SHELLABLE POSETS ARISING FROM EVEN SUBGRAPHS OF A GRAPH

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Abstract. The concept of a poset of even subgraphs of a graph was firstly considered by S. Choi and H. Park to compute the rational Betti numbers of a real toric manifold associated with a simple graph. S. Choi and the authors extended this to a graph allowing multiple edges, motivated by the work on the pseudograph associahedron of Carr, Devadoss and Forcey. In this paper, we completely characterize the graphs (allowing multiple edges) whose posets of even subgraphs are always shellable.

1. Introduction

Throughout this paper, a graph permits multiple edges but not a loop, and when a simple graph is considered, we always mention that it is 'simple'. We only consider a finite poset and a finite graph.

Shellability is a combinatorial property of simplicial complexes with strong topological and algebraic consequences, and so it is one of crucial concepts in the theory of poset topology, which provides a deep and fundamental link between combinatorics and other branches of mathematics, see [16]. It has been an important research issue to study shellable simplicial complexes and investigate their topological properties. Among them, many simplicial complexes arising from graphs are beautiful objects with a rich topological structure and may hence be considered as interesting in their own right and see [9]. See also [8, 13, 15, 17, 18] for results on some simplicial complexes arising from graphs.

The main purpose of the paper is to characterize graphs G always having shellable posets of even subgraphs. Here, we consider shellability for a nonpure poset developed by Björner and Wachs [3, 4].

A maximal set of multiple edges which have the same pair of endpoints is called a *bundle*. A graph H is an *induced* (respectively, *semi-induced*) subgraph of G if H is a subgraph that includes all edges (respectively, at least one edge) between every pair of vertices in H, if such edges exist in G. A graph H is a *partial underlying induced graph* of a graph G if H can be obtained from an induced subgraph of G by replacing some bundles with simple edges. Let $\mathcal{A}^*(G)$ be the set of all pairs (H, C) of a partial underlying induced graph H of G and an admissible collection C of H, where an admissible collection C of H is defined to be a set consisting of an even number of vertices of H and an even positive number of multiple edges in each bundle of H with a certain property, see Definition 3.1. For each $(H, C) \in \mathcal{A}^*(G)$, $\mathcal{P}^{\text{even}}_{H,C}$ is a poset whose elements are all semi-induced subgraphs of H such that each component of I has an even number of elements in C, including both $\emptyset$ and H, ordered by subgraph containment.

In [6], the notion of $\mathcal{P}^{\text{even}}_{H,C}$ for a simple graph H was firstly introduced to study the topology of a real toric manifold associated with a simple graph, and it was shown that $\mathcal{P}^{\text{even}}_{H,C}$ is CL-shellable for every simple graph H. The work of [6] was generalized to a graph (allowing multiple edges) in [7], motivated by the work on the pseudograph associahedron in [5]. It was shown in [7] that the shellability of the poset $\mathcal{P}^{\text{even}}_{H,C}$ gives a direct consequence of the rational Betti numbers of a real toric manifold associated with a graph, and also asked to characterize all graphs G such that $\mathcal{P}^{\text{even}}_{H,C}$ is shellable for each pair (H, C) in $\mathcal{A}^*(G)$. Our main result is the following:

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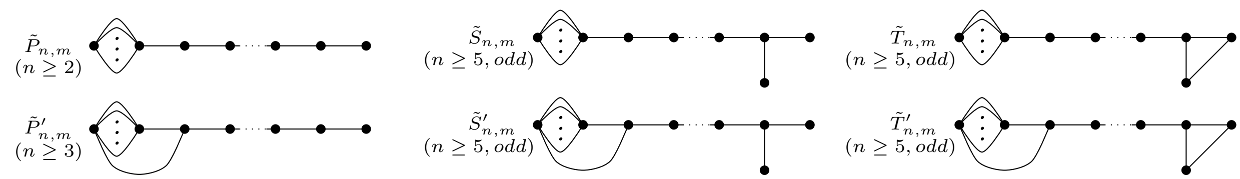
Theorem 1.1 (Main result)**.** *Let G be a graph. Then $\mathcal{P}^{\text{even}}_{H,C}$ is shellable for every $(H,C) \in A^*(G)$ if and only if each component of G is either a simple graph or one of the graphs in the following figure.*



Non-simple connected graphs with n vertices and m multiple edges $(m \geq 2)$

The proof of the 'if' part of our main result relies on finding a recursive atom ordering of a poset. We also provide some applications of the main result to the topology of a real toric manifold associated with a graph. We believe that this paper has a contribution to studying combinatorial and topological properties of a poset arising from a graph.

This paper is organized as follows. Section 2 collects some basic definitions and important facts about a poset and its shellability. Section 3 gives the definition of the poset $\mathcal{P}^{\text{even}}_{G,C}$ of the C-even subgraphs of a graph G, and then explains the main theorem of this paper. Section 4 proves a necessary condition of the main theorem, which gives a possible list of graphs G such that $\mathcal{P}^{\text{even}}_{H,C}$ is shellable for every $(H,C) \in \mathcal{A}^*(G)$. Section 5 proves a sufficient condition of the main theorem, which shows the CL-shellability of each $\mathcal{P}^{\text{even}}_{H,C}$ for a graph G in the list and $(H,C) \in \mathcal{A}^*(G)$. In Section 6, as an important application of our result, we explain how to compute the rational Betti numbers of a real toric manifold associated with a graph using our result, and then study it for the graph $\tilde{P}_{n,2}$ in the figure of the main result. Section 7 gives some further questions.

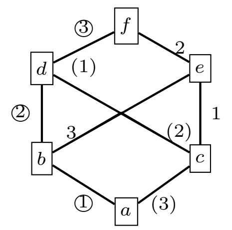
2. Preliminaries: A poset and its shellability

In this section, we prepare some notions and basic facts about a poset and its shellability. See [16] for more detailed explanation about this section.

We only consider a finite poset in this paper. Let $\mathcal{P}$ be a poset (partially ordered set). For two elements $x, y \in \mathcal{P}$, we say y *covers* x, denoted by $x \lessdot y$, if $x < y$ and there is no z such that $x < z < y$. We also call it a cover $x \lessdot y$. One represents $\mathcal{P}$ as a mathematical diagram, called a *Hasse diagram*, in a way that a point in the plane is drawn for each element of $\mathcal{P}$, and a line segment or curve is drawn upward from x to y whenever y covers x. A *chain* of $\mathcal{P}$ is a totally ordered subset σ of $\mathcal{P}$, and we say the *length* $\ell(\sigma)$ of σ is $|\sigma| - 1$. We say $\mathcal{P}$ is *pure* if all maximal chains have the same length. The *length* $\ell(\mathcal{P})$ of $\mathcal{P}$ is the length of a longest chain of $\mathcal{P}$. For $x \leq y$ in $\mathcal{P}$, let $[x, y]$ denote the (closed) interval $\{z \in \mathcal{P} \colon x \leq z \leq y\}$. We say $\mathcal{P}$ is *semimodular* if for all $x, y \in \mathcal{P}$ that cover $a \in \mathcal{P}$, there is an element $b \in \mathcal{P}$ that covers both x and y. If every closed interval of $\mathcal{P}$ is semimodular, then $\mathcal{P}$ is said to be *totally semimodular*. If $\mathcal{P}$ has a unique minimum element, it is usually denoted by $\hat{0}$ and referred to as the bottom element. Similarly, the unique maximum element, if it exists, is denoted by $\hat{1}$ and referred to as the top element. An element of $\mathcal{P}$ that covers the bottom element, if it exists, is called an *atom*. We say $\mathcal{P}$ is *bounded* if it has the elements $\hat{0}$ and $\hat{1}$. The *order complex* of $\mathcal{P}$, denoted by $\Delta(\mathcal{P})$, is an abstract simplicial complex whose faces are the chains of $\mathcal{P}$. Note that if $\mathcal{P}$ has either $\hat{0}$ or $\hat{1}$, then $\Delta(\mathcal{P})$ is contractible, hence we usually remove the top and bottom elements, and then study the topology of the remaining part. The *proper part* of a bounded poset $\mathcal{P}$ with length at least one is defined to be $\overline{\mathcal{P}} := \mathcal{P} - \{\hat{0}, \hat{1}\}$.

The notion of shellability was firstly appeared in the middle of the nineteenth century in the computation of the Euler characteristic of a convex polytope [11], and in this paper shellability refers to the general notion of nonpure shellability introduced in [3]. A simplicial complex K is *shellable* if its facets can be arranged in linear order $F_1, F_2, \dots, F_t$ in such a way that the subcomplex $(\sum_{i=1}^{k-1} \overline{F_i}) \cap \overline{F_k}$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, \dots, t$. Such an ordering of the facets is called a *shelling*. A poset $\mathcal{P}$ is said to be *shellable* if its order complex $\Delta(\mathcal{P})$ is shellable.

A chain-lexicographic shellability (CL-shellability for short) was introduced by Björner and Wachs to establish the shellability of Bruhat order on a Coxeter group [2]. It is known that CL-shellability is stronger than shellability, that is, if a bounded poset is CL-shellable, then it is shellable, but the converse is not true, see [14]. Let $\mathcal{P}$ be a bounded poset. We denote by $\mathcal{ME}(\mathcal{P})$ the set of pairs $(\sigma, x \lessdot y)$ consisting of a maximal chain σ and a cover $x \lessdot y$ along that chain. For $x, y \in \mathcal{P}$ and a maximal chain r of $[\hat{0}, x]$, the closed rooted interval $[x, y]_r$ of $\mathcal{P}$ is a subposet of $\mathcal{P}$ obtained from $[x, y]$ adding the chain r. A *chain-edge labeling* of $\mathcal{P}$ is a map $\lambda \colon \mathcal{ME}(\mathcal{P}) \to \Lambda$, where Λ is some poset satisfying; if two maximal chains coincide along their bottom d covers, then their labels also coincide along these covers. A *chain-lexicographic labeling* (*CL-labeling* for short) of a bounded poset $\mathcal{P}$ is a *chain-edge labeling* such that for each closed rooted interval $[x, y]_r$ of $\mathcal{P}$, there is a unique strictly increasing maximal chain, which lexicographically precedes all other maximal chains of $[x, y]_r$. A poset that admits a CL-labeling is said to be *CL-shellable*. Figure 1 shows an example of a CL-shellable poset.



Labeling of the covers in chain $a < b < d < f$ is 1, 2, 3 (marked as ①, ②, ③).
Labeling of the covers in chain $a < b < e < f$ is 1, 3, 2 (marked as ①, 3, 2).
Labeling of the covers in chain $a < c < d < f$ is 3, 2, 1 (marked as (3), (2), (1)).
Labeling of the covers in chain $a < c < e < f$ is 3, 1, 2 (marked as (3), 1, 2).

FIGURE 1. A chain-edge labeling of a poset with four maximal chains (same example in [16])

We recall well-known properties on shellability and CL-shellability which we will use. The *product* $\mathcal{P} \times \mathcal{Q}$ of two posets $\mathcal{P}$ and $\mathcal{Q}$ is the new poset with partial order given by $(a, b) \le (c, d)$ if and only if $a \le c$ (in $\mathcal{P}$) and $b \le d$ (in $\mathcal{Q}$).

Theorem 2.1 ([1,3,4])**.** *The following hold:*

(1) *Every (closed) interval of a shellable (respectively, CL-shellable) poset is shellable (respectively, CL-shellable).*
(2) *The product of bounded posets is shellable (respectively, CL-shellable) if and only if each of the posets is shellable (respectively, CL-shellable).*
(3) *A bounded poset is pure and totally semimodular, then it is CL-shellable.*

It is worthy to note that the homotopy type of $\Delta(\overline{\mathcal{P}})$ is known when a bounded poset $\mathcal{P}$ has a CL-labeling $\lambda \colon \mathcal{ME}(\mathcal{P}) \to \Lambda$. A *falling chain* $\sigma : x_0 \lessdot x_1 \lessdot \cdots \lessdot x_\ell$ of $\mathcal{P}$ is a maximal chain such that $\lambda(\sigma, x_{i-1} \lessdot x_i) \ge_\Lambda \lambda(\sigma, x_i \lessdot x_{i+1})$ in Λ for every $1 \le i < \ell(\sigma)$.

Theorem 2.2 ([3])**.** *If a bounded poset $\mathcal{P}$ is CL-shellable, then $\Delta(\overline{\mathcal{P}})$ has the homotopy type of a wedge of spheres. Furthermore, for any fixed CL-labeling, the reduced ith Betti number of $\Delta(\overline{\mathcal{P}})$ is equal to the number of falling chains of length $i + 2$.*

The poset in Figure 1 has exactly one falling chain $a < c < d < f$, and $\Delta(\overline{P})$ is homotopy equivalent to S^1.

A recursive atom ordering is an alternative approach to lexicographic shellability, which is known to be an equivalent concept of CL-shellability.

Definition 2.3. A bounded poset $\mathcal{P}$ is said to *admit a recursive atom ordering* if its length $\ell(\mathcal{P})$ is 1, or $\ell(\mathcal{P}) > 1$ and there is an ordering $\alpha_1, \dots, \alpha_t$ of the atoms $\mathcal{P}$ that satisfying the following:

(1) For all $j = 1, \dots, t$, the interval $[\alpha_j, \hat{1}]$ admits a recursive atom ordering in which the atoms of $[\alpha_j, \hat{1}]$ that belong to $[\alpha_i, \hat{1}]$ for some $i < j$ come first.
(2) For all i, j with $1 \le i < j \le t$, if $\alpha_i, \alpha_j < y$ then there exist an integer k and an atom z of $[\alpha_j, \hat{1}]$ such that $1 \le k < j$ and $\alpha_k < z \le y$.

