_{top}$ are not isomorphic as ribbon graphs, the "number of triples leading to meander" in the numerator of (1.3) is equal to the weighted number of pillowcase covers as above which in addition have a single vertical cylinder. When $\mathcal{T}_{bottom}$ and $\mathcal{T}_{top}$ are isomorphic, the "number of triples leading to a meander" is twice the weighted number of pillowcase covers as above that have a single vertical cylinder.

Thus, the limit in Theorem 4 is the asymptotic fraction of pillowcase covers having a single horizontal cylinder of height $\frac{1}{2}$ corresponding to the separatrix diagram $\mathcal{D}$ of horizontal saddle connections, and a single vertical cylinder of width $\frac{1}{2}$ among all pillowcase covers with a single horizontal cylinder of height $\frac{1}{2}$ corresponding to the separatrix diagram $\mathcal{D}$ of horizontal saddle connections.

Theorem 1.19 in [DGZZ] asserts that such limit exists and that the "horizontal and vertical cylinder decompositions are asymptotically uncorrelated", so the above limit coincides with the asymptotic fraction of pillowcase covers having a single vertical cylinder of width $\frac{1}{2}$ among all pillowcase covers – we can omit the conditions on the horizontal foliation. This proves existence of the limit in Theorem 4. By definition, the latter asymptotic fraction is precisely the quantity $\mathrm{P}_1(\mathcal{Q}(\nu, -1^{|\nu|+4}))$ which proves the second statement in Theorem 4 together with formula (3.5). The remaining formula (3.6) for the quantity $cyl_1(\mathcal{Q}(\nu, -1^{|\nu|+4}))$ introduced in (2.6) will be derived in Theorem 6 of section 4. This completes the proof of Theorem 4. □

Proof of Theorem 2. Theorem 2 is a particular case of the Theorem 4 when the plane trees $\mathcal{T}_{bottom}, \mathcal{T}_{top}$ are trivalent and have the total number p of leaves (vertices of valence one). In this situation $\nu = [1^{p-4} \ -1^p]$. By Theorem 4 we have

$$\lim_{N\to+\infty} \mathrm{P}_{connected}(\mathcal{T}_{bottom}, \mathcal{T}_{top}; N) = \mathrm{P}_1(\mathcal{Q}(1^{p-4}, -1^p))\,.$$

We apply now formula (4.6) proved in Corollary 4.2 in section 4 which states that

$$\mathrm{P}_1(\mathcal{Q}(1^{p-4}, -1^p)) = \frac{cyl_1(\mathcal{Q}(1^{p-4}, -1^p))}{\operatorname{Vol}\mathcal{Q}(1^{p-4}, -1^p)}\,.$$

It remains to apply formula (4.3) for the numerator and equation (2.2) for the denominator of the latter quantity to complete the proof. We have proved Theorem 2 conditional to Corollaries 4.1 and 4.2 left to section 4. □

3.5. **Counting meanders of special combinatorial types.** In this section we return to plane meanders with exception for the Proof of Theorem 3 at the very end of the section, where we work with meanders on the sphere.

We state now an analog of Theorem 1, where instead of the number of minimal arcs (pimples) we use the partition ν as a combinatorial passport of the meander.

Theorem 5. *For any partition $\nu = [1^{\nu_1} 2^{\nu_2} 3^{\nu_3} \dots]$, the number $\mathcal{M}^+_\nu(N)$ (respectively $\mathcal{M}^-_\nu(N)$) of closed plane meanders of type ν, with (respectively without) a maximal arc and with at most $2N$ crossings has the following asymptotics as $N \to +\infty$:*

$$\mathcal{M}^+_\nu(N) = 2(|\nu|+4) \cdot \frac{cyl_{1,1}\big(\mathcal{Q}(\nu, -1^{|\nu|+4})\big)}{(|\nu|+4)! \cdot \prod_j \nu_j!} \cdot \frac{N^{\ell(\nu)+|\nu|+2}}{2\ell(\nu)+2|\nu|+4} + o\big(N^{\ell(\nu)+|\nu|+2}\big)\,, \tag{3.7}$$

$$\mathcal{M}^-_\nu(N) = \frac{2\, cyl_{1,1}\big(\mathcal{Q}(\nu, 0, -1^{|\nu|+4})\big)}{(|\nu|+4)! \cdot \prod_j \nu_j!} \cdot \frac{N^{\ell(\nu)+|\nu|+3}}{2\ell(\nu)+2|\nu|+6} + o\big(N^{\ell(\nu)+|\nu|+3}\big)\,, \tag{3.8}$$

where

$$cyl_{1,1}\big(\mathcal{Q}(\nu, 0, -1^{|\nu|+4})\big) = 2 \cdot cyl_{1,1}\big(\mathcal{Q}(\nu, -1^{|\nu|+4})\big) \tag{3.9}$$

and

$$cyl_{1,1}\big(\mathcal{Q}(\nu, -1^{|\nu|+4})\big) = \frac{4}{\operatorname{Vol} \mathcal{Q}(\nu, -1^{|\nu|+4})} \cdot \left(\sum_{\iota_1=0}^{\nu_1} \sum_{\iota_2=0}^{\nu_2} \sum_{\dots}^{\dots} \binom{\nu_1}{\iota_1} \binom{\nu_2}{\iota_2} \cdots \binom{|\nu|+4}{|\iota|+2} \right)^2 \tag{3.10}$$

Note that contrary to the original Theorem 1, where the setting is somewhat misleading, in the setting of Theorem 5 we get more natural formula $\mathcal{M}^+_\nu(N) = o\big(\mathcal{M}^-_\nu(N)\big)$ as $N \to +\infty$.

Up to now we performed the exact count. The Lemma below gives the term with dominating contribution to the asymptotic count when the bound $2N$ for the number of squares in the pillowcase cover tends to infinity.

Lemma 3.6. *The following limits hold*

$$\lim_{N\to+\infty} \frac{1}{\mathcal{P}_{1^{p-4}}(N)} \cdot \mathcal{P}_p(N) = 1\,, \tag{3.11}$$

$$\lim_{N\to+\infty} \frac{1}{2\,\mathcal{P}_{1^{p-4},0}(N)} \cdot \left(\sum_{j=0}^{p-4} (j+2) \cdot \mathcal{P}_{p,j}(N) \right) = 1\,, \tag{3.12}$$

$$\lim_{N\to+\infty} \frac{1}{2\,\mathcal{P}_{\nu,0}(N)} \cdot \left(\sum_{j=0}^{|\nu|} (j+2) \cdot \mathcal{P}_{\nu,j}(N) \right) = 1\,. \tag{3.13}$$

Proof. Let $\nu = [1^{\nu_1} 2^{\nu_2} \dots]$, where $|\nu| = p-4$ be a partition of the number $p-4$ into the sum of positive integers $j_1, \dots, j_r$, where $j_1 + \dots + j_r = p-4$. By Theorem 1.19 in [DGZZ],

$$\mathcal{P}_\nu(N) = \frac{cyl_{1,1}\big(\mathcal{Q}(\nu, -1^p)\big)}{p! \cdot \prod_j \nu_j!} \cdot \frac{N^d}{2d} + o(N^d)\,, \text{ when } N \to +\infty\,, \tag{3.14}$$

where the constant $cyl_{1,1}\left(\mathcal{Q}(\nu, -1^p)\right)$ defined in (2.7) is positive, and

$$d = \dim_{\mathbb{C}} \mathcal{Q}(\nu, -1^{|\nu|+4}) = \dim_{\mathbb{C}} \mathcal{Q}(j_1, \dots, j_r, -1^p) = r + p - 2\,. \tag{3.15}$$

For a given number $p \geq 4$ of simple poles, the only stratum of the maximal dimension is the principal stratum $\mathcal{Q}(1^{p-4}, -1^p)$, where all zeroes are simple. This is the only stratum which contributes a term of order N^{2p-6} to $\mathcal{P}_p(N)$, where $2p-6 = \dim_{\mathbb{C}} \mathcal{Q}(1^{p-4}, -1^p)$. This proves equation (3.11).

