EQUIDISTRIBUTIONS OF MAHONIAN STATISTICS OVER PATTERN AVOIDING PERMUTATIONS

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ABSTRACT. A Mahonian *d*-function is a Mahonian statistic that can be expressed as a linear combination of vincular pattern statistics of length at most *d*. Babson and Steingrímsson classified all Mahonian 3-functions up to trivial bijections and identified many of them with well-known Mahonian statistics in the literature. We prove a host of Mahonian 3-function equidistributions over pattern avoiding sets of permutations. Tools used include block decomposition, Dyck paths and generating functions.

1. INTRODUCTION

A combinatorial statistic on a set S is a map stat : $S \to \mathbb{N}$. The distribution of stat over S is given by the coefficients of the generating function $\sum_{\sigma \in S} q^{\text{stat}(\sigma)}$. Let S_n be the set of permutations $\sigma = a_1 a_2 \cdots a_n$ of the letters $[n] = \{1, 2, \ldots, n\}$ and let $\sigma(k)$ denote the entry a_k . Let $S = \bigcup_{n \ge 0} S_n$. The inversion set of $\sigma \in S_n$ is defined by $\text{Inv}(\sigma) = \{(i, j) : i < j \text{ and } \sigma(i) > \sigma(j)\}$. A particularly well-studied statistic on S_n is inv : $S_n \to \mathbb{N}$, given by $\text{inv}(\sigma) = |\text{Inv}(\sigma)|$. An elegant formula for the distribution of the inversion statistic was found in 1839 by Rodrigues [25]

$$\sum_{\sigma \in \mathcal{S}_n} q^{\mathrm{inv}(\sigma)} = [n]_q!$$

where $[n]_q! = [1]_q[2]_q \cdots [n]_q$ and $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$. The descent set of σ is defined by $\text{Des}(\sigma) = \{i : \sigma(i) > \sigma(i+1)\}$. In 1915 MacMahon [23] showed that inv has the same distribution as another statistic, now called the major index (due to MacMahon's profession as a major in the british army), given by $\text{maj}(\sigma) = \sum_{i \in \text{Des}(\sigma)} i$. We also write $\text{imaj}(\sigma) = \text{maj}(\sigma^{-1})$. In honor of MacMahon any permutation statistic with the same distribution as maj is called Mahonian. Mahonian statistics are well-studied in the literature. Since MacMahon's initial work many new Mahonian statistics have been identified. Babson and Steingrímsson[1] showed that almost all (at the time) known Mahonian statistics can be expressed as linear combinations of statistics counting occurrences of vincular patterns. They further made several conjectures regarding new vincular-pattern based Mahonian statistics. These have since been proved and reproved at various levels of refinement by a number of authors (see e.g. [16, 4, 31, 6]).

Two distinct sequences of integers $a_1a_2 \cdots a_n$ and $b_1b_2 \cdots b_n$ are said to be order isomorphic provided $a_i < a_j$ if and only if $b_i < b_j$ for all $1 \le i < j \le n$. A vincular pattern (also known as generalized pattern) of length m is a pair (π, X) where π is a permutation in S_m and $X \subseteq \{0\} \cup [m]$ is a set of adjacencies. Notationwise, adjacencies are indicated by underlining the adjacent entries in π (see Example 1.1). If $0 \in X$ $(m \in X)$, then we denote this by adding a square bracket at the beginning

(end) of the pattern π . If $X = \emptyset$, then (π, X) coincides with the definition of a classical pattern. A permutation $\sigma = a_1 a_2 \cdots a_n \in S_n$ contains the vincular pattern (π, X) if there is a *m*-tuple $1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n$ such that the following three criteria are satisfied

- $a_{i_1}a_{i_2}\cdots a_{i_m}$ is order-isomorphic to π , $i_{j+1} = i_j + 1$ for each $j \in X \setminus \{0, m\}$ and
- $i_1 = 1$ if $0 \in X$ and $i_m = n$ if $m \in X$.

We also say that $a_{i_1}a_{i_2}\cdots a_{i_m}$ is an *occurrence* of π in σ . We say that σ avoids π if σ contains no occurrences of π . We denote the set of permutations in S_n avoiding the pattern π by $\mathcal{S}_n(\pi)$. Moreover if Π is a set of patterns, then we set $\mathcal{S}_n(\Pi) = \bigcap_{\pi \in \Pi} \mathcal{S}_n(\pi).$

In this paper we shall also need an additional generalization of vincular patterns, allowing us to restrict occurrences to particular value requirements. Let $v = (v_1, \ldots, v_m)$ where $v_i \in \mathbb{N} \sqcup \{-\}$. Define a value restricted vincular pattern to be a triple (π, X, v) where (π, X) is a vincular pattern. We use the notation $(\pi, X)|_{v}$ to denote such a pattern. We say that $a_{i_1}a_{i_2}\cdots a_{i_m}$ is an occurrence of $(\pi, X)|_{v}$ in σ if it is an occurrence of the vincular pattern (π, X) and $a_{i_j} = v_j$ whenever $v_j \in \mathbb{N}$ for j = 1, ..., m. Note in particular that $(\pi, X)|_{(-,...,-)} = (\pi, X)$. Every value restricted vincular pattern $(\pi, X)|_{v}$ gives rise to a permutation statistic (denoted with a bracket around the pattern) counting the number of occurrences of $(\pi, X)|_{v}$ in σ (see Example 1.1).

Example 1.1. Let $\sigma = 246153$.

Pattern π	X	Occurrences in σ
231	Ø	241, 261, 461, 463, 453
[231	$\{0\}$	241,261
231	{1}	241,461,463
2 <u>31</u>	$\{2\}$	261, 461, 453
231	$\{1, 2\}$	461
2 <u>31</u>]	$\{2, 3\}$	453
$\underline{231} _{(-,6,-)}$	{1}	461, 463

We also have $(231)\sigma = 5$, $(231)\sigma = 2$, $(231)\sigma = 3$, $(231)\sigma = 3$, $(231)\sigma = 1$, $(2\underline{31}]\sigma = 1$ and $(\underline{231})|_{(-,6,-)}\sigma = 2$. On the other hand the permutation $\sigma = 215346$ avoids the pattern $\pi = 231$ (and hence all the patterns in the table above).

In this paper we mainly study equidistributions of the form

$$\sum_{\sigma \in \mathcal{S}_n(\Pi_1)} q^{\operatorname{stat}_1(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(\Pi_2)} q^{\operatorname{stat}_2(\sigma)}$$
(1.1)

where Π_1, Π_2 are sets of patterns and stat₁, stat₂ are permutation statistics. We will almost exclusively focus on the case where Π_i consists of a single classical pattern of length three and stat_i is a Mahonian statistic. Although Mahonian statistics are equidistributed over S_n , they need not be equidistributed over pattern avoiding sets of permutations. For instance maj and inv are not equidistributed over $\mathcal{S}_n(\pi)$ for any classical pattern $\pi \in \mathcal{S}_3$. Neither do the existing bijections in the literature for proving equidistribution over \mathcal{S}_n necessarily restrict to bijections over $\mathcal{S}_n(\pi)$. Therefore whenever such an equidistribution is present, we must usually seek a new bijection which simultaneously preserves statistic and pattern avoidance. Another motivation for studying equidistributions over permutations avoiding a classical pattern of length three, is that $|S_n(\pi)| = C_n$ for all $\pi \in S_3$ where $C_n = \frac{1}{n+1} {2n \choose n}$ is the *n*th Catalan number (see [20]). Therefore equidistributions of this kind induce equidistributions between statistics on other Catalan objects (and vice versa) whenever we have bijections where the statistics translate in an appropriate fashion. We prove several results in this vein where an exchange between statistics on $S_n(\pi)$, Dyck paths and polyominoes takes place. In general, studying the generating function (1.1) provides a rich source of interesting *q*-analogues to well-known sequences enumerated by pattern avoidance and raises new questions about the coefficients of such polynomials.

Equidistributions such as (1.1) have been studied in the past. For instance, Burstein and Elizable proved the following result involving the Mahonian *Denert statistic*

$$den(\sigma) = inv(Exc(\sigma)) + inv(NExc(\sigma)) + \sum_{\substack{i \in [n] \\ \sigma(i) > i}} i,$$

where $\operatorname{Exc}(\sigma) = (\sigma(i))_{\sigma(i) > i}$ and $\operatorname{NExc}(\sigma) = (\sigma(i))_{\sigma(i) \le i}$.

Theorem 1.1 (Burstein-Elizalde [5]). For any $n \ge 1$,

$$\sum_{\sigma \in \mathcal{S}_n(231)} q^{\operatorname{maj}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(321)} q^{\operatorname{den}(\sigma)}$$

Two sets of patterns Π_1 and Π_2 are said to be *Wilf-equivalent* if $|S_n(\Pi_1)| = |S_n(\Pi_2)|$ for all $n \ge 0$. Sagan and Savage [26] coined a *q*-analogue of this concept. Two sets of patterns Π_1 and Π_2 are said to be *st-Wilf equivalent* with respect to the statistic st : $S \to \mathbb{N}$ if (1.1) holds with stat₁ = st = stat₂ for any $n \ge 0$. Let $[\Pi]_{st}$ denote the st-Wilf class of the set Π . This concept have been studied explicitly and implicitly at several places in the literature. An overview of the st-Wilf classification of single and multiple classical patterns of length three may be found in the table below.

\mathbf{st}	Reference					
maj, inv	Dokos-Dwyer-Johnson-Sagan-Selsor [13]					
charge	Killpatrick [18]					
${ m fp, exc, des}$	Elizalde [14, 15]					
peak, valley	Baxter $[2]$					
peak, valley, head, last, lir, rir, lrmin, rank, comp, ldr	Claesson-Kitaev [10]					

In particular it was shown in [13] that $I_n(132;q) = I_n(213;q) = C_n(q)$ and $I_n(231;q) = I_n(312;q) = \tilde{C}_n(q)$ where

$$I_n(\pi;q) = \sum_{\sigma \in S_n(\pi)} q^{\text{inv}(\sigma)},$$

$$C_n(q) = \sum_{k=0}^{n-1} q^{(k+1)(n-k)} C_k(q) C_{n-k-1}(q), \quad C_0(q) = 1,$$

$$\tilde{C}_n(q) = \sum_{k=0}^{n-1} q^k \tilde{C}_k(q) \tilde{C}_{n-k-1}(q), \quad \tilde{C}_0(q) = 1.$$

The polynomial $C_n(q)$ is known as the *Carlitz-Riordan q-analogue of the Catalan* numbers and have been studied by numerous authors (though no explicit formula is known). Similar recursions for maj have been studied in [13, 7].

To decompose pattern avoiding permutations we will require some effective notation. Given permutations $\tau \in S_k$ and $\sigma_1, \sigma_2, \ldots, \sigma_k \in S$, the *inflation* of τ by $\sigma_1, \sigma_2, \ldots, \sigma_k$ is the permutation $\tau[\sigma_1, \sigma_2, \ldots, \sigma_k]$ obtained by replacing each entry $\tau(i)$ by a block of length $|\sigma_i|$ order isomorphic to σ_i for $i = 1, \ldots, k$ such that the blocks are externally order-isomorphic to τ .

Example 1.2. 231[21, 1, 213] = 546213.

Let $\sigma \in S_n$. Recall that the descent set of σ is given by $\text{Des}(\sigma) = \{i : \sigma(i) > \sigma(i+1)\}$. The set of descent bottoms and descent tops of σ are given respectively by $\text{DB}(\sigma) = \{\sigma(i+1) : i \in \text{Des}(\sigma)\}$ and $\text{DT}(\sigma) = \{\sigma(i) : i \in \text{Des}(\sigma)\}$. Likewise the ascent set of σ is given by $\text{Asc}(\sigma) = \{i : \sigma(i) < \sigma(i+1)\}$ and we define the set of ascent bottoms and ascent tops of σ to be $\text{AB}(\sigma) = \{\sigma(i) : i \in \text{Asc}(\sigma)\}$ and $\text{AT}(\sigma) = \{\sigma(i+1) : i \in \text{Asc}(\sigma)\}$ respectively. An entry $\sigma(j)$ is called a left-to-right maxima if $\sigma(j) > \sigma(i)$ for all i < j. Let $\text{LRMax}(\sigma)$ denote the set of left-to-right minima if $\sigma(j) < \sigma(i)$ for all i < j. Let $\text{LRMin}(\sigma)$ denote the set of left-to-right minima in σ and let $\text{Irmin}(\sigma) = |\text{LRMin}(\sigma)|$. We call $\sigma(i)$ a peak if $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ and $\sigma(i)$ a valley if $\sigma(i-1) > \sigma(i) < \sigma(i+1)$.

If $\sigma = a_1 a_2 \cdots a_{n-1} a_n$, then the reverse of σ is given by $\sigma^r = a_n a_{n-1} \cdots a_2 a_1$ and the complement of σ by $\sigma^c = (n - a_1 + 1)(n - a_2 + 1) \cdots (n - a_{n-1} + 1)(n - a_n + 1)$. The inverse of σ (in the group theoretical sense) is denoted by σ^{-1} . The operations complement, reverse and inverse are often referred to as trivial bijections and together they generate a group isomorphic to the Dihedral group D_4 of order 8 acting on S_n . If π is a classical pattern and $g \in D_4$, then it is not difficult to see that $\sigma \in S_n(\pi)$ if and only if $\sigma^g \in S_n(\pi^g)$. However if π is a non-classical pattern, then there is no such corresponding statement for inverse. Therefore taking the inverse should not be viewed as a 'trivial bijection' in the same sense as complement and reverse when it comes to vincular patterns.