For example, for the poset in Figure 1, if we order the atoms of each interval by an alphabetical order (for the atoms of $[a, f]$, the ordering is $b \prec c$, for the atoms of $[b, f]$, the ordering is $d \prec e$, and for the atoms of $[c, f]$, the ordering is $d \prec e$), then it is a recursive atom ordering.

We note that any atom ordering of a pure totally semimodular bounded poset is a recursive atom ordering, which implies (3) of Theorem 2.1. We finish the section by introducing a sketch of the proof shown in [3] that the existence of a recursive atom ordering implies CL-shellability.

Theorem 2.4 ([3]). *A bounded poset admits a recursive atom ordering if and only if it is CL-shellable.*

Sketch of proof of the 'only if' part. Let us give an integer labeling λ of the bottom covers of $\mathcal{P}$ such that $\lambda(\hat{0}, \alpha_i) < \lambda(\hat{0}, \alpha_j)$ for all $i < j$. For each j, let $F(\alpha_j)$ be the set of all atoms of $[\alpha_j, \hat{1}]$ that cover some α_i where $i < j$. We label the bottom covers of $[\alpha_j, \hat{1}]$ consistently with the atom ordering of $[\alpha_j, \hat{1}]$ and satisfying

$$x \in F(\alpha_j) \Rightarrow \lambda(\alpha_j, x) < \lambda(\hat{0}, \alpha_j) \qquad \text{and} \qquad x \notin F(\alpha_j) \Rightarrow \lambda(\alpha_j, x) > \lambda(\hat{0}, \alpha_j),$$

where λ denotes the labeling of the bottom covers of $[\alpha_j, \hat{1}]$ as well as the original labeling of the bottom covers of $\mathcal{P}$. This labeling inductively extends to an integer CL-labeling of $[\alpha_j, \hat{1}]$. Choosing such an extension at each α_j, we obtain a chain-edge labeling λ of $\mathcal{P}$ which is a CL-labeling of $[\alpha_j, \hat{1}]$ for all $j = 1, \dots, t$, and hence for every rooted interval whose bottom element is not $\hat{0}$, and which extends the original labeling of the bottom covers of $\mathcal{P}$. Then one can show that the unique lexicographically first maximal chain of each interval $[0, y]$ is the only increasing maximal chain of that interval. Hence the labeling λ is an integer CL-labeling on $\mathcal{P}$. □

3. A poset $\mathcal{P}_{G,C}^{even}$ of C-even subgraphs of a graph G and the main result

In this section, we give basic definitions related to graphs and then define the poset $\mathcal{P}_{G,C}^{\text{even}}$. Here, the poset $\mathcal{P}_{G,C}^{\text{even}}$ arose from the computation of the rational Betti numbers of a real toric manifold associated with a graph G in [7][1], which gives a strong motivation for the main question of this paper. At this moment, we do not review the definition of pseudograph associahedron or its corresponding real toric manifold (see [7, Sections 2 and 3], for readers to find a much more detailed account of results of pseudograph associahedra). In Section 6, we simply explain how the main result of this paper is useful in computing the rational Betti numbers of a real toric manifold.

For a graph $G = (V, E)$, an element of V and an element of E are called a *vertex* and an *edge* respectively, and we only consider a finite graph not allowing a loop, an edge whose endpoints are the same. An edge e is said to be *multiple* if there exists another edge e' which has the same endpoints as e. An edge which is not a multiple edge is said to be *simple*. A *bundle* is a maximal set of multiple edges which have the same endpoints. A *simple graph* is a graph having neither bundles nor loops.

Let G be a graph. A subgraph H of G is an *induced* (respectively, *semi-induced*) subgraph of G if H includes all edges (respectively, at least one edge) between every pair of vertices in H if such edges exist in G. A graph H is a *partial underlying graph* of G if H can be obtained from G by replacing some bundles with simple edges, that is, the set of all the bundles of H is a subset of that of G. A graph H is a *partial underlying induced* graph (*PI*-graph for short) of G if H is an induced subgraph of some partial underlying graph of G. For example, see the graph G with two bundles $\{a, b\}$ and $\{c, d, e\}$ in

[1] In [5, 7], a graph allowing multiple edges is called a *pseudograph*.

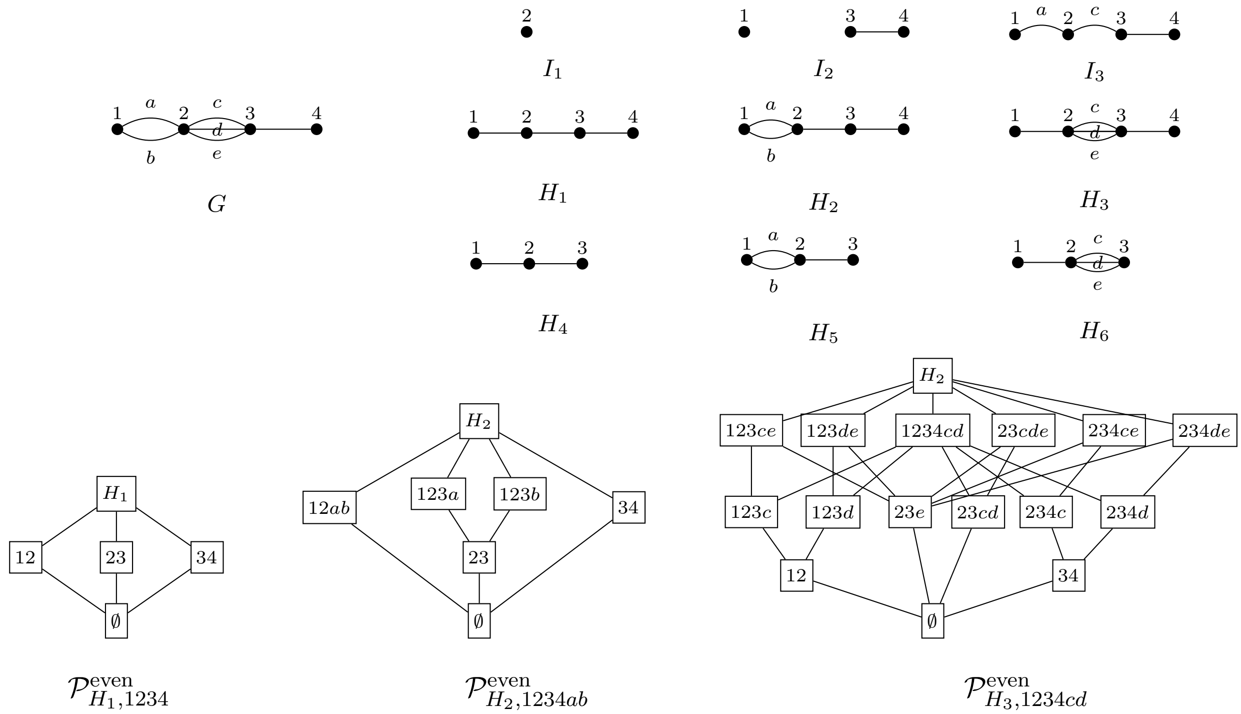


FIGURE 2. Examples for PI-graphs of G and the posets $\mathcal{P}^{\text{even}}_{H,C}$

Figure 2, where I_1, I_2, and I_3 are semi-induced subgraphs of G, H_1, H_2, and H_3 are partial underlying graphs of G, and all H_i's and the subgraph I_1 are PI-graphs of G.

Before stating the definitions of the main notions of the paper, we need to explain a way to denote a subgraph of a graph by a set. For a graph G, we label the vertices and the multiple edges of a graph G and we set $\mathcal{C}_G = V(G) \cup B_1 \cup \cdots \cup B_k$, where $B_1, \ldots, B_k$ are the bundles of G. For instance, $\mathcal{C}_G = \{1,2,3,4,a,b,c,d,e\}$ and $\mathcal{C}_{H_3} = \{1,2,3,4,c,d,e\}$ for the graphs G and H_3 in Figure 2. A subgraph I of G will be written as the set of the vertices of I and the edges of I in a bundle of G. For instance, the three subgraphs $I_1 \sim I_3$ of G in Figure 2 are expressed as $I_1 = \{2\}$, $I_2 = \{1,3,4\}$, and $I_3 = \{1,2,3,4,a,c\}$. It should be noted that for a semi-induced subgraph I, this set expression makes sense because I is distinguishable by the corresponding set. In the same sense, for a semi-induced subgraph I, we say $\alpha \in I$ if α is a vertex of I or an edge of I which is a multiple edge of G. For simplicity, we omit the braces and commas to denote a subset of $\mathcal{C}_G$ and we always denote it in a way that the vertices precede the multiple edges. For the semi-induced subgraphs I_i's in Figure 2,

$$I_1 = 2, \quad I_2 = 134, \quad I_3 = 1234ac.$$

We remark that when we consider a subgraph I of a graph G, the labels of I are inherited from the labels of G. Thus if a graph I is considered as a subgraph of a graph G, then I may have a labeled simple edge, which is not in a bundle of I (actually, it is in a bundle of G).

Definition 3.1. For a connected graph H, a subset C of $\mathcal{C}_H$ is *admissible* to H if the following hold:

(1) $|C \cap V(H)| \equiv 0 \pmod 2$ and each vertex incident to only simple edges of H is contained in C,
(2) $B \cap C \neq \emptyset$ and $|B \cap C| \equiv 0 \pmod 2$, for each bundle B of H.

For a disconnected graph H, $C \subset \mathcal{C}_H$ is *admissible* to H if $\mathcal{C}_{H'} \cap C$ is admissible to H' for each component H' of H.

We denote by $\mathcal{A}(H)$ the set of all the admissible collections of H. For each $C \in \mathcal{A}(H)$, a semi-induced subgraph I of H is said to be C-*even* if $|I' \cap C|$ is even for each component I' of I. Now we define the poset $\mathcal{P}^{\text{even}}_{H,C}$ by the poset consisting of all C-even semi-induced subgraphs of H ordered by subgraph containment, including both $\emptyset$ and H. Note that if $\mathcal{A}(H) = \emptyset$ then $\mathcal{P}^{\text{even}}_{H,C}$ is defined to be the null poset, and if $\mathcal{A}(H) \neq \emptyset$ then $\mathcal{P}^{\text{even}}_{H,C}$ is a bounded poset. For the graphs H_i's in Figure 2,

$$\begin{array}{lll} \mathcal{A}(H_1) = \{1234\}, & \mathcal{A}(H_2) = \{34ab, 1234ab\}, & \mathcal{A}(H_3) = \{14cd, 14ce, 14de, 1234cd, 1234ce, 1234de\}, \\ \mathcal{A}(H_4) = \emptyset, & \mathcal{A}(H_5) = \{12ab, 23ab\}, & \mathcal{A}(H_6) = \{12cd, 12ce, 12de, 13cd, 13ce, 13de\}. \end{array}$$

Figure 2 shows the posets $\mathcal{P}^{\text{even}}_{H_1,1234}$, $\mathcal{P}^{\text{even}}_{H_2,1234ab}$ and $\mathcal{P}^{\text{even}}_{H_3,1234cd}$. Note that the first two posets are shellable but the last one is not. For more examples of $\mathcal{P}^{\text{even}}_{H,C}$, see also Figure 10.

Let H be a simple graph. Then $\mathcal{A}(H) = \{H\}$ if each component of H has an even number of vertices, and $\mathcal{A}(H) = \emptyset$ otherwise. Thus we write $\mathcal{P}^{\text{even}}_{H}$ instead of $\mathcal{P}^{\text{even}}_{H,H}$. In [6], it is shown that $\mathcal{P}^{\text{even}}_{H}$ is always shellable.

Theorem 3.2 ([6]). *Let H be a simple graph such that each component has an even number of vertices. Then $\mathcal{P}^{\text{even}}_{H}$ is pure and totally semimodular, and so it is shellable.*

Since a pure and totally semimodular poset is CL-shellable by (3) of Theorem 2.1, $\mathcal{P}^{\text{even}}_{H}$ is CL-shellable when H is a simple graph such that each component has an even number of vertices. Thus Theorem 3.2 says that for any induced subgraph H of a simple graph G such that $\mathcal{A}(H) \neq \emptyset$, the poset $\mathcal{P}^{\text{even}}_{H}$ is CL-shellable.

Remark 3.3. In [6], Theorem 3.2 is used to determine the homotopy type of the order complex $\Delta(\overline{\mathcal{P}^{\text{even}}_{H}})$. Finally, $\Delta(\overline{\mathcal{P}^{\text{even}}_{H}})$ is homotopy equivalent to a wedge of the same dimensional spheres, and the Möbius invariant $\mu(\mathcal{P}^{\text{even}}_{H})$[2] is equal to the $(\ell-2)$th Betti number of $\Delta(\overline{\mathcal{P}^{\text{even}}_{H}})$, where ℓ is the length of the poset $\mathcal{P}^{\text{even}}_{H}$. For example, when H is a simple path graph P_{2n} with $2n$ vertices, $\mu(\mathcal{P}^{\text{even}}_{H}) = (-1)^n C_n$, where C_n is the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$, and hence $\Delta(\overline{\mathcal{P}^{\text{even}}_{H}})$ is homotopy equivalent to $\bigvee_{C_n} S^{n-2}$.