For $j \geq 1$ the quantity $\mathcal{P}_{\nu,j}(N)$ counts pillowcase covers with a marked zero of order j in the stratum $\mathcal{Q}(\nu, -1^{|\nu|+4})$. Hence, it has the asymptotic growth rate of the same order as the quantity $\mathcal{P}_\nu(N)$ counting pillowcase covers in the same stratum without any marking, i. e. it grows like N^d, where $d = \dim_{\mathbb{C}} \mathcal{Q}(\nu, -1^{|\nu|+4})$. The dimensional count as above implies that the contribution of any term $\mathcal{P}_{\nu,j}(N)$ with $j \geq 1$ to the sum in the right-hand side of (3.12) has the order at most N^{2p-6}.

Let us now analyse the contribution of various strata to $\mathcal{P}_{p,0}(N)$. It follows from Theorem 1.19 in [DGZZ] that

$$\mathcal{P}_{\nu,0}(N) = \frac{cyl_{1,1}\big(\mathcal{Q}(\nu, 0, -1^{|\nu|+4})\big)}{(|\nu|+4)! \cdot \prod_j \nu_j!} \cdot \frac{N^{\ell(\nu)+|\nu|+3}}{2\ell(\nu) + 2|\nu| + 6} + \\ + \; o\big(N^{\ell(\nu)+|\nu|+3}\big)\,, \text{ when } N \to +\infty\,, \tag{3.16}$$

where the constant $cyl_{1,1}\big(\mathcal{Q}(\nu, 0, -1^{|\nu|+4})\big)$ is positive. Here we used the same notation as in formula (2.4) for the dimension of the stratum $\mathcal{Q}(\nu, 0, -1^{|\nu|+4})$.

This implies that for any partition ν of $p-4$ different from 1^{p-4}, its contribution $\mathcal{P}_{\nu,0}$ also has order at most N^{2p-6}, which means that $\mathcal{P}_{p,0}(N)$ behaves like N^{2p-5} for N large, and that the only stratum which gives a contribution of this order is the principal stratum with a marked point $\mathcal{Q}(1^{p-4}, 0, -1^p)$. This proves equality (3.12).

By the same reason the summand $2\,\mathcal{P}_{\nu,0}(N)$ dominates in the sum in the right-hand side of (3.13). It is the only term whose contribution is of order N^{d+1}, where $d = \dim_{\mathbb{C}} \mathcal{Q}(\nu, -1^{|\nu|+4})$. The asymptotics of other terms in the sum have lower orders in N as $N \to +\infty$. This proves equality (3.13). □

Now we have everything for the proofs of Theorem 1 and of Theorem 5.

Proof of Theorem 1. The chain of relations including (3.1) from Lemma 3.3, (3.11) from Lemma 3.11 and (3.14) yields

$$\mathcal{M}_p^+(N) = 2(p+1) \cdot \mathcal{P}_{p+1}(N) = 2(p+1) \cdot \mathcal{P}_{1^{p-3}}(N) + o(N^{2p-4}) \; = \\ = 2(p+1) \cdot \frac{cyl_{1,1}\big(\mathcal{Q}(1^{p-3}, -1^{p+1})\big)}{(p+1)!\,(p-3)!} \cdot \frac{N^{2p-4}}{4p-8} + o(N^{2p-4}) \text{ when } N \to +\infty\,.$$

This proves the first equality in (1.1). The constant $cyl_{1,1}\big(\mathcal{Q}(1^{p-3}, -1^{p+1})\big)$ is expressed by our main formula (2.8) in terms of $cyl_1\big(\mathcal{Q}(1^{p-3}, -1^{p+1})\big)$ computed in Corollary 4.1 in section 4 and in terms of the Masur–Veech volume of the stratum $\mathcal{Q}(1^{p-3}, -1^{p+1})$ given by formula (2.2).

Similarly, the chain of relations including (3.2) from Lemma 3.3, (3.12) from Lemma 3.11 and (3.16) implies

$$\mathcal{M}_p^-(N) = \sum_{j=0}^{p-4}(j+2)\cdot\mathcal{P}_{p,j}(N) \;-\; \frac{1}{2}\,\mathcal{M}_{p-1}^+(N) \;=\; 2\,\mathcal{P}_{1^{p-4},0}(N) + o(N^{2p-5}) \;=$$

$$= \frac{2\,cyl_{1,1}\big(\mathcal{Q}(1^{p-4},0,-1^p)\big)}{p!\,(p-4)!}\cdot\frac{N^{2p-5}}{4p-10} + o(N^{2p-5})\,,\ \text{when } N\to+\infty\,.$$

This proves the first equality in (1.2). The constant $cyl_{1,1}\big(\mathcal{Q}(1^{p-4},0,-1^p)\big)$ is expressed by our main formula (2.8) in terms of $cyl_1\big(\mathcal{Q}(1^{p-4},0,-1^p)\big)$ computed in Corollary 4.1 of Section 4 and in terms of the Masur–Veech volume of the stratum $\mathcal{Q}(1^{p-4},0,-1^p)$ given by formula (2.2).

Thus, the proof of Theorem 1 is conditional subject to the explicit count of $cyl_1\big(\mathcal{Q}(1^{p-3},-1^{p+1})\big)$ and of $cyl_1\big(\mathcal{Q}(1^{p-4},0,-1^p)\big)$ performed in Corollary 4.1 below. □

Proof of Theorem 5. The proof of Theorem 5 is completely analogous to the proof of Theorem 1.

Composing relation (3.3) from Lemma 3.4 with relation (3.14), we get

$$\mathcal{M}_\nu^+(N) = 2(|\nu|+4)\cdot\mathcal{P}_\nu(N) \;=$$

$$=\; 2(|\nu|+4)\cdot\frac{cyl_{1,1}\big(\mathcal{Q}(\nu,-1^{|\nu|+4})\big)}{(|\nu|+4)!\cdot\prod_j \nu_j!}\cdot\frac{N^{\ell(\nu)+|\nu|+2}}{2\ell(\nu)+2|\nu|+4} \;+$$

$$+\; o\big(N^{\ell(\nu)+|\nu|+2}\big)\,,\ \text{when } N\to+\infty\,,$$

where we translated formula (3.15) for the dimension d of the stratum $\mathcal{Q}(\nu,-1^{|\nu|+4})$ to notations (2.4). This proves formula (3.7) in Theorem 5.

