arXiv:1705.05268v1 [math.CO] 15 May 2017

GORENSTEIN SIMPLICES WITH A GIVEN δ-POLYNOMIAL

TAKAYUKI HIBI, AKIYOSHI TSUCHIYA, AND KOUTAROU YOSHIDA

ABSTRACT. It is fashionable among the study on convex polytopes to classify the lattice polytopes with a given δ-polynomial. As a basic challenges toward the classification problem, we achieve the study on classifying lattice simplices with a given δ-polynomial of the form $1+t^{k+1}+\cdots+t^{(v-1)(k+1)}$, where $k \geq 0$ and $v > 0$ are integers. The lattice polytope with the above δ-polynomial is necessarily Gorenstein. A complete classification is already known, when v is prime. In the present paper, a complete classification will be performed, when v is either p^2 or pq, where p and q are prime integers with $p \neq q$. Moreover, we focus on the number of Gorenstein simplices, up to unimodular equivalence, with the expected δ-polynomial.

INTRODUCTION

It is fashionable among the study on convex polytopes to classify the lattice polytopes with a given δ-polynomial. A *lattice polytope* is a convex polytope $\mathscr{P} \subset \mathbb{R}^d$ all of whose vertices have integer coordinates. Recall from [4, Part II] what the *δ-polynomial* of $\mathscr{P}$ is.

Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d and define $\delta(\mathscr{P},t)$ by the formula

$$(1-t)^{d+1}\delta(\mathscr{P},t) = 1+\sum_{n=1}^{\infty} |n\mathscr{P} \cap \mathbb{Z}^d| t^n,$$

where $n\mathscr{P} = \{n\mathbf{a} : \mathbf{a} \in \mathscr{P}\}$, the dilated polytopes of $\mathscr{P}$. It follows that $\delta(\mathscr{P},t)$ is a polynomial in t of degree at most d. We say that $\delta(\mathscr{P},t)$ is the *δ-polynomial* of $\mathscr{P}$. Let $\delta(\mathscr{P},t) = \delta_0+\delta_1 t+\cdots+\delta_d t^d$. Then $\delta_0 = 1$, $\delta_1 = |\mathscr{P} \cap \mathbb{Z}^d| - (d+1)$ and $\delta_d = |(\mathscr{P} \setminus \partial\mathscr{P}) \cap \mathbb{Z}^d|$, where $\partial\mathscr{P}$ is the boundary of $\mathscr{P}$, and each $\delta_i \geq 0$. When $\delta_d \neq 0$, one has $\delta_i \geq \delta_1$ for $1 \leq i \leq d$. Moreover, $\delta(\mathscr{P},1) = \sum_{i=0}^{d} \delta_i$ coincides with the *normalized volume* $\mathrm{Vol}(\mathscr{P})$ of $\mathscr{P}$.

A lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ of dimension d is called *reflexive* if the origin of $\mathbb{R}^d$ belongs to the interior of $\mathscr{P}$ and the dual polytope ([4, pp. 103–104]) of $\mathscr{P}$ is again a lattice polytope. A lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ of dimension d is called *Gorenstein of index r* if $r\mathscr{P}$ is unimodularly equivalent to a reflexive polytope. It is known that $\mathscr{P} \subset \mathbb{R}^d$ is Gorenstein if and only if the δ-polynomial $\delta(\mathscr{P},t) = \delta_0+\delta_1 t+\cdots+\delta_s t^s$, where $\delta_s \neq 0$ is symmetric, i.e., $\delta_i = \delta_{s-i}$ for each $0 \leq i \leq \lfloor s/2 \rfloor$.

Gorenstein polytopes are of interest in commutative algebra, mirror symmetry and tropical geometry ([1, 7]). In each dimension, there exist only finite many Gorenstein polytopes up to unimodular equivalence ([9]) and, in addition, Gorenstein polytopes are

2010 *Mathematics Subject Classification.* 52B12, 52B20.
Key words and phrases. Lattice polytope, Gorenstein polytope, δ-polynomial, empty simplex.
The second author was partially supported by Grant-in-Aid for JSPS Fellows 16J01549.

completely classified up to dimension 4 ([8]). Recently certain classification results of higher-dimensional Gorenstein polytopes are obtained by [3, 6, 12].

The final goal of one of our research projects is to classify the Gorenstein simplices with given δ-polynomials. In [12, Corollary 2.4] it is shown that if Δ is a Gorenstein simplex whose normalized volume $\mathrm{Vol}(\Delta)$ is a prime number p, then its δ-polynomial is of the form

$$\delta(\Delta,t) = 1 + t^{k+1} + \cdots + t^{(p-1)(k+1)},$$

where $k \geq 0$ is an integer (Proposition 3.1). Once the fact became known, we cannot escape from the temptation to achieve the study on the following problem:

Problem 0.1. *Given integers $k \geq 0$ and $v > 0$, classify the Gorenstein simplices with the δ-polynomial $1 + t^{k+1} + \cdots + t^{(v-1)(k+1)}$.*

A lattice simplex is called *empty* if it possesses no lattice point except for its vertices. A lattice simplex Δ with $\delta(\Delta,t) = \delta_0 + \delta_1 t + \cdots$ is empty if and only if $\delta_1 = 0$. In particular, in Problem 0.1, when $k > 0$, its target is Gorenstein empty simplices.

The present paper is organized as follows. Section 1 consists of the review of fundamental materials on lattice simplices and the collection of indispensable lemmata. We devote Section 2 to discuss a lower bound on the dimensions of Gorenstein simplices with a given δ-polynomial of Problem 0.1 and, in addition, to classify the Gorenstein simplices when the lower bound is hold (Theorem 2.1). The highlight of this paper is Section 3, where a complete answer of Problem 0.1 when v is either p^2 or pq, where p and q are distinct prime integers (Theorems 3.4 and 3.5). Finally, in Section 4, we will discuss on the number of Gorenstein simplices, up to unimodular equivalence, with a given δ-polynomial of Problem 0.1.

1. Preliminaries

In this section, we recall basic materials on lattice simplices and we prepare essential lemmata in this paper.