In [7], there was an effort to extend results of [6] for a simple graph to a graph allowing multiple edges. Almost all results of [6] except for Theorem 3.2 were well-extended by using $\mathcal{P}^{\text{even}}_{H,C}$ where H is a PI-graph of G and $C \in \mathcal{A}(H)$. As the poset $\mathcal{P}^{\text{even}}_{H_3,1234cd}$ in Figure 2 is not shellable, Theorem 3.2 cannot be generalized to $\mathcal{P}^{\text{even}}_{H,C}$. Hence it is natural to ask which $\mathcal{P}^{\text{even}}_{H,C}$ is shellable. From an interest of the topology of a real toric manifold associated with a graph, the following Question 3.4 was asked in [7], instead of asking the conditions on (H, C) to give a shelling of $\mathcal{P}^{\text{even}}_{H,C}$. For a graph G, let $\mathcal{A}^*(G) = \{(H, C) \mid H \text{ is a PI-graph of } G \text{ and } C \in \mathcal{A}(H)\}$.

Question 3.4 ([7]). Find all graphs G such that $\mathcal{P}^{\text{even}}_{H,C}$ is shellable for every $(H, C) \in \mathcal{A}^*(G)$.

For simplicity, throughout the paper, let $\mathcal{G}^*$ be the family of all graphs G such that $\mathcal{P}^{\text{even}}_{H,C}$ is shellable for every $(H, C) \in \mathcal{A}^*(G)$. We will give some remark that $\mathcal{G}^*$ is distinct from the set of all graphs G such that $\mathcal{P}^{\text{even}}_{G,C}$ is shellable for each $C \in \mathcal{A}(G)$ in Section 7. Clearly, the family $\mathcal{G}^*$ contains all simple graphs by Theorem 3.2. The answer to Question 3.4 is the following, which restates Theorem 1.1.

[2]The Möbius function μ, introduced by Rota in [10], is inductively defined as follows: for a poset $\mathcal{P}$, for elements x and y in $\mathcal{P}$,

$$\mu_{\mathcal{P}}(x, y) = \begin{cases} 1 & \text{if } x = y \\ -\displaystyle\sum_{z\,:\,x \le z < y} \mu_{\mathcal{P}}(x, z) & \text{if } x < y \\ 0 & \text{otherwise.} \end{cases}$$

For a bounded poset $\mathcal{P}$, the *Möbius invariant* is defined as $\mu(\mathcal{P}) = \mu_{\mathcal{P}}(\hat{0}, \hat{1})$. See [12] for various techniques for computing the Möbius function of a poset.

Theorem 1.1 (Main result) *A graph G is in $\mathcal{G}^*$ if and only if each component of G is either a simple graph or one of the graphs in Figure 3.*

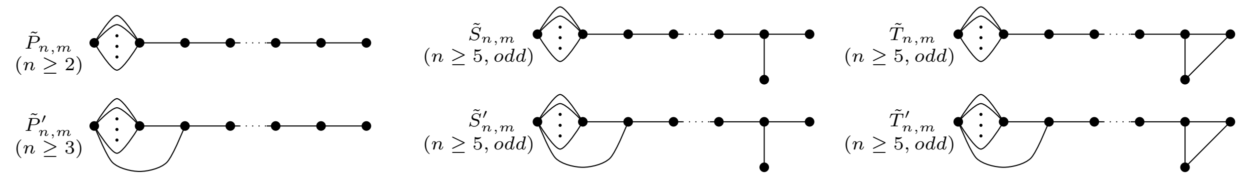


FIGURE 3. Non-simple connected graphs in $\mathcal{G}^*$ with n vertices and m multiple edges ($m \geq 2$)

As an immediate consequence of Section 5, we also get a generalization of Theorem 3.2 as follows.

Theorem 3.5. *For every $G \in \mathcal{G}^*$, each $\mathcal{P}^{\text{even}}_{H,C}$ is CL-shellable for every $(H, C) \in \mathcal{A}^*(G)$.*

We finish the section by giving a remark that it is sufficient to consider a connected graph to prove Theorem 1.1 and Theorem 3.5. To see why, let $G_1, \ldots, G_k$ be the components of a graph G. Note that for a subgraph H of G and $C \in \mathcal{C}_H$, $(H, C) \in \mathcal{A}^*(G)$ if and only if $(H \cap G_i, C \cap \mathcal{C}_{G_i}) \in \mathcal{A}^*(G_i)$ for each i. Thus for each $(H, C) \in \mathcal{A}^*(G)$, $\mathcal{P}^{\text{even}}_{H,C}$ is isomorphic to the product $\mathcal{P}^{\text{even}}_{H_1,C_1} \times \cdots \times \mathcal{P}^{\text{even}}_{H_k,C_k}$, where $H_i = H \cap G_i$ and $C_i = C \cap \mathcal{C}_{G_i}$ for each i. By (2) of Theorem 2.1, $\mathcal{P}^{\text{even}}_{H,C}$ is shellable if and only if $\mathcal{P}^{\text{even}}_{H_i,C_i}$ is shellable for each i. Thus $G \in \mathcal{G}^*$ if and only if $G_i \in \mathcal{G}^*$ for each i.

4. Graphs which admit a non-shellable poset $\mathcal{P}^{\text{even}}_{H,C}$

In this section, we give the 'only if' part of Theorem 1.1. We will see that almost all graphs do not belong to the family $\mathcal{G}^*$. The results of this section are obtained from the following basic observation.

Lemma 4.1. *Let $\mathcal{P}_0$ be a poset in Figure 4 and $\mathcal{Q}$ be its subposet which has two chains of length 3, one contains a or b, and the other contains a' or b'. Then $\mathcal{Q}$ is not shellable.*

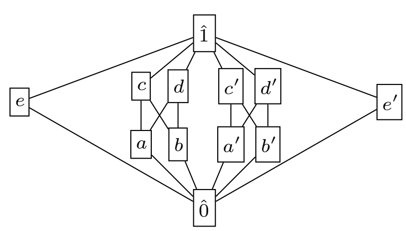


FIGURE 4. The poset $\mathcal{P}_0$

Theorem 4.2. *Let G be a connected non-simple graph in $\mathcal{G}^*$. Then G is one of the graphs in Figure 3.*

Before starting the proof, recall that we often drop the braces and commas to denote a subset of $\mathcal{C}_G$.

Proof. Suppose that G is a connected non-simple graph in $\mathcal{G}^*$. If $|V(G)| = 2$, then $G = \tilde{P}_{2,m}$ in Figure 3 for some m. Assume that $|V(G)| \geq 3$ and G has a bundle B whose endpoints are 1 and 2.

Claim 4.3. *The graph G has exactly one bundle B.*

Proof of Claim 4.3. Suppose that G has a bundle B' other than B. Take a shortest path Q in G whose starting vertex is an endpoint of B and whose terminal vertex is an endpoint of B'. Let $Q := (v_1, \ldots, v_k)$, where $k \geq 1$, and let $v_1 = 2$ without loss of generality. Let H be a PI-graph of G such that $V(H) = V(Q) \cup \{1, 2\} \cup \{\text{endpoints of } B'\}$ and H has exactly two bundles B and B'. Let $a, b \in B$ and $a', b' \in B'$.

(Case 1) Suppose that $k = 1$. Then $|V(H)| = 3$, so we set $V(H) = \{1, 2, 3\}$. Then $C := 23aba'b'$ belongs to $\mathcal{A}(H)$. Setting $I = 123aba'b'$ (the dotted edge in Figure 5 is a simple edge or does not exist), we see $I \cap C = C$ and hence I is an element of $\mathcal{P}^{\text{even}}_{H,C}$. Let $I' = 1$ and consider the interval $\mathcal{I} = [I', I]$ of $\mathcal{P}^{\text{even}}_{H,C}$. Then $\mathcal{I}$ is a subposet of $\mathcal{P}_0$ in Figure 4 as in Figure 5. By Lemma 4.1, $\mathcal{I}$ is not shellable, a contradiction to (1) of Theorem 2.1.

FIGURE 5. A graph I and the interval $\mathcal{I}$

(Case 2) Suppose that $k \geq 2$. Let the endpoints of B' be labeled by 3 and 4, and $v_k = 3$. Let

$$C = \begin{cases} (V(H) \setminus \{1\}) \cup aba'b' & \text{if } k \text{ is odd;} \\ (V(H) \setminus \{1, 2\}) \cup aba'b' & \text{if } k \text{ is even.} \end{cases}$$

Note that $C \in \mathcal{A}(H)$. Let $I' = V(Q) \setminus \{v_k\}$, and $I = I' \cup 134aba'b'$. Then $I' \cap C = \{v_1, \dots, v_{k-1}\}$ (if k is odd) or $I' \cap C = \{v_2, \dots, v_{k-1}\}$ (if k is even). Then they have the form in Figure 6 (the dotted edges are simple edges or do not exist), and both I' and I are elements of $\mathcal{P}^{\text{even}}_{H,C}$. Consider the interval $\mathcal{I} = [I', I]$ in $\mathcal{P}^{\text{even}}_{H,C}$. Thus $\mathcal{I}$ is a subposet of $\mathcal{P}_0$ in Figure 4 as in Figure 6. Note that $I' \cup 134aa'$, $I' \cup 134ab'$, $I' \cup 134ba'$, $I' \cup 134ba'$ are elements in $\mathcal{I}$, and both $I' \cup 13a$ and $I' \cup 13b$ are also elements in $\mathcal{I}$. By Lemma 4.1, $\mathcal{I}$ is not shellable, a contradiction to (1) of Theorem 2.1. □

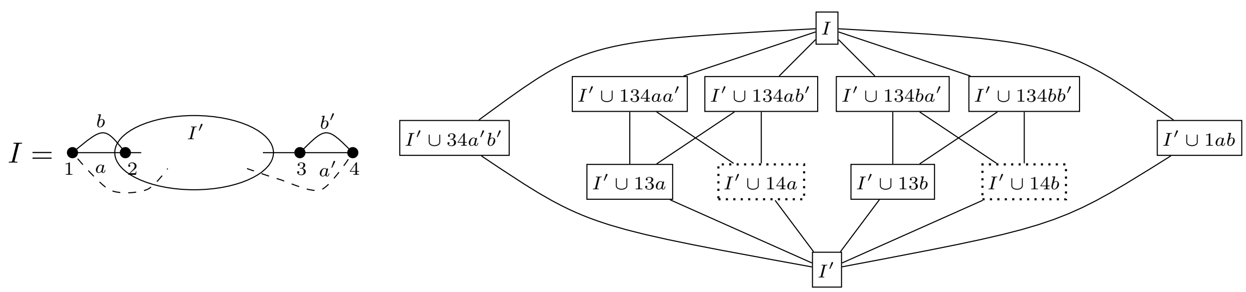


FIGURE 6. A graph I and the interval $\mathcal{I}$ where the dotted boxes may be in $\mathcal{I}$

Hence G has the only one bundle B. If $|V(G)| = 3$, then clearly G is one of the graphs in Figure 3. Now assume that $|V(G)| \geq 4$. For each vertex i, we let $N^*(i) = N_G(i) \setminus \{1, 2\}$, where $N_G(i)$ is the set of vertices which are adjacent to i in G.

Claim 4.4. $|N^*(1) \cup N^*(2)| = 1$.

Proof of Claim 4.4. Since $|V(G)| \geq 3$ and G is connected, $|N^*(1) \cup N^*(2)| \geq 1$. Suppose that $|N^*(1) \cup N^*(2)| \geq 2$, and $3, 4 \in N^*(1) \cup N^*(2)$. Let H be a PI-graph of G such that $V(H) = \{1, 2, 3, 4\}$ and H has the bundle B. Let $C = 1234ab$ for some $a, b \in B$. Note that $C \in \mathcal{A}(H)$. Let $I = 1234ab$, and consider the interval $\mathcal{I} = [\emptyset, I]$ in $\mathcal{P}^{\text{even}}_{H,C}$. Then I is a subgraph of a complete graph of four vertices with exactly one bundle of size two, and $\mathcal{I}$ is a subposet of $\mathcal{P}_0$ as in Figure 7. Note $123a$, $123b$, $124a$, $124b$ are elements of $\mathcal{I}$. Since the vertex 3 is a neighbor of 1 or 2, at least one of 13 and 23 is an element of

$\mathcal{I}$. Similarly, since the vertex 4 is also a neighbor of 1 or 2, at least one of 14 and 24 is an element of $\mathcal{I}$. By Lemma 4.1, $\mathcal{I}$ is not shellable, a contradiction to (1) of Theorem 2.1. □

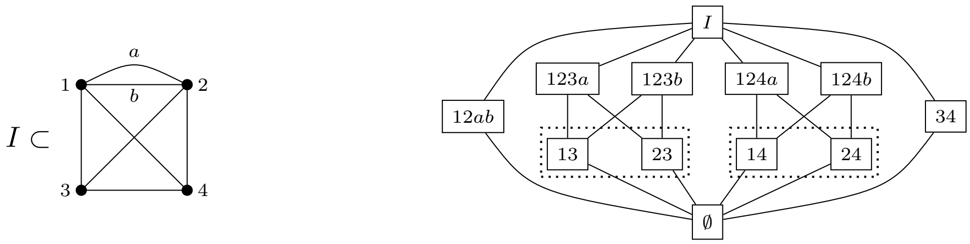


FIGURE 7. A graph containing I and the interval $\mathcal{I}$ where at least one of the elements in each dotted box is in $\mathcal{I}$

From now on, we set $N^*(1) \cup N^*(2) = N^*(2) = \{3\}$.

Claim 4.5. *For each vertex i other than 1 or 2, let Q_i be a shortest path of G from 3 to i. Then*

$$|N^*(i) \setminus V(Q_i)| \leq 2,$$

where the equality holds if and only if $|V(Q_i)|$ is odd and $V(G) = V(Q_i) \cup \{1,2\} \cup N^*(i)$.