Composing relation (3.4) from Lemma 3.4 with relation (3.13), followed by (3.16) we get

$$\mathcal{M}_\nu^-(N) = \sum_{j=0}^{|\nu|}(j+2)\cdot\mathcal{P}_{\nu,j}(N) \;-\; \frac{1}{2}\,\mathcal{M}_\nu^+(N) = 2\,\mathcal{P}_{\nu,0}(N) + o\big(N^{2\ell(\nu)+2|\nu|+3}\big) =$$

$$= \frac{2\,cyl_{1,1}\big(\mathcal{Q}(\nu,0,-1^{|\nu|+4})\big)}{(|\nu|+4)!\cdot\prod_j \nu_j!}\cdot\frac{N^{\ell(\nu)+|\nu|+3}}{2\ell(\nu)+2|\nu|+6} \;+$$

$$+\; o\big(N^{\ell(\nu)+|\nu|+3}\big)\,,\ \text{when } N\to+\infty\,.$$

This proves formula (3.8) in Theorem 5.

We have proved Theorem 5 conditional to expressions (3.9) and (3.10) for the quantities $cyl_{1,1}(\mathcal{Q}(\nu,0,-1^{|\nu|+4}))$ and $cyl_{1,1}(\mathcal{Q}(\nu,-1^{|\nu|+4}))$ left to Theorem 6 in section 4. □

We conclude this section with the proof of Theorem 3.

Proof of Theorem 3. The number of reduced plane trees with fixed number p of leaves is finite. By definition of the ratio $\mathrm{P}_{connected}(p;N)$ its numerator is the sum of the numerators of (1.3) over all such pairs of trees, and the denominator of the ratio $\mathrm{P}_{connected}(p;N)$ is the sum of the denominators of (1.3) over all such pairs of trees. Applying the dimensional argument as in the proof of Theorem 1 we conclude that contributions of pairs of trees, where at least one of the trees is not trivalent, to the numerator or to the denominator of the above ratio is of the order $o(N^{2p-6})$ for such contributions correspond to strata $\mathcal{Q}(\nu,-1^p)$ of meromorphic quadratic differentials in genus zero different from the principal one. Thus, for large N these contributions are negligible compared to the contribution of any pair of trivalent trees. The contribution of any pair of trivalent trees is of the order N^{2p-6}, where $2p-6 = \dim_{\mathbb{C}} \mathcal{Q}(1^{p-4},-1^p)$ is the dimension of the principal stratum. Thus, studying the asymptotics of the ratio $\mathrm{P}_{connected}(p;N)$ we can ignore the pairs of trees where at least one of the trees is not trivalent.

To complete the proof of Theorem 3 it remains to notice that by Theorem 2 for any pair of *trivalent* plane trees $\mathcal{T}_{bottom}, \mathcal{T}_{top}$, the ratio of positive quantities (1.3) has the same limit $\mathrm{P}_1(\mathcal{Q}(1^{p-4},-1^p))$.

□

4. Computations for pillowcase covers

Let $\nu = [0^{\nu_0}1^{\nu_1}2^{\nu_2}\dots]$ be a (generalized) partition of a natural number $|\nu|$ into the sum of nonnegative integer numbers (in this section we allow entries 0):

$$|\nu| := \underbrace{0+\dots+0}_{\nu_0} + \underbrace{1+\dots+1}_{\nu_1} + \underbrace{2+\dots+2}_{\nu_2} + \dots$$

The common convention on Masur–Veech volumes of the strata of meromorphic quadratic differentials with at most simple poles suggests to label (give names) to all zeroes and poles. Denote by $\mathcal{P}_\nu^{labeled}(N)$ the number of pillowcase covers with labeled zeroes and poles in the stratum $\mathcal{Q}(\nu,-1^{|\nu|+4})$ in genus zero tiled with at most $2N$ squares with the side $\frac{1}{2}$ and having a single horizontal cylinder of height $\frac{1}{2}$ and a single vertical cylinder of width $\frac{1}{2}$. It is easy to see that a pillowcase cover as above cannot have any symmetries. Convention 3.2 on weights with which we count pillowcase covers with non-labeled zeroes and poles is designed to assure the following relation between the two counts valid for any $N \in \mathbb{N}$:

$$\mathcal{P}_\nu^{labeled}(N) = \left(\prod_{j=0}^{\infty} \nu_j!\right) \cdot (|\nu|+4)! \cdot \mathcal{P}_\nu(N)\,, \tag{4.1}$$

where the product above contains, actually, only finite number of factors.

Theorem 6. *The number $\mathcal{P}_\nu^{labeled}(N)$ of pillowcase covers with labeled zeroes and poles in the stratum $\mathcal{Q}(\nu,-1^{|\nu|+4})$ tiled with at most $2N$ squares $(\frac{1}{2})\times(\frac{1}{2})$ and having a single horizontal cylinder of height $\frac{1}{2}$ and a single vertical cylinder of width $\frac{1}{2}$ has the following asymptotics as $N \to +\infty$:*

$$\mathcal{P}_\nu^{labeled}(N) = cyl_{1,1}\left(\mathcal{Q}(\nu,-1^{|\nu|+4})\right) \cdot \frac{N^d}{2d} + o\left(N^d\right) \text{ as } N \to +\infty\,,$$

were

$$cyl_{1,1}\left(\mathcal{Q}(\nu,-1^{|\nu|+4})\right) = \frac{\left(cyl_1\left(\mathcal{Q}(\nu,-1^{|\nu|+4})\right)\right)^2}{\operatorname{Vol}\mathcal{Q}(\nu,-1^{|\nu|+4})}$$

and

$$cyl_1\left(\mathcal{Q}(\nu,-1^{|\nu|+4})\right) = 2 \cdot \sum_{\iota_0=0}^{\nu_0} \sum_{\iota_1=0}^{\nu_1} \sum_{\iota_2=0}^{\nu_2} \sum_{\dots}^{\dots} \binom{\nu_0}{\iota_0}\binom{\nu_1}{\iota_1}\binom{\nu_2}{\iota_2}\cdots\binom{|\nu|+4}{|\iota|+2}. \tag{4.2}$$

Here $\iota = [0^{\iota_0}1^{\iota_1}2^{\iota_2}\dots]$ and $d = \dim_{\mathbb{C}} \mathcal{Q}(\nu,-1^{|\nu|+4}) = \ell(\nu)+|\nu|+2$.

Before proving Theorem 6 we prove the following Corollary 4.1.