At first, we introduce the associated finite abelian groups of lattice simplices. For a lattice simplex $\Delta \subset \mathbb{R}^d$ of dimension d whose vertices are $v_0,\ldots,v_d \in \mathbb{Z}^d$ set

$$\Lambda_\Delta = \{(\lambda_0,\ldots,\lambda_d) \in (\mathbb{R}/\mathbb{Z})^{d+1} : \sum_{i=0}^{d} \lambda_i(v_i,1) \in \mathbb{Z}^{d+1}\}.$$

The collection Λ_Δ forms a finite abelian group with addition defined as follows: For $(\lambda_0,\ldots,\lambda_d) \in (\mathbb{R}/\mathbb{Z})^{d+1}$ and $(\lambda'_0,\ldots,\lambda'_d) \in (\mathbb{R}/\mathbb{Z})^{d+1}$, $(\lambda_0,\ldots,\lambda_d)+(\lambda'_0,\ldots,\lambda'_d) = (\lambda_0+\lambda'_0,\ldots,\lambda_d+\lambda'_d) \in (\mathbb{R}/\mathbb{Z})^{d+1}$. We denote the unit of Λ_Δ by $\mathbf{0}$, and the inverse of $\mathbf{x}$ by $-\mathbf{x}$, and also denote $\underbrace{\mathbf{x}+\cdots+\mathbf{x}}_{j}$ by $j\mathbf{x}$ for an integer $j>0$ and $\mathbf{x} \in \Lambda_\Delta$. For $\mathbf{x} = (x_0,\ldots,x_d) \in \Lambda_\Delta$, we set $\mathrm{ht}(\mathbf{x}) = \sum_{i=0}^{d} x_i \in \mathbb{Z}$ and $\mathrm{ord}(\mathbf{x}) = \min\{\ell \in \mathbb{Z}_{>0} : \ell\mathbf{x} = \mathbf{0}\}$.

It is well known that the δ-polynomial of the lattice simplex Δ can be computed as follows:

Lemma 1.1. *Let Δ be a lattice simplex of dimension d whose δ-polynomial equals $1 + \delta_1 t + \cdots + \delta_d t^d$. Then for each i, we have $\delta_i = \sharp\{\lambda \in \Lambda_\Delta : \mathrm{ht}(\lambda) = i\}$.*

Recall that a matrix $A \in \mathbb{Z}^{d\times d}$ is *unimodular* if $\det(A) = \pm 1$. For lattice polytopes $\mathscr{P}, \mathscr{Q} \subset \mathbb{R}^d$ of dimension d, $\mathscr{P}$ and $\mathscr{Q}$ are called *unimodularly equivalent* if there exist a unimodular matrix $U \in \mathbb{Z}^{d\times d}$ and an integral vector $\mathbf{w} \in \mathbb{R}^d$ such that $\mathscr{Q} = f_U(\mathscr{P}) + \mathbf{w}$, where f_U is the linear transformation in $\mathbb{R}^d$ defined by U, i.e., $f_U(\mathbf{v}) = \mathbf{v}U$ for all $\mathbf{v} \in \mathbb{R}^d$.

In [2], it is shown that there is a bijection between unimodular equivalence classes of d-dimensional lattice simplices with a chosen ordering of their vertices and finite subgroups of $(\mathbb{R}/\mathbb{Z})^{d+1}$ such that the sum of all entries of each element is an integer. In particular, two lattice simplices Δ and Δ' are unimodularly equivalent if and only if there exists an ordering of their vertices such that $\Lambda_\Delta = \Lambda_{\Delta'}$.

For a lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ of dimension d, the *lattice pyramid* over $\mathscr{P}$ is defined by $\mathrm{conv}(\mathscr{P} \times \{0\}, (0,\dots,0,1)) \subset \mathbb{R}^{d+1}$. We denote this by $\mathrm{Pyr}(\mathscr{P})$. We can characterize lattice pyramids in terms of the associated finite abelian groups by using the following lemma.

Lemma 1.2 ([10, Lemma 12]). *Let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d. Then Δ is a lattice pyramid if and only if there is $i \in \{0,\dots,d\}$ such that $\lambda_i = 0$ for all $(\lambda_0,\dots,\lambda_d) \in \Lambda_\Delta$.*

For a lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ of dimension d, one has $\delta(\mathscr{P},t) = \delta(\mathrm{Pyr}(\mathscr{P}),t)$. Therefore, it is essential that we characterize polytopes which are not lattice pyramids over any lower-dimensional lattice simplex.

Finally, we give some lemmata. These lemmata are characterizations of some Gorenstein simlpices in terms of the associated finite abelian groups.

Lemma 1.3 ([12, Theorem 3.2]). *Let p be a prime integer and $\Delta \subset \mathbb{R}^d$ a d-dimensional lattice simplex whose normalized volume equals p^2. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then Δ is Gorenstein of index r if and only if one of the followings is satisfied:*

(1) There exists an integer s with $0 \le s \le d-1$ such that $rp^2 - 1 = (d-s) + ps$ and Λ_Δ is generated by $\left(\underbrace{1/p,\dots,1/p}_{s},\underbrace{1/p^2,\dots,1/p^2}_{d-s+1}\right)$ for some ordering of the vertices of Δ;

(2) $d = rp-1$ and there exist integers $0 \le a_0,\dots,a_{d-2} \le p-1$ with $p \mid (a_0+\cdots+a_{d-2}-1)$ such that Λ_Δ is generated by $((a_0+1)/p,\dots,(a_{d-2}+1)/p,0,1/p)$ and $((p-a_0)/p,\dots,(p-a_{d-2})/p,1/p,0)$ for some ordering of the vertices of Δ.

Lemma 1.4 ([12, Theorem 3.3]). *Let p and q be prime integers with $p \neq q$ and $\Delta \subset \mathbb{R}^d$ a d-dimensional lattice simplex whose normalized volume equals pq. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then Δ is Gorenstein of index r if and only if there exist nonnegative integers s_1,s_2,s_3 with $s_1+s_2+s_3 = d+1$ such that the following conditions are satisfied:*

(1) $rpq = s_1q + s_2p + s_3$;

(2) Λ_Δ *is generated by* $\left(\underbrace{1/p,\dots,1/p}_{s_1},\underbrace{1/q,\dots,1/q}_{s_2},\underbrace{1/(pq),\dots,1/(pq)}_{s_3}\right)$ *for some ordering of the vertices of* Δ.

2. EXISTENCE

In this section, we prove that for integers $k \geq 0$ and $v > 0$, there exists a lattice simplex with the δ-polynomial $1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(v-1)(k+1)}$. Moreover, we give a lower bound and an upper bound on the dimension of such a lattice simplex which is not a lattice pyramid. In fact, we obtain the following theorem.