Proof of Claim 4.5. Suppose that there is a vertex $i \in V(G) \setminus \{1,2\}$ satisfying one of the following:

(1) $|N^*(i) \setminus V(Q_i)| \geq 3$;
(2) $|N^*(i) \setminus V(Q_i)| = 2$ and $|V(Q_i)|$ is even;
(3) $|N^*(i) \setminus V(Q_i)| = 2$, $|V(Q_i)|$ is odd, and $V(G) \neq V(Q_i) \cup \{1,2\} \cup N^*(i)$.

If $|V(Q_i)|$ is even then we set $I' = Q_i$, and if $|V(Q_i)|$ is odd then we set $I' = Q_i \cup \{w\}$ by taking some vertex $w \in N^*(i) \setminus V(Q_i)$. Then $3 \in I'$, $I' \cap \{1,2\} = \emptyset$, $|I'|$ is even, and I' is a connected subgraph of G. Furthermore, there are two vertices x and y in $V(G) \setminus (I' \cup \{1,2\})$ such that both $I' \cup x$ and $I' \cup y$ are connected. More precisely, for the cases of (1) and (2), x and y are selected from $N^*(i) \setminus V(I')$. For the case of (3), x is selected from $N^*(i) \setminus V(I')$ and y is a vertex in $V(G) \setminus (V(Q_i) \cup \{1,2\} \cup N^*(i))$ which is closest to the vertex 1 or 2. Let H be a PI-graph such that $V(H) = I' \cup 12xy$ and B is the bundle of H. Let $C = V(H) \cup ab$ and $I = C$ for some $a, b \in B$. Note that $C \in \mathcal{A}(H)$ and I is the graph in the left of Figure 8 (the dotted edges are simple edges or do not exist). Consider the interval $\mathcal{I} = [I', I]$ in $\mathcal{P}^{\text{even}}_{H,C}$, and then $\mathcal{I}$ is a subposet of $\mathcal{P}_0$ as in Figure 8. Note that $I' \cup 12xa$, $I' \cup 12xb$, $I' \cup 12ya$, and $I' \cup 12yb$ are elements in $\mathcal{I}$. Moreover, both $I' \cup 2x$ and $I' \cup 2y$ are in $\mathcal{I}$. By Lemma 4.1, $\mathcal{I}$ is not shellable, a contradiction to (1) of Theorem 2.1. □

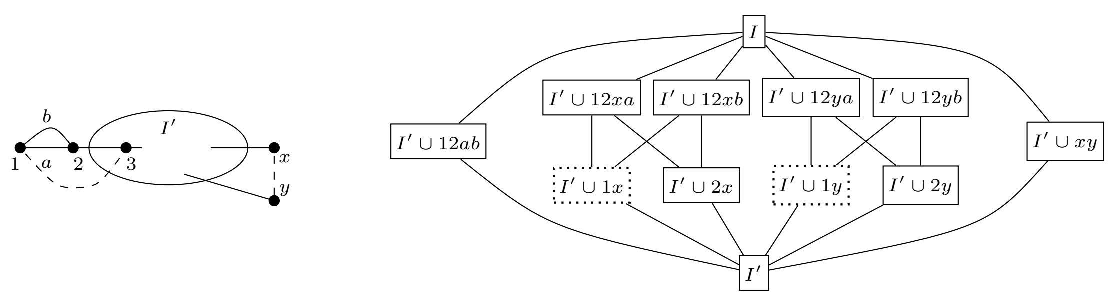


FIGURE 8. A graph containing I and the poset containing $\mathcal{I}$ where the dotted boxes may be in $\mathcal{I}$

Since $|V(G)| \geq 4$, we have $|N^*(3)| \geq 1$. Since $N^*(3) \setminus V(Q_3) = N^*(3)$, we see $|N^*(3)| \leq 2$ by Claim 4.5. If $|N^*(3)| = 2$, then the equality part of Claim 4.5 says that G is one of $\tilde{S}_{5,m}$, $\tilde{S}'_{5,m}$, $\tilde{T}_{5,m}$, and $\tilde{T}'_{5,m}$ in Figure 3 for some m. Suppose that $|N^*(3)| = 1$, and let $N^*(3) = \{4\}$. Since $N^*(4) \setminus V(Q_4) = N^*(4) \setminus \{3\}$, we see $|N^*(4) \setminus \{3\}| \leq 1$ by Claim 4.5. If $|N^*(4) \setminus \{3\}| = 0$, then G is one of $\tilde{P}_{4,m}$, and $\tilde{P}'_{4,m}$ in Figure 3 for some m. Suppose that $|N^*(4) \setminus \{3\}| = 1$, and let $N^*(4) \setminus \{3\} = \{5\}$. Then consider $N^*(5) \setminus V(Q_5)$. Repeating the argument through the vertices one by one completes the proof. □

5. Shellability of $\mathcal{P}^{\text{even}}_{G,C}$

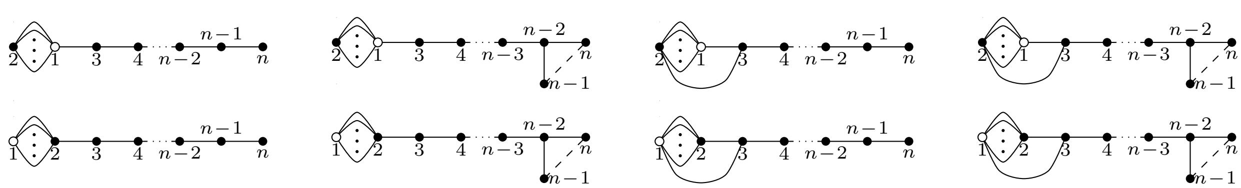
In this section, we show that the poset $\mathcal{P}^{\text{even}}_{H,C}$ is shellable for every $(H, C) \in \mathcal{A}^*(G)$ if G is a graph in Figure 3. Note that a connected PI-graph of G in Figure 3 is a simple graph or a graph in Figure 3. Thus it is sufficient to show that when G is a graph in Figure 3, $\mathcal{P}^{\text{even}}_{G,C}$ is shellable for every $C \in \mathcal{A}(G)$.

5.1. **Definition of an ordering $\prec^I_{\text{atm}}$ for the atoms of $[I, G]$.** Let G be a graph in Figure 3, and $C \in \mathcal{A}(G)$. We let $V = \{1, 2, \ldots, n\}$ $(n \geq 2)$ be the set of vertices of G, and 1 and 2 be the endpoints of the bundle B. By the definition of an admissible collection, note that $C \cap B \neq \emptyset$ and $|C \cap B|$ is even, and so we let $B \cap C = \{a_1, \ldots, a_{2m}\}$ $(m \geq 1)$, and $B \setminus C = \{b_1, \ldots, b_\ell\}$. Here, $B \setminus C$ may be the empty set. In addition, there are three cases:

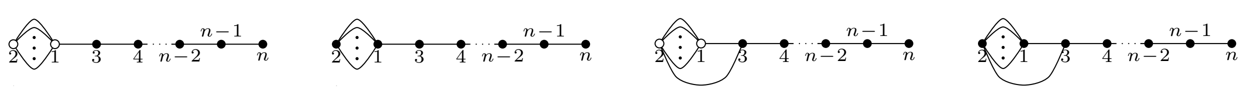
- $|V|$ is odd and $V \cap C = V \setminus \{w\}$ for some $w \in \{1, 2\}$;
- $|V|$ is even and $V \cap C = V \setminus \{1, 2\}$;
- $|V|$ is even and $V \cap C = V$.

We label the vertices not the endpoints of B in a way that each $i \in \{3, \ldots, n\}$ is closest to the vertex $i - 1$. We relabel the endpoints of B so that $1 \notin C$ if $|V|$ is odd, and so that 13 is an edge if $|V|$ is even. See (A) of Figure 9 for all the possible labelings when $|V|$ is odd. We illustrate all the possible labelings when $|V|$ is even in (B) of Figure 9. See Figure 10 for examples of $\mathcal{P}^{\text{even}}_{G,C}$ under this labeling. We also assume that there is a total ordering between the vertices: $1 \prec 2 \prec \cdots \prec n$. Thus for $I \subset V$, the minimum of I, denoted by $\min(I)$, means the frontmost one in the ordering.

We define the type of a cover $I \lessdot J$ in $\mathcal{P}^{\text{even}}_{G,C}$ according to the size of $J \setminus I$ and the intersection with $B \setminus C$. A cover $I \lessdot J$ has type (Ei) if $|J \setminus I| = i$ for $1 \leq i \leq 4$ and $J \setminus I$ has no element of $B \setminus C$; and $I \lessdot J$ has type (Ei') if $|J \setminus I| = i$ for $1 \leq i \leq 3$ and $J \setminus I$ contains some elements of $B \setminus C$. See Table 1. Note that (E1$'$)$\sim$(E3$'$) occurs when $B \setminus C \neq \emptyset$. It should be noted that there is no cover $I \lessdot J$ of $\mathcal{P}^{\text{even}}_{G,C}$

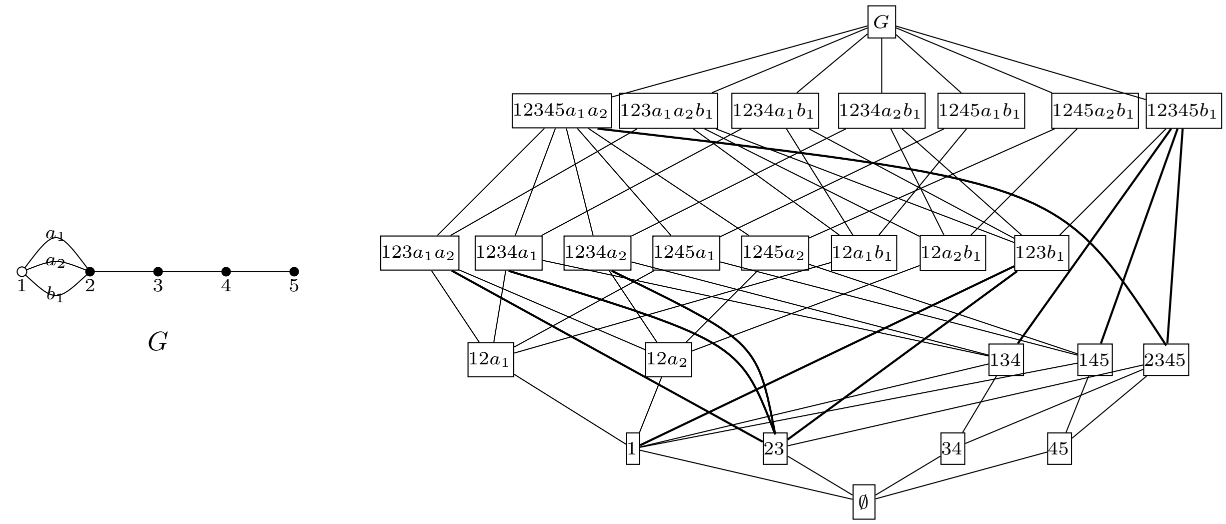


(A) Labeling of the vertices, where the hollow vertex does not belong to C, when n is odd.

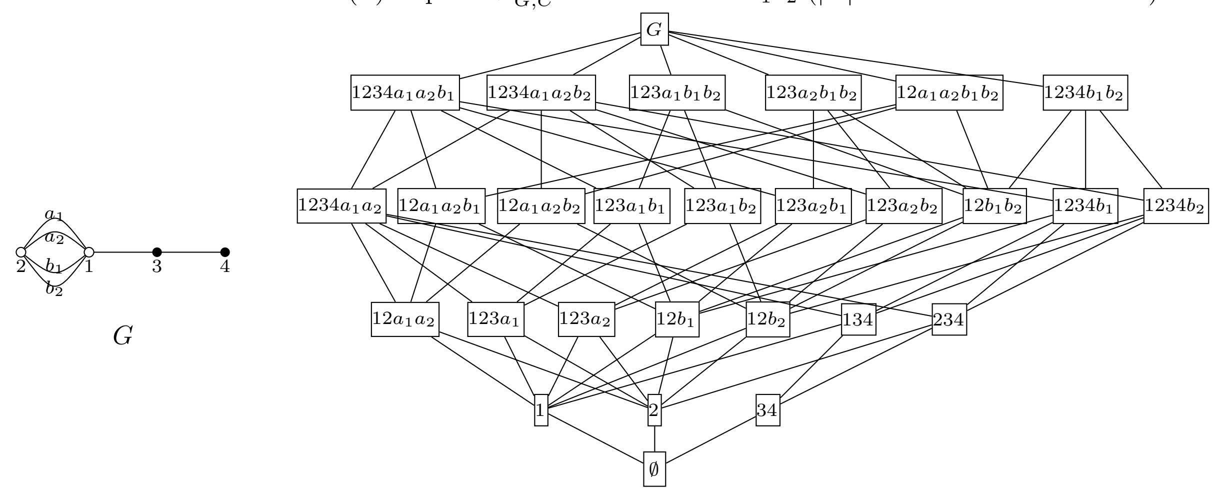


(B) Labeling of the vertices, where the hollow vertices do not belong to C, when n is even.