Corollary 4.1. *The number $\mathcal{P}_{1^k}^{labeled}(N)$ of pillowcase covers with labeled zeroes and poles in the stratum $\mathcal{Q}(1^k,-1^{k+4})$ tiled with at most $2N$ squares and having a single horizontal cylinder of height $\frac{1}{2}$ and a single vertical cylinder of width $\frac{1}{2}$ has the following asymptotics as $N\to+\infty$:*

$$\mathcal{P}_{1^k}^{labeled}(N) = cyl_{1,1}\left(\mathcal{Q}(1^k,-1^{k+4})\right)\cdot\frac{N^{2k+2}}{4k+4} + o\left(N^{2k+2}\right) \ \text{ as } N\to+\infty\,,$$

were

$$cyl_{1,1}\left(\mathcal{Q}(1^k,-1^{k+4})\right) = \frac{\left(cyl_1\left(\mathcal{Q}(1^k,-1^{k+4})\right)\right)^2}{4\left(\dfrac{\pi^2}{2}\right)^{k+1}}$$

and

$$cyl_1\left(\mathcal{Q}(1^k,-1^{k+4})\right) = 2\cdot\binom{2k+4}{k+2} \tag{4.3}$$

The number $\mathcal{P}_{1^k,0}^{labeled}(N)$ of pillowcase covers as above with a marked regular vertex of the tiling has the following asymptotics as $N\to+\infty$:

$$\mathcal{P}_{1^k,0}^{labeled}(N) = 2\cdot cyl_{1,1}\left(\mathcal{Q}(1^k,-1^{k+4})\right)\cdot\frac{N^{2k+3}}{4k+6} + o\left(N^{2k+3}\right) \ \text{ as } N\to+\infty\,,$$

Proof. By (2.2) we have

$$\operatorname{Vol}\mathcal{Q}(1^k,-1^{k+4}) = 2\pi^2\cdot\left(\frac{\pi^2}{2}\right)^k = 4\cdot\left(\frac{\pi^2}{2}\right)^{k+1}$$

To prove (4.3) we apply the following combinatorial identity to simplify formula (4.2) in the particular case when $\nu=[1^k]$:

$$\sum_{\iota_1=0}^{k}\binom{k}{\iota_1}\binom{k+4}{\iota_1+2} = \binom{2k+4}{k+2}\,,$$

see (3.20) in [Gd].

It remains to prove that

$$cyl_{1,1}\left(\mathcal{Q}(1^k,0,-1^{k+4})\right) = 2\cdot cyl_{1,1}\left(\mathcal{Q}(1^k,-1^{k+4})\right)\,. \tag{4.4}$$

By (2.8) we have

$$cyl_{1,1}\left(\mathcal{Q}(1^k,0,-1^{k+4})\right) = \frac{\Big(cyl_1\left(\mathcal{Q}(1^k,0,-1^{k+4})\right)\Big)^2}{\operatorname{Vol}\mathcal{Q}(1^k,0,-1^{k+4})}\,.$$

Equation 4.2 implies that

$$cyl_1\left(\mathcal{Q}(1^k,0,-1^{k+4})\right) = 2\cdot cyl_1\left(\mathcal{Q}(1^k,-1^{k+4})\right)$$

Finally, by (2.2) we have

$$\operatorname{Vol}\mathcal{Q}(1^k,0,-1^{k+4}) = 2\operatorname{Vol}\mathcal{Q}(1^k,-1^{k+4})\,.$$

and (4.4) follows. □

We also prove the following elementary technical Corollary of Theorem 6.

Corollary 4.2. *Consider a (generalized) partition $\nu=[0^{\nu_0}1^{\nu_1}2^{\nu_2}\dots]$ and its subpartition $\nu'=[1^{\nu_1}2^{\nu_2}\dots]$ obtained by suppressing all zero entries. The following formulae are valid:*

$$cyl_1\big(\nu,-1^{|\nu|+4}\big)=2^{\nu_0}\cdot cyl_1\big(\nu',-1^{|\nu'|+4}\big) \tag{4.5}$$

$$\mathrm{P}_1\left(\nu,-1^{|\nu|+4}\right)=\mathrm{P}_1\left(\nu',-1^{|\nu'|+4}\right). \tag{4.6}$$

Proof. Note that $|\nu'|=|\nu|$. Similarly, having any subpartion $\iota'\subset\iota$ obtained from a partition ι by suppressing all zero entries we have $|\iota'|=|\iota|$. Thus we can rewrite formula (4.2) as

$$cyl_1\left(\mathcal{Q}(\nu,-1^{|\nu|+4})\right)=2\cdot\sum_{\iota_0=0}^{\nu_0}\sum_{\iota_1=0}^{\nu_1}\sum_{\iota_2=0}^{\nu_2}\sum_{\dots}^{\dots}\binom{\nu_0}{\iota_0}\binom{\nu_1}{\iota_1}\binom{\nu_2}{\iota_2}\cdots\binom{|\nu|+4}{|\iota|+2}=$$

$$=\left(\sum_{\iota_0=0}^{\nu_0}\binom{\nu_0}{\iota_0}\right)\cdot\left(2\sum_{\iota_1=0}^{\nu_1}\sum_{\iota_2=0}^{\nu_2}\sum_{\dots}^{\dots}\binom{\nu_1}{\iota_1}\cdots\binom{|\nu'|+4}{|\iota'|+2}\right)=2^{\nu_0}cyl_1\left(\mathcal{Q}(\nu',-1^{|\nu'|+4})\right),$$

which proves (4.5). To prove (4.6) it suffices to note that by formula (2.2), we have

$$\operatorname{Vol}\mathcal{Q}(\nu,-1^{|\nu|+4})=(f(0))^{\nu_0}\operatorname{Vol}\mathcal{Q}(\nu',-1^{|\nu'|+4})=2^{\nu_0}\operatorname{Vol}\mathcal{Q}(\nu',-1^{|\nu'|+4}).$$

Passing to the ratios

$$\mathrm{P}_1\left(\mathcal{Q}(\nu,-1^{|\nu|+4})\right):=\frac{cyl_1\left(\mathcal{Q}(\nu,-1^{|\nu|+4})\right)}{\operatorname{Vol}\left(\mathcal{Q}(\nu,-1^{|\nu|+4})\right)}=$$

$$=\frac{cyl_1\left(\mathcal{Q}(\nu',-1^{|\nu'|+4})\right)}{\operatorname{Vol}\left(\mathcal{Q}(\nu',-1^{|\nu'|+4})\right)}=:\mathrm{P}_1\left(\mathcal{Q}(\nu',-1^{|\nu'|+4})\right)$$

we get the desired equation (4.5). □

Recall that a type $\iota=[0^{\iota_0}1^{\iota_1}2^{\iota_2}\dots]$ of a plane tree $\mathcal{T}$ records the number ι_j of vertices of valence $j+2$ for $j=0,1,2,\dots$. Note that in section 4 we allow to the tree have several vertices of valence 2. Recall also that $|\nu|$ denotes the sum of the entries of the partition $\nu=[0^{\nu_0}1^{\nu_1}2^{\nu_2}\dots]$; by $\ell(\nu)$ we denote the length of ν, where this time we count the entries 0 if any:

$$|\nu|:=1\cdot\nu_1+2\cdot\nu_2+3\cdot\nu_3+\dots$$
$$\ell(\nu):=\nu_0+\nu_1+\nu_2+\nu_3+\dots$$

In the Lemma below we reproduce formula (2.2) from Proposition (2.2) in [DGZZ] adapting it to the language of the current paper.