Theorem 2.1. *Let v be a positive integer and k a nonnegative integer. Then there exists a lattice simplex $\Delta \subset \mathbb{R}^d$ of dimension d whose δ-polynomial is $1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(v-1)(k+1)}$. Furthermore, if Δ is not a lattice pyramid over any lower-dimensional lattice simplex, then one has $v(k+1)-1 \leq d \leq 4(v-1)(k+1)-2$. In particular, the lower bound holds if and only if Λ_Δ is generated by $(1/v,\dots,1/v)$.*

Proof. We assume that there exists a lattice simplex $\Delta \subset \mathbb{R}^d$ of dimension d whose δ-polynomial is $1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(v-1)(k+1)}$. Let $\mathbf{x}=(x_0,\dots,x_d) \in \Lambda_\Delta$ be an element such that $\mathrm{ht}(\mathbf{x})=(v-1)(k+1)$. Then we have that $\mathrm{ht}(-\mathbf{x}) \geq k+1$. Hence since $\mathrm{ht}(\mathbf{x})+\mathrm{ht}(-\mathbf{x}) \leq d+1$, we obtain $d \geq v(k+1)-1$. From [10, Theorem 7], if Δ is not a lattice pyramid over any lower-dimensional lattice simplex, then one has $d \leq 4(v-1)(k+1)-2$. Now, we assume that $d = v(k+1)-1$. Since for each i, one has $0 \leq x_i \leq (v-1)/v$, we obtain $\mathrm{ht}(\mathbf{x}) \leq (d+1)(v-1)/v = (v-1)(k+1)$. Hence for each i, it follows that $x_i = (v-1)/v$. Therefore Λ_Δ is generated by $(1/v,\dots,1/v)$. Then it is easy to show that $\delta(\Delta,t) = 1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(v-1)(k+1)}$, as desired. □

3. CLASSIFICATION

In this section, we give a complete ansewer of 0.1 for the case that v is the product of two prime integers. First, we consider the case where v is a prime integer. The following proposition motivates us to consider Problem 0.1.

Proposition 3.1 ([12, Corollary 2.4])**.** *Let p be a prime integer and $\Delta \subset \mathbb{R}^d$ a Gorenstein simplex of index r whose normalized volume equals p. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then $d = rp-1$ and Λ_Δ is generated by $(1/p,\dots,1/p)$. Furthermore, one has $\delta(\Delta,t) = 1+t^r+t^{2r}+\cdots+t^{(p-1)r}$.*

This theorem says that for each integers $k \geq 0$ and $v > 0$, if v is a prime integer, then there exists just one lattice simplex up to unimodular equivalence such that its δ-polynomial equals $1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(v-1)(k+1)}$. By the following proposition, we know that if v is not a prime integer, then there exist at least two such simplices up to unimodular equivalence.

Proposition 3.2. *Given integers $k \geq 0$, $v > 0$ and a proper divisor u of v, let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d such that Λ_Δ is generated by*

$$\left(\underbrace{u/v,\ldots,u/v}_{(v-1)(k+1)},\underbrace{1/v,\ldots,1/v}_{u(k+1)}\right) \in (\mathbb{R}/\mathbb{Z})^{(v+u-1)(k+1)}.$$

Then one has $\delta(\Delta,t) = 1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(v-1)(k+1)}$.

Proof. Set $\mathbf{x} = \left(\underbrace{u/v,\ldots,u/v}_{(v-1)(k+1)},\underbrace{1/v,\ldots,1/v}_{u(k+1)}\right)$ and $\mathbf{y} = (v/u)\mathbf{x} = \left(\underbrace{0,\ldots,0}_{(v-1)(k+1)},\underbrace{1/u,\ldots,1/u}_{u(k+1)}\right)$. Then we obtain $\mathrm{ht}(\mathbf{x}) = u(k+1)$ and $\mathrm{ht}(\mathbf{y}) = k+1$. Moreover, it follows that

$$\Lambda_\Delta = \{i\mathbf{x}+j\mathbf{y} \in (\mathbb{R}/\mathbb{Z})^{d+1} : i = 0,\ldots,v/u-1,\ j = 0,\ldots,u-1\}.$$

For any integers $0 \leq i \leq v/u-1$ and $0 \leq j \leq u-1$, one has

$$\mathrm{ht}(i\mathbf{x}+j\mathbf{y}) = i\,\mathrm{ht}(\mathbf{x})+j\,\mathrm{ht}(\mathbf{y}) = (iu+j)(k+1).$$

Hence, it follows from Lemma 1.1 that $\delta(\Delta,t) = 1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(v-1)(k+1)}$, as desired. □

Furthermore, the following proposition can immediately be obtained by Lemma 1.1.

Proposition 3.3. *Given integers $v_1, v_2 > 0$ and $k \geq 0$, let $\Delta_1 \subset \mathbb{R}^{d_1}$ and $\Delta_2 \subset \mathbb{R}^{d_2}$ be lattice simplices of dimension d_1 and d_2 such that $\delta(\Delta_1,t) = 1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(v_1-1)(k+1)}$ and $\delta(\Delta_2,t) = 1+t^{v_1(k+1)}+t^{2v_1(k+1)}+\cdots+t^{v_1(v_2-1)(k+1)}$. Let $\Delta \subset \mathbb{R}^{d_1+d_2+1}$ be a lattice simplex of dimension d_1+d_2+1 such that*

$$\Lambda_\Delta = \{(\mathbf{x},\mathbf{y}) \in (\mathbb{R}/\mathbb{Z})^{d_1+d_2+2} : \mathbf{x} \in \Lambda_{\Delta_1}, \mathbf{y} \in \Lambda_{\Delta_2}\}.$$

Then one has $\delta(\Delta,t) = 1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(v_1v_2-1)(k+1)}$. In particular, if neither Δ_1 nor Δ_2 is not a lattice pyramid, then Δ is not a lattice pyramid.

Now, we consider Problem 0.1 for the case that v is p^2 or pq, where p and q are prime integers with $p \neq q$. The following theorems are the main results of the present paper.

Theorem 3.4. *Let p be a prime integer and k a nonnegative integer, and let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d whose δ-poynomial is $1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(p^2-1)(k+1)}$. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then one of the followings is satisfied:*

(1) $d = p^2(k+1)-1$;
(2) $d = p^2(k+1)+(p-1)(k+1)-1$;
(3) $d = p^2(k+1)+p(k+1)-1$.