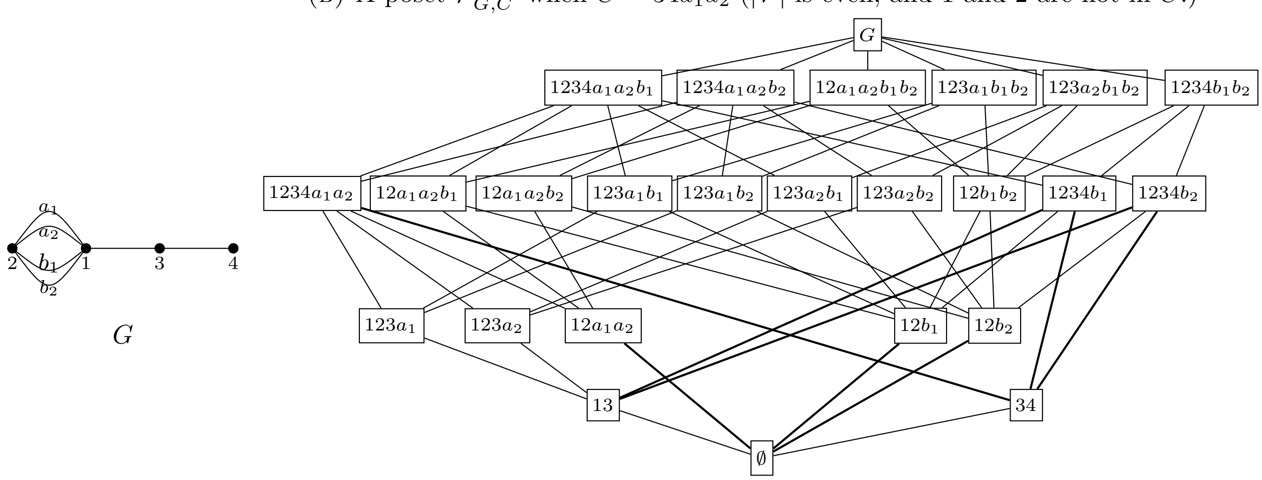
Figure 9. Labeling of the vertices



(A) A poset $\mathcal{P}^{\text{even}}_{G,C}$ when $C = 2345a_1a_2$ ($|V|$ is odd and 1 is not in C.)



(B) A poset $\mathcal{P}^{\text{even}}_{G,C}$ when $C = 34a_1a_2$ ($|V|$ is even, and 1 and 2 are not in C.)



(C) A poset $\mathcal{P}^{\text{even}}_{G,C}$ when $C = 1234a_1a_2$ ($|V|$ is even)

FIGURE 10. Examples of posets $\mathcal{P}^{\text{even}}_{G,C}$

Type		$J \setminus I$		
		When $\|V\|$ is odd	When $\|V\|$ is even	
		$C \cap \{1,2\} = \{2\}$	$C \cap \{1,2\} = \emptyset$	$\{1,2\} \subset C$
(E1)		1	1 or 2	-
(E2)		cc'	cc'	cc'
(E3)		$1ac$	$1ac$ or $2ac$	-
(E4)		-	-	$12aa'$
(E1$'$)		b	b	b
(E2$'$)		$1b$	$1b$ or $2b$	-
(E3$'$)	(E3$'$-1)	$2vb$	-	$1vb$ or $2vb$
	(E3$'$-2)	-	-	$12b$

TABLE 1. Types of $I \lessdot J$ in $\mathcal{P}^{\text{even}}_{G,C}$, where $a,\ a' \in B \cap C$, $b \in (B \setminus C)$, $c,\ c' \in C$, $v = \min(V \setminus (I \cup \{1,2\}))$

such that $J \setminus I$ contains both a multiple edge in $B \cap C$ and a multiple edge in $B \setminus C$. Moreover, for each cover $I \lessdot J$, the elements in $J \setminus I$ belong to a same component of J.

Let $I \lessdot J$ have type (E3$'$). Since $J \setminus I$ has a multiple edge in $B \setminus C$ and $|J \setminus I| > 1$, $J \setminus I$ contains at least one vertex in $\{1,2\}$, and so we divide the type (E3$'$) into two subtypes:

- $I \lessdot J$ has type (E3$'$-1) if $I \lessdot J$ has type (E3$'$) and $|(J \setminus I) \cap \{1,2\}| = 1$;
- $I \lessdot J$ has type (E3$'$-2) if $I \lessdot J$ has type (E3$'$) and $|(J \setminus I) \cap \{1,2\}| = 2$.

When $I \lessdot J$ is of (E3$'$-1), $J \setminus I$ contains the vertex $\min(V \setminus (I \cup \{1,2\}))$. To see why, let $J \setminus I = wcb$ for some $w \in \{1,2\} \cap C$, $c \in V \setminus \{1,2\}$, and $b \in B \setminus C$. From the structure of G, it is sufficient to check the case where $I \cup wb = 123 \cdots (n-2)b$ and G is none of $\tilde{P}_{n,m}$ and $\tilde{P}'_{n,m}$. In this case, $I = V \setminus \{w, n-1, n\}$, $|V|$ is odd, and $1 \notin C$. Hence $|I \cap C| = |V| - 4$ is odd, a contradiction. Therefore, if $I \lessdot J$ is of (E3$'$-1), then the vertex $\min(V \setminus (I \cup \{1,2\}))$ lies in $J \setminus I$.

Proposition 5.1. *Let G be a graph in Figure 3 and C be an admissible collection of G. Then the lengths of maximal chains of $\mathcal{P}^{\text{even}}_{G,C}$ are*

$$\begin{cases} \frac{|C|}{2} + |B \setminus C| + 1 \quad \text{or} \quad \frac{|C|}{2} + |B \setminus C| & \text{if } |V| \text{ is odd,} \\ \frac{|C|}{2} + |B \setminus C| + 1 & \text{if } |V| \text{ is even and } C \cap V \neq V, \\ \frac{|C|}{2} + |B \setminus C| \quad \text{or} \quad \frac{|C|}{2} + |B \setminus C| - 1 & \text{if } |V| \text{ is even and } C \cap V = V. \end{cases}$$

Proof. Recall that $|V| = n$, $|B \cap C| = 2m$, and $|B \setminus C| = \ell$. Note that $2m + n \geq 4$. For a maximal chain $\sigma\colon I_0 \lessdot I_1 \lessdot \cdots \lessdot I_p$ of $\mathcal{P}^{\text{even}}_{G,C}$, let I_k be the first element of σ containing a multiple edge. Suppose that $|V|$ is odd. Note that if σ contains a cover of (E1), then it can contain a cover of neither (E3) nor (E2$'$). As $I_{k-1} \lessdot I_k$ is of (E2), (E3), (E2$'$), or (E3$'$-1), we have the following table for the length of σ:

$I_{k-1} \lessdot I_k$	The types of the covers in σ	Length of σ
(E2)	one (E1), ℓ (E1$'$)s, $\frac{2m+n-1}{2}$ (E2)s	$\frac{2m+n+1}{2} + \ell$
(E3)	one (E3), ℓ (E1$'$)s, $\frac{2m+n-3}{2}$ (E2)s	$\frac{2m+n-1}{2} + \ell$
(E2$'$)	one (E2$'$), $(\ell-1)$ (E1$'$)s, $\frac{2m+n-1}{2}$ (E2)s	
(E3$'$-1)	one (E1), one (E3$'$-1), $(\ell-1)$ (E1$'$)s, $\frac{2m+n-3}{2}$ (E2)s	

Since $|C| = 2m + n - 1$, every maximal chain has the length either $\frac{|C|}{2} + \ell + 1$ or $\frac{|C|}{2} + \ell$, and hence the poset $\mathcal{P}^{\text{even}}_{G,C}$ is nonpure. Note that if 2 and 3 are not adjacent in G then there is no cover of (E3), and if $B \setminus C = \emptyset$ then there is no cover of (E2′) or (E3′). Hence if 2 and 3 are not adjacent in G and $B \setminus C = \emptyset$, then $\mathcal{P}_{G,C}^{\text{even}}$ is a pure poset of length $\frac{|C|}{2} + 1$.

Suppose that $|V|$ is even. When $C \cap \{1,2\} = \emptyset$, $I_{k-1} \lessdot I_k$ is of (E3) or (E2′). When C contains $\{1,2\}$, $I_{k-1} \lessdot I_k$ is one of (E2), (E4), and (E3′). Thus we have the following table for the length of σ:

$C \cap \{1,2\}$	$I_{k-1} \lessdot I_k$	The types of the covers in σ	Length of σ
$\emptyset$	(E3)	one (E1), one (E3), ℓ (E1′)s, $\frac{2m+n-4}{2}$ (E2)s	$\frac{2m+n}{2} + \ell$
	(E2′)	one (E1), one (E2′), $(\ell-1)$ (E1′)s, $\frac{2m+n-2}{2}$ (E2)s	
$\{1,2\}$	(E2)	$\frac{2m+n}{2}$ (E2)s, ℓ (E1′)s	$\frac{2m+n}{2} + \ell$
	(E4)	one (E4), ℓ (E1′), $\frac{2m+n-4}{2}$ (E2)s	$\frac{2m+n-2}{2} + \ell$
	(E3′)	one (E3′), $(\ell-1)$ (E1′)s, $\frac{2m+n-2}{2}$ (E2)s	

If $C \cap \{1,2\} = \emptyset$, then $|C| = 2m + n - 2$, and so every maximal chain has the length $\frac{|C|}{2} + \ell + 1$. If $\{1,2\} \subset C$, then $|C| = 2m+n$, and so every maximal chain has the length either $\frac{|C|}{2}+\ell$ or $\frac{|C|}{2}+\ell-1$. $\square$

We shall show that $\mathcal{P}_{G,C}^{\text{even}}$ admits a recursive atom ordering. We first define the lexicographic order $\prec_{\text{lex}}^{I}$ on $V \cap B$ for each $I \in \mathcal{P}_{G,C}^{\text{even}}$ and then define the atom ordering $\prec_{\text{atm}}^{I}$ for $[I, G]$.

Definition 5.2. Let $I \in \mathcal{P}_{G,C}^{\text{even}}$. We define the lexicographic order $\prec_{\text{lex}}^{I}$ on $V \cup B$ as follows:

- If $B \cap I = \emptyset$, then
$$\prec_{\text{lex}}^{I}\colon\ 1, 2, 3, \dots, n, a_1, \dots, a_{2m}, b_1, \dots, b_\ell.$$
- If $B \cap I \neq \emptyset$ and $(B \setminus C) \cap I = \emptyset$, then let $k := \max\{i \mid a_i \in B \cap C \cap I\}$ and
$$\prec_{\text{lex}}^{I}\colon\ 1, 2, a_1, \dots, a_k, 3, \dots, n, a_{k+1}, \dots, a_{2m}, b_1, \dots, b_\ell.$$
- If $(B \setminus C) \cap I \neq \emptyset$, then let $k := \max\{i \mid b_i \in (B \setminus C) \cap I\}$ and
$$\prec_{\text{lex}}^{I}\colon\ 1, 2, a_1, \dots, a_{2m}, b_1, \dots, b_k, 3, \dots, n, b_{k+1}, \dots, b_\ell.$$

Then for two atoms J and J' of $[I, G]$, we define $J \prec_{\text{atm}}^{I} J'$ if one of the following holds:

(A1) $|(J \setminus I) \cap \{1,2\}| = 1$ and $|(J' \setminus I) \cap \{1,2\}| = 2$; or
(A2) $J \setminus I \prec_{\text{lex}}^{I} J' \setminus I$, where the elements of $J \setminus I$ and $J' \setminus I$ are arranged in the lexicographic order $\prec_{\text{lex}}^{I}$, respectively.

Note that (A1) is considered only when $\mathcal{P}_{G,C}^{\text{even}}$ admits a cover of (E4) or (E3′-2), that is, $|V|$ is even and C contains $\{1,2\}$.

Here is an example. Let G be the graph $\tilde{P}_{6,5}$ in Figure 3. Suppose that $C = V \cup \{a_1, a_2, a_3, a_4\}$. Then the atoms of $\mathcal{P}_{G,C}^{\text{even}}$ are ordered as follows:

$$\prec_{\text{atm}}^{\emptyset}\colon\ 13,\ 12a_1a_2,\ 12a_1a_3,\ 12a_1a_4,\ 12a_2a_3,\ 12a_2a_4,\ 12a_3a_4,\ 12b_1,\ 34,\ 45,\ 56.$$

For $I = 12a_1a_3$, $\prec_{\text{lex}}^{I}$: $1, 2, a_1, \boldsymbol{a_2}, a_3, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \boldsymbol{a_4}, \boldsymbol{b_1}$, and the atoms of $[I, G]$ are ordered as follows:

$$\prec_{\text{atm}}^{I}\colon\ 12\mathbf{3}a_1\boldsymbol{a_2}a_3,\ 12a_1\boldsymbol{a_2}a_3\boldsymbol{a_4},\ 12\mathbf{34}a_1a_3,\ 12\mathbf{3}a_1a_3\boldsymbol{a_4},\ 12\mathbf{45}a_1a_3,\ 12\mathbf{56}a_1a_3,\ 12a_1a_3\boldsymbol{b_1}$$

because $a_23 \prec_{\text{lex}}^{I} a_2a_4 \prec_{\text{lex}}^{I} 34 \prec_{\text{lex}}^{I} 3a_4 \prec_{\text{lex}}^{I} 45 \prec_{\text{lex}}^{I} 56 \prec_{\text{lex}}^{I} b_1$, where the bold letters indicate the elements not in I.

The following is the main theorem of this section, whose proof is given in Subsection 5.2.