Lemma 4.3. *Consider a separatrix diagram $\mathcal{D}=\mathcal{T}(\iota)\sqcup\mathcal{T}(\nu-\iota)$ represented by a non-labeled pair of plane trees $\mathcal{T}(\iota)$ and $\mathcal{T}(\nu-\iota)$ with profiles $\iota\subset\nu$ and $\nu-\iota$ respectively. The number of pillowcase covers with labeled zeroes and poles, tiled with at most $2N$ squares and having a single horizontal cylinder of height $\frac{1}{2}$ representing a given separatrix diagram $\mathcal{D}$ has the following asymptotics when $N\to+\infty$*

$$cyl_1(\mathcal{D})\cdot\frac{N^d}{2d}+o(N^d), \tag{4.7}$$

where the dimension d of the ambient stratum $\mathcal{Q}(\nu,-1^{|\nu|+4})$ is defined by equation (2.4) *and*

$$cyl_1(\mathcal{D}) = \frac{4}{|\operatorname{Aut}(\mathcal{D})|}\cdot\frac{(|\nu|+4)!\cdot\nu_0!\cdot\nu_1!\cdot\nu_2!\cdots}{\big(|\iota|+\ell(\iota)\big)!\cdot\big(|\nu-\iota|+\ell(\nu-\iota)\big)!}\,. \tag{4.8}$$

Remark 4.1. In this paper we denote by $cyl_1(\mathcal{D})$ the coefficient of the leading term in the asymptotics of the number of pillowcase covers tiled with at most $2N$ squares and having a single horizontal cylinder of *minimal possible* height $\frac{1}{2}$. In the companion paper [DGZZ] we used a similar notation $c_1(\mathcal{D})$ for the coefficient in asymptotics where we made no restriction on the height of the cylinder. It is easy to see that the two coefficients differ by the factor $\zeta(d)$, namely,

$$c_1(\mathcal{D}) = \zeta(d)\cdot cyl_1(\mathcal{D})\,,$$

where $d = \dim_{\mathbb{C}}\mathcal{Q}(\nu,-1^{|\nu|+4})$ is given by formula (2.4).

Proof. The number of edges m of $\mathcal{T}(\iota)$, and n of $\mathcal{T}(\nu-\iota)$ are expressed as

$$m = |\iota| + \ell(\iota) + 1$$
$$n = |\nu-\iota| + \ell(\nu-\iota) + 1$$

and the dimension d of the stratum satisfies relation $d = m + n$.

Consider any pillowcase cover having the diagram $\mathcal{D}$ as the diagram of horizontal saddle connections. Cut it open along all horizontal saddle connections. By definition of $\mathcal{D}$ it has m pairs of saddle connections on one side of the cylinder; n pairs of saddle connections on the other side of the cylinder, all saddle connection has its twin on the same side.

The proof now follows line by line the second part of the proof of the more general Proposition 2.2 in [DGZZ]. Note that the parameter l used in Proposition 2.2 to denote the number of saddle connections which after the surgery as above appear on both sides of the cylinder is equal to zero in genus zero. One extra simplification comes from the fact that in the proof of Proposition (2.2) in [DGZZ] we sum over various possible heights of the horizontal cylinder, while in our context it equals to $\frac{1}{2}$, see Remark (4.1). As a result we do not get the extra factor $\zeta(d)$ present in the original expression (2.2) in Proposition (2.2) in [DGZZ]. □

Consider a separatrix diagram $\mathcal{D} = \mathcal{T}(\iota)\sqcup\mathcal{T}(\nu-\iota)$ as above. Defining the automorphism group $\operatorname{Aut}(\mathcal{D})$ we assume that none of the vertices, edges, or boundary components of the ribbon graph $\mathcal{D}$ is labeled; however, we assume that the orientation of the ribbons is fixed. Thus

$$|\operatorname{Aut}(\mathcal{D})| = |\operatorname{Aut}(\mathcal{T}(\iota))|\cdot|\operatorname{Aut}(\mathcal{T}(\nu-\iota))|\cdot\begin{cases}2 & \text{if } \mathcal{T}(\iota)\simeq\mathcal{T}(\nu-\iota)\\ 1 & \textit{otherwise}\end{cases} \tag{4.9}$$

Here $\simeq$ stands for an isomorphism of plane ("ribbon") trees.

The following counting Theorem for plane trees is well known; see, for example, [Mo, 2, p.6]. It is the last element needed for proof of Theorem 6.

Theorem. *For any partition $\iota = [0^{\iota_0}1^{\iota_1}2^{\iota_2}\dots]$ the following expression holds*

$$\sum_{\mathcal{T}(\iota)}\frac{1}{|\operatorname{Aut}(\mathcal{T}(\iota))|} = \frac{\big(|\iota|+\ell(\iota)\big)!}{\big(|\iota|+2\big)!\cdot\iota_0!\cdot\iota_1!\cdot\iota_2!\cdots}\,,$$

where we sum over all plane trees corresponding to a partition ι and $|\operatorname{Aut}(\mathcal{T}(\iota))|$ is the order of the automorphism group of the tree $\mathcal{T}(\iota)$.

Proof of Theorem 6. The first two statements of Theorem 6 are, a particular case of Theorem 1.19 in [DGZZ] which, morally, claims that "horizontal and vertical decompositions of pillowcase covers are asymptotically uncorrelated".

It only remains to prove expression (4.2).

Combining equation (4.8) with the above Theorem we conclude that the sum of $cyl_1(\mathcal{D})$ over all realizable one-cylinder separatrix diagrams $\mathcal{D}$ in any given stratum $\mathcal{Q}(\nu, -1^{|\nu|+4})$ in genus zero can be expressed as follows

$$cyl_1\left(\mathcal{Q}(\nu,-1^{|\nu|+4})\right) = \sum_{\mathcal{D}} cyl_1(\mathcal{D}) = \frac{1}{2}\sum_{\iota\subset\nu}\left(\frac{4\cdot(|\nu|+4)!\cdot\nu_0!\cdot\nu_1!\cdot\nu_2!\cdots}{\big(|\iota|+\ell(\iota)\big)!\cdot\big(|\nu-\iota|+\ell(\nu-\iota)\big)!}\right)\cdot$$
$$\cdot\left(\frac{\big(|\iota|+\ell(\iota)\big)!}{\big(|\iota|+2\big)!\cdot\iota_0!\cdot\iota_1!\cdots}\right)\cdot\left(\frac{\big(|\nu-\iota|+\ell(\nu-\iota)\big)!}{\big(|\nu-\iota|+2\big)!\cdot(\nu_0-\iota_0)!\cdot(\nu_1-\iota_1)!\cdots}\right)=$$
$$= 2\sum_{\iota\subset\nu}\binom{|\nu|+4}{|\iota|+2}\binom{\nu_0}{\iota_0}\binom{\nu_1}{\iota_1}\binom{\nu_2}{\iota_2}\cdots$$

□

Appendix A. Meanders and pairs of arc systems satisfying additional combinatorial constraints

In this appendix we count the frequency of meanders among all pairs of arc systems imposing additional constraints on combinatorics of the pair of arc systems.

In section 1.2 we have assigned to any closed meander two arc systems on discs (considered as hemispheres). Passing to the one-point compactification of the plane we place our closed meander curve on the resulting sphere. By construction, this meander curve is the simple closed curve on the sphere obtained from the two arc systems by identifying the two hemispheres along the common equator. By convention we call the equator the *horizontal* curve and the simple closed meander curve — the *vertical curve* on the resulting sphere.