Moreover, in each case, a system of generators of the finite abelian group Λ_Δ is the set of row vectors of the matrix which can be written up to permutation of the columns as follows:

(1) $(1/p^2\ \cdots\ 1/p^2) \in (\mathbb{R}/\mathbb{Z})^{1\times p^2(k+1)}$;

(2) $\left(\underbrace{1/p\ \cdots\ 1/p}_{(p^2-1)(k+1)}\ \underbrace{1/p^2\ \cdots\ 1/p^2}_{p(k+1)}\right) \in (\mathbb{R}/\mathbb{Z})^{1\times(p^2+p-1)(k+1)}$;

(3) $\begin{pmatrix} 1/p\ \cdots\ 1/p & 0\ \cdots\ 0 \\ \underbrace{0\ \cdots\ 0}_{p(k+1)} & \underbrace{1/p\ \cdots\ 1/p}_{p^2(k+1)} \end{pmatrix} \in (\mathbb{R}/\mathbb{Z})^{2\times p(p+1)(k+1)}$.

Theorem 3.5. *Let p and q be prime integers with $p \neq q$ and k a nonnegative integer, and let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d whose δ-poynomial is $1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(pq-1)(k+1)}$. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then one of the followings is satisfied:*

(1) $d = pq(k+1)-1$;
(2) $d = pq(k+1)+p(k+1)-1$;
(3) $d = pq(k+1)+q(k+1)-1$;
(4) $d = pq(k+1)+(p-1)(k+1)-1$;
(5) $d = pq(k+1)+(q-1)(k+1)-1$.

Moreover, in each case, the finite abelian group Λ_Δ is generated by one element which can be written up to permutation of the coordinates as follows:

(1) $(1/(pq),\ldots,1/(pq)) \in (\mathbb{R}/\mathbb{Z})^{pq(k+1)}$;

(2) $\left(\underbrace{1/p,\ldots,1/p}_{p(k+1)},\underbrace{1/q,\ldots,1/q}_{pq(k+1)}\right) \in (\mathbb{R}/\mathbb{Z})^{p(q+1)(k+1)}$;

(3) $\left(\underbrace{1/q,\ldots,1/q}_{q(k+1)},\underbrace{1/p,\ldots,1/p}_{pq(k+1)}\right) \in (\mathbb{R}/\mathbb{Z})^{(p+1)q(k+1)}$;

(4) $\left(\underbrace{1/q,\ldots,1/q}_{(pq-1)(k+1)},\underbrace{1/(pq),\ldots,1/(pq)}_{p(k+1)}\right) \in (\mathbb{R}/\mathbb{Z})^{(pq+p-1)(k+1)}$;

(5) $\left(\underbrace{1/p,\ldots,1/p}_{(pq-1)(k+1)},\underbrace{1/(pq),\ldots,1/(pq)}_{q(k+1)}\right) \in (\mathbb{R}/\mathbb{Z})^{(pq+q-1)(k+1)}$.

Remark 3.6. The lattice simplices in Theorems 3.4 and 3.5 can be constructed by Propositions 3.2 and 3.3.

Before proving these theorems, we give the vertex representations of Gorenstein simplices in Theorems 3.4 and 3.5. Given a sequence $A = (a_1,\ldots,a_d)$ of integers, let $\Delta(A) \subset$

$\mathbb{R}^d$ be the convex hull of the origin of $\mathbb{R}^d$ and all row vectors of the following matrix:

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ a_d - a_1 & \cdots & a_d - a_{d-1} & a_d \end{pmatrix} \in \mathbb{Z}^{d \times d},$$

where the rest entries are all 0. Given sequences $B = (b_1, \ldots, b_s)$ and $C = (c_1, \ldots, c_d)$ of integers with $1 \leq s < d$, let $\Delta(B,C) \subset \mathbb{R}^d$ be the convex hull of the origin of $\mathbb{R}^d$ and all row vectors of the following matrix:

$$\begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ b_s - b_1 & \cdots & b_s - b_{s-1} & b_s & & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ c_d - c_1 & \cdots & \cdots & \cdots & \cdots & \cdots & c_d - c_{d-1} & c_d \end{pmatrix} \in \mathbb{Z}^{d \times d},$$

where the rest entries are all 0.

Corollary 3.7. *Let p be a prime integer and k a nonnegative integer, and let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d whose δ-poynomial is $1 + t^{k+1} + t^{2(k+1)} + \cdots + t^{(p^2-1)(k+1)}$. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then Δ is unimodularly equivalent to one of $\Delta(A_1)$, $\Delta(A_2)$ and $\Delta(B,C)$, where*

(1) $A_1 = (\underbrace{1, \ldots, 1}_{p^2(k+1)-2}, p^2)$;

(2) $A_2 = \left(\underbrace{1, \ldots, 1}_{p(k+1)-1}, \underbrace{p, \ldots, p}_{(p^2-1)(k+1)-1}, p^2\right)$;

(3) $B = \left(\underbrace{1, \ldots, 1}_{p(k+1)-1}, p\right)$, $C = \left(\underbrace{p, \ldots, p}_{p(k+1)}, \underbrace{1, \ldots, 1}_{p^2(k+1)-2}, p\right)$.

Corollary 3.8. *Let p and q be prime integers with $p \neq q$ and k a nonnegative integer, and let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d whose δ-poynomial is $1 + t^{k+1} + t^{2(k+1)} + \cdots + t^{(pq-1)(k+1)}$. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then Δ is unimodularly equivalent to one of $\Delta(A_1)$, $\Delta(A_2)$, $\Delta(A_3)$, $\Delta(B_1,C_1)$ and $\Delta(B_2,C_2)$, where*

(1) $A_1 = (\underbrace{1, \ldots, 1}_{pq(k+1)-2}, pq)$;

(2) $B_1 = \left(\underbrace{1,\ldots,1}_{p(k+1)-1}, p\right)$, $C_1 = \left(\underbrace{q,\ldots,q}_{p(k+1)}, \underbrace{1,\ldots,1}_{pq(k+1)-2}, q\right)$;

(3) $B_2 = \left(\underbrace{1,\ldots,1}_{q(k+1)-1}, q\right)$, $C_2 = \left(\underbrace{p,\ldots,p}_{q(k+1)}, \underbrace{1,\ldots,1}_{pq(k+1)-2}, p\right)$;

(4) $A_2 = \left(\underbrace{1,\ldots,1}_{p(k+1)-1}, \underbrace{p,\ldots,p}_{(pq-1)(k+1)-1}, pq\right)$;

(5) $A_3 = \left(\underbrace{1,\ldots,1}_{q(k+1)-1}, \underbrace{q,\ldots,q}_{(pq-1)(k+1)-1}, pq\right)$.