Theorem 5.3. *Let G be a connected graph in Figure 3 and C be an admissible collection of G. Then $\mathcal{P}_{G,C}^{\text{even}}$ admits a recursive atom ordering, and hence $\mathcal{P}_{G,C}^{\text{even}}$ is CL-shellable.*

Remark 5.4. We insist that the ordering $\prec_{\text{atm}}^{I}$ is essential. Suppose that we consider a lexicographic order $\prec^*$ (it may be natural) given by $1, 2, a_1, a_2, \ldots, a_{2m}, 3, 4, \ldots, n, b_1, \ldots, b_\ell$, and define $\prec_{\text{atm}}^{*}$ by replacing (A2) of Definition 5.2 with the fixed ordering $\prec^*$. For the posets in Figure 10, $\prec_{\text{atm}}^{*}$ gives a recursive atom ordering. However, it fails to be a recursive atom ordering in general. For example, let G be a graph in Figure 3 with $|V| = 4$ and $|B| = 6$, and let $C = V \cup B$. Then $C \in \mathcal{A}(G)$. Let $I = 12a_1a_3$, and consider the atoms $J_1 = 12a_1a_3\boldsymbol{a}_5\boldsymbol{a}_6$ and $J_2 = 12\mathbf{3}a_1a_3\boldsymbol{a}_5$ of $[I, G]$, where the bold letters indicate the elements not in I. Then the atoms of $[\emptyset, G]$ preceding I in $\prec_{\text{atm}}^{*}$ are 13 and $12a_1a_2$. However, $J_1 \prec_{\text{atm}}^{*} J_2$, J_2 contains the atom 13, and J_1 does not contain any atom of $[\emptyset, G]$ preceding I, and so (2) of Definition 2.3 fails.

5.2. **Proof of Theorem 5.3.** For a subset $X \subset V \cup B$, $\min^I(X)$ and $\max^I(X)$ denote the minimum and the maximum of X with respect to $\prec_{\text{lex}}^{I}$, respectively. The proof of the following will be given later.

Lemma 5.5. *Let I_j be an atom of $[I, G]$, not the first in $\prec_{\text{atm}}^{I}$. There is an element $x(I \lessdot I_j) \in V \cup B$ such that an atom J of $[I_j, G]$ belongs to $[I_k, G]$ for some $I_k \prec_{\text{atm}}^{I} I_j$ if and only if $\min^{I_j}(J \setminus I_j) \prec_{\text{lex}}^{I_j} x(I \lessdot I_j)$.*

We first prove Theorem 5.3 by using Lemma 5.5.

Proof of Theorem 5.3. We will show that the ordering $\prec_{\text{atm}}^{I}$ ($I \in \mathcal{P}_{G,C}^{\text{even}}$) is a recursive atom ordering. Lemma 5.5 inserts that $\prec_{\text{atm}}^{I}$ satisfies (1) of Definition 2.3. Let us check (2) of Definition 2.3. Let I_i and I_j be atoms of $[I, G]$ such that $I_i \prec_{\text{atm}}^{I} I_j$. Suppose that there is an element K of $[I, G]$ such that $I_i, I_j < K$. We need to find an atom K_* of $[I_j, G]$ and an atom I_* of $[I, G]$ such that $K_* \le K$, $K_* \in [I_*, G]$, and $I_* \prec_{\text{atm}}^{I} I_j$. Let $K_0 = I_i \cup I_j$ for simplicity. Note that $K_0 \subset K$.

Suppose that K_0 is not a semi-induced subgraph of G. Then $I_i \setminus I$ and $I_j \setminus I$ contain exactly one of the endpoints of B, not the same. More precisely, letting $v = \min(V \setminus (I \cup \{1, 2\}))$, the following hold:

$$\begin{aligned} I_i \setminus I &= 1 &&\text{and} && I_j \setminus I = 2v && \text{if } C \cap \{1,2\} = \{2\}, \\ I_i \setminus I &= 1 &&\text{and} && I_j \setminus I = 2 && \text{if } C \cap \{1,2\} = \emptyset, \text{ and} \\ I_i \setminus I &= 1v &&\text{and} && I_j \setminus I = 2v && \text{if } C \cap \{1,2\} = \{1,2\}. \end{aligned}$$

Note that $I_j \setminus I = 2v$ occurs only when 2 and 3 are adjacent in G. In addition, $1 \notin C$ if and only if $|K_0 \cap C| \equiv |(K_0 \setminus I) \cap C| \equiv 0 \pmod 2$. Since K contains $\{1, 2\}$, K contains a multiple edge e. Let H be the component of K containing e and so $e \in H \setminus K_0$. If $|C \cap \{1, e\}| \equiv 0 \pmod 2$, then $K_* = K_0 \cup e$ and $I_* = I_i$. Now assume that $|C \cap \{1, e\}| = 1$. Then $|K_0 \cap C| \equiv |H \cap K_0 \cap C| \equiv |(H \setminus K_0) \cap C| \pmod 2$, where the first equivalence is from the definition of H and the second equivalence is from $|H \cap C| \equiv 0 \pmod 2$. Hence $1 \notin C$ if and only if $|(H \setminus K_0) \cap C|$ is even. For simplicity, let $X = (H \setminus K_0) \cap C$. If $|X|$ is even, then $1 \notin C$ and $e \in C$ and therefore, $|X \setminus \{e\}| \ge 1$. If $|X|$ is odd, then $1 \in C$ and $e \notin C$ and therefore, $|X \setminus \{e\}| = |X| \ge 1$. Hence we can take an element $c \in X \setminus \{e\}$ so that $K_* = K_0 \cup ce$ and $I_* = I_i$. More precisely, either c is the vertex $\min(V \setminus (I_j \cup \{1, 2\}))$ or belongs to $(B \cap C) \setminus e$.

Assume that K_0 is a semi-induced subgraph of G. Note that $|(I_i \setminus I) \cap (I_j \setminus I) \cap C|$ is possible from zero to three. If $|(I_i \setminus I) \cap (I_j \setminus I) \cap C|$ is even, then $|K_0 \cap C|$ is even and so $K_* = K_0$ and $I_* = I_i$. Now we assume that $|(I_i \setminus I) \cap (I_j \setminus I) \cap C|$ is odd. Then $|(I_i \setminus I) \cap (I_j \setminus I) \cap C|$ is one or three. Since $|K_0 \cap C|$ is odd, K_0 has exactly one component H_0 such that $|H_0 \cap C|$ is odd. Let H be the component of K containing H_0. Note that since $|H \cap C|$ is even, it holds that $|(H \setminus H_0) \cap C| \ge 1$.

If H_0 contains a multiple edge, then there exists an element $c \in (H \setminus H_0) \cap C$ such that $K_* = K_0 \cup c$ and $I_* = I_i$; more precisely, if $(H \setminus H_0) \cap B \cap C \neq \emptyset$, then $c \in B \cap C$; otherwise, c is the vertex $\min(V \setminus H_0)$.

Suppose that H_0 has no multiple edge. Then both $I_i \setminus I$ and $I_j \setminus I$ consist of two vertices in C, and $|(I_i \setminus I) \cap (I_j \setminus I) \cap C| = 1$. Since H_0 is a semi-induced subgraph of G and $|H_0 \cap C|$ is odd, $(H \setminus H_0) \cap V \neq \emptyset$. If $(H \setminus H_0) \cap V$ has a vertex in $\{3, ..., n\}$, then by the structure of G, it is easy to see that there is a vertex v in $(H \setminus H_0) \cap \{3, ..., n\}$ such that $K_* = K_0 \cup v$ and $I_* = I_i$. Hence we only need to consider the case in which $(H \setminus H_0) \cap V \subset \{1, 2\}$. If $(H \setminus H_0) \cap V = \{1, 2\}$, then

$$\begin{cases} K_* = I_j \cup 1 \text{ and } I_* = I \cup 1 & \text{if } 1 \notin C \\ K_* = K_0 \cup 1 \text{ and } I_* = I_i & \text{if } 1 \in C. \end{cases}$$

It remains to consider the case where $(H \setminus H_0) \cap V = \{1\}$ or $\{2\}$. Let $(H \setminus H_0) \cap V = \{w_1\}$, and let w_2 be the other vertex in $\{1, 2\}$. If H_0 does not contain w_2, then $H = H_0 \cup w_1$ and $w_1 \in C$ (and therefore, w_1 must be a neighbor of 3 since H is an element of $\mathcal{P}^{\text{even}}_{G,C}$) and hence $K_* = K_0 \cup w_1$ and $I_* = I_i$. Suppose that H_0 contains w_2. Then H contains a multiple edge, that is, $H \cap B = (H \setminus H_0) \cap B \neq \emptyset$. Moreover, since $|(H \setminus H_0) \cap C|$ is odd and $(H \setminus H_0) \cap V = \{w_1\}$, if $w_1 \notin C$, then $|H \cap B \cap C| \geq 1$; and if $w_1 \in C$ and $H \cap (B \setminus C) = \emptyset$, then $|H \cap B \cap C| \geq 2$. Hence if $w_1 \notin C$, then $K_* = K_0 \cup w_1 a$ and $I_* = I_i$ for some $a \in H \cap B \cap C$. Now assume that $w_1 \in C$, and we prove the remaining part by dividing two subcases whether $w_2 \in I$ or not. If $w_2 \in I$, then

$$\begin{cases} K_* = K_0 \cup w_1 b \text{ and } I_* = I_i \text{ for some } b \in H \cap (B \setminus C) & \text{if } H \cap (B \setminus C) \neq \emptyset, \\ K_* = I_j \cup w_1 a a' \text{ and } I_* = I \cup w_1 a \text{ for some } a, a' \in H \cap B \cap C & \text{if } H \cap (B \setminus C) = \emptyset. \end{cases}$$

Suppose that $w_2 \notin I$. Then $w_2 \in (I_i \setminus I) \cup (I_j \setminus I)$. Since $I_i \prec^I_{\text{atm}} I_j$, $w_2 \in I_i$. Moreover, by the structure of G, $I_i \setminus I = w_2 v$ and $I_j \setminus I = vv'$ for $v = \min(V \setminus (I \cup \{1, 2\}))$ and $v' = \min(V \setminus (I \cup \{1, 2, v\}))$. Hence

$$\begin{cases} K_* = I_j \cup 12b \text{ and } I_* = I \cup 12b \text{ for some } b \in H \cap (B \setminus C) & \text{if } H \cap (B \setminus C) \neq \emptyset, \\ K_* = I_j \cup 12aa' \text{ and } I_* = I \cup 12aa' \text{ for some } a, a' \in H \cap B \cap C & \text{if } H \cap (B \setminus C) = \emptyset. \end{cases}$$

This completes the proof. □

For an element I of $\mathcal{P}^{\text{even}}_{G,C}$, a multiple edge e is called a *big* (respectively, *small*) edge of I if $n \prec^I_{\text{lex}} e$ (respectively, $e \prec^I_{\text{lex}} 3$). Now we prove Lemma 5.5.

Proof of Lemma 5.5. Let I_j be an atom of $[I, G]$, not the first in $\prec^I_{\text{atm}}$. We will show that an atom J of $[I_j, G]$ belongs to $[I_*, G]$ for some atom I_* of $[I, G]$ with $I_* \prec^I_{\text{atm}} I_j$ if and only if the following hold:

- **(1)** $\min^{I_j}(J \setminus I_j) \prec^{I_j}_{\text{lex}} 2$, if $I_j \setminus I \subset V$ and $(I_j \setminus I) \cap \{1, 2\} \neq \emptyset$;
- **(2)** $\min^{I_j}(J \setminus I_j) \prec^{I_j}_{\text{lex}} \min^{I_j}\{v, b_1\}$, if $I_j \setminus I = va$ for $v \in V$, $a \in B \cap C$, and $a \prec^I_{\text{lex}} n$;
- **(3)** $\min^{I_j}(J \setminus I_j) \preceq^{I_j}_{\text{lex}} n$, if $I_j \setminus I$ consists of only big edges of I;
- **(4)** $\min^{I_j}(J \setminus I_j) \preceq^{I_j}_{\text{lex}} \min(V \setminus I_j)$, if $V \setminus I_j \neq \emptyset$, $I_j \setminus I$ has an element $c \preceq^I_{\text{lex}} n$ and a big edge of I, and $|(I_j \setminus I) \cap \{1, 2\}| \equiv 0 \pmod 2$,
- **(5)** $\min^{I_j}(J \setminus I_j) \prec^{I_j}_{\text{lex}} \max^{I_j}(I_j \setminus I)$, otherwise.

Whenever we show the 'if' part of each case, we finish the proof when we find a proper atom I_* of $[I, G]$, that is, I_* is an atom of $[I, G]$ such that $J \in [I_*, G]$ and $I_* \prec^I_{\text{atm}} I_j$.

(1) Suppose that $I_j \setminus I \subset V$ and $(I_j \setminus I) \cap \{1, 2\} \neq \emptyset$. Note that $\prec^I_{\text{lex}}$: $1, 2, \ldots, n, a_1, \ldots, a_{2m}, b_1, \ldots, b_\ell$. From the assumption, if $1 \in I_j \setminus I$, then I_j is the first atom of $[I, G]$. Thus $2 \in I_j \setminus I$. Suppose $1 \in J \setminus I_j$. Then for some $a \in B \cap C$, $c \in C$, $b \in B \setminus C$, and $v = \min(V \setminus (I_j \cup \{1\}))$,

$$J \setminus I_j = \begin{cases} 1ac \text{ or } 1b, & \text{if } 1 \notin C, \\ 1a \text{ or } 1vb & \text{if } 1 \in C. \end{cases}$$

Then I_* is either $I \cup 1$ or $I \cup 1v$ in each case, which proves the 'if' part. If $1 \notin J \setminus I_j$, then $J \setminus I_j$ cannot have a multiple edge and so it consists of vertices greater than $\max^I(I_j \setminus I)$. Therefore, I_j is the first atom of $[I, G]$, which proves the 'only if' part.