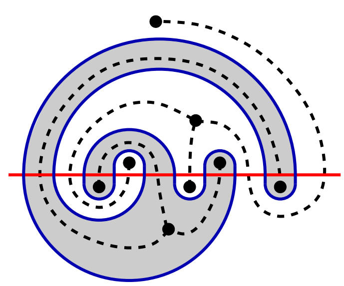


Figure 5. We can change the roles of the horizontal and vertical curves and construct the reduced dual trees for the horizontal curve of a closed meander. The tree in the bounded region (which is shaded in the picture) is denoted by $\mathcal{T}_0^*$. The tree in the complementary unbounded region is denoted by $\mathcal{T}_\infty^*$.

We have constructed two reduced dual graphs to the two arc systems, see the right picture in Figure 2. We can change the roles of the horizontal and vertical curves and consider the vertical curve as the new "equator" of the sphere. Then, the horizontal curve (former equator) takes the role of the meander curve and defines a pair of arc systems and reduced dual trees, see Figure 5. The meander cuts the plane in two regions: one bounded and one unbounded. We denote the reduced dual tree as above staying in the bounded domain of the plane by $\mathcal{T}_0^*$ and the one in the complementary unbounded domain — by $\mathcal{T}_\infty^*$.

In the setting where a pair of transverse labeled simple closed curves on a sphere comes from arc systems on two hemispheres, we do not have the distinction between "bounded" and "unbounded" domains. In this case the graph $\mathcal{D}^* = \mathcal{T}_0^* \sqcup \mathcal{T}_\infty^*$ obtained as a disjoint union of the trees $\mathcal{T}_0^*$ and $\mathcal{T}_\infty^*$ does not have any canonical labeling of connected components.

As before, consider a pair of arc systems on two hemispheres containing the same number of arcs and identify them along the equator matching the endpoints of arcs. We get a simple closed curve on the sphere and a multicurve transverse to it. Recall that we associate to any transverse connected pair of multicurves its type $\nu = [1^{\nu_1} 2^{\nu_2} 3^{\nu_3} \dots]$, where the entry ν_j records the number of $(2j+4)$-gons, for $j \in \mathbb{N}$, among the faces in which the pair of multicurves cuts the sphere.

When the two multicurves are simple closed curves, the entry ν_j also records the number of vertices of valence $j+2$ in the graph $\mathcal{T}_0^* \sqcup \mathcal{T}_\infty^*$, and in the graph $\mathcal{T}_{bottom} \sqcup \mathcal{T}_{top}$. We will call ν the *type* of the graphs $\mathcal{T}_{bottom} \sqcup \mathcal{T}_{top}$ and $\mathcal{T}_0^* \sqcup \mathcal{T}_\infty^*$.

By duality, if we specify the trees $\mathcal{T}_0^*$ and $\mathcal{T}_\infty^*$ instead of the trees $\mathcal{T}_{bottom}$ and $\mathcal{T}_{top}$ in the setting of Theorem 4 we get the completely parallel statement.

We can state the following more detailed version of Theorem 4. This time we chose two pairs of trees: $\mathcal{T}_{bottom}, \mathcal{T}_{top}$ and $\mathcal{T}_0^*, \mathcal{T}_\infty^*$, such that the graphs $\mathcal{D}^* := \mathcal{T}_0^* \sqcup \mathcal{T}_\infty^*$ and $\mathcal{T}_{bottom} \sqcup \mathcal{T}_{top}$ share any given type ν. The two connected components of the graph $\mathcal{T}_{bottom} \sqcup \mathcal{T}_{top}$ are labeled; the two components of the graph $\mathcal{D}^*$ — not.

As before, we consider all possible triples

$$(n\text{-arcs system of type } \mathcal{T}_{top};\ n\text{-arcs system of type } \mathcal{T}_{bottom};\ \text{identification})$$

as described above for all $n \le N$. In analogy with (1.3) we define the fraction $p(\mathcal{T}_{bottom}, \mathcal{T}_{top}; \mathcal{T}_0^*, \mathcal{D}^*; N)$ of triples as above which lead to a meander with the graph $\mathcal{D}^*$ dual to the equator, among all triples as above.

Proposition 2. *For any two pairs of plane trees $\mathcal{T}_{bottom}, \mathcal{T}_{top}$ and $\mathcal{T}_0^*, \mathcal{T}_\infty^*$, such that the graphs $\mathcal{D}^* := \mathcal{T}_0^* \sqcup \mathcal{T}_\infty^*$ and $\mathcal{T}_{bottom} \sqcup \mathcal{T}_{top}$ share any given type ν, the following limit exists and has the following strictly positive value:*

$$\lim_{N \to +\infty} p(\mathcal{T}_{bottom}, \mathcal{T}_{top}; \mathcal{D}^*; N) = \mathrm{P}_1(\mathcal{D}^*)\,.$$

The limit $\mathrm{P}_1(\mathcal{D}^*)$ is expressed by the following formula

$$\mathrm{P}_1(\mathcal{D}^*) = \frac{cyl_1(\mathcal{D}^*)}{\operatorname{Vol}\mathcal{Q}(\nu, -1^{|\nu|+4})} \tag{A.1}$$

Denote by ι and $\nu - \iota$ profiles of the plane trees $\mathcal{T}_0^*$ and $\mathcal{T}_\infty^*$. The coefficient $cyl_1(\mathcal{D}^*)$ in the above formula is given by equation

$$cyl_1(\mathcal{D}(\iota)) = \frac{4}{|\operatorname{Aut}(\mathcal{D}^*)|} \cdot \frac{(|\nu|+4)! \cdot \nu_0! \cdot \nu_1! \cdot \nu_2! \cdots}{\big(|\iota| + \ell(\iota)\big)! \cdot \big(|\nu - \iota| + \ell(\nu - \iota)\big)!} \tag{A.2}$$

Here $|\operatorname{Aut}(\mathcal{D}^*)|$ *denotes the order of the automorphism group of the ribbon graph* $\mathcal{D}^*$*, where neither of connected components, edges or vertices of* $\mathcal{D}^*$ *are labeled, but the orientation of the ribbon graph is fixed.*

Proof. As in Theorem 4 we consider labled pairs of arc systems on oriented discs assuming that the number of arcs in two systems is the same. "All possible triples" in the denominator of the ratio $p(\mathcal{T}_{bottom}, \mathcal{T}_{top}; \mathcal{D}^*; N)$ is exactly the same as above: the graph $\mathcal{D}^*$ does not carry any information in the definition of the "possible triples". Computing the numerator we impose now a given separatrix diagram $\mathcal{D}^*$ as the graph of vertical saddle connections of the single vertical cylinder.

Thus, the limit in Proposition 2 is the asymptotic fraction of pillowcase covers having a single horizontal cylinder of height $\frac{1}{2}$ corresponding to the separatrix diagram $\mathcal{D} = \mathcal{T}_{bottom} \sqcup \mathcal{T}_{top}$ of horizontal saddle connections, and a single vertical cylinder of width $\frac{1}{2}$ corresponding to the separatrix diagram $\mathcal{D}^*$ of vertical saddle connections among all pillowcase covers having a single horizontal cylinder of height $\frac{1}{2}$ corresponding to the separatrix diagram $\mathcal{D}$ of horizontal saddle connections.

Theorem 1.19 in [DGZZ] asserts that such limit exists and that "horizontal and vertical cylinder decompositions are asymptotically uncorrelated", so the above limit coincides with the asymptotic fraction of pillowcase covers having a single vertical cylinder of width $\frac{1}{2}$ corresponding to the separatrix diagram $\mathcal{D}^*$ among all pillowcase covers. As before we can omit conditions on the horizontal foliation. This proves existence of the limit.