In Table 1 we list the vincular pattern specifications of the Mahonian statistics that we shall consider from [1]. See the references in Table 1 for the original definitions of these statistics. According to [1], Table 1 is the complete list of *Mahonian 3-functions* (up to trivial bijections) i.e. Mahonian statistics that can be written as a sum of vincular pattern statistics of length at most three. Since some of these statistics have received no conventional name in the literature we will take the liberty of naming them according to the initials of the authors who first proved their Mahonity.

2. Equidistributions via block decomposition

The equidistributions proved in this section are shown by directly exhibiting a bijection. The bijections are based on standard decompositions of pattern avoiding permutations, or rely on specifying data by which pattern avoiding permutations

Name	Vincular pattern statistic	Reference					
maj	$(1\underline{32}) + (2\underline{31}) + (3\underline{21}) + (\underline{21})$	MacMahon [23]					
inv	$(\underline{23}1) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	MacMahon [23]					
mak	$(1\underline{32}) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	Foata-Zeilberger [17]					
makl	$(1\underline{32}) + (2\underline{31}) + (\underline{32}1) + (\underline{21})$	Clarke-Steingrímsson-Zeng [12]					
mad	$(2\underline{31}) + (2\underline{31}) + (\underline{31}2) + (\underline{21})$	Clarke-Steingrímsson-Zeng [12]					
bast	$(\underline{13}2) + (\underline{21}3) + (\underline{32}1) + (\underline{21})$	Babson-Steingrímsson[1]					
bast'	$(\underline{13}2) + (\underline{31}2) + (\underline{32}1) + (\underline{21})$	Babson-Steingrímsson[1]					
bast''	$(1\underline{32}) + (3\underline{12}) + (3\underline{21}) + (2\underline{1})$	Babson-Steingrímsson[1]					
foze	$(\underline{21}3) + (\underline{321}) + (\underline{13}2) + (\underline{21})$	Foata-Zeilberger [16]					
foze'	$(1\underline{32}) + (2\underline{31}) + (2\underline{31}) + (\underline{21})$	Foata-Zeilberger [16]					
$\mathrm{foze}^{\prime\prime}$	$(\underline{23}1) + (\underline{31}2) + (\underline{31}2) + (\underline{21})$	Foata-Zeilberger [16]					
sist	$(\underline{13}2) + (\underline{13}2) + (2\underline{13}) + (\underline{21})$	Simion-Stanton [26]					
sist'	$(\underline{13}2) + (\underline{13}2) + (2\underline{31}) + (\underline{21})$	Simion-Stanton [26]					
sist''	$(\underline{13}2) + (\underline{231}) + (\underline{231}) + (\underline{21})$	Simion-Stanton [26]					

TABLE 1. Mahonian 3-functions.

are uniquely determined. In many cases we are able to find a more refined equidistribution. We begin by proving that maj and mak are related via the inverse map over certain pattern avoiding sets of permutations. This may seem unexpected given that vincular patterns do not behave as straightforwardly under the inverse map as they do under complement and reverse.

Proposition 2.1. Let $\sigma \in S_n(\pi)$ where $\pi \in \{132, 213, 231, 312\}$. Then

$$mak(\sigma) = imaj(\sigma).$$

Moreover for any $n \geq 1$,

$$\sum_{\sigma\in\mathcal{S}_n(\pi)}q^{\operatorname{maj}(\sigma)}t^{\operatorname{des}(\sigma)} = \sum_{\sigma\in\mathcal{S}_n(\pi^{-1})}q^{\operatorname{mak}(\sigma)}t^{\operatorname{des}(\sigma)}$$

Proof. Let $\sigma \in S_n(132)$. We argue by induction on n. Consider the inflation form $\sigma = 231[\sigma_1, 1, \sigma_2]$ where $\sigma_1 \in S_k(132)$ and $\sigma_2 \in S_{n-k-1}(132)$. Suppose $\sigma_2 = \emptyset$. Then σ and hence σ^{-1} ends with n. Since neither maj nor mak counts any occurrences ending with n we may restrict attention to σ_1 , so by induction

$$mak(\sigma) = mak(\sigma_1) = imaj(\sigma_1) = imaj(\sigma).$$

We may therefore consider a decomposition $\sigma = 21[\sigma_1, \sigma_2]$ where $\sigma_1, \sigma_2 \neq \emptyset$. Then maj and mak satisfy the recursions

$$\begin{split} \mathrm{maj}(\sigma) &= |\sigma_1| + |\sigma_1| \operatorname{des}(\sigma_2) + \mathrm{maj}(\sigma_1) + \mathrm{maj}(\sigma_2), \\ \mathrm{mak}(\sigma) &= |\sigma_2| + |\sigma_2| \operatorname{des}(\sigma_1) + \mathrm{mak}(\sigma_1) + \mathrm{mak}(\sigma_2), \end{split}$$

respectively. By an inductive argument it is easy to see that $\operatorname{des}(\tau^{-1}) = \operatorname{des}(\tau)$ for all $\tau \in \mathcal{S}(132)$ by comparing the decompositions $\tau^{-1} = 231[\tau_2^{-1}, 1, \tau_1^{-1}]$ and

 $\tau = 231[\tau_1, 1, \tau_2]$. Hence by induction

 σ

$$\begin{aligned} \operatorname{imaj}(\sigma) &= \operatorname{maj}(21[\sigma_2^{-1}, \sigma_1^{-1}]) \\ &= |\sigma_2^{-1}| + |\sigma_2^{-1}| \operatorname{des}(\sigma_1^{-1}) + \operatorname{maj}(\sigma_1^{-1}) + \operatorname{maj}(\sigma_2^{-1}) \\ &= |\sigma_2| + |\sigma_2| \operatorname{des}(\sigma_1) + \operatorname{mak}(\sigma_1) + \operatorname{mak}(\sigma_2) \\ &= \operatorname{mak}(\sigma). \end{aligned}$$

The statement is proved similarly for remaining patterns and are therefore omitted. $\hfill \Box$

Remark 2.2. By Proposition 2.1 and [30, Corollary 4.1] it follows that

$$\sum_{\in \mathcal{S}_n(231)} q^{\operatorname{maj}(\sigma) + \operatorname{mak}(\sigma)} = \frac{1}{[n+1]_q} \begin{bmatrix} 2n\\n \end{bmatrix}_q$$
(2.1)

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$. The right hand side of (2.1) is known as *MacMahon's q*-analogue of the Catalan numbers [24].

The following lemma regarding the structure of $S_n(321)$ is part of folklore pattern avoidance (see e.g. [20]).

Lemma 2.3. We have $\sigma \in S_n(321)$ if and only if the elements of $[n] \setminus \text{LRMax}(\sigma)$ form an increasing subsequence of σ .

Theorem 2.4. For any $n \ge 1$,

$$\sum_{\sigma \in \mathcal{S}_n(321)} q^{\operatorname{maj}(\sigma)} \mathbf{x}^{\operatorname{DB}(\sigma)} \mathbf{y}^{\operatorname{DT}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(321)} q^{\operatorname{mak}(\sigma)} \mathbf{x}^{\operatorname{DB}(\sigma)} \mathbf{y}^{\operatorname{DT}(\sigma)},$$
$$\sum_{\sigma \in \mathcal{S}_n(123)} q^{\operatorname{maj}(\sigma)} \mathbf{x}^{\operatorname{AB}(\sigma)} \mathbf{y}^{\operatorname{AT}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(123)} q^{\operatorname{mak}(\sigma)} \mathbf{x}^{\operatorname{AB}(\sigma)} \mathbf{y}^{\operatorname{AT}(\sigma)}.$$

Proof. Let $\sigma \in S_n(321)$. By Lemma 2.3 we may decompose σ as

$$\sigma = u_1 v_1 u_2 v_2 \cdots u_t v_t,$$

where u_1, \ldots, u_t are non-empty factors of left-to-right maxima in σ and v_1, \ldots, v_t are non-empty factors (except possibly v_t) such that $v_1v_2\cdots v_t$ is an increasing subword. Assume first that $v_t \neq \emptyset$. Let $M_i = \max(u_i)$ and $m_i = \min(v_i)$ for $i = 1, \ldots, t$. Clearly $\text{DB}(\sigma) = \{m_i : 1 \leq i \leq t\}$ and $\text{DT}(\sigma) = \{M_i : 1 \leq i \leq t\}$. Let $\bar{u}_i = u_i \setminus M_i$ and $\bar{v}_i = v_i \setminus m_i$ for $i = 1, \ldots, t$. Write $\bar{u} = \bar{u}_1 \cdots \bar{u}_t$ and $\bar{v} = \bar{v}_1 \cdots \bar{v}_t$.

We now define an involution

$$\phi: \mathcal{S}_n(321) \to \mathcal{S}_n(321) \tag{2.2}$$

such that $\operatorname{maj}(\phi(\sigma)) = \operatorname{mak}(\sigma)$, preserving all pairs of descent top and descent bottoms. For convenience, set $M_0 = -\infty$ and $M_{t+1} = \infty$. Let u'_k denote the unique increasing word of the letters in the set

$$\left\{\alpha \in \bar{v} : M_{k-1} < \alpha < M_k\right\},\,$$

with M_k adjoined at the end and let v'_k denote the unique increasing word of the letters in the set

$$\{\beta \in \bar{u} : m_k < \beta < M_{k+1}\},\$$

with m_k adjoined at the beginning for $k = 1, \ldots, t$. Define

$$\phi(\sigma) = \begin{cases} u_1' v_1' \cdots u_t' v_t' & \text{if } v_t \neq \emptyset \\ \phi(u_1 v_1 \cdots u_{t-1} v_{t-1}) u_t & \text{if } v_t = \emptyset \end{cases}$$

Thus ϕ effectively swaps $\bar{u} = \text{LRMax}(\sigma) \setminus \text{DT}(\sigma)$ with $\bar{v} = [n] \setminus (\text{LRMax}(\sigma) \cup \text{DB}(\sigma))$ (when $v_t \neq \emptyset$) and $\text{DB}(\phi(\sigma)) = \text{DB}(\sigma)$, $\text{DT}(\phi(\sigma)) = \text{DT}(\sigma)$. Hence ϕ is an involution. We have

$$(2\underline{31})\sigma = \sum_{\beta \in \overline{u}} (2\underline{31})|_{(\beta, \neg, \neg)}\sigma$$

=
$$\sum_{\beta \in \overline{u}} (\max\{k : m_k < \beta\} - \min\{k : M_k > \beta\} + 1)$$

=
$$\sum_{\beta \in \overline{u}} (\max\{k : m_k < \phi(\beta)\} - \min\{k : M_k > \phi(\beta)\} + 1)$$

=
$$\sum_{\beta \in \overline{u}} (\underline{312})|_{(\neg, \neg, \phi(\beta))}\phi(\sigma)$$

=
$$(\underline{312})\phi(\sigma),$$

since under the involution ϕ , each $\beta \in \text{LRMax}(\sigma) \setminus \text{DT}(\sigma)$ precisely passes the number of descent bottoms that are less than it to its right. Therefore β is involved in the same number of 2<u>31</u> occurrences in σ as $\phi(\beta)$ is involved in <u>31</u>2 occurrences in $\phi(\sigma)$. Hence

$$\begin{aligned} \max(\phi(\sigma)) &= ((\underline{132}) + (\underline{321}) + (\underline{21}))\phi(\sigma) + (\underline{312})\phi(\sigma) \\ &= \sum_{\alpha \in \mathrm{DB}(\phi(\sigma))} \alpha + (\underline{312})\phi(\sigma) \\ &= \sum_{\alpha \in \mathrm{DB}(\sigma)} \alpha + (\underline{231})\sigma \\ &= \mathrm{maj}(\sigma). \end{aligned}$$

The statement is proved analogously over $\mathcal{S}(123)$.

Example 2.1. Let ϕ be the involution (2.2) in Theorem 2.4 and let $\sigma = 341625978 \in S_9(321)$. Then

where the black letters indicate the fixed pairs of descent tops and descent bottoms, red letters denote non-descent top left-to-right maxima and blue letters denote nondescent bottom non-left-to-right maxima. The involution swaps the role of red and blue letters while keeping consecutive pairs of black letters together in the same relative order.

Proposition 2.5. We have

$$\begin{split} & [123]_{mak} = \{123\}, \\ & [321]_{mak} = \{321\}, \\ & [132]_{mak} = \{132, 312\} = [312]_{mak}, \\ & [213]_{mak} = \{213, 231\} = [231]_{mak}, \end{split}$$

Proof. As shown in [13, Theorem 2.6] the map $\phi : S_n(132) \to S_n(231)$ recursively defined by

$$\phi(231[\sigma_1, 1, \sigma_2]) = 132[\phi(\sigma_1), 1, \phi(\sigma_2)],$$

is a descent preserving bijection implying that $[132]_{maj} = [231]_{maj}$. Thus by Proposition 2.1 we have

$$\sum_{\sigma \in \mathcal{S}_n(132)} q^{\max(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(132)} q^{\max(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(231)} q^{\max(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(312)} q^{\max(\sigma)}.$$

Hence $[132]_{mak} = [312]_{mak}$. The remaining mak-Wilf equivalence is proved similarly invoking Proposition 2.1. The inequivalences between the four classes is easily verified by hand or with computer.

Remark 2.6. The charge statistic is also a Mahonian statistic related to maj via trivial bijections by $maj(\sigma) = charge(((\sigma^r)^c)^{-1})$ (see [18]). It is worth noting that the mak-Wilf classes in Proposition 2.5 coincide with the charge-Wilf classes identified in [18].

Remark 2.7. It can be checked that maj, inv and mak are the only statistics in Table 1 with non-singleton st-Wilf classes for single classical patterns of length three.

The bijection (2.2) in Theorem 2.4 induces an interesting equidistribution on shortened polyominoes. A shortened polyomino is a pair (P,Q) of N (north), E (east) lattice paths $P = (P_i)_{i=1}^n$ and $Q = (Q_i)_{i=1}^n$ satisfying

- (i) P and Q begin at the same vertex and end at the same vertex.
- (ii) P stays weakly above Q and the two paths can share E-steps but not N-steps.