(2) Suppose that $I_j \setminus I = va$ for $v \in V$, $a \in B \cap C$ and $a \preceq^I_{\text{lex}} n$. The existence of a small edge of I implies $\{1,2\} \subset I$ and $v = \min(V \setminus I)$. Then $\prec^I_{\text{lex}} = \prec^{I_j}_{\text{lex}}$ and they are either

$$\prec^I_{\text{lex}} = \prec^{I_j}_{\text{lex}}:\ 1, 2, a_1, a_2, \ldots, a_k, 3, \ldots, n, a_{k+1}, \ldots, a_{2m}, b_1, \ldots, b_\ell, \quad \text{or}$$
$$\prec^I_{\text{lex}} = \prec^{I_j}_{\text{lex}}:\ 1, 2, a_1, a_2, \ldots, a_{2m}, b_1, \ldots, b_k, 3, \ldots, n, b_{k+1}, \ldots, b_\ell.$$

If $J \setminus I_j$ has an element $a' \prec^{I_j}_{\text{lex}} \min^{I_j}\{v, b_1\}$, then $a' \in B \cap C$, and $I_* = I \cup aa'$, which proves the 'if' part. Suppose that $\min^{I_j}\{v, b_1\} \preceq^{I_j}_{\text{lex}} \min^{I_j}(J \setminus I_j)$. If $J \setminus I_j = b$ for some $b \in B \setminus C$, then $[I, J]$ has only two atoms I_j and $I \cup b$, and I_j is the first. If $J \setminus I_j = cc' \subset C$ for some $c, c' \succeq^{I_j}_{lex} v$, then $I_j \setminus I$ consists of the first two smallest elements of $J \setminus I$ and so I_j is the first atom of $[I, J]$. This proves the 'only if' part.

(3) Suppose that $I_j \setminus I$ consists of only big edges of I. From the hypothesis, $I \cap B \neq \emptyset$ and either $I_j \setminus I = aa'$ or $I_j \setminus I = b$, where $a, a' \in B \cap C$ and $b \in B \setminus C$.
(Case 1) $I_j \setminus I = aa'$. Note that the existence of a big edge of I in $B \cap C$ implies that $I \cap (B \setminus C) = \emptyset$, and the lexicographic orders $\prec^I_{\text{lex}}$ and $\prec^{I_j}_{\text{lex}}$ are as follows:

$$\prec^I_{\text{lex}}:\ 1, 2, a_1, \ldots, a_k, 3, \ldots, n, a_{k+1}, \ldots, a, \ldots, a'(= a_t), \ldots, a_{2m}, b_1, \ldots, b_\ell$$
$$\prec^{I_j}_{\text{lex}}:\ 1, 2, a_1, \ldots, a_k, \ldots, a, \ldots, a'(= a_t), 3, \ldots, n, a_{t+1}, \ldots, a_{2m}, b_1, \ldots, b_\ell.$$

Set $x := \min^{I_j}(J \setminus I_j)$. Suppose that $x \preceq^{I_j}_{\text{lex}} n$. Then $I_j \lessdot J$ is of (E2), and we can set $J \setminus I_j = xx'$, where $x \prec^{I_j}_{\text{lex}} x'$. Note that $x \prec^I_{\text{lex}} a'$ since $x \preceq^{I_j}_{\text{lex}} n$. If $x \preceq^{I_j}_{\text{lex}} \min(V \setminus I)$, then $I_* = I \cup ax$. If $\min(V \setminus I) \prec^{I_j}_{\text{lex}} x \preceq^{I_j}_{\text{lex}} n$, that is, x is a vertex different from $\min(V \setminus I)$, then x' is also a vertex and $I_* = I \cup xx'$. This proves the 'if' part. To prove the 'only if' part, suppose that $n \prec^{I_j}_{\text{lex}} x$. Then either $J \setminus I_j = xx' \subset B \cap C$ or $J \setminus I_j = x \in B \setminus C$. Since $a \prec^I_{\text{lex}} a' \prec^I_{\text{lex}} x$, I_j is the first atom of $[I, J]$ in $\prec^I_{\text{atm}}$.
(Case 2) $I_j \setminus I = b$. Note that $\prec^{I_j}_{\text{lex}}$: $1, 2, a_1, a_2, \ldots, a_{2m}, b_1, \ldots, b(= b_k), 3, 4, \ldots, n, b_{k+1}, \ldots, b_\ell$. Suppose that $J \setminus I_j$ contains an element e with $e \preceq^{I_j}_{\text{lex}} n$. Note that $e \prec^I_{\text{lex}} b$. If $e \in B \setminus C$, then $I_* = I_j \cup e$. If $e \in B \cap C$, then $J \setminus I_j = ec$ for some $c \in C$, which implies that $I_* = I \cup ec$. If $\min^{I_j}(J \setminus I_j)$ is a vertex, then $J \setminus I_j$ consists of two vertices, and $I_* = I \cup (J \setminus I_j)$. This proves the 'if' part. If $n \prec^{I_j}_{\text{lex}} \min^{I_j}(J \setminus I_j)$, then $J \setminus I_j = \{b'\}$ for some $b' \in B \setminus C$ with $b \prec^{I_j}_{\text{lex}} b'$, and hence $[I, J]$ has only two atoms I_j and $I \cup b'$, where I_j is the first atom in $\prec^I_{\text{atm}}$.

Now, to show **(4)** and **(5)**, we need the following claim.

Claim 5.6. *Suppose that $I \cap (B \setminus C) = \emptyset$, $I_j \setminus I$ has both an element $c \preceq^I_{\text{lex}} n$ and a big edge of I, and $|(I_j \setminus I) \cap \{1,2\}| \equiv 0 \pmod 2$. Then an atom J of $[I_j, G]$ belongs to $[I_*, G]$ for some atom I_* of $[I, G]$ with $I_* \prec^I_{\text{atm}} I_j$ if and only if one of the following holds:*

(i) $\min^{I_j}(J \setminus I_j) \preceq^{I_j}_{\text{lex}} \min(V \setminus I_j)$ *if* $V \setminus I_j \neq \emptyset$
(ii) $\min^{I_j}(J \setminus I_j) \prec^{I_j}_{\text{lex}} \max^{I_j}(I_j \setminus I)$ *if* $V \setminus I_j = \emptyset$.

Proof of Claim 5.6. From the hypotheses, one of ①∼④ holds in the following table, where $a, a' \in B \cap C$ with $a' \prec^I_{\text{lex}} a$, $b \in B \setminus C$, $v = \min(V \setminus I)$:

$I_j \setminus I$	The lexicographic orders
① $12a'a$	$\prec^I_{\mathrm{lex}}$: $1, 2, \ldots, n, a_1, \ldots, a', \ldots, a(= a_k), \ldots, a_{2m}, b_1, \ldots, b_\ell$
	$\prec^{I_j}_{\mathrm{lex}}$: $1, 2, a_1, \ldots, a', \ldots, a(= a_k), 3, \ldots, n, a_{k+1}, \ldots, a_{2m}, b_1, \ldots, b_\ell$
② $12b$	$\prec^I_{\mathrm{lex}}$: $1, 2, \ldots, n, a_1, \ldots, a_{2m}, b_1, \ldots, b(= b_k), \ldots, b_\ell$
	$\prec^{I_j}_{\mathrm{lex}}$: $1, 2, a_1, \ldots, a_{2m}, b_1, \ldots, b(= b_k), 3, \ldots, n, b_{k+1}, \ldots, b_\ell$
③ va	$\prec^I_{\mathrm{lex}}$: $1, 2, a_1, \ldots, a_j, 3, \ldots, v, \ldots, n, a_{j+1}, \ldots, a(= a_k), \ldots, a_{2m}, b_1, \ldots, b_\ell$
	$\prec^{I_j}_{\mathrm{lex}}$: $1, 2, a_1, \ldots, a'(= a_k), 3, \ldots, v, \ldots, n, a_{k+1}, \ldots, a_{2m}, b_1, \ldots, b_\ell$
④ $a'a$	$\prec^I_{\mathrm{lex}}$: $1, 2, a_1, \ldots, a', \ldots, a_j, 3, \ldots, n, a_{j+1}, \ldots, a(= a_k), \ldots, a_{2m}, b_1, \ldots, b_\ell$
	$\prec^{I_j}_{\mathrm{lex}}$: $1, 2, a_1, \ldots, a', \ldots, a(= a_k), 3, \ldots, n, a_{k+1}, \ldots, a_{2m}, b_1, \ldots, b_\ell$

Note that the cases of ① and ② can occur only when $\{1, 2\} \subset C$. Let $v_* := \min(V \setminus I_j)$, provided $V \setminus I_j \neq \emptyset$. In the cases of ①∼④, if $v_* \in J \setminus I_j$, then I_*'s are $I \cup 1v_*$, $I \cup 1v_*$, $I \cup vv_*$, and $I \cup a'v_*$, respectively. Now assume that $v_* \notin J \setminus I_j$ and $\min^{I_j}(J \setminus I_j) \prec^{I_j}_{\mathrm{lex}} \max^{I_j}(I_j \setminus I)$. Set $x := \min^{I_j}(J \setminus I_j)$. Since $\max^{I_j}(I_j \setminus I)$ is a small edge of I_j, x is also a multiple edge and hence $J \setminus I_j$ consists of multiple edges. In the case of ②, $I_* = I \cup 12 \cup (J \setminus I_j)$. For the other cases, $x \prec^I_{\mathrm{lex}} a$ and $x, a \in B \cap C$. Hence I_* is obtained from I_j by replacing a with x. This proves the 'if' part.

To prove the 'only if' part, first suppose that $V \setminus I_j \neq \emptyset$ and $\min^{I_j}(J \setminus I_j) \succ^{I_j}_{\mathrm{lex}} v_*$. Then $\min^{I_j}(J \setminus I_j)$ is either a vertex greater than v_* or a big edge of I_j. Hence $J \setminus I_j$ consists of either two vertices greater than v_* or only big edges of I_j. Note that a big edge of I_j is a big edge of I. Hence I_j is the first atom of $[I, J]$ in $\prec^I_{\mathrm{atm}}$. If $V \setminus I_j = \emptyset$ and $\min^{I_j}(J \setminus I_j) \succ^{I_j}_{\mathrm{lex}} \max^{I_j}(I_j \setminus I)$, then $\min^{I_j}(J \setminus I_j)$ is a big edge of I_j and hence $I_j \setminus I$ consists of only big edges of I_j, and so I_j is the first atom of $[I, J]$ in $\prec^I_{\mathrm{atm}}$. □

By Claim 5.6, **(4)** follows and **(5)** partially follows. We exclude the cases of **(1)**∼**(4)** and the case shown by Claim 5.6. We divide the remaining part into two cases according to the existence of a big edge of I in $I_j \setminus I$.

(Case 1) $I_j \setminus I$ has no big edge of I. By excluding **(1)** and **(2)**, we get one of the following:

① $I_j \setminus I = b$ where $b \in B \setminus C$ and b is a small edge of I;
② $I_j \setminus I = aa'$ where both $a, a' \in B \cap C$ are small edges of I; or
③ $I_j \setminus I = vv'$ where $v, v' \in V \setminus \{1, 2\}$.

In each case, the 'only if' part follows easily, that is, if $\max^{I_j}(I_j \setminus I) \prec^{I_j}_{\mathrm{lex}} \min^{I_j}(J \setminus I_j)$, then $I_j \setminus I$ has the first $|I_j \setminus I|$ smallest elements of $J \setminus I$ (in $\prec^I_{\mathrm{lex}}$), and so I_j is the first atom of $[I, J]$ in $\prec^I_{\mathrm{atm}}$. Let us prove the 'if' part of each case. We note that $\prec^I_{\mathrm{lex}} = \prec^{I_j}_{\mathrm{lex}}$.

① From the existence of a small edge in $B \setminus C$, it follows that $I \cap (B \setminus C) \neq \emptyset$ and

$$\prec^I_{\mathrm{lex}} = \prec^{I_j}_{\mathrm{lex}}: 1, 2, a_1, a_2, \ldots, a_{2m}, b_1, \ldots, b, \ldots, b_k, 3, \ldots, n, b_{k+1}, \ldots, b_\ell.$$

If $\min^{I_j}(J \setminus I_j) \prec^{I_j}_{\mathrm{lex}} b$, then $I_* = I \cup (J \setminus I_j)$.

② Let $a \prec^{I_j}_{\mathrm{lex}} a'$, and hence $a' = \max^{I_j}(J \setminus I_j)$. If $\min^{I_j}(J \setminus I_j) \prec^{I_j}_{\mathrm{lex}} a'$, then $J \setminus I$ contains a multiple edge a'' with $a'' \prec^I_{\mathrm{lex}} a'$, and so $I_* = I \cup aa''$.

③ Let $v \prec^{I_j}_{\mathrm{lex}} v'$, and hence $v' = \max^{I_j}(I_j \setminus I)$. Suppose that $\min^{I_j}(J \setminus I_j) \prec^{I_j}_{\mathrm{lex}} v'$. If $J \setminus I_j$ has an element $b \in B \setminus C$, then for some $w \in \{1, 2\}$ and for the vertex $v'' = \min(V \setminus (I_j \cup \{1, 2\}))$,

$$\begin{cases} J \setminus I_j = b & \text{if } \{1, 2\} \subset I, \\ J \setminus I_j = 12b & \text{if } \{1, 2\} \cap I = \emptyset, \text{ or} \\ J \setminus I_j = wb \text{ or } wv''b & \text{if } |\{1, 2\} \cap I| = 1. \end{cases}$$

Then I_*'s are $I \cup b$, $I \cup 12b$, $I \cup wb$, and $I \cup wv_*b$ in the order, where $v_* = \min\{v, v''\}$.