By definition, the latter asymptotic fraction is the quantity $\mathrm{P}_1(\mathcal{D}^*)$. This proves the first equality in formula (A.1). The value of $cyl_1(\mathcal{D}^*)$ is computed in equation (4.8) in Lemma 4.7, which completes the proof of Proposition 2. □

Appendix B. Arc systems as linear involutions

We have seen that every closed meander in the plane defines a pair of arc systems as in Figure 2. A pair of arc systems can be encoded by a *linear involution* (see [DaN] and [BL]) generalizing an interval exchange transformation. In the example of Figure 2 we get the linear involution

$$\begin{pmatrix} A, B, B, C, C, A \\ D, E, E, F, F, D, G, G \end{pmatrix},$$

see Figure 6.

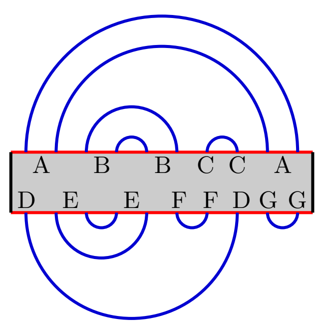


Figure 6. Pair of arc systems as a linear involution.

We define the distance between two consecutive intersections of the horizontal segment with the meander to be $\frac{1}{2}$. Thus, in our example we assign the following lengths to the intervals under exchange:

$$|A| = 1,\ |B| = 1,\ |C| = \frac{1}{2},\ |D| = \frac{1}{2},\ |E| = 1,\ |F| = \frac{1}{2},\ |G| = \frac{1}{2}\,.$$

Consider the trees $\mathcal{T}_0^*$ and $\mathcal{T}_\infty^*$ dual to the horizontal line as in Figure 5. Follow each half-edge of these trees from the vertex till the first intersection with the horizontal line. The resulting intersection points are exactly the extremities of the intervals under exchange.

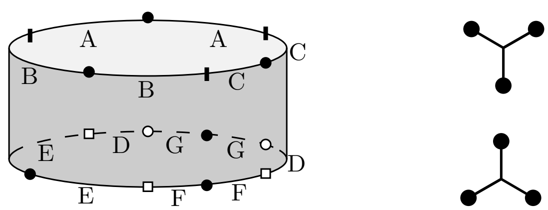


Figure 7. Jenkins–Strebel differential with a single horizontal cylinder on $\mathbb{CP}^1$ associated to a linear involution and the ribbon graph of its horizontal saddle connections.

Reciprocally, every linear involution of intervals of lengths in $\frac{1}{2}\mathbb{N}$ and such that twins of the top interval are located on top and twins of the bottom intervals are located on the bottom naturally defines a pair of arc systems in the plane and a pillowcase cover of genus zero having a single horizontal cylinder. Namely, consider a rectangle of height $\frac{1}{2}$ and with a horizontal side having the same length as the base segment of the linear involution (see the shadowed rectangle in the middle of Figure 6). Identify the two vertical sides of the rectangle by a parallel translation and identify the subsegments of the horizontal sides as prescribes the linear involution, see Figure 7. We get a Jenkins–Strebel differential with a single horizontal cylinder.

Appendix C. Meanders of low combinatorial complexity

In this section we present an explicit formula for meanders whose underlying pillowcase cover belongs to $\mathcal{Q}(-1^4)$. We also present some numerical experiments for meanders leading to pillowcase covers with 5 poles (stratum $\mathcal{Q}(1, -1^5)$) and with 6 poles (strata $\mathcal{Q}(1, -1^6)$ and $\mathcal{Q}(2, -1^6)$). The reader can see that these experiments provide strong numerical evidence for our asymptotic results.

Let us recall from Theorem 1 that $\mathcal{M}_p^+(N)$ (respectively $\mathcal{M}_p^-(N)$) counts the number of plane meanders with at most N arcs (or, equivalently, with at most $2N$ crossings) and with (resp. without) a maximal arc and p minimal arcs. In the language of flat surfaces, $\mathcal{M}_{p-1}^+(N)$ and $\mathcal{M}_p^-(N)$ correspond to pillowcases covers in one of the strata $\mathcal{Q}(\nu, -1^p)$ where $\nu = [1^{\nu_1} 2^{\nu_2} \ldots]$ is a partition of $p - 4$ (see Lemma 3.3).

We consider here a refined counting. Namely, let $M_{n,p}$ be the number of meanders with exactly n arcs (or, equivalently, with exactly $2n$ crossings) and either with p minimal arcs and no maximal arcs, or with $p - 1$ minimal arcs and a maximal arc. In other words, $M_{n,p}$ denotes the number of meanders that correspond to pillowcase covers of degree n whose associated quadratic differential has exactly p

poles. We have the following relation

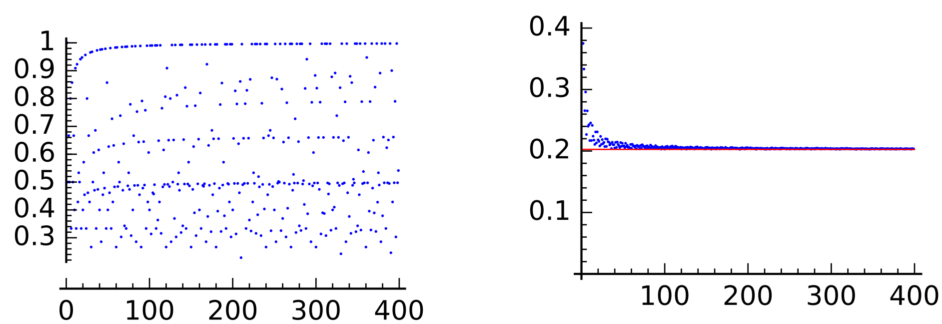
$$\mathcal{M}^{+}_{p-1}(N) + \mathcal{M}^{-}_{p}(N) = \sum_{n=1}^{N} M_{n,p}.$$

Note that by equation (1.1) we have $\mathcal{M}^{+}_{p-1}(N) = O(N^{2p-6})$ while by equation (1.2) $\mathcal{M}^{-}_{p}(N) = O(N^{2p-5})$. Thus, the contribution of $\mathcal{M}^{+}_{p-1}(N)$ becomes negligible for large N.

The following array presents the values $M_{n,p}$ where the number of arcs n ranges from 1 to 9 and p ranges from 4 to 8. These values were obtained by listing all meanders and filtering them by the total number p of minimal and maximal arcs.

p \ n	1	2	3	4	5	6	7	8	9
4	1	2	6	8	20	12	4	32	54
5	0	0	0	16	40	168	280	544	1152
6	0	0	2	16	110	416	1470	4128	9102
7	0	0	0	0	60	576	3276	13632	45468
8	0	0	0	2	30	462	4228	26424	130410

The sum of entries $\sum_{p=4}^{+\infty} M_{n,p}$ in each column of this biinfinite array is the mysterious number of meanders with n arcs: 1, 2, 8, 42, 262, 1828, 13820, 110954, 933458, ... (sequence A005315 from [OEIS]). Here we study this array by lines.



(A) Number of meanders with exactly n arcs divided by n^2, i. e. the function $n \mapsto \phi(n)/n$.