Denote the set of shortened polyominoes with |P| = |Q| = n by \mathcal{H}_n . For $(P, Q) \in \mathcal{H}_n$, let $\operatorname{Proj}_P^Q(i)$ denote the step $j \in [n]$ of P that is the projection of the i^{th} step of Q on P. Let

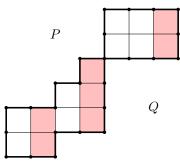
$$Valley(Q) = \{i : Q_i Q_{i+1} = EN\}$$

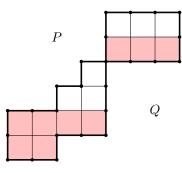
denote the set of indices of the *valleys* in Q and let nval(Q) = |Valley(Q)|. Moreover for each $i \in [n]$ define

 $\operatorname{area}_{(P,Q)}(i) = \#$ squares between the i^{th} step of Q and the j^{th} step of P,

where $j = \operatorname{Proj}_{P}^{Q}(i)$. Consider the statistics valley-column area and valley-row area of (P, Q) given by

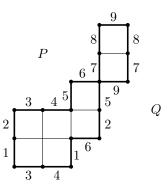
$$\operatorname{vcarea}(P,Q) = \sum_{i \in \operatorname{Valley}(Q)} \operatorname{area}_{(P,Q)}(i),$$
$$\operatorname{vrarea}(P,Q) = \sum_{i \in \operatorname{Valley}(Q)} \operatorname{area}_{(P,Q)}(i+1).$$





(a) vcarea(P, Q) = 2 + 3 + 2 = 7

(b) vrarea(P, Q) = 2 + 4 + 3 = 9



The bijection Υ . Here $\Upsilon(P,Q) = 341625978 \in \mathcal{S}_9(321)$.

Theorem 2.8. For any $n \ge 1$,

$$\sum_{(P,Q)\in\mathcal{H}_n} q^{\operatorname{vcarea}(P,Q)} t^{\operatorname{nval}(Q)} = \sum_{(P,Q)\in\mathcal{H}_n} q^{\operatorname{vrarea}(P,Q)} t^{\operatorname{nval}(Q)}.$$

Proof. We begin by recalling a bijection $\Upsilon : \mathcal{H}_n \to \mathcal{S}_n(321)$ due to Cheng-Eu-Fu [8]. Given $(P, Q) \in \mathcal{H}_n$, set $\text{Label}_P(i) = i$ and $\text{Label}_Q(i) = \text{Label}_P(\text{Proj}_P^Q(i))$. Then

$$\Upsilon(P,Q) = \operatorname{Label}_Q(1) \cdots \operatorname{Label}_Q(n) \in \mathcal{S}_n(321)$$

is a bijection.

Let $(P,Q) \in \mathcal{H}_n$ and $i \in \text{Valley}(Q)$. The definition of Υ immediately gives

 $Valley(P,Q) = Des(\Upsilon(P,Q)).$

In particular Label_Q(i + 1) < Label_Q(i). Let $s = \operatorname{Proj}_P^Q(i + 1)$ and $t = \operatorname{Proj}_P^Q(i)$. Then s < t and

$$\begin{aligned} \operatorname{area}_{(P,Q)}(i) &= |\{j: P_j = N, s \leq j \leq t\}| \\ &= |\{j: \operatorname{Label}_Q(i+1) \leq \operatorname{Label}_Q(j) < \operatorname{Label}_Q(i), j > i\}| \\ &= 1 + (\underline{312})|_{(\operatorname{Label}_Q(i), \operatorname{Label}_Q(i+1), -)} \Upsilon(P,Q). \end{aligned}$$

Similarly,

$$\begin{aligned} \operatorname{area}_{(P,Q)}(i+1) &= |\{j: P_j = E, \, s \leq j \leq t\}| \\ &= |\{j: \operatorname{Label}_Q(i+1) < \operatorname{Label}_Q(j) \leq \operatorname{Label}_Q(i), \, j \leq i\}| \\ &= 1 + (2\underline{31})|_{(\text{-}, \operatorname{Label}_Q(i), \operatorname{Label}_Q(i+1))} \Upsilon(P,Q). \end{aligned}$$

Let $\phi : S_n(321) \to S_n(321)$ be the bijection (2.2) from Theorem 2.4. Recall that $(\underline{312})\phi(\sigma) = (2\underline{31})\sigma$ and $\operatorname{des}(\phi(\sigma)) = \operatorname{des}(\sigma)$ for all $\sigma \in S_n(321)$. Let $\Phi : \mathcal{H}_n \to \mathcal{H}_n$ be the bijection

$$\Phi = \Upsilon^{-1} \circ \phi \circ \Upsilon,$$

and set
$$(P', Q') = \Phi(P, Q)$$
. Then

$$\operatorname{vcarea}(\Phi(P, Q)) = \sum_{i \in \operatorname{Valley}(Q')} \operatorname{area}_{(P',Q')}(i)$$

$$= \sum_{i \in \operatorname{Valley}(Q')} \left(1 + (\underline{312})\big|_{(\operatorname{Label}_{Q'}(i), \operatorname{Label}_{Q'}(i+1), -)} \Upsilon(P', Q')\right)$$

$$= \sum_{i \in \operatorname{Des}(\phi(\Upsilon(P,Q)))} \left(1 + (\underline{312})\big|_{(\phi(\operatorname{Label}_{Q}(i)), \phi(\operatorname{Label}_{Q}(i+1)), -)} \phi(\Upsilon(P,Q))\right)$$

$$= (\operatorname{des} + (\underline{312}))\phi(\Upsilon(P,Q))$$

$$= (\operatorname{des} + (\underline{231}))\Upsilon(P,Q)$$

$$= \sum_{i \in \operatorname{Valley}(Q)} \left(1 + (\underline{231})\big|_{(-, \operatorname{Label}_{Q}(i), \operatorname{Label}_{Q}(i+1))} \Upsilon(P,Q)\right)$$

$$= \sum_{i \in \operatorname{Valley}(Q)} \operatorname{area}_{(P,Q)}(i+1)$$

$$= \operatorname{vrarea}(P,Q).$$

Since $\operatorname{Valley}(P,Q) = \operatorname{Des}(\Upsilon(P,Q))$ and $\operatorname{des}(\phi(\sigma)) = \operatorname{des}(\sigma)$ it follows that $\operatorname{nval}(Q') = \operatorname{nval}(Q)$. This concludes the proof. \Box

Below we provide a brief account for a well-known lemma due to Simion and Schmidt which will be used to justify the bijection in the next theorem.

Lemma 2.9 (Simion-Schmidt [27]). A permutation $\sigma \in S(132)$ is uniquely determined by the values and positions of its left-to-right minima.

Proof. It is clear that the left-to-right minima are positioned in decreasing order relative to each other. Now fill in the remaining numbers from left to right, for each empty position i choosing the smallest remaining entry that is larger than the closest left-to-right minima m in position before i. If the remaining numbers are not entered in this unique way and y is placed before x where y > x, then myx is an occurrence of the pattern 132.

Theorem 2.10. For any $n \geq 1$,

$$\sum_{\sigma \in \mathcal{S}(132)} q^{\operatorname{maj}(\sigma)} \mathbf{x}^{\operatorname{LRMin}(\sigma)} = \sum_{\sigma \in \mathcal{S}(132)} q^{\operatorname{foze}(\sigma)} \mathbf{x}^{\operatorname{LRMin}(\sigma)}$$

Proof. Let $\sigma \in S_n(132)$. It is not difficult to see that $\operatorname{LRMin}(\sigma) = \operatorname{DB}(\sigma) \cup \{\sigma(1)\}$. Indeed if $\sigma(i) \in \operatorname{DB}(\sigma)$ and $\sigma(j) < \sigma(i)$ for some j < i, then $\sigma(j)\sigma(i-1)\sigma(i)$ is an occurrence of 132. Hence by Lemma 2.9 we have that σ is uniquely determined equivalently by its first letter, $\operatorname{Des}(\sigma)$ and $\operatorname{DB}(\sigma)$. We define a map $\phi : S_n(132) \to S_n(132)$ by requiring

$$\begin{split} \phi(\sigma)(1) &= \sigma(1), \\ \mathrm{DB}(\phi(\sigma)) &= \mathrm{DB}(\sigma), \\ \mathrm{Des}(\phi(\sigma)) &= \{n - \sigma(i) + 1 : i \in \mathrm{Des}(\sigma)\}. \end{split}$$

We claim that a permutation $\phi(\sigma) \in \mathcal{S}_n(132)$ with the above requirements exists. If the claim holds, then the image of σ is uniquely determined by the data above and therefore ϕ is well-defined. It also immediately follows that ϕ is a bijection.

Let $i_1 < \cdots < i_m$ be the descents of σ . Suppose

$$n - \sigma(i_{j_1}) + 1 < \dots < n - \sigma(i_{j_m}) + 1$$

To show that ϕ is well-defined we show that the insertion procedure from Lemma 2.9 is always valid. Given a descent bottom (i.e. left-to-right minima) $\sigma(i_k + 1)$ in position $n - \sigma(i_{j_k}) + 2$ we must show that there exists enough remaining numbers greater than $\sigma(i_k + 1)$ to fill in the gap to the next descent bottom $\sigma(i_{k+1} + 1)$. Within the filling procedure, next after the descent bottom $\sigma(i_k + 1)$, there exists

$$n - \sigma(i_k + 1) - (n - \sigma(i_{j_k}) + 1) = \sigma(i_{j_k}) - \sigma(i_k + 1) - 1$$

numbers remaining that are greater than $\sigma(i_k + 1)$. There are

$$(n - \sigma(i_{j_{k+1}}) + 2) - (n - \sigma(i_{j_k}) + 2) - 1 = \sigma(i_{j_k}) - \sigma(i_{j_{k+1}}) - 1$$

positions to fill in the gap between the descent bottoms $\sigma(i_k + 1)$ and $\sigma(i_{k+1} + 1)$. By minimality

$$\sigma(i_{j_k}) - \sigma(i_{j_{k+1}}) \le \sigma(i_{j_k}) - \sigma(i_k) \le \sigma(i_{j_k}) - \sigma(i_k+1),$$

so there are enough numbers remaining to fill in the gap. Hence ϕ is well-defined. Finally,

$$\operatorname{maj}(\phi(\sigma)) = \sum_{i \in \operatorname{Des}(\phi(\sigma))} i$$
$$= \sum_{i \in \operatorname{Des}(\sigma)} (n - \sigma(i) + 1)$$
$$= \sum_{\alpha \in \operatorname{DT}(\sigma)} (n - \alpha) + \operatorname{des}(\sigma)$$
$$= ((\underline{213}) + (\underline{321}))\sigma + (\underline{21})\sigma$$
$$= \operatorname{foze}(\sigma).$$

Since also $\phi(\operatorname{LRMin}(\sigma)) = \operatorname{LRMin}(\sigma)$, the theorem follows.

Below we provide an additional list of information uniquely determining permutations in $\mathcal{S}_n(231)$.

Lemma 2.11. A permutation $\sigma \in S_n(231)$ is uniquely determined by any of the following data:

- (i) The values and positions of right-to-left minima.
- (ii) The last letter, ascents and ascent bottoms.
- (iii) The pairs $P(\sigma) = \{(p, v) : p \text{ peak and } v \text{ its following valley}\}.$
- (iv) The pairs $Q(\sigma) = \{(\alpha, \beta) : \alpha \text{ descent top and } \beta \text{ its following descent bottom}\}.$ (v) The pairs $R(\sigma) = \left\{ \left(\alpha, (\underline{132}) \Big|_{(-,\alpha,-)} \sigma \right) : \alpha \text{ descent top} \right\}.$

Proof.

- (i) Suppose the values and positions of right-to-left minima are fixed in σ . Then $\sigma^r \in S_n(132)$ and the values and positions of the left-to-right minima in σ^r are fixed. By Lemma 2.9 this information uniquely determines σ^r . Hence σ is uniquely determined.
- (ii) Follows directly from (i) since the positions and values of right-to-left minima are given by the positions and values of the ascents and ascent bottoms together with the last letter.
- (iii) Consider the peak-valley decomposition

$$\sigma = a_1 p_1 d_1 v_1 \cdots a_{m-1} p_{m-1} d_{m-1} v_{m-1} a_m p_k d_m$$

where p_i and v_i are peaks resp. valleys and a_i and d_i are (possibly empty) increasing resp. decreasing words for i = 1, ..., m.

We claim that the pairs in P are relatively positioned in increasing order of the valleys. Indeed let $(p, v), (p', v') \in P(\sigma)$. Without loss assume v < v'. Suppose (for a contradiction) that (p', v') is ordered before (p, v)in σ . Note that v' < p, otherwise $v'\alpha p$ is an occurrence of 231, where α is the ascent top following v'. This in turn implies that v'pv is an occurrence of 231 giving a contradiction. Therefore (p, v) is ordered before (p', v')proving the claim.

Next we claim that the decreasing words d_j are uniquely determined. Going from right to left, let d_j be the unique decreasing word of all remaining letters (in value) between p_j and v_j for $j = m, \ldots, 1$. If we do not insert the letters this way and $v_j < \sigma_i < p_j$, where σ_i is positioned before p_j (and hence v_j) then $\sigma_i p_j v_j$ is an occurrence of 231 which is forbidden.

Finally we show that the increasing words a_j are uniquely determined. Suppose a_j contains a letter α such that $\alpha > v_j$. Since $\alpha < p_j$ it follows that $\alpha p_j v_j$ is an occurrence of 231. Therefore all letters of a_i are smaller than v_j . Hence a_j is given by the unique increasing word of all letters α such that $v_{j-1} < \alpha < v_j$ for $j = 1, \ldots, m$. Hence σ is uniquely determined.