In order to prove Theorems 3.4 and 3.5, we use the following lemma.

Lemma 3.9. *Let v be a positive integer and k a nonnegative integer, and let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d whose δ-polynomial equals $1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(v-1)(k+1)}$. Assume that $\mathbf{x} \in (\mathbb{R}/\mathbb{Z})^{d+1}$ is an element of Λ_Δ such that* $\mathrm{ht}(\mathbf{x}) = k+1$ *and set* $m = \mathrm{ord}(\mathbf{x})$. *Then by reordering the coordinates, we obtain* $\mathbf{x} = \left(\underbrace{1/m,\ldots,1/m}_{s}, \underbrace{0,\ldots,0}_{d-s+1}\right)$ *for some positive integer s.*

Proof. Since $m = \mathrm{ord}(\mathbf{x})$, $\mathbf{x}$ must be of a form $(k_1/m,\ldots,k_s/m,0,\ldots,0)$ for a positive integer s and integers $1 \le k_1,\ldots,k_s \le m-1$ by reordering the coordinates. If there exists an integer $k_i \ge 2$ for some $1 \le i \le s$, then one has $k_i(m-1)/m \ge 1$. Therefore, we obtain $\mathrm{ht}((m-1)\mathbf{x}) < (m-1)\mathrm{ht}(\mathbf{x}) = (m-1)(k+1)$. Since $m = \mathrm{ord}(\mathbf{x})$, $(m-1)\mathbf{x}$ is different from $\mathbf{0},\mathbf{x},\ldots,(m-2)\mathbf{x}$. We remark that for any $\mathbf{a}$, $\mathbf{b} \in (\mathbb{R}/\mathbb{Z})^{d+1}$, one has $\mathrm{ht}(\mathbf{a}+\mathbf{b}) \le \mathrm{ht}(\mathbf{a}) + \mathrm{ht}(\mathbf{b})$. This fact and the supposed δ-polynomial imply that $\mathrm{ht}(t\mathbf{x}) = t\,\mathrm{ht}(\mathbf{x}) = t(k+1)$ for any $1 \le t \le m-1$. This is a contradiction, as desired. □

Finally, we prove Theorem 3.4 and Theorem 3.5.

Proof of Theorem 3.4. By Lemma 1.3, Δ is unimodularly equivalent to either Δ_1 or Δ_2, where Δ_1 and Δ_2 are lattice simplices such that each system of generators of Λ_{Δ_1} and Λ_{Δ_2} is the set of vectors of matrix as follows:

(i) $\left(\underbrace{1/p \cdots 1/p}_{d-s+1}\ \underbrace{1/p^2 \ldots 1/p^2}_{s}\right) \in (\mathbb{R}/\mathbb{Z})^{1\times(d+1)}$;

(ii) $\begin{pmatrix} (a_0+1)/p & \cdots & (a_{d-2}+1)/p & 0 & 1/p \\ (p-a_0)/p & \cdots & (p-a_{d-2})/p & 1/p & 0 \end{pmatrix} \in (\mathbb{R}/\mathbb{Z})^{2\times(d+1)}$,

where s is a positive integer and $0 \le a_0,\ldots,a_{d-2} \le p-1$ are integers.

At first, we assume that Δ is unimodularly equivalent to Δ_1. If $s = d+1$, then one has $(d+1)/p^2 = k+1$, hence, $d = p^2(k+1)-1$. This is the case (1). Now, we suppose that

$s \neq d+1$. Let $\mathbf{x}$ be an element of Λ_{Δ_1} with $\mathrm{ht}(\mathbf{x}) = k+1$. Then by Lemma 3.9, one has $\mathbf{x} = \left(\underbrace{0,\dots,0}_{d-s+1},\underbrace{1/p,\dots,1/p}_{s}\right)$, hence $s = p(k+1)$. Set $\mathbf{y} = \left(\underbrace{1/p,\dots,1/p}_{d-s+1},\underbrace{1/p^2,\dots,1/p^2}_{s}\right)$. Since for any $1 \leq m \leq p-1$, $\mathrm{ht}(m\mathbf{x}) = m(k+1)$, we have $\mathrm{ht}(\mathbf{y}) = p(k+1)$. Hence it follows that $d-s+1 = p^2(k+1)-(k+1)$, namely, $d = p^2(k+1)+(p-1)(k+1)-1$. This is the case (2).

Next, we assume that Δ is unimodularly equivalent to Δ_2. By Lemma 3.9, it follows that for any $0 \leq i \leq d-2$, $a_i \in \{0, p-1\}$. Hence by reordering the coordinates of Λ_{Δ_2}, we can assume that Λ_{Δ_2} is generated by

$$\mathbf{x}_1 = \left(\underbrace{1/p,\dots,1/p}_{s},\underbrace{0,\dots,0}_{d-s+1}\right), \mathbf{x}_2 = \left(\underbrace{0,\dots,0}_{s},\underbrace{1/p,\dots,1/p}_{d-s+1}\right),$$

where $1 \leq s \leq \lfloor (d+1)/2 \rfloor$. Then since $\mathrm{ht}(\mathbf{x}_1) = k+1$, one has $s = p(k+1)$. Moreover, since $\mathrm{ht}(\mathbf{x}_2) = p(k+1)$, we have $d-s+1 = p^2(k+1)$, namely, $d = p^2(k+1)+p(k+1)-1$. Therefore, This is the case (3).