(B) Number of meanders with at most N arcs divided by N^3 (represented by blue points) and the asymptotic value $\frac{2}{\pi^2}$ (red line).

FIGURE 8. The number $M_{n,4}$ of meanders with 4 minimal arcs or, equivalently, the number of meanders whose associated pillowcase covers belongs to $\mathcal{Q}(-1^4)$.

For $p = 4$, there is only one stratum $\mathcal{Q}(-1^4)$ and the corresponding generalized interval exchange transformations (see Appendix B) are reduced to rotations. It is then easy to deduce the following

Lemma C.1. *We have* $M_{n,4} = n\phi(n)$ *where* ϕ *is the Euler totient function. In particular*

$$\sum_{n=1}^{N} M_{n,4} = \mathcal{M}_3^+(N) + \mathcal{M}_4^-(N) \sim \frac{2N^3}{\pi^2}.$$

This result is coherent with the one suggested by formula (1.2) for $\mathcal{M}_4^-(N)$; see also Figure 8 for graphics related to $M_{n,4}$.

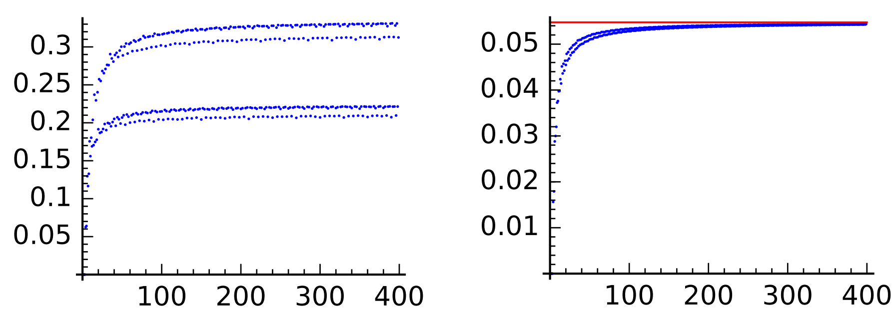
We do not hope to get a closed formula for $M_{n,5}$. However, it is not hard to compute these numbers using generalized interval exchanges and Rauzy induction. We were able to compute 400 of these numbers represented in the second line of the above table and the list starts with

0, 0, 0, 16, 40, 168, 280, 544, 1152, 1560, 2640, 3504, 5824, 6552, 12000, 11456, 19176, 18648, 31312, 30640, 50064, 43736, 71392, 62304, 104800, 87672, 141048, 121968, 191632, 154200, 255192, 209536, ...

Figure 9 represents first 400 terms of this sequence. It agrees with the value

$$\mathrm{P}_1\left(\mathcal{Q}(1,-1^5)\right) \; = \; \frac{16}{3\pi^4}$$

predicted by formula (1.2) for $\mathcal{M}_5^-(N)$.



(A) Normalized number of meanders $\frac{M_{5,n}}{n^4}$ with exactly n arcs

(B) Normalized number of meanders $\frac{\sum_{n=6}^{N} M_{5,n}}{N^5}$ with at most N arcs (blue points) and the asymptotic value $\frac{16}{3\pi^4}$ (red line).

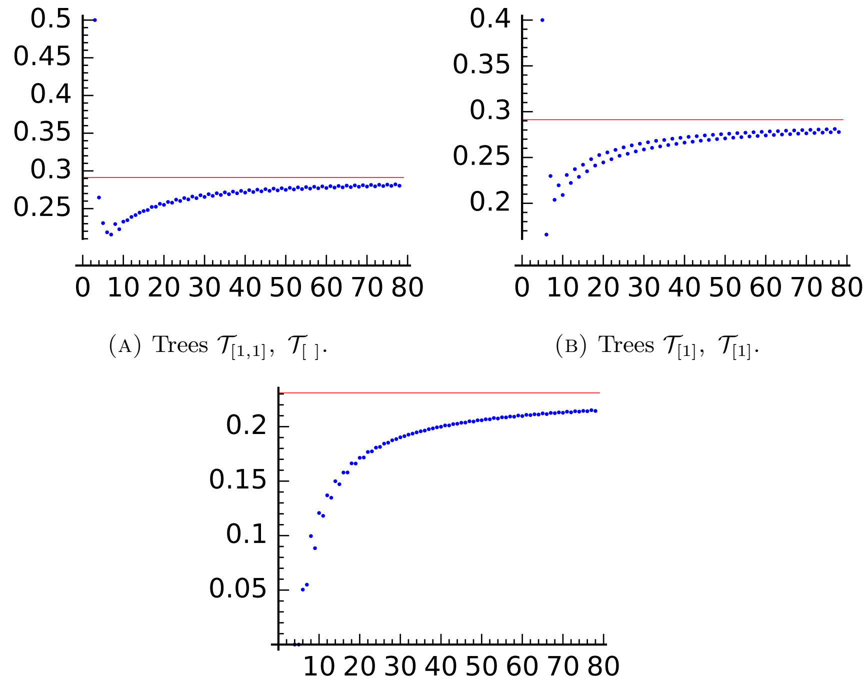
FIGURE 9. The number $M_{n,5}$ of meanders whose associated pillowcase covers belongs to $\mathcal{Q}(1,-1^5)$.

We now present some numerical evidence for the theoretical prediction of Theorem 4. There are three pairs of trees with 6 univalent vertices in total. For these three pairs of trees Theorem 4 gives

$$\mathrm{P}_{connected}(\mathcal{T}_{[2]}, \mathcal{T}_{[\,]}) = \mathrm{P}_1\left(\mathcal{Q}(2,-1^6)\right) = \frac{45}{2\pi^4} \sim 0.231$$

$$\mathrm{P}_{connected}(\mathcal{T}_{[1]}, \mathcal{T}_{[1]}) = \mathrm{P}_{connected}(\mathcal{T}_{[1,1]}, \mathcal{T}_{[\,]}) = \mathrm{P}_1\left(\mathcal{Q}(1^2,-1^6)\right) = \frac{280}{\pi^6} \sim 0.291\,.$$

These limit ratios are represented by horizontal lines dominating the plots of exact values of $\mathrm{P}_{connected}(\mathcal{T}_{top}, \mathcal{T}_{bottom}; N)$ in Figure 10. We denote by $\mathcal{T}_{[]}$, $\mathcal{T}_{[1]}$, $\mathcal{T}_{[2]}$ and $\mathcal{T}_{[1,1]}$ respectively the unique trees with no internal vertices, with a single internal vertex of valence 3, with a single internal vertex of valence 4 and with two internal vertices of valence 3.



(A) Trees $\mathcal{T}_{[1,1]}$, $\mathcal{T}_{[\,]}$.

(B) Trees $\mathcal{T}_{[1]}$, $\mathcal{T}_{[1]}$.

(C) Trees $\mathcal{T}_{[2]}, \mathcal{T}_{[\,]}$.

FIGURE 10. Proportion $\mathrm{P}_{connected}(\mathcal{T}_{top}, \mathcal{T}_{bottom}; N)$ of pairs of arcs systems which lead to meanders among all pairs of arc systems with at most N arcs. We consider all pairs of trees $(\mathcal{T}_{top}, \mathcal{T}_{bottom})$ with 6 leaves.

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