- (iv) Partition the letters in $DB(\sigma) \cup DT(\sigma)$ into maximal consecutive decreasing subwords d_1, \ldots, d_m based on the pairs in $Q(\sigma)$. The top element of each decreasing subword d_i must be a peak and the bottom element valley. This information uniquely determines σ as per part (iii).
- (v) Note that $\alpha \in DT(\sigma)$ is the largest letter in an occurrence of <u>132</u> in σ if and only if α is a peak. Therefore the peaks are the descent tops α with $(\underline{132})|_{(-,\alpha,-)}\sigma > 0$. Given a peak p and the closest valley v to its right, any letter σ_i such that $v < \sigma_i < p$ must be in position after v, otherwise $\sigma_i p v$ is an occurrence of 231. Hence $(\underline{132})|_{(-,p,-)}\sigma$ precisely represents the difference between p and v. In other words $v = p - (\underline{132})|_{(-,p,-)}\sigma$. Hence σ is uniquely determined by part (iii).

Theorem 2.12. *For* $n \ge 1$ *,*

$$\sum_{\sigma \in S_n(231)} q^{\max(\sigma)} t^{\operatorname{des}(\sigma)} = \sum_{\sigma \in S_n(231)} q^{\operatorname{foze}(\sigma)} t^{\operatorname{des}(\sigma)}$$

Proof. Let $\sigma \in S_n(231)$. Note that for $\alpha \in DT(\sigma)$ we have $(\underline{132})|_{(-,\alpha,-)} \sigma \leq \alpha - 2$ since there are at most $\alpha - 2$ numbers between α and its immediately preceding

ascent bottom (if present). Thus the function

$$f_{\sigma} : \mathrm{DT}(\sigma) \to [n]$$
$$\alpha \mapsto (n - \alpha + 2) + (\underline{132})|_{(-,\alpha,-)}\sigma$$

is well-defined.

We claim that f_{σ} is injective by induction on n. Consider the inflation form $\sigma = 132[\sigma_1, 1, \sigma_2]$ where $\sigma_1 \in S_k(231)$ and $\sigma_2 \in S_{n-k-1}(231)$. Let $\mathrm{DT}_{\leq k}(\sigma) = \{\alpha \in \mathrm{DT}(\sigma) : \alpha \leq k\}$ and $\mathrm{DT}_{>k}(\sigma) = \{\alpha \in \mathrm{DT}(\sigma) : \alpha > k\}$. By induction $f_{\sigma_1} : \mathrm{DT}(\sigma_1) \to [k]$ is injective and $f_{\sigma}(\alpha) = n - k + f_{\sigma_1}(\alpha)$ for every $\alpha \in \mathrm{DT}_{\leq}(\sigma)$. Hence $f_{\sigma}|_{\mathrm{DT}_{\leq k}(\sigma)}$ is injective. By induction $f_{\sigma_2} : \mathrm{DT}(\sigma_2) \to [n-k-1]$ is injective and $f_{\sigma}(\alpha) = 1 + f_{\sigma_2}(\alpha - k)$ for every $\alpha \in \mathrm{DT}_{>k}(\sigma)$. Hence $f_{\sigma}|_{\mathrm{DT}_{>k}(\sigma)}$ is injective. Finally note that $f_{\sigma}(n) = 2 + |\sigma_2|$ if $\sigma_1 \neq \emptyset$ and $f_{\sigma}(n) = 2$ if $\sigma_1 = \emptyset$. Therefore for all $\alpha \in \mathrm{DT}_{\leq k}(\sigma)$ and $\beta \in \mathrm{DT}_{>k}(\sigma)$ we have

$$f_{\sigma}(\alpha) \ge (n-k+2) > f_{\sigma}(n) > n-k \ge f_{\sigma}(\beta),$$

if $\sigma_1 \neq \emptyset$ and

$$f_{\sigma}(\alpha) \ge (n-k+2) > f_{\sigma}(\beta) > 2 = f_{\sigma}(n),$$

if $\sigma_1 = \emptyset$. Hence f_{σ} is injective on all of $DT(\sigma)$.

Define a map $\phi : S_n(231) \to S_n(231)$ by setting the pairs of descent tops and descent bottoms in $\phi(\sigma)$ to $Q(\phi(\sigma)) = \{(f_{\sigma}(\alpha), n - \alpha + 1) : \alpha \in \mathrm{DT}(\sigma)\}$. By Lemma 2.11 (iv) this data uniquely determines $\phi(\sigma)$. Note that the pairs are well-defined since f_{σ} is injective and $f_{\sigma}(\alpha) > n - \alpha + 1$ for all $\alpha \in \mathrm{DT}(\sigma)$.

We claim that ϕ is a bijection. By Lemma 2.11 (iv) we may uniquely associate σ with a set of pairs $R(\sigma) = \left\{ \left(\alpha, (\underline{132}) \Big|_{(-,\alpha,-)} \sigma \right) : \alpha \in \mathrm{DT}(\sigma) \right\}$. It suffices to show that ϕ is injective. Let $\pi_1, \pi_2 \in \mathcal{S}_n(231)$ such that $\pi_1 \neq \pi_2$. If $\mathrm{DT}(\pi_1) \neq \mathrm{DT}(\pi_2)$, then $\mathrm{DB}(\phi(\pi_1)) \neq \mathrm{DB}(\phi(\pi_2))$, so $\phi(\pi_1) \neq \phi(\pi_2)$. Assume therefore $\mathrm{DT}(\pi_1) = \mathrm{DT}(\pi_2)$. Since $\pi_1 \neq \pi_2$ we have by uniqueness that $R(\pi_1) \neq R(\pi_2)$. Therefore there exists $\alpha \in \mathrm{DT}(\pi_1) = \mathrm{DT}(\pi_2)$ such that $f_{\pi_1}(\alpha) \neq f_{\pi_2}(\alpha)$. Thus $Q(\phi(\pi_1)) \neq Q(\phi(\pi_2))$ which again implies that $\phi(\pi_1) \neq \phi(\pi_2)$. Hence ϕ is injective and therefore a bijection.

It remains to show that $mak(\phi(\sigma)) = foze(\sigma)$. Note that

(

$$((1\underline{32}) + (\underline{32}1) + (\underline{21}))\sigma = \sum_{\beta \in \mathrm{DB}(\sigma)} \beta.$$

Since there are no occurrences of $2\underline{31}$ in σ by assumption, the letters between each pair of descent top and descent bottom occur to the right of the pair. Therefore the number of occurrences of $\underline{312}$ in σ is given precisely by

$$\sum_{\alpha,\beta)\in Q(\sigma)} (\alpha - \beta - 1).$$

Hence

$$\operatorname{mak}(\sigma) = \sum_{\beta \in \operatorname{DT}(\sigma)} (\alpha - 1).$$

On the other hand note that

$$((\underline{213}) + (\underline{321}) + (\underline{21}))\sigma = \sum_{\alpha \in DT(\sigma)} (n - \alpha + 1).$$

Thus

$$foze(\sigma) = \sum_{\alpha \in DT(\sigma)} (n - \alpha + 1) + (\underline{132})\sigma$$
$$= \sum_{\alpha \in DT(\sigma)} \left(n - \alpha + 1 + (\underline{132}) \Big|_{(-,\alpha,-)}\sigma \right)$$
$$= \sum_{\alpha \in DT(\sigma)} (f_{\sigma}(\alpha) - 1).$$

Hence

$$\max(\phi(\sigma)) = \sum_{\alpha' \in \mathrm{DT}(\phi(\sigma))} (\alpha' - 1) = \sum_{\alpha \in \mathrm{DT}(\sigma)} (f(\alpha) - 1) = \operatorname{foze}(\sigma).$$

Finally since $des(\phi(\sigma)) = des(\sigma)$, the theorem follows.

Remark 2.13. Via Proposition 2.1 we may deduce further equidistributions between maj and foze, see Table 2 in §5 for a summary.

3. Equidistributions via Dyck paths

A Dyck path of length 2n is a lattice path in \mathbb{Z}^2 between (0,0) and (2n,0) consisting of up-steps (1,1) and down-steps (1,-1) which never go below the x-axis. For convenience we denote the up-steps by U and the down-steps by D enabling us to encode a Dyck path as a Dyck word (we will refer to the two notions interchangeably). Let \mathcal{D}_n denote the set of all Dyck paths of length 2n and set $\mathcal{D} = \bigcup_{n\geq 0} \mathcal{D}_n$. For $P \in \mathcal{D}_n$, let |P| = 2n denote the length of P. There are many statistics associated with Dyck paths in the literature. Here we will consider several Dyck path statistics that are intimately related with the inv statistic on pattern avoiding permutations.

Let $P = s_1 \cdots s_{2n} \in \mathcal{D}_n$. A double rise in P is a subword UU and a double fall in P a subword DD. Let dr(P) and df(P) respectively denote the number of double rises and double falls in P. A peak in P is an up-step followed by a down-step, in other words, a subword of the form UD. Let $\text{Peak}(P) = \{p : s_p s_{p+1} = UD\}$ denote the set of indices of the peaks in P and $\operatorname{npea}(P) = |\operatorname{Peak}(P)|$. For $p \in \operatorname{Peak}(P)$ define the *position* of p, $pos_P(p)$, resp. the *height* of p, $ht_P(p)$, to be the x resp. y-coordinate of its highest point. A valley in P is a down step followed by an up step, in other words, a subword of the form DU. Let $Valley(P) = \{v : s_v s_{v+1} = v \}$ DU denote the set of indices of the valleys in P and nval(P) = |Valley(P)|. For $v \in \text{Valley}(P)$ define the position of v, $\text{pos}_P(v)$, resp. the height of v, $\text{ht}_P(v)$, to be the x resp. y-coordinate of its lowest point. For each $v \in Valley(P)$, there is a corresponding *tunnel* which is the subword $s_i \cdots s_v$ of P where i is the step after the first intersection of P with the line $y = ht_P(v)$ to the left of step v (see Figure 2). The length, v - i, of a tunnel is always an even number. Let Tunnel(P) = { $(i, j) : s_i \cdots s_j$ tunnel in P} denote the set of pairs of beginning and end indices of the tunnels in P. Cheng et.al. [7] define the statistics sumpeaks and

sumtunnels given respectively by

spea
$$(P) = \sum_{p \in \operatorname{Peak}(P)} (\operatorname{ht}_P(p) - 1),$$

stun $(P) = \sum_{(i,j) \in \operatorname{Tunnel}(P)} (j - i)/2.$

Let $\operatorname{Up}(P) = \{i : s_i = U\}$ denote the indices of the set of U-steps in P and $\operatorname{Down}(P) = \{i : s_i = D\}$ the set of indices of the D-steps in P. Given $i \in [2n]$ define the *height* of the step i in P, $\operatorname{ht}_P(i)$, to be the y-coordinate of its lowest point. Define the statistics sumups and sumdowns by

$$\sup(P) = \sum_{i \in Up(P)} \lceil \operatorname{ht}_P(i)/2 \rceil$$
$$\operatorname{sdow}(P) = \sum_{i \in \operatorname{Down}(P)} \lfloor \operatorname{ht}_P(i)/2 \rfloor$$

Define the *area* of P, denoted area(P), to be the number of complete $\sqrt{2} \times \sqrt{2}$ tiles that fit between P and the x-axis (cf [19]).

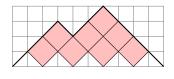


FIGURE 1. $\operatorname{area}(P) = 8$.

Burstein and Elizalde [5] define a statistic which they call the 'mass' of P. We will define two versions of it, one pertaining to the U-steps and one to the D-steps. For each $i \in \text{Up}(P)$ define the mass of i, $\text{mass}_P(i)$, as follows. If $s_{i+1} = D$, then $\text{mass}_P(i) = 0$. If $s_{i+1} = U$, then P has a subword of the form $s_i UP_1 DP_2 D$ where P_1, P_2 are Dyck paths and we define $\text{mass}_P(i) = |P_2|/2$. In other words, the mass is half the number of steps between the matching D-steps of two consecutive U-steps. The part of the Dyck path P contributing to the mass of each of the first three U-steps is highlighted with matching colours in Figure 2. Define

$$\operatorname{mass}_{\mathrm{U}}(P) = \sum_{i \in \operatorname{Up}(P)} \operatorname{mass}_{P}(i).$$

The statistic mass_U coincides with the mass statistic defined by Burstein and Elizable [5]. Analogously if $i \in \text{Down}(P)$, define mass_P(i) = 0 if $s_{i-1} = U$. If $s_{i-1} = D$, then P has a subword of the form $UP_1UP_2Ds_i$ where P_1, P_2 are Dyck paths and we define mass_D $(s) = |P_1|/2$. In other words, the mass is half the number of steps between the matching U-steps of two consecutive D-steps. Define

$$\operatorname{mass}_{\mathcal{D}}(P) = \sum_{i \in \operatorname{Down}(P)} \operatorname{mass}_{P}(i).$$

Next we give a description of the various Dyck path bijections that will be referenced. The standard bijection $\Delta : S_n(231) \to \mathcal{D}_n$ can be defined recursively by

$$\Delta(\sigma) = U\Delta(\sigma_1)D\Delta(\sigma_2),$$

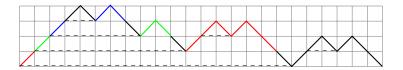


FIGURE 2. The tunnel lengths of a Dyck path (indicated with dashes) and the mass associated with the first three up-steps is highlighted with matching colours.