Conversely, in each case, it is easy to show that $\delta(\Delta,t) = 1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(p^2-1)(k+1)}$, as desired. $\square$

Proof of Theorem 3.5. By Lemma 1.4, we can suppose that Λ_Δ is generated by

$$\mathbf{x} = \left(\underbrace{1/p,\dots,1/p}_{s_1},\underbrace{1/q,\dots,1/q}_{s_2},\underbrace{1/(pq),\dots,1/(pq)}_{s_3}\right),$$

where $s_1+s_2+s_3 = d+1$ with nonnegative integers s_1, s_2, s_3. If $s_1 = s_2 = 0$, since $\mathrm{ht}(\mathbf{x}) = k+1$, one has $d = pq(k+1)-1$. This is the case (1). If $s_3 = 0$, we can assume that Λ_Δ is generated by

$$\mathbf{x}_1 = \left(\underbrace{1/p,\dots,1/p}_{s_1},\underbrace{0,\dots,0}_{s_2}\right), \mathbf{x}_2 = \left(\underbrace{0,\dots,0}_{s_1},\underbrace{1/q,\dots,1/q}_{s_2}\right),$$

with $s_1, s_2 > 0$. Then it follows that $\mathrm{ht}(\mathbf{x}_1) = k+1$ and $\mathrm{ht}(\mathbf{x}_2) = p(k+1)$, or $\mathrm{ht}(\mathbf{x}_1) = q(k+1)$ and $\mathrm{ht}(\mathbf{x}_2) = k+1$. Assume that $\mathrm{ht}(\mathbf{x}_1) = k+1$ and $\mathrm{ht}(\mathbf{x}_2) = p(k+1)$. Then one has $s_1 = p(k+1)$ and $s_2 = pq(k+1)$. Hence since $d = pq(k+1)+p(k+1)-1$, this is the case (2). Similarly, we can show the case (3).

Next we suppose that $s_1, s_2, s_3 > 0$. Let $\mathbf{a}$ be an element of Λ_Δ such that $\mathrm{ht}(\mathbf{a}) = k+1$. By Lemma 3.9, we know that $\mathrm{ord}(\mathbf{a}) \neq pq$. Hence, it follows that $\mathrm{ord}(\mathbf{a})$ equals p or q. Now we assume that $\mathrm{ord}(\mathbf{a}) = p$. By Lemma 3.9 again, $\mathbf{a}$ must be of a form $\left(\underbrace{1/p,\dots,1/p}_{s_1},\underbrace{0,\dots,0}_{s_2},\underbrace{1/p,\dots,1/p}_{s_3}\right)$. Let $\mathbf{b} = (b_1,\dots,b_{d+1})$ be an element of Λ_Δ such that $\mathrm{ht}(\mathbf{b}) = p(k+1)$. If there exists an index $1 \leq i \leq s_1$ such that $b_i = n/p$ with an integer $1 \leq n \leq p-1$, then $\mathrm{ht}(\mathbf{b}+(p-1)\mathbf{a}) < \mathrm{ht}(\mathbf{b})+(p-1)\mathrm{ht}(\mathbf{a})$. Since $\mathbf{b}+(p-1)\mathbf{a}$ is different from $\mathbf{0}, \mathbf{a}, 2\mathbf{a}, \dots, (p-1)\mathbf{a}, \mathbf{b}, \mathbf{b}+\mathbf{a}, \dots, \mathbf{b}+(p-2)\mathbf{a}$, this contradicts to that

$\delta_\Delta(t) = 1 + t^{k+1} + t^{2(k+1)} + \cdots + t^{(pq-1)(k+1)}$. Hence one obtains $b_i = 0$ for any $1 \le i \le s_1$. Therefore, we can assume that $\mathbf{b} = \left(\underbrace{0,\ldots,0}_{s_1},\underbrace{\ell/q,\ldots,\ell/q}_{s_2},\underbrace{m/q,\ldots,m/q}_{s_3}\right)$ for some positive integers ℓ, m. Then whenever $(g_1,h_1) \neq (g_2,h_2)$ with $0 \le g_1, g_2 \le p-1$ and $0 \le h_1, h_2 \le q-1$, $g_1\mathbf{a} + h_1\mathbf{b}$ and $g_2\mathbf{a} + h_2\mathbf{b}$ are different elements of Λ_Δ. Hence since $\delta_\Delta(t) = 1 + t^{k+1} + t^{2(k+1)} + \cdots + t^{(pq-1)(k+1)}$, one has

$$\mathrm{ht}(g\mathbf{a} + h\mathbf{b}) = g\mathrm{ht}(\mathbf{a}) + h\mathrm{ht}(\mathbf{b})$$

for any $0 \le g \le p-1$ and $0 \le h \le q-1$. This implies that $\ell = m = 1$. However since $(p-1)/p + (q-1)/q > 1$, we have $\mathrm{ht}((p-1)\mathbf{a} + (q-1)\mathbf{b}) < (p-1)\mathrm{ht}(\mathbf{a}) + (q-1)\mathrm{ht}(\mathbf{b})$, a contradiction. Therefore, it does not follow $s_1, s_2, s_3 > 0$.

Finally, we assume that $s_1 = 0$ and $s_2 > 0$. Then one has $\mathrm{ht}(q\mathbf{x}) = k+1$, hence, $s_3 = p(k+1)$. Moreover, since $\mathrm{ht}(\mathbf{x}) = p(k+1)$, we obtain $s_2 = (pq-1)(k+1)$. Therefore, this is the case (4). Similarly, we can show the case (5).

Conversely, in each case, it is easy to see that $\delta(\Delta,t) = 1 + t^{k+1} + t^{2(k+1)} + \cdots + t^{(pq-1)(k+1)}$, as desired. □

4. The number of Gorenstein simplices

In [5, Section 4], we asked how many reflexive polytopes which have the same δ-polynomial exist. Analogy to this question, in this section, we consider how many Gorenstein simplices which have a given δ-polynomial of Problem 0.1.

Given integers $v \ge 1$ and $k \ge 0$, let $N(v,k)$ denote the number of Gorenstein simplices, up to unimodular equivalence, which are not lattice pyramids over any lower-dimensional lattice simplex and whose δ-polynomials equal $1 + t^{k+1} + t^{2(k+1)} + \cdots + t^{(v-1)(k+1)}$. For example, from Proposition 3.1, $N(p,k) = 1$ for any prime integer p. Moreover, from Theorems 3.4 and 3.5, $N(p^2,k) = 3$ and $N(pq,k) = 5$ for any distinct prime integers p and q. However, in other case, it is hard to determine $N(v,k)$. Therefore, our aim of this section is to construct more examples of Gorenstein simplices of Problem 0.1 and to give a lower bound on $N(v,k)$.

The following theorem gives us more examples of Gorenstein simplces of Problem 0.1.