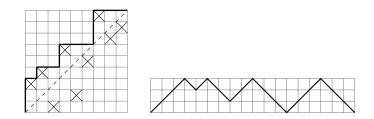


FIGURE 3. The Dyck path $\Gamma(\sigma)$ corresponding to $\sigma = 341625978$.

where $\sigma = 213[1, \sigma_1, \sigma_2]$. We will also (with abuse of notation) define the standard bijection $\Delta : S_n(312) \to \mathcal{D}_n$ recursively by

$$\Delta(\sigma) = \Delta(\sigma_1) U \Delta(\sigma_2) D,$$

where $\sigma = 132[\sigma_1, \sigma_2, 1]$. There is also a non-recursive description of Δ due to Krattenthaler, see [21].

We now define another well-known map $\Gamma : S_n(321) \to \mathcal{D}_n$ due to Krattenthaler [21] which also appears in a slightly different form in the work of Elizalde [14]. Let $\sigma \in S_n(321)$ and consider an $n \times n$ array with crosses in positions (i, π_i) for $1 \leq i \leq n$, where the first coordinate is the column number, increasing from left to right, and the second coordinate is the row number, increasing from bottom to top. Consider the path with north and east steps from the lower-left corner to the upper-right corner of the array, whose right turns occur at the crosses (i, σ_i) with $\sigma_i \geq i$. Define $\Gamma(\sigma)$ to be the Dyck path obtained from this path by reading a *U*-step for every north step and a *D*-step for every east step of the path. The bijection is illustrated in Figure 3.

Theorem 3.1 (Krattenthaler [21], Elizalde [14]). For each $n \ge 1$ the map $\Gamma : S_n(321) \to D_n$ is a bijection.

Theorem 3.2 (Cheng-Elizalde-Kasraoui-Sagan [7]). We have $inv(\sigma) = spea(\Gamma(\sigma))$ and $lrmax(\sigma) = npea(\Gamma(\sigma))$ for all $\sigma \in S_n(321)$.

Next we define a Dyck path bijection $\Psi : \mathcal{D}_n \to \mathcal{D}_n$ due to Cheng et.al. [7] that is weight preserving between the statistics spea and stun.

First we define a bijection $\delta : \bigsqcup_{k=0}^{n-1} \mathcal{D}_k \times \mathcal{D}_{n-k-1} \to \mathcal{D}_n$ as follows. Given two Dyck paths

$$Q = U^{a_1} D^{b_1} U^{a_2} D^{b_2} \cdots U^{a_s} D^{b_s} \in \mathcal{D}_k$$
 and $R = U^{c_1} D^{d_1} U^{c_2} D^{d_2} \cdots U^{c_t} D^{d_t} \in \mathcal{D}_{n-k-1}$

where all exponents are positive, define $\delta(Q, R)$ by

$$\delta(Q,R) = U^{a_1+1} D^{b_1+1} U^{a_2} D^{b_2} \cdots U^{a_s} D^{b_s}$$

if $R = \emptyset$ and define

$$\delta(Q,R) = U^{a_1+1} D U^{a_2} D^{b_1} U^{a_3} D^{b_2} \cdots U^{a_s} D^{b_{s-1}} U^{c_1} D^{b_s+d_1} U^{c_2} D^{d_2} \cdots U^{c_t} D^{d_t}$$

if $R \neq \emptyset$. If $Q = \emptyset$ the same definition works with the convention that $a_1 = b_1 = 0$. Let $P \in \mathcal{D}_n$ and $(Q, R) = \delta^{-1}(P)$. Define $\Psi(\emptyset) = \emptyset$ and for $n \ge 1$

$$\Psi(P) = \begin{cases} UD\Psi(Q) & \text{if } R = \emptyset \\ U\Psi(R)D & \text{if } Q = \emptyset, \\ U\Psi(Q)D\Psi(R) & \text{otherwise.} \end{cases}$$

Theorem 3.3 (Cheng-Elizalde-Kasraoui-Sagan [7]). The map $\Psi : \mathcal{D}_n \to \mathcal{D}_n$ is a bijection such that $\operatorname{spea}(P) = \operatorname{stun}(\Psi(P))$ and $\operatorname{npea}(P) = n - \operatorname{nval}(\Psi(P))$ for all $P \in \mathcal{D}_n$. In particular

$$\sum_{P \in \mathcal{D}_n} q^{\operatorname{spea}(P)} t^{\operatorname{npea}(P)} = \sum_{P \in \mathcal{D}_n} q^{\operatorname{stun}(P)} t^{n - \operatorname{nval}(P)}$$

for all $P \in \mathcal{D}_n$.

We will now interpret mad over both $S_n(231)$ and $S_n(312)$ in terms of Dyck path statistics under Δ . The following theorem is a straightforward modification of Theorem 3.11 in [5].

Theorem 3.4. For all $\sigma \in S_n(231)$, $\pi \in S_n(312)$ and $P \in D_n$ we have

- (i) $\operatorname{mad}(\sigma) = \operatorname{mass}_{\mathrm{U}}(\Delta(\sigma)) + \operatorname{dr}(\Delta(\sigma)),$
- (*ii*) $\operatorname{mad}(\pi) = 2 \operatorname{mass}_{D}(\Delta(\pi)) + \operatorname{df}(\Delta(\pi)),$

(iii) a bijection $\Theta : \mathcal{D}_n \to \mathcal{D}_n$ such that $\operatorname{sups}(P) = \operatorname{mass}_{\mathrm{U}}(\Theta(P)) + \operatorname{dr}(\Theta(P))$.

Proof.

(i) Let $\sigma \in S_n(231)$ and decompose $\sigma = 213[1, \sigma_1, \sigma_2]$. If we assume $\sigma_1 \neq \emptyset$, then we may further decompose σ_1 and write $\sigma = 42135[1, 1, \sigma_3, \sigma_4, \sigma_2]$. In particular $(\underline{312})|_{(\sigma(1), \sigma(2), -)}\sigma = |\sigma_4|$. Since

$$\Delta(\sigma) = UU\Delta(\sigma_3)D\Delta(\sigma_4)D\Delta(\sigma_2),$$

we have by induction that

$$\begin{aligned} \max_{\mathrm{U}}(\Delta(\sigma)) &= \max_{\mathrm{U}}(\Delta(\sigma_3)) + \max_{\mathrm{U}}(\Delta(\sigma_4)) + \max_{\mathrm{U}}(\Delta(\sigma_2)) + |\Delta(\sigma_4)|/2 \\ &= (\underline{312})\sigma_3 + (\underline{312})\sigma_4 + (\underline{312})\sigma_2 + |\sigma_4| \\ &= (\underline{312})\sigma. \end{aligned}$$

and

$$dr(\Delta(\sigma)) = dr(\Delta(\sigma_1)) + dr(\Delta(\sigma_2)) + 1$$
$$= des(\sigma_1) + des(\sigma_2) + 1$$
$$= des(\sigma).$$

Hence $\operatorname{mass}_{\mathrm{U}}(\Delta(\sigma)) + \operatorname{dr}(\Delta(\sigma)) = \operatorname{mad}(\sigma).$

(ii) Let $\pi \in S_n(312)$ and decompose $\pi = 132[\pi_1, \pi_2, 1]$. Assuming $\pi_2 \neq \emptyset$ we may write $\pi = 13542[\pi_1, \pi_3, \pi_4, 1, 1]$. In particular $(2\underline{31})|_{(-,\pi(n-1),\pi(n))}\pi = |\pi_3|$. Since

$$\Delta(\pi) = \Delta(\pi_1) U \Delta(\pi_3) U \Delta(\pi_4) DD,$$

it follows by an induction similar to part (i) that $\operatorname{mass}_{D}(\Delta(\pi)) = (2\underline{31})\pi$ and $\operatorname{df}(\Delta(\pi)) = \operatorname{des}(\pi)$. Hence $2 \operatorname{mass}_{D}(\Delta(\pi)) + \operatorname{df}(\Delta(\pi)) = \operatorname{mad}(\pi)$.

(iii) Construct a recursive bijection $\Theta : \mathcal{D}_n \to \mathcal{D}_n$ as follows. Let $P \in \mathcal{D}_n$. If $P = P_1 \cdots P_r$ where P_i is a Dyck path returning to the x-axis for the first time at its endpoint, then define $\Theta(P) = \Theta(P_1) \cdots \Theta(P_r)$. Assume therefore r = 1 and write

$$P = UUQ_1DUQ_2D\cdots UQ_sDD,$$

provided $P \neq UD$, where Q_1, \ldots, Q_s are Dyck paths. Define

$$\Theta(P) = \begin{cases} \emptyset & \text{if } P = \emptyset, \\ UD & \text{if } P = UD, \\ U^{s+1}D\Theta(Q_1)D\Theta(Q_2)D\cdots\Theta(Q_s)D & \text{otherwise.} \end{cases}$$

The map Θ is clearly a bijection. Note that

$$\operatorname{sups}(P) = \sum_{i=1}^{s} \operatorname{sups}(Q_i) + \frac{1}{2} \sum_{i=1}^{s} |Q_i| + s,$$

$$\operatorname{mass}_{\mathrm{U}}(\Theta(P)) + \operatorname{dr}(\Theta(P)) = \sum_{i=1}^{s} (\operatorname{mass}_{\mathrm{U}}(\Theta(Q_i)) + \operatorname{dr}(\Theta(Q_i)) + \frac{1}{2} \sum_{i=1}^{s} |\Theta(Q_i)| + s.$$

Hence by induction it follows that $\operatorname{mass}_{\mathrm{U}}(\Theta(P)) + \operatorname{dr}(\Theta(P)) = \operatorname{sups}(P)$.

Theorem 3.5. There exists a bijection $\Phi : \mathcal{D}_n \to \mathcal{D}_n$ such that $\operatorname{stun}(P) = \operatorname{mass}_{\mathrm{U}}(\Phi(P)) + \operatorname{dr}(\Phi(P))$. In particular, for any $n \ge 1$,

$$\sum_{P \in \mathcal{D}_n} q^{\operatorname{stun}(P)} = \sum_{P \in \mathcal{D}_n} q^{\operatorname{mass}_{\operatorname{U}}(P) + \operatorname{dr}(P)}.$$

Proof. Let $P \in \mathcal{D}_n$ and consider the decomposition

$$P = UP_1D\cdots UP_{m-1}DUP_mD,$$

where $P_1, \ldots, P_{m-1}, P_m$ are (possibly empty) Dyck paths. Define $\Phi : \mathcal{D}_n \to \mathcal{D}_n$ recursively by

$$\Phi(P) = \begin{cases} \emptyset, & \text{if } P = \emptyset \\ UD\Phi(P_1), & \text{if } m = 1 \\ UUU^{m-2}D^{m-2}D\Phi(P_1)\cdots\Phi(P_{m-1})D\Phi(P_m), & \text{if } m > 1 \end{cases}$$

It is not difficult to verify by induction that Φ is a bijection from the recursion. It remains to show that $\operatorname{stun}(P) = \operatorname{mass}_{\mathrm{U}}(\Phi(P)) + \operatorname{dr}(\Phi(P))$. We argue by induction on *n*. The statement holds for $P = \emptyset$. If m = 1, then by induction

$$stun(P) = stun(P_1)$$

= mass_U($\Phi(P_1)$) + dr($\Phi(P_1)$)
= mass_U($UD\Phi(P_1)$) + dr($UD\Phi(P_1)$)
= mass_U($\Phi(P)$) + dr($\Phi(P)$).

Suppose m > 1. Note that

$$mass_{U}(UUP_{0}DP_{1}\cdots P_{m-1}DP_{m}) = \sum_{i=0}^{m} mass_{U}(P_{i}) + \sum_{i=1}^{m-1} |P_{i}|/2$$

and that $\text{mass}_{\mathcal{U}}(U^kD^k) = 0$ for all $k \ge 0$. Hence by induction

$$\begin{aligned} \operatorname{stun}(P) &= \operatorname{stun}(P_m) + \sum_{i=1}^{m-1} (\operatorname{stun}(P_i) + (|P_i| + 2)/2) \\ &= \operatorname{mass}_{\mathrm{U}}(\Phi(P_m)) + \operatorname{dr}(\Phi(P_m)) + \sum_{i=1}^{m-1} [(\operatorname{mass}_{\mathrm{U}}(\Phi(P_i)) + \operatorname{dr}(\Phi(P_i)) + (|P_i| + 2)/2)] \\ &= \left(\operatorname{mass}_{\mathrm{U}}(U^{m-2}D^{m-2}) + \sum_{i=1}^{m} \operatorname{mass}_{\mathrm{U}}(\Phi(P_i)) + \sum_{i=1}^{m-1} |\Phi(P_i)|/2 \right) \\ &+ \left((m-1) + \sum_{i=1}^{m} \operatorname{dr}(\Phi(P_i)) \right) \\ &= \operatorname{mass}_{\mathrm{U}}(\Phi(P)) + \operatorname{dr}(\Phi(P)), \end{aligned}$$

as required.

Corollary 3.6. For any $n \ge 1$,

$$\sum_{\sigma \in \mathcal{S}_n(231)} q^{\mathrm{mad}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(321)} q^{\mathrm{inv}(\sigma)}.$$

Proof. By Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4 (i) and Theorem 3.5 we have the following diagram of weight preserving bijections

$$\begin{array}{ccc} (\mathcal{S}_n(321), \mathrm{inv}) & \stackrel{\Gamma}{\longrightarrow} (\mathcal{D}_n, \mathrm{spea}) & \stackrel{\Psi}{\longrightarrow} (\mathcal{D}_n, \mathrm{stun}) \\ & \downarrow^{\phi} & & \downarrow^{\Phi} \\ (\mathcal{S}_n(231), \mathrm{mad}) & \stackrel{\Delta}{\longrightarrow} (\mathcal{D}_n, \mathrm{mass}_{\mathrm{U}} + \mathrm{dr}) \end{array}$$

Thus

$$\phi = \Delta^{-1} \circ \Phi \circ \Psi \circ \Gamma$$

is our sought bijection with $inv(\sigma) = mad(\phi(\sigma))$ for all $\sigma \in S_n(321)$.

The following corollary answers a question of Burstein and Elizalde in [5].

Corollary 3.7. There exists a bijection $\Lambda : \mathcal{D}_n \to \mathcal{D}_n$ such that $\operatorname{spea}(P) = \operatorname{sups}(\Lambda(P))$. In particular for any $n \ge 1$,

$$\sum_{P \in \mathcal{D}_n} q^{\operatorname{spea}(P)} = \sum_{P \in \mathcal{D}_n} q^{\operatorname{sups}(P)}.$$

Proof. By Theorem 3.3, Theorem 3.4 (iii) and Theorem 3.5 we have the following diagram of weight preserving bijections

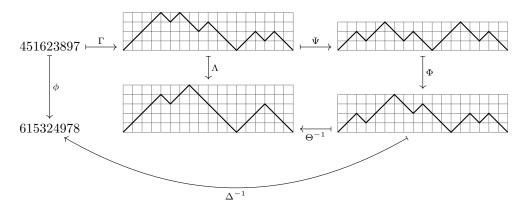
$$\begin{array}{ccc} (\mathcal{D}_n, \mathrm{spea}) & & \stackrel{\Psi}{\longrightarrow} (\mathcal{D}_n, \mathrm{stun}) \\ & & & \downarrow^{\Phi} \\ (\mathcal{D}_n, \mathrm{sups}) & \stackrel{\Theta}{\longrightarrow} (\mathcal{D}_n, \mathrm{mass}_{\mathrm{U}} + \mathrm{dr}) \end{array}$$

Hence

$$\Lambda = \Theta^{-1} \circ \Phi \circ \Psi$$

is the required bijection.

Example 3.1. The below diagram shows an example of the intermediate images under the bijections ϕ and Λ from Corollary 3.6 and Corollary 3.7.



For each Dyck path $P \in \mathcal{D}_n$, Kim et.al. [19] construct two bijections $DTS(P, \cdot)$ and $DTR(P, \cdot)$ from the set of linear extensions of the chord poset of P to the set of cover-inclusive Dyck tilings with lower path P (see [19] for terminology). In the special case where $P = (UD)^n$ and the set of linear extensions is restricted to $\mathcal{S}_n(312)$, it follows from [19, Theorem 2.3] that $DTS(P, \cdot)$ and $DTR(P, \cdot)$ induce bijections $\theta_{DTS} : \mathcal{S}_n(312) \to \mathcal{D}_n$ and $\theta_{DTR} : \mathcal{S}_n(312) \to \mathcal{D}_n$. We remark that the restriction is over $\mathcal{S}_n(231)$ in [19] due to difference in notation. By [19, Theorem 2.4] and [19, Theorem 6.1] it moreover follows that

$$\operatorname{inv}(\sigma) = \operatorname{area}(\theta_{\mathrm{DTS}}(\sigma)), \tag{3.1}$$

$$\operatorname{mad}(\sigma) = \operatorname{area}(\theta_{\mathrm{DTR}}(\sigma))$$
 (3.2)

for all $\sigma \in \mathcal{S}_n(312)$. Therefore we get a bijection $\theta : \mathcal{S}_n(312) \to \mathcal{S}_n(312)$ given by

$$\theta = \theta_{\rm DTS}^{-1} \circ \theta_{\rm DTR},$$

satisfying $\operatorname{mad}(\theta(\sigma)) = \operatorname{inv}(\sigma)$. Hence we obtain the following theorem.

Theorem 3.8 (Kim-Mésáros-Panova-Wilson [19]). For any $n \ge 1$,

$$\sum_{\substack{\in \mathcal{S}_n(312)}} q^{\mathrm{mad}(\sigma)} = \sum_{\substack{\sigma \in \mathcal{S}_n(312)}} q^{\mathrm{inv}(\sigma)}$$

Corollary 3.9. For any $n \ge 1$,

$$\sum_{P \in \mathcal{D}_n} q^{\operatorname{area}(P)} = \sum_{P \in \mathcal{D}_n} q^{2 \operatorname{mass}_{\mathrm{D}}(P) + \operatorname{df}(P)}.$$

Proof. Combine Theorem 3.4 (ii) with (3.2).

Below we find an interpretation of Theorem 1.1 in terms of Dyck path statistics. Part of the answer is given by a bijection $\Omega : S_n(231) \to \mathcal{D}_n$ due to Stump [30] which we now define. Let $\sigma \in S_n(231)$. Suppose $\text{Des}(\sigma) = \{i_1, \ldots, i_k\}$ and $\text{iDes} = \{i'_j \in \text{Des}(\sigma^{-1})\}$ such that $i_1 < \cdots < i_k$ and $i'_1 < \cdots < i'_k$ (recall that $\text{des}(\sigma) = \text{des}(\sigma^{-1})$) via e.g. the argument in Proposition 2.1). For notational purposes set $i_{k+1} = n = i'_{k+1}$. Define a Dyck path $\Omega(\sigma)$ by starting with i'_1 U-steps, followed

by i_1 *D*-steps, followed by $i'_2 - i'_1$ *U*-steps, followed by $i_2 - i_1$ *D*-steps, followed by $i'_3 - i'_2$ *U*-steps, and so on, ending with $i_{k+1} - i_k$ *D*-steps. Define the statistic

$$\beta(P) = \sum_{v \in \text{Valley}(P)} |\{j \le \text{pos}_P(v) : s_j = D\}|,$$

for each Dyck path $P = s_1 \cdots s_{2n} \in \mathcal{D}_n$.

Theorem 3.10 (Stump [30]). The map $\Omega : S_n(231) \to \mathcal{D}_n$ is a well-defined bijection such that $\operatorname{maj}(\sigma) = \beta(\Omega(\sigma))$ for all $\sigma \in S_n(231)$.

Proposition 3.11. For all $\sigma \in S_n(231)$ and $\pi \in S_n(321)$ we have

$$\operatorname{maj}(\sigma) = \sum_{v \in \operatorname{Valley}(\Omega(\sigma))} \frac{\operatorname{pos}_{\Omega(\sigma)}(v) - \operatorname{ht}_{\Omega(\sigma)}(v)}{2},$$
$$\operatorname{den}(\pi) = \operatorname{npea}(\Gamma(\pi)) + \sum_{p \in \operatorname{Peak}(\Gamma(\pi))} \frac{\operatorname{pos}_{\Gamma(\pi)}(p) - \operatorname{ht}_{\Gamma(\pi)}(p)}{2}.$$

Proof. As in [5, Theorem 2.5], observe that

$$\operatorname{den}(\pi) = \sum_{\substack{i \in [n]\\\pi(i) > i}} i,$$

for all $\pi \in S_n(321)$. In the definition of Krattenthaler's bijection Γ , each $i \in [n]$ such that $\pi(i) > i$ corresponds to a column i in the array containing a box above the main diagonal. In other words it corresponds to the number of east steps in the lattice path that occur to the left of the box. In the Dyck path $\Gamma(\pi) = s_1 \cdots s_{2n}$ this is reflected in the statistic

$$|\{j \le \text{pos}_{\Gamma(\pi)}(p) : s_j = D\}| + 1,$$

associated with each $p \in \text{Peak}(\Gamma(\pi))$. We have the following two obvious relations

$$|\{j \le \text{pos}_{\Gamma(\pi)}(p) : s_j = U\}| - |\{j \le \text{pos}_{\Gamma(\pi)}(p) : s_j = D\}| = \text{ht}_{\Gamma(\pi)}(p), |\{j \le \text{pos}_{\Gamma(\pi)}(p) : s_j = U\}| + |\{j \le \text{pos}_{\Gamma(\pi)}(p) : s_j = D\}| = \text{pos}_{\Gamma(\pi)}(p),$$

for each $p \in \text{Peak}(\Gamma(\pi))$. Hence

$$den(\pi) = \sum_{p \in Peak(\Gamma(\pi))} (|\{j \le pos_{\Gamma(\pi)}(p) : s_j = D\}| + 1)$$
$$= npea(\Gamma(\pi)) + \sum_{p \in Peak(\Gamma(\pi))} \frac{pos_{\Gamma(\pi)}(p) - ht_{\Gamma(\pi)}(p)}{2}.$$

The first statement in the proposition follows from Theorem 3.10 and a similar observation to above. $\hfill \Box$

Remark 3.12. By Theorem 1.1 the Dyck path statistics in Proposition 3.11 are equidistributed over \mathcal{D}_n .

4. Equidistributions via generating functions

In this section we use generating functions to derive equidistributions (albeit nonbijectively) between Mahonian statistics over $S_n(\pi)$. We also provide a recursion for a more general statistic involving arbitrary linear combinations of vincular pattern statistics of length three. This recursion generalizes for instance the recursions in [13].

Theorem 4.1. We have

$$\sum_{\sigma \in \mathcal{S}(231)} q^{\mathrm{mad}(\sigma)} z^{|\sigma|} = \sum_{\sigma \in \mathcal{S}(132)} q^{\mathrm{sist}(\sigma)} z^{|\sigma|} = \frac{1}{1 - \frac{z}{1 - \frac{qz}{1 - \frac{qz}{1 - \frac{q^2z}{1 - \frac$$

$$\sum_{\sigma \in \mathcal{S}(312)} q^{\mathrm{mad}(\sigma)} z^{|\sigma|} = \sum_{\sigma \in \mathcal{S}(213)} q^{\mathrm{sist}(\sigma)} z^{|\sigma|} = \frac{1}{1 - \frac{z}{1 - \frac{qz}{1 - \frac{q^2 z}{1 - \frac{q^3 z}{1 - \frac{q^3 z}{1 - \frac{q^4 z$$

Proof. Note that over S(231) we have mad $= (\underline{31}2) + (\underline{21})$. The reverse of sist (i.e. the statistic obtained by reversing all vincular patterns) is given by rsist $= (\underline{31}2) + (\underline{12})$. Hence (4.1) is equivalent to proving

$$\sum_{\sigma \in \mathcal{S}(231)} q^{\mathrm{mad}(\sigma)} z^{|\sigma|} = \sum_{\sigma \in \mathcal{S}(231)} q^{\mathrm{rsist}(\sigma)} z^{|\sigma|}.$$

Let $\sigma \in \mathcal{S}(231)$ and decompose $\sigma = 213[1, \sigma_1, \sigma_2]$. Then we obtain the recursion

$$\operatorname{rsist}(\sigma) = [12)\sigma_1 + \delta_{\sigma_2 \neq \emptyset} + \operatorname{rsist}(\sigma_1) + \operatorname{rsist}(\sigma_2),$$
$$[12)\sigma = |\sigma_2|,$$

where δ denotes the Kronecker delta. Let $F(q, t, z) = \sum_{\sigma \in S(231)} q^{\text{rsist}(\sigma)} t^{[12)\sigma} z^{|\sigma|}$. Then

$$\begin{split} F(q,t,z) &= 1 + z \left(\sum_{\sigma_1 \in \mathcal{S}(231)} q^{\operatorname{rsist}(\sigma_1)} q^{[12)\sigma_1} z^{|\sigma_1|} \right) \\ &+ q z \left(\sum_{\sigma_1 \in \mathcal{S}(231)} q^{\operatorname{rsist}(\sigma_1)} q^{[12)\sigma_1} z^{|\sigma_1|} \right) \left(\sum_{\sigma_2 \in \mathcal{S}(231)} q^{\operatorname{rsist}(\sigma_2)} (zt)^{|\sigma_2|} - 1 \right) \\ &= 1 + z F(q,q,z) + q z F(q,q,z) (F(q,1,zt) - 1). \end{split}$$

Substituting t = 1 and t = q we obtain the equation system

$$\begin{cases} F(q,1,z) = 1 + zF(q,q,z) + qzF(q,q,z)(F(q,1,z)-1) \\ F(q,q,z) = 1 + zF(q,q,z) + qzF(q,q,z)(F(q,1,qz)-1) \end{cases}$$

Eliminating F(q, q, z) and solving for F(q, 1, z) we obtain

$$F(q, 1, z) = \frac{1}{1 - \frac{z}{1 - qzF(q, 1, qz)}}$$

which gives the continued fraction in the theorem. Similarly putting $G(q, z, t) = \sum_{\sigma \in S(231)} q^{\text{mad}(\sigma)} t^{[12)} z^{|\sigma|}$ we obtain the recursive relation

$$G(q,t,z) = 1 + zG(q,1,zt) + qzG(q,1,zt)(G(q,q,z)-1).$$

Substituting t = 1 and t = q as before and solving for G(q, 1, z) we obtain the same continued fraction expansion as above, proving the desired equidistribution.

The second statement in the theorem is proved similarly. Over S(312) we have mad = $(2\underline{31}) + (2\underline{31}) + (\underline{21})$. Let $\sigma \in S(312)$ and decompose $\sigma = 132[\sigma_1, \sigma_2, 1]$. Then we obtain the recursion

$$\operatorname{mad}(\sigma) = 2 \cdot (12]\sigma_2 + \delta_{\sigma_2 \neq \emptyset} + \operatorname{mad}(\sigma_1) + \operatorname{mad}(\sigma_2),$$
$$(12]\sigma = |\sigma_1|.$$

Letting $F(q,t,z) = \sum_{\sigma \in \mathcal{S}(312)} q^{\text{mad}(\sigma)} t^{(12]\sigma} z^{|\sigma|}$ we thus obtain

$$F(q, t, z) = 1 + zF(q, 1, zt) + qzF(q, 1, zt)(F(q, q^2, z) - 1).$$

Putting t = 1 and $t = q^2$, eliminating $F(q, q^2, z)$ from the resulting equation system and solving for F(q, 1, z) we obtain the continued fraction expansion in the theorem.