Theorem 4.1. *Given a positive integer v, let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d such that Λ_Δ is generated by*

$$\left(\underbrace{1/v_1,\ldots,1/v_1}_{s_1},\underbrace{1/v_2,\ldots,1/v_2}_{s_2},\ldots,\underbrace{1/v_t,\ldots,1/v_t}_{s_t}\right) \in (\mathbb{R}/\mathbb{Z})^{d+1},$$

where $1 < v_1 < \cdots < v_t = v$ and for any $1 \le i \le t-1$, $v_i \mid v_{i+1}$ and $s_1,\ldots,s_t$ are positive integers. Then $\delta(\Delta,t) = 1 + t^{k+1} + t^{2(k+1)} + \cdots + t^{(v-1)(k+1)}$ with a nonnegative integer k if and only if

$$s_i = \begin{cases} \left(\dfrac{v_t}{v_{i-1}} - \dfrac{v_t}{v_{i+1}}\right)(k+1), & 1 \le i \le t-1 \\ \dfrac{v_t}{v_{t-1}}(k+1), & i = t, \end{cases}$$

where $v_0 = 1$.

Proof. Let

$$\mathbf{x}_0 = \Bigg(\underbrace{1/v_1,\dots,1/v_1}_{s_1},\underbrace{1/v_2,\dots,1/v_2}_{s_2},\dots,\underbrace{1/v_t,\dots,1/v_t}_{s_t}\Bigg) \in (\mathbb{R}/\mathbb{Z})^{d+1},$$

and for $i = 1,\dots,t-1$, we set $\mathbf{x}_i = v_i\mathbf{x}_0$. Then it follows that

$$\Lambda_\Delta = \left\{\sum_{i=0}^{t-1} c_i\mathbf{x}_i : c_i \in \mathbb{Z}_{\geq 0}, 0 \leq c_i \leq v_{i+1}/v_i - 1 \text{ for } i = 0,\dots,t-1\right\}.$$

Moreover, we obtain $\mathrm{ht}(\mathbf{x}_i) = \sum_{j=1}^{t-i} \dfrac{v_i}{v_{i+j}} s_{i+j}$ for $i = 0,\dots,t-1$. Since

$$\mathrm{ht}(\mathbf{x}_i) = \mathrm{ht}\Big(\frac{v_i}{v_{i-1}}\mathbf{x}_{i-1}\Big) = \frac{v_i}{v_{i-1}}\mathrm{ht}(\mathbf{x}_{i-1}) - s_i$$

for any $1 \leq i \leq t-1$, it follows that for any $1 \leq i \leq t-1$, $s_i = \Big(\dfrac{v_t}{v_{i-1}} - \dfrac{v_t}{v_{i+1}}\Big)(k+1)$ and $s_t = \dfrac{v_t}{v_{t-1}}(k+1)$ if and only if for any $0 \leq i \leq t-1$, $\mathrm{ht}(\mathbf{x}_i) = \dfrac{v_t}{v_{i+1}}(k+1)$. Hence we should prove that $\delta(\Delta,t) = 1 + t^{k+1} + t^{2(k+1)} + \cdots + t^{(v-1)(k+1)}$ if and only if for any $0 \leq i \leq t-1$, $\mathrm{ht}(\mathbf{x}_i) = \dfrac{v_t}{v_{i+1}}(k+1)$.

At first, we assume that $\delta(\Delta,t) = 1 + t^{k+1} + t^{2(k+1)} + \cdots + t^{(v-1)(k+1)}$. By Lemma 3.9, one has $\mathrm{ht}(\mathbf{x}_{t-1}) = k+1$. Suppose that for any $n \leq i \leq t-1$, $\mathrm{ht}(\mathbf{x}_i) = \dfrac{v_t}{v_{i+1}}(k+1)$ with an integer $1 \leq n \leq t-1$. Then since $\mathrm{ht}(\sum_{i=n}^{t-1}(v_{i+1}/v_i - 1)\mathbf{x}_i) = (v_t/v_n - 1)(k+1)$, there exists an integer m with $0 \leq m \leq n-1$ such that $\mathrm{ht}(\mathbf{x}_m) = \dfrac{v_t}{v_n}(k+1)$. Now, we assume that $m < n-1$. Set

$$\Lambda' = \left\{c_m\mathbf{x}_m + \sum_{i=n}^{t-1} c_i\mathbf{x}_i : 0 \leq c_i \leq v_{i+1}/v_i - 1 \text{ for } i = m,n,n+1,\dots,t-1\right\}.$$

Then one has $\{\mathrm{ht}(\mathbf{x}) : \mathbf{x} \in \Lambda'\} = \{j(k+1) : j = 0,\dots,(v_{m+1}v_t)/(v_m v_n) - 1\}$. However,

$$\mathrm{ht}(\mathbf{x}_{m+1}) = \mathrm{ht}\Big(\frac{v_{m+1}}{v_m}\mathbf{x}_m\Big) < \frac{v_{m+1}}{v_m}\mathrm{ht}(\mathbf{x}_m) = \Big(\frac{v_{m+1}v_t}{v_m v_n}\Big)(k+1).$$

and $\mathbf{x}_{m+1}$ is not in Λ', a contradiction. Hence we obtain $\mathrm{ht}(\mathbf{x}_{i-1}) = \dfrac{v_t}{v_i}(k+1)$ for any $0 \leq i \leq t-1$.

Conversely, we assume that for any $0 \leq i \leq t-1$, $\mathrm{ht}(\mathbf{x}_i) = \dfrac{v_t}{v_{i+1}}(k+1)$. Since for any c_i with $0 \leq c_i \leq v_{i+1}/v_i - 1$, $\mathrm{ht}(\sum_{i=0}^{t-1} c_i\mathbf{x}_i) = \sum_{i=0}^{t-1} c_i\mathrm{ht}(\mathbf{x}_i)$, one has $\delta(\Delta,t) = 1 + t^{k+1} + t^{2(k+1)} + \cdots + t^{(v-1)(k+1)}$, as desired. □

By Theorems 4.1 and [12, Theorem 2.2], we can answer to Problem 0.1 when v is a power of a prime integer and the associated finite abelian group is cyclic, namely, it is generated by one element.