A similar argument for rsist over S(312) gives a matching continued fraction expansion. We leave the details to the reader.

Remark 4.2. In [7, Corollary 8.6] it was proved that the continued fraction expansion of the generating function of inv over S(321) matches that of (4.1). This gives an alternative proof of Corollary 3.6.

Remark 4.3. For mad, the continued fractions (4.1) and (4.2) may also be deduced from the following more refined continued fraction in [11, Theorem 22] by specializing (x, y, p, q) = (1, q, 0, q) = 1 resp. $(x, y, p, q) = (1, p, p^2, 0)$ and using the fact that $\sigma \in \mathcal{S}(231)$ if and only if $\sigma \in \mathcal{S}(231)$ (see [9, Lemma 2]),

$$\sum_{\sigma \in \mathcal{S}} x^{\delta_{\sigma \neq \emptyset} + (\underline{12})\sigma} y^{(\underline{21})\sigma} p^{(\underline{231})\sigma} q^{(\underline{312})\sigma} z^{|\sigma|} = \frac{1}{1 - \frac{x[1]_{p,q} z}{1 - \frac{y[1]_{p,q} z}{1 - \frac{x[2]_{p,q} z}{1 - \frac{x[2]_{p,q} z}{1 - \frac{y[2]_{p,q} z}{1 - \frac{x[3]_{p,q} z}{1 - \frac{x[3]$$

where $[n]_{p,q} = q^{n-1} + pq^{n-2} + \dots + p^{n-2}q + p^{n-1}$ and δ denotes the Kronecker delta.

Using almost identical arguments to Theorem 4.1 we may moreover prove the following equidistributions.

Theorem 4.4. For any $n \ge 1$

$$\sum_{\substack{\sigma \in \mathcal{S}_n(231)}} q^{\mathrm{mad}(\sigma)} = \sum_{\substack{\sigma \in \mathcal{S}_n(132)}} q^{\mathrm{sist}'(\sigma)} = \sum_{\substack{\sigma \in \mathcal{S}_n(231)}} q^{\mathrm{sist}''(\sigma)},$$
$$\sum_{\substack{\sigma \in \mathcal{S}_n(312)}} q^{\mathrm{mad}(\sigma)} = \sum_{\substack{\sigma \in \mathcal{S}_n(132)}} q^{\mathrm{foze}'(\sigma)} = \sum_{\substack{\sigma \in \mathcal{S}_n(231)}} q^{\mathrm{sist}'(\sigma)} = \sum_{\substack{\sigma \in \mathcal{S}_n(132)}} q^{\mathrm{sist}''(\sigma)}.$$

By combining Theorem 4.1 and Theorem 4.4 with Theorem 3.8 and Corollary 3.6 we may deduce further equidistributions between inv and the statistics foze', sist, sist' and sist", see Table 2 in §5 for a summary.

For each $k \geq 1$, let $\iota_{k-1} = (12\cdots k)$ denote the statistic that counts the number of increasing subsequences of length k in a permutation. Define ι_{-1} by $\iota_{-1}(\sigma) = 1$ for all $\sigma \in S$ (i.e. we declare all permutations to have exactly one subsequence of length 0). We will now find a statistic expressed as a linear combination of ι_k 's which is equidistributed with the continued fraction (4.1). We will derive this statistic using the Catalan continued fraction framework of Brändén-Claesson-Steingrímsson[3]. Let

 $\mathcal{A} = \{A : \mathbb{N} \times \mathbb{N} \to \mathbb{Z} : A_{nk} = 0 \text{ for all but finitely many } k \text{ for each } n\}$

be the ring of infinite matrices with a finite number of non-zero entries in each row. Note in particular that the matrices in \mathcal{A} are indexed starting from 0. With each $A \in \mathcal{A}$ associate a family of statistics $\{\langle \iota, A_k \rangle\}_{k \geq 0}$ where $\iota = (\iota_0, \iota_1, \ldots), A_k$ is the k^{th} column of A, and

$$\langle \boldsymbol{\iota}, A_k \rangle = \sum_{i=0}^{\infty} A_{ik} \iota_i.$$

Let $\mathbf{q} = (q_0, q_1, \dots)$, where q_0, q_1, \dots are indeterminates. For each $A \in \mathcal{A}$ define

$$F_{A}(\mathbf{q}) = \sum_{\sigma \in \mathcal{S}(132)} \prod_{k \ge 0} q_{k}^{\langle \iota, A_{k} \rangle}(\sigma),$$

$$C_{A}(\mathbf{q}) = \frac{1}{1 - \frac{\prod q_{k}^{A_{0k}}}{1 - \frac{\prod q_{k}^{A_{1k}}}{1 - \frac{\prod q_{k}^{A_{2k}}}{1 - \frac{\prod q_{k}^{A_{2k}}}{1 - \frac{\prod q_{k}^{A_{3k}}}{1 - \frac{\prod q_{k}^{A_{3k}}}{1 - \frac{\prod q_{k}^{A_{4k}}}{1 - \frac{\prod q_{k}^{A_{4k}}}}}}$$

Theorem 4.5 (Brändén-Claesson-Steingrímsson[3]). Let $A \in \mathcal{A}$ and $B = \left(\binom{i}{j}\right)_{i,j\geq 0}$. Then

$$F_A(\mathbf{q}) = C_{BA}(\mathbf{q}),$$

and conversely

$$C_A(\mathbf{q}) = F_{B^{-1}A}(\mathbf{q}).$$

In particular, all continued fractions $C_A(\mathbf{q})$ are generating functions of statistics on $\mathcal{S}(132)$ expressed as (possibly infinite) linear combinations of ι_k 's.

Define the permutation statistic

inc =
$$\iota_1 + \sum_{k=2}^{\infty} (-1)^{k-1} 2^{k-2} \iota_k.$$

Note that inc is not a Mahonian statistic.

Proposition 4.6. We have

$$\sum_{\sigma \in \mathcal{S}(132)} q^{\text{inc}(\sigma)} z^{|\sigma|} = \frac{1}{1 - \frac{z}{1 - \frac{qz}{1 - \frac{qz}{1 - \frac{qz}{1 - \frac{q^2z}{1 - \frac{q}{1 - \frac{q^2z}{1 - \frac{q^2z}{1 - \frac{q^2z}{1 - \frac{q^2z}{1 - \frac{q}{1 - \frac{q}{1 - \frac{q}{1 - \frac{q}{1 - \frac{q}{1 - \frac{q}{1 - \frac{q}{1$$

Proof. Comparing (4.1) with the definition of $C_A(\mathbf{q})$ we get

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Note that $B^{-1} = \left((-1)^{i-j} {i \choose j}\right)_{i,j\geq 0}$. In $B^{-1}A$ we see that columns 2, 3, ... are zero columns and that column 1 is equal to $(1,0,0,\ldots)^T$ since $\sum_{k\geq 0} (-1)^{n-k} {n \choose k} = \delta_{n0}$ where δ_{ij} denotes the Kronecker delta. The entries $(B^{-1}A)_{k0}$ in column 0 are given by

$$(B^{-1}A)_{n0} = \sum_{i \ge 0} \lfloor (i+1)/2 \rfloor (-1)^{k-i} \binom{k}{i} = \begin{cases} 0, & \text{if } k = 0\\ 1, & \text{if } k = 1\\ (-1)^{k-1}2^{k-2}, & \text{if } k > 1. \end{cases}$$

Hence the proposition follows from Theorem 4.5.

Remark 4.7. Applying the same argument to the continued fraction (4.2) it is easy to see that Theorem 4.5 gives equidistribution with

$$\sum_{\sigma \in \mathcal{S}(132)} q^{\iota_1(\sigma)} z^{|\sigma|} = \sum_{\sigma \in \mathcal{S}(312)} q^{\mathrm{inv}(\sigma)} z^{|\sigma|}.$$

Corollary 4.8. For any $n \ge 1$,

$$\sum_{\sigma \in \mathcal{S}_n(132)} q^{\operatorname{inc}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(321)} q^{\operatorname{inv}(\sigma)}$$

Proof. Follows by combining Corollary 3.6, Theorem 4.1 and Proposition 4.6. \Box

Proposition 4.9. Let $\Delta : \mathcal{S}(132) \to \mathcal{D}$ denote the standard bijection defined by $\Delta(\sigma) = U\Delta(\sigma_1)D\Delta(\sigma_2)$ where $\sigma = 231[\sigma_1, 1, \sigma_2] \in \mathcal{S}(132)$. Then

$$\operatorname{inc}(\sigma) = \operatorname{sdow}(\Delta(\sigma))$$

for all $\sigma \in \mathcal{S}(132)$.

Proof. In [21] (see also [3]) Krattenthaler shows that

$$\iota_k(\sigma) = \sum_{i \in \text{Down}(\Delta(\sigma))} \binom{\text{ht}_{\Delta(\sigma)}(i) - 1}{k},$$

for all $\sigma \in \mathcal{S}(132)$. Hence

$$\operatorname{inc}(\sigma) = \sum_{i \in \operatorname{Down}(\Delta(\sigma))} \left(\binom{\operatorname{ht}_{\Delta(\sigma)}(i) - 1}{1} + \sum_{k=2}^{\infty} (-1)^{k-1} 2^{k-2} \binom{\operatorname{ht}_{\Delta(\sigma)}(i) - 1}{k} \right)$$
$$= \sum_{i \in \operatorname{Down}(\Delta(\sigma))} \lfloor \operatorname{ht}_{\Delta(\sigma)}(i)/2 \rfloor$$
$$= \operatorname{sdow}(\Delta(\sigma)),$$

for all $\sigma \in \mathcal{S}(132)$.

Since the Mahonian statistics in Table 1 are linear combinations of vincular patterns of length at most three, it is natural to consider the following more general statistic.

Definition 4.1. Let $\mathcal{P} = \{ \underline{abc} : \underline{abc} \in \mathcal{S}_3 \} \cup \{ \underline{abc} : \underline{abc} \in \mathcal{S}_3 \} \cup \{ \underline{21} \}$ and $\boldsymbol{\alpha} = (\alpha_{\rho}) \in \mathcal{S}_3 \}$ $\mathbb{N}^{\mathcal{P}}$. Define the statistic stat_{α} : $\mathcal{S} \to \mathbb{N}$ by

$$\operatorname{stat}_{\alpha}(\sigma) = \sum_{\rho \in \mathcal{P}} \alpha_{\rho}(\rho)\sigma,$$

for all $\sigma \in \mathcal{S}$.

Let *head* and *last* be the statistics defined by $head(\sigma) = \sigma(1)$ and $last(\sigma) = \sigma(n)$ for all $\sigma \in S_n$. We associate to stat_{α} the following generating function for each set Π of patterns

$$F_n(\Pi, \boldsymbol{\alpha}; q, t, u, v) = \sum_{\sigma \in \mathcal{S}_n(\Pi)} q^{\operatorname{stat}_{\boldsymbol{\alpha}}(\sigma)} t^{\operatorname{des}(\sigma)} u^{\operatorname{head}(\sigma)} v^{\operatorname{last}(\sigma)}.$$

Theorem 4.10. We have

$$\begin{split} F_{n}(312, \boldsymbol{\alpha}; q, t, u, v) \\ &= q^{C(0)} uv F_{n-1} \left(312, \boldsymbol{\alpha}; q, q^{A_{2}(0)}t, q^{B_{2}}, v \right) + q^{C(n-1)} tuv F_{n-1} \left(312, \boldsymbol{\alpha}; q, q^{A_{1}(n-1)}t, u, q^{B_{1}} \right) \\ &+ \sum_{k=1}^{n-2} q^{C(k)} tuv^{k} F_{k} \left(312, \boldsymbol{\alpha}; q, q^{A_{1}(k)}t, u, q^{B_{1}} \right) F_{n-k-1} \left(312, \boldsymbol{\alpha}; q, q^{A_{2}(k)}t, q^{B_{2}}, v \right), \end{split}$$

where

$$\begin{split} A_1(k) &= \alpha_{\underline{321}} - \alpha_{\underline{231}} + (n-k-1) \left(\alpha_{\underline{213}} - \alpha_{\underline{123}} \right), \\ A_2(k) &= (k+1) \left(\alpha_{\underline{132}} - \alpha_{\underline{123}} \right), \\ B_1 &= \alpha_{\underline{231}} - \alpha_{\underline{321}}, \\ B_2 &= \alpha_{\underline{132}} - \alpha_{\underline{123}}, \\ C(k) &= (k\alpha_{\underline{123}} - \alpha_{\underline{213}})(n-k-1) - \delta_{k< n-1}\alpha_{\underline{132}} + \delta_{k>0}(n-k-1)\alpha_{\underline{213}} \\ &+ \delta_{k>0}(k-1)\alpha_{\underline{231}} + \delta_{k< n-1}(k+1)(n-k-2)\alpha_{\underline{123}} \\ &+ \delta_{k< n-1}k\alpha_{\underline{213}} - \delta_{k>0}\alpha_{\underline{231}} + k\alpha_{\underline{321}} + \delta_{k>0}\alpha_{\underline{21}}, \end{split}$$

and δ denotes the Kronecker delta.

Proof. Let $\sigma \in S_n(312)$ and consider the inflation form $\sigma = 213[\sigma_1, 1, \sigma_2]$ where $\sigma_1 \in S_k(312)$ and $\sigma_2 \in S_{n-k-1}(312)$. Then for each $\rho \in \mathcal{P}$ we get the recursive relations

$$(\rho)\sigma = (\rho)\sigma_1 + (\rho)\sigma_2 + m_\rho,$$

where

$$\begin{split} m_{\underline{123}} &= [12)\sigma_2 + |\sigma_2|(\underline{12})\sigma_1, & m_{\underline{123}} &= (|\sigma_1| + 1)(\underline{12})\sigma_2, \\ m_{\underline{132}} &= [21)\sigma_2, & m_{\underline{132}} &= (|\sigma_1| + 1)(\underline{12})\sigma_2, \\ m_{\underline{213}} &= (\underline{(21)} + \delta_{\sigma_1 \neq \emptyset})|\sigma_2|, & m_{\underline{213}} &= |\sigma_1|\delta_{\sigma_2 \neq \emptyset}, \\ m_{\underline{231}} &= (\underline{12})\sigma_1, & m_{\underline{231}} &= (\underline{12}]\sigma_1, \\ m_{\underline{321}} &= (\underline{21})\sigma_1, & m_{\underline{321}} &= (\underline{21}]\sigma_1 \end{split}$$

and $m_{\underline{21}} = \delta_{\sigma_1 \neq \emptyset}$. It follows that stat_{α} satisfies the following recursion

$$\operatorname{stat}_{\alpha}(\sigma) = \operatorname{stat}_{\alpha}(\sigma_1) + \operatorname{stat}_{\alpha}(\sigma_1) + \sum_{\rho \in \mathcal{P}} m_{\rho}.$$

We note that $|\sigma_1| = k$, $|\sigma_2| = n - k - 1$, $(\underline{21})\sigma = \operatorname{des}(\sigma)$, $(\underline{12})\sigma = \delta_{\sigma\neq\emptyset}(|\sigma| - 1) - \operatorname{des}(\sigma)$, $[21)\sigma = \operatorname{head}(\sigma) - \delta_{\sigma\neq\emptyset}$, $[12)\sigma = |\sigma| - \operatorname{head}(\sigma)$, $(12]\sigma = \operatorname{last}(\sigma) - \delta_{\sigma\neq\emptyset}$ and $(21]\sigma = |\sigma| - \operatorname{last}(\sigma)$ for all $\sigma \in S_n(312)$. Converting these statements into generating functions proves the theorem.

Remark 4.11. If $\alpha_{231} = \alpha_{312} = \alpha_{321} = \alpha_{21} = 1$ and $\alpha_{\rho} = 0$ otherwise, then $\operatorname{stat}_{\boldsymbol{\alpha}} = \operatorname{inv}$ and $F(312, \boldsymbol{\alpha}; q, 1, 1, 1) = I_n(q) = \tilde{C}_n(q)$. Similarly if we choose $\boldsymbol{\alpha}$ such that $\operatorname{stat}_{\boldsymbol{\alpha}} = \operatorname{maj}$, then we recover the recursion in [13, Theorem 3.4] via the recursion for $F(312, \boldsymbol{\alpha}; q, t, 1, 1)$ in Theorem 4.10.

Recall the Simion-Schmidt bijection $\phi : S_n(123) \to S_n(132)$ which maps $\sigma \in S_n(123)$ to the unique permutation in $S_n(132)$ with the same left-to-right minima in the same positions as σ (cf Lemma 2.9). As explicitly noted by Claesson and Kitaev [10] this bijection clearly preserves the head statistic and hence $[123]_{\text{head}} = [132]_{\text{head}}$. Although head is not a Mahonian statistic we complete its st-Wilf classification below for all subsets of S_3 of size at most three. Equivalences for subsets of larger size can easily be found using similar analysis on the inflation forms. These are less interesting and omitted for brevity. We note in particular that the single pattern distributions with respect to the head statistic are given by well-known refinements of the Catalan numbers.

Proposition 4.12. We have

$$\begin{split} & [123]_{head} = \{123, 132\} = [132]_{head}, \\ & [321]_{head} = \{321, 312\} = [312]_{head}, \\ & [231]_{head} = \{213, 231\} = [213]_{head}, \\ & [123, 213]_{head} = \{\{123, 213\}, \{132, 213\}, \{132, 231\}\} \\ & [231, 321]_{head} = \{\{231, 321\}, \{213, 312\}, \{231, 312\}\} \\ & [213, 231, 321]_{head} = \{\{213, 231, 321\}, \{213, 231, 312\}\} \\ & [132, 213, 231]_{head} = \{\{132, 213, 231\}, \{132, 213, 231\}\} \\ & [132, 213, 321]_{head} = \{\{132, 213, 321\}, \{132, 213, 312\}\} \\ & [132, 213, 321]_{head} = \{\{132, 213, 321\}, \{132, 213, 312\}\} \\ & [132, 231, 312]_{head} = \{\{132, 213, 321\}, \{123, 213, 312\}, \{132, 231, 321\}, \\ & \{132, 231, 312\}, \{123, 213, 312\}\}. \end{split}$$

Remaining subsets $\Pi \subseteq S_3$ of size at most three have singleton head-Wilf class. Moreover for any $n \ge 1$

$$\sum_{\sigma \in \mathcal{S}_n(123)} q^{\text{head}(\sigma)} = \sum_{k=1}^n C_{n-1,k-1} q^k,$$
$$\sum_{\sigma \in \mathcal{S}_n(213)} q^{\text{head}(\sigma)} = \sum_{k=1}^n C_{k-1} C_{n-k} q^k$$
$$\sum_{\sigma \in \mathcal{S}_n(123,213)} q^{\text{head}(\sigma)} = q + \sum_{k=2}^n 2^{k-2} q^k.$$

where $C_n = \frac{1}{n+1} \binom{2n}{n}$ and $C_{n,k} = \frac{n-k+1}{n+1} \binom{n+k}{n}$ (A009766[29]).

Proof. The map $\psi : S_n(321) \to S_n(312)$ given by $\psi(\sigma) = \phi(\sigma^c)^c$, where $\phi : S_n(123) \to S_n(132)$ is the Simion-Schmidt bijection, clearly satisfies head $(\psi(\sigma)) =$ head (σ) . Hence $[321]_{\text{head}} = [312]_{\text{head}}$. Let $\sigma = a_1 a_2 \cdots a_n \in S_n(132)$. According to the non-recursive description of the standard bijection $\Delta : S_n(132) \to \mathcal{D}_n$ (due to Krattenthaler [21]), when a_i is read from left to right we adjoin as many U-steps as necessary to the path obtained thus far to reach height $h_j + 1$, followed by a D-step to height h_j . Here h_j is the number of letters in $a_{j+1} \cdots a_n$ which are larger than a_j . As such, the number of permutations $\sigma \in S_n(132)$ with head $(\sigma) = k$ is given by the number of Dyck paths starting with exactly n - k + 1 number of U-steps. These are equivalently enumerated by the number of lattice paths with steps (1,0) and (0,1) from (1, n - k + 1) to (n, n) staying weakly above the line y = x. By [22, Theorem 10.3.1] the number of such paths are given by

$$\binom{n+n-1-(n-k+1)}{n-(n-k+1)} - \binom{n+n-1-(n-k+1)}{n-1+1} = C_{n-1,k-1}.$$

The map $\varphi : S_n(231) \to S_n(213)$ recursively defined by $\varphi(213[1,\sigma_1,\sigma_2]) = 231[1,\varphi(\sigma_1),\varphi(\sigma_2)]$, where $\sigma_1 \in S_{k-1}(231)$ and $\sigma_2 \in S_{n-k}(231)$, is clearly a head-preserving bijection. Hence $[231]_{\text{head}} = [213]_{\text{head}}$. Since $|S_k(231)| = C_k$ it follows from the inflation form that there are $C_{k-1}C_{n-k}$ permutations $\sigma \in S_n(231)$ with head $(\sigma) = k$.

If $\sigma \in S_n(132, 231)$, then σ is either decomposed as $12[\sigma_1, 1]$ or as $21[1, \sigma_1]$ where $\sigma_1 \in S_{n-1}(132, 231)$. Thus the letters $1, 2, \ldots, n$ are in reverse order recursively placed at the beginning or at the end of the permutation. For σ to have head $(\sigma) = k$,

the letters k + 1, ..., n must be placed in increasing order at the end and k at the beginning. Remaining k - 1 letters may be placed on either end giving two choices each (except for the last letter). Hence there exists 2^{k-2} permutations $\sigma \in S_n(132, 231)$ with head $(\sigma) = k$ for k > 1.

Let $\iota_k = 12 \cdots k$ and $\delta_k = k \cdots 21$ for $k \ge 1$. If $\sigma \in \mathcal{S}_n(123, 213)$ and head $(\sigma) = k$, then $\sigma = 231[1, \delta_{n-k}, \sigma_1]$ for some $\sigma_1 \in \mathcal{S}_{k-1}(123, 213)$. It is easy to see that $|\mathcal{S}_k(123, 213)| = 2^{k-1}$ by induction. Hence $[132, 231]_{\text{head}} = [123, 213]_{\text{head}}$.

If $\sigma \in S_n(132, 213)$, then $\sigma = 231[1, \iota_{n-k}, \sigma_1]$ where $\sigma_1 \in S_{k-1}(132, 213)$. The map $\chi : S_n(132, 213) \to S_n(123, 213)$ recursively given by $\chi(231[1, \iota_{n-k}, \sigma_1]) =$ $231[1, \delta_{n-k}, \chi(\sigma_1)]$ is clearly a head-preserving bijection. Hence $[132, 213]_{\text{head}} =$ $[123, 213]_{\text{head}}$. Remaining equivalences and their distributions may be deduced from the fact that head(σ^c) = n – head(σ) + 1. The equivalences between the size three subsets can be proved similarly via bijections between their corresponding inflation forms (the inflation forms can be referenced in [13]). The details for these are left to the reader.

5. Summary and conjectures

In Table 2 we summarize the equidistributions proved in this paper (listed in black). In a given cell corresponding to stat_{row} and stat_{col} , a pair of patterns π_1, π_2 denotes the equidistribution

$$\sum_{\sigma \in \mathcal{S}_n(\pi_1)} q^{\operatorname{stat_{row}}(\sigma)} \sum_{\sigma \in \mathcal{S}_n(\pi_2)} q^{\operatorname{stat_{col}}(\sigma)}.$$

The equidistributions corresponding to $\operatorname{stat}_{\operatorname{row}} = \operatorname{maj} = \operatorname{stat}_{\operatorname{col}}$ and $\operatorname{stat}_{\operatorname{row}} = \operatorname{inv} = \operatorname{stat}_{\operatorname{col}}$ were proved in [13]. The equidistributions between maj, bast' and bast" can be proved in a similar way to Proposition 2.1, since the inverse map is the right bijection in two of the cases and the rest can be deduced via the maj-Wilf equivalences from [13]. Remaining equidistributions were either proved directly or follow by combining equidistributions proved in this paper. For instance $\sum_{\sigma \in S_n(213)} q^{\operatorname{maj}(\sigma)} = \sum_{\sigma \in S_n(231)} q^{\operatorname{foze}(\sigma)}$ is deduced by combining Proposition 2.1 and Theorem 2.12.

Conjecture 5.1. Table 2 is the complete table of Mahonian 3-function equidistributions over permutations avoiding a single classical pattern of length three.

We have verified all entries in Table 2 up to n = 10 by computer. Other than than the entries in Table 2 there are no additional equidistributions (over permutations avoiding a single classical pattern of length three) between the statistics in Table 1.

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_	maj	inv	mak	makl	mad	bast	bast'	bast''	foze	foze'	foze"	sist	sist'	sist''
maj	132,231 213,312		$\begin{array}{c} 123,123\\ 132,132\\ 132,312\\ 213,213\\ 213,231\\ 231,132\\ 231,312\\ 312,213\\ 312,231\\ 312,231\\ 321,321 \end{array}$	$\begin{array}{c} 132,231\\ 213,312\\ 231,231\\ 312,312\\ 321,321\\ \end{array}$		132, 213 213, 231 231, 213 312, 231	132, 132 231, 132	213, 231 312, 231	132, 132 213, 231 231, 132 312, 231					
$_{ m inv}$	•	$132,213\\231,312$			231,312 312,312 321,231					231, 132 312, 132 321, 213	231,132 312,132 321,213	231, 213 312, 213 321, 132	231,231 312,231 321,132	231, 132 312, 132 321, 231
mak	•	•	132,312 213,231	132, 231 213, 312 231, 312 312, 231 321, 321		$\begin{array}{c} 132,213\\ 213,231\\ 231,231\\ 312,213 \end{array}$	132, 132 312, 132	213, 231 231, 231	$\begin{array}{c} 132,132\\ 213,231\\ 231,231\\ 312,132 \end{array}$					
makl	•	•	•			132, 132 231, 213 312, 231	231,132	312, 231	$\begin{array}{c} 132,213\\ 231,132\\ 312,231 \end{array}$					
mad	•	•	•	•						$\frac{231,213}{312,132}$	$231,213 \\ 312,132$	$231, 132 \\ 312, 213$	132,213 231,132 312,231	213, 213 231, 231 312, 132
bast	•	•	•	•	•		213, 132	231, 231	$\begin{array}{c} 123,123\\ 213,132\\ 132,213\\ 231,231\\ 312,312\\ 321,321\\ \end{array}$					
bast'	•	•	•	•	•	•			132, 132					
bast''	•	٠	•	•	•	•	•		231, 231					
foze	•	•	•	•	•	•	•	•			132,132	132,213	132,231	132,132
foze'	•	•	•	•	•	•	•	•	•		132, 132 213, 213	213,132	213, 132	213,231
foze''	•	•	•	•	•	•	•	•	•	•		$213, 132 \\ 132, 213$	$213, 132 \\ 132, 231$	132, 132 213, 231
sist	•	•	•	•	•	•	•	•	•	•	•		132,132 213,231 312,312	132, 231 213, 132 231, 312
sist'	•	•	•	•	•	•	•	•	•	•	•	•		132,231 231,132
sist''	٠	٠	٠	٠	٠	٠	٠	٠	٠	٠	٠	٠	٠	

TABLE 2. Established equidistributions in black and conjectured equidistributions in red.

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