Corollary 4.2. *Let p be a prime integer, ℓ a positive integer and k a nonnegative integer, and let $\Delta \subset \mathbb{R}^d$ be a lattice simplex of dimension d such that Λ_Δ is cyclic and $\delta(\Delta,t) = 1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(p^\ell-1)(k+1)}$. Suppose that Δ is not a lattice pyramid over any lower-dimensional lattice simplex. Then there exist positive integers $0<\ell_1<\cdots<\ell_t=\ell$ and $s_1,\ldots,s_t$ such that the following conditions are satisfied:*

- *It follows that*
$$s_i = \begin{cases} (p^{\ell-\ell_{i-1}} - p^{\ell-\ell_{i+1}})(k+1), & 1 \le i \le t-1 \\ p^{\ell-\ell_{t-1}}(k+1), & i = t, \end{cases}$$
where $\ell_0 = 0$;
- *Λ_Δ is generated by*
$$\left(\underbrace{1/p^{\ell_1},\ldots,1/p^{\ell_1}}_{s_1},\underbrace{1/p^{\ell_2},\ldots,1/p^{\ell_2}}_{s_2},\ldots,\underbrace{1/p^{\ell_t},\ldots,1/p^{\ell_t}}_{s_t}\right) \in (\mathbb{R}/\mathbb{Z})^{d+1}$$
for some ordering of the vertices of Δ.

The lattice simplices in Theorems 3.4 and 3.5 can be constructed by Propositions 3.2 and 3.3. We conjecture the following.

Conjecture 4.3. *Given integers $v \ge 1$ and $k \ge 0$, let $\Delta \subset \mathbb{R}^d$ be a Gorenstein simplex of dimension d with the δ-polynomial $1+t^{k+1}+\cdots+t^{(v-1)(k+1)}$ which is not a lattice pyramid over any lower-dimensional lattice simplex. Then Δ can be constructed by repeatedly using Proposition 3.3 and Theorem 4.1.*

Now, we consider to give a lower bound on $N(v,k)$. Given integers $v \ge 1$ and $k \ge 0$, let $M(v,k)$ denote the number of Gorenstein simplices, up to unimodular equivalence, which are appeared in Theorem 4.1. Then one has $N(v,k) \ge M(v,k)$. By Theorem 4.1, we can determine $M(v,k)$ in terms of the divisor lattice of v. Given a positive integer v, let D_v the set of all divisors of v, ordered by divisibility. Then D_v is a partially ordered set, in particular, a lattice, called the *divisor lattice* of v. We call subset $C \subset D_v$ a *chain* of D_v if C is a totally ordered subset with respect to the induced order.

Corollary 4.4. *Let v be a positive integer and k a nonnegative integer. Then $M(v,k)$ equals the number of chains from a non-least element to the greatest element in D_v. In particular, one has $M(v,k) = \sum_{n \in D_v \setminus \{v\}} M(n,k)$.*

This corollary says that $M(v,k)$ depends on only the divisor lattice D_v. In particular, letting $v = p_1^{a_1}\cdots p_t^{a_t}$ with distinct prime integers $p_1,\ldots,p_t$ and positive integers $a_1,\ldots,a_t$, $M(v,k)$ depends on only $(a_1,\ldots,a_t)$.

Finally, we give examples of $M(v,k)$.

Example 4.5. (1) Let $v = p^\ell$ with a prime integer p and a positive integer ℓ. Then from Corollary 4.4, we know that $M(v,k)$ equals the number of subsets of $\{1,\ldots,\ell-1\}$. Hence one has $M(v,k) = 2^{\ell-1}$.

(2) Let $v = p_1\cdots p_t$, where $p_1,\ldots,p_t$ are distinct prime integers. From Corollary 4.4, we know that $M(v,k)$ depends on only t. Now, let $a(t) = M(v,k)$, where we define $a(0) =$

$M(1,k) = 1$. Then one has

$$a(t) = M(v,k) = \sum_{n \in D_v \setminus \{v\}} M(n,k) = 1 + \sum_{i=1}^{t-1} \binom{t}{i} M(p_1 \cdots p_i, k) = \sum_{i=0}^{t-1} \binom{t}{i} a(i).$$

We remark that $a(t)$ is the well-known recursive sequence ([11, A000670]) which is called the *ordered Bell numbers* or *Fubini numbers*.

REFERENCES

[1] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, *J. Algebraic Geom.* **3**(1994), 493–535.

[2] V. V. Batyrev and J. Hofscheier, Lattice polytopes, finite abelian subgroups in $\mathrm{SL}(n,\mathbb{C})$ and coding theory, arXiv:1309.5312.

[3] V. Batyrev and D. Juny, Classification of Gorenstein toric del Pezzo varieties in arbitrary dimension, *Mosc. Math. J.*, **10**(2010), 285–316.

[4] T. Hibi, "Algebraic Combinatorics on Convex Polytopes," Carslaw Publications, Glebe NSW, Australia, 1992.

[5] T. Hibi and A. Tsuchiya, Facets and volume of Gorenstein Fano polytopes, *Math. Nachr.*, to appear.

[6] A. Higashitani, B. Nill and A. Tsuchiya, Gorenstein polytopes with trinomial h^*-polynomials, arXiv:1503.05685.

[7] M. Joswig and K. Kulas, Tropical and ordinary convexity combined, *Adv. Geom.* **10**(2010), 335-352.

[8] M. Kreuzer and H. Skarke, Complete classification of reflexive polyhedra in four dimensions, *Adv. Theor. Math. Phys.* **4**(2000), 1209–1230.

[9] J. C. Lagarias and G. M. Ziegler, Bounds for lattice polytopes containing a fixed number of interior points in a sublattice, *Canad. J. Math.* **43**(1991), 1022–1035.

[10] B. Nill, Lattice polytopes having h^*-polynomials with given degree and linear coefficient, *European J. Combin.*, **29**, 1596–1602, 2008.

[11] N. J. A. Sloane, On-line encyclopedia of integer sequences, Webpage, `https://oeis.org/A000670`

[12] A. Tsuchiya, Gorenstein simplices and the associated finite abelian groups, arXiv:1702. 02704.

(Takayuki Hibi) DEPARTMENT OF PURE AND APPLIED MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY, OSAKA UNIVERSITY, SUITA, OSAKA 565-0871, JAPAN

E-mail address: `hibi@math.sci.osaka-u.ac.jp`

(Akiyoshi Tsuchiya) DEPARTMENT OF PURE AND APPLIED MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY, OSAKA UNIVERSITY, SUITA, OSAKA 565-0871, JAPAN

E-mail address: `a-tsuchiya@ist.osaka-u.ac.jp`

(Koutarou Yoshida) DEPARTMENT OF PURE AND APPLIED MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY, OSAKA UNIVERSITY, SUITA, OSAKA 565-0871, JAPAN

E-mail address: `kt-yoshida@ist.osaka-u.ac.jp`