If $J \setminus I_j$ contains a multiple edge in $B \cap C$, then $J \setminus I_j$ is either aa', $12aa'$, $v''a$, or wac, where $a, a' \in B \cap C$, $w \in \{1,2\}$, $c \in C$ and $v'' = \min(V \setminus I_j)$. If $J \setminus I_j$ is either aa' or $12aa'$, then $I_* = I \cup (J \setminus I_j)$. If $J \setminus I_j = v''a$ and $v'' \notin \{1,2\}$, then $I_* = I \cup v_*a$ where $v_* = \min\{v, v''\}$. If $J \setminus I_j = v''a$ and $v'' \in \{1,2\}$, then $I_* = I \cup (J \setminus I_j)$. If $J \setminus I_j = wac$, then $I_* = I \cup wac_*$, where $c_* = \min^{I_j}\{v, c\}$. If $J \setminus I_j$ consists of only vertices, then $J \setminus I$ consists of only vertices and $I_* = I \cup xy$, where x and y are the first two smallest elements of $J \setminus I$. This completes the proof of the 'if' part.

(Case 2) $I_j \setminus I$ has a big edge of I. By excluding **(3)** and **(4)**, we get one of ①∼⑤ in the following table, where $w \in \{1,2\}$, $a, a' \in B \cap C$ with $a' \prec_{\text{lex}}^{I_j} a$, $b \in B \setminus C$, and $v = \min(V \setminus (I \cup \{1,2\}))$:

	$I_j \setminus I$	The lexicographic order $\prec_{\text{lex}}^{I_j}$	$\max^{I_j}(I_j \setminus I)$
$w \notin C$	① $wa'a$	$\prec_{\text{lex}}^{I_j}$: $1, 2, a_1, \ldots, a', \ldots, a(= a_k), 3, \ldots, n, a_{k+1}, \ldots, a_{2m}, b_1, \ldots, b_\ell$	a
	② wva	$\prec_{\text{lex}}^{I_j}$: $1, 2, a_1, \ldots, a(= a_k), 3, \ldots, v, \ldots, n, a_{k+1}, \ldots, a_{2m}, b_1, \ldots, b_\ell$	v
	③ wb	$\prec_{\text{lex}}^{I_j}$: $1, 2, a_1, \ldots, a_{2m}, b_1, \ldots, b(= b_k), 3, \ldots, n, b_{k+1}, \ldots, b_\ell$	b
$w \in C$	④ wa	$\prec_{\text{lex}}^{I_j}$: $1, 2, a_1, \ldots, a(= a_k), 3, \ldots, n, a_{k+1}, \ldots, a_{2m}, b_1, \ldots, b_\ell$	a
	⑤ wvb	$\prec_{\text{lex}}^{I_j}$: $1, 2, a_1, \ldots, a_{2m}, b_1, \ldots, b(= b_k), 3, \ldots, v, \ldots, n, b_{k+1}, \ldots, b_\ell$	v

Note that, in any case, the lexicographic ordering $\prec_{\text{lex}}^{I}$ on $V \cup B$ is given by

$$\prec_{\text{lex}}^{I}:\ 1, 2, 3, \ldots, n, a_1, \ldots, \ldots, a_{2m}, b_1, \ldots, b_\ell,$$

and any atom of $[I, J]$ containing the element w has a multiple edge. If $\max^{I_j}(I_j \setminus I) \prec_{\text{lex}}^{I_j} \min^{I_j}(J \setminus I_j)$, then $J \setminus I_j$ cannot have a multiple edge less than $\max^{I}(B \cap (I_j \setminus I))$ in $\prec_{\text{lex}}^{I}$, and hence I_j is the first atom of $[I, J]$ in $\prec_{\text{atm}}^{I}$. This proves the 'only if' part. To see the 'if' part, suppose that $\min^{I_j}(J \setminus I_j) \prec_{\text{lex}}^{I_j} \max^{I_j}(I_j \setminus I)$. In the cases of ①, ②, and ④, $J \setminus I_j$ contains a multiple edge a'' with $a'' \prec_{\text{lex}}^{I_j} a$, which impliles that I_* can be obtained from I_j by replacing a with a''. In the cases of ③ and ⑤, $I_j \setminus I$ contains a multiple edge $e \prec_{\text{lex}}^{I_j} b$. If $e \in C$, then $J \setminus I_j$ is ec for some $c \in C$, and hence I_*'s are $I \cup wec$ and $I \cup we$, repsectively. If $e \notin C$, then I_*'s are $I \cup we$ and $I \cup wve$, repsectively. □

We remark that **(1)**∼**(5)** of the proof above are useful to figure out a falling chain of $\mathcal{P}_{G,C}^{\text{even}}$, which will be discussed in the next section.

6. Applications of shellable posets of even subgraphs

6.1. **Falling chains and the order complex of a poset.** Throughout this subsection, for a graph H in Figure 3 and its admissible collection C, the labeling of the vertices follows the way shown in Figure 9, and so the labels of the endpoints of the bundle are changed according to C.

Recall that if a bounded poset $\mathcal{P}$ admits a recursive atom ordering, then we can find the CL-labeling λ as in the sketch of the proof of Theorem 2.4. Furthermore the ith reduced Betti number of the order complex $\Delta(\overline{\mathcal{P}})$ equals the number of falling chains of length $i+2$ from Theorem 2.2. For a graph $G \in \mathcal{G}^*$, if G is simple, then the homotopy type of $\Delta(\overline{\mathcal{P}_G^{\text{even}}})$ is already known as noted in Remark 3.3. If G is a graph in Figure 3, then as we seen in Section 5, the order $\prec_{\text{atm}}^{I}$ in Definition 5.2 gives a recursive atom ordering of $\mathcal{P}_{G,C}^{\text{even}}$ for every $C \in \mathcal{A}(G)$, and so we can determine the homotopy type of $\Delta(\overline{\mathcal{P}_{G,C}^{\text{even}}})$ by considering the CL-labeling λ obtained from the recursive atom order on $\mathcal{P}_{G,C}^{\text{even}}$.

Corollary 6.1. *Let G be a graph in Figure 3, and let V and B be the set of vertices and the bundle of G, respectively. For each $C \in \mathcal{A}(G)$, the order complex $\Delta(\overline{\mathcal{P}_{G,C}^{\text{even}}})$ has the homotopy type of a wedge* *of spheres of dimensions*

$$\begin{cases} \frac{|C|}{2} + |B \setminus C| - 2, & \textit{if } |V| \textit{ is odd;} \\ \frac{|C|}{2} + |B \setminus C| - 1, & \textit{if } |V| \textit{ is even and } C \cap V \neq V; \\ \frac{|C|}{2} - 2 \textit{ or } \frac{|C|}{2} - 3, & \textit{if } C = V \cup B, \textit{ and the vertices } 2 \textit{ and } 3 \textit{ are adjacent; or} \\ \frac{|C|}{2} + |B \setminus C| - 3, & \textit{otherwise.} \end{cases}$$

Proof. Let σ be a longest maximal chain of $\mathcal{P}^{\text{even}}_{G,C}$. By Proposition 5.1, we only need to show that σ cannot be a falling chain if one of the following holds: (i) $|V|$ is odd, (ii) $C \cap V = V$ and the vertices 2 and 3 are not adjacent, (iii) $C \cap V = V$ and $B \setminus C \neq \emptyset$.

If $|V|$ is odd, then σ has the cover $I \lessdot I \cup 1$. Since $I \cup 1$ is the first atom of $[I, G]$, σ cannot be a falling chain. Hence every falling chain has the length $\frac{|C|}{2} + |B \setminus C|$. This proves case (i).

Now assume that $|V|$ is even and $C \cap V = V$. Then σ has the covers

$$I \lessdot I \cup 1v \text{ and } J \lessdot J \cup 2a, \qquad \text{or} \qquad I \lessdot I \cup 2v \text{ and } J \lessdot J \cup 1a,$$

where $I < J$, $v = \min(V \setminus (I \cup \{1, 2\}))$, and $a \in B \cap C$. If σ has the covers $I \lessdot I \cup 1v$ and $J \lessdot J \cup 2a$, then $I \lessdot I \cup 1v$ is the first atom of $[I, G]$. Hence σ cannot be a falling chain, which proves case (ii). Now assume that $B \setminus C \neq \emptyset$ and σ has the covers $I \lessdot I \cup 2v$ and $J \lessdot J \cup 1a$. Note that this occurs only when the vertices 2 and 3 are adjacent. Then $J = I \cup 2v$ by (1) in the proof of Lemma 5.5. Then σ must have the cover $K \lessdot K \cup b$ for some $b \in B \setminus C$ such that $K \cap (B \setminus C) = \emptyset$. Hence σ cannot be a falling chain. This proves case (iii). ☐

Example 6.2. Let us go back to the posets $\mathcal{P}^{\text{even}}_{G,C}$ in Figure 10. The posets in (A) and (C) are nonpure but none of the longest maximal chains of (A) and (C) are falling chains. In (A), (B), and (C), there are four, three, and four falling chains, respectively:

(A) $\emptyset < \mathbf{23} < \mathbf{1}23\boldsymbol{b_1} < 123\mathbf{4}\boldsymbol{a_1}b_1 < 123\mathbf{4}5a_1\boldsymbol{a_2}b_1$ $\qquad$ $\emptyset < \mathbf{23} < \mathbf{1}23\boldsymbol{b_1} < 123\mathbf{4}\boldsymbol{a_2}b_1 < 1234\mathbf{5}\boldsymbol{a_1}a_2b_1$

$\emptyset < \mathbf{34} < \mathbf{2}34\mathbf{5} < \mathbf{1}2345\boldsymbol{b_1} < 12345\boldsymbol{a_1}\boldsymbol{a_2}b_1$ $\qquad$ $\emptyset < \mathbf{45} < \mathbf{23}45 < \mathbf{1}2345\boldsymbol{b_1} < 12345\boldsymbol{a_1}\boldsymbol{a_2}b_1$

(B) $\emptyset < \mathbf{2} < \mathbf{1}2\boldsymbol{b_2} < 12\boldsymbol{b_1}b_2 < 12\mathbf{3}\boldsymbol{a_1}b_1b_2 < 123\mathbf{4}a_1\boldsymbol{a_2}b_1b_2$ $\qquad$ $\emptyset < \mathbf{2} < \mathbf{1}2\boldsymbol{b_2} < 12\boldsymbol{b_1}b_2 < 12\mathbf{3}\boldsymbol{a_2}b_1b_2 < 123\mathbf{4}\boldsymbol{a_1}a_2b_1b_2$

$\emptyset < \mathbf{34} < \mathbf{2}34 < \mathbf{1}234\boldsymbol{b_2} < 1234\boldsymbol{b_1}b_2 < 1234\boldsymbol{a_1}\boldsymbol{a_2}b_1b_2$

(C) $\emptyset < \mathbf{12}\boldsymbol{b_2} < 12\boldsymbol{a_1}\boldsymbol{a_2}b_2 < 12a_1a_2\boldsymbol{b_1}b_2 < 12\mathbf{34}a_1a_2b_1b_2$ $\qquad$ $\emptyset < \mathbf{34} < \mathbf{12}34\boldsymbol{b_2} < 1234\boldsymbol{b_1}b_2 < 1234\boldsymbol{a_1}\boldsymbol{a_2}b_1b_2$

$\emptyset < \mathbf{12}\boldsymbol{b_2} < 12\boldsymbol{b_1}b_2 < 12\mathbf{3}\boldsymbol{a_1}b_1b_2 < 123\mathbf{4}a_1\boldsymbol{a_2}b_1b_2$ $\qquad$ $\emptyset < \mathbf{12}\boldsymbol{b_2} < 12\boldsymbol{b_1}b_2 < 12\mathbf{3}\boldsymbol{a_2}b_1b_2 < 123\mathbf{4}\boldsymbol{a_1}a_2b_1b_2$

Hence the order complexes $\Delta(\overline{\mathcal{P}^{\text{even}}_{G,C}})$ of the proper parts of the posets $\mathcal{P}^{\text{even}}_{G,C}$ in Figure 10 are homotopy equivalent to $\bigvee_4 S^2$, $\bigvee_3 S^3$, and $\bigvee_4 S^2$, respecively.

In the rest of this subsection, we consider the graph $H = \tilde{P}_{n,m}$ in Figure 3. Let $V = \{1, 2, \ldots, n\}$ be the set of vertices and $B = \{a_1, \ldots, a_m\}$ be the unique bundle of H. Recall that we follow the labeling of the vertices shown in Figure 9.

Let $C \in \mathcal{A}(H)$ such that $B \subset C$. Note that from the proof of Corollary 6.1, if $\sigma : I_0 \lessdot I_1 \lessdot \cdots \lessdot I_{p+1}$ is a falling chain of $\mathcal{P}^{\text{even}}_{H,C}$, then there exists i such that $|I_i \setminus I_{i-1}| \geq 3$. More precisely, if $V \cap C = V$, then $12 \subset I_i \setminus I_{i-1}$, and if $V \cap C \neq V$, then $I_i \setminus I_{i-1}$ is $1ac$, where $1 \notin C$, $a \in B$, and $c \in C$. Then the number of falling chains of $\mathcal{P}^{\text{even}}_{H,C}$ for $C = V \cup B$ is equal to

$$\sum_{I \subset V \setminus \{1,2\}} (\#\text{ falling chains of } [I \cup 12aa', H] \text{ for some } a, a' \in B) \times (\#\text{ falling chains of } [\emptyset, I]). \tag{6.1}$$

If $V \cap C \neq V$, then the number of falling chains of $\mathcal{P}^{\text{even}}_{H,C}$ is equal to

$$\sum_{\substack{I \subset V \setminus \{1\} \\ 2 \in I}} (\#\text{ falling chains of } [I \cup 1ac, H] \text{ for some } a \in B, c \in C) \times (\#\text{ falling chains of } [\emptyset, I]). \tag{6.2}$$

Proposition 6.3. *Let $H = \tilde{P}_{n,2}$ in Figure 3. For $C \in \mathcal{A}(H)$, the number of falling chains of $\mathcal{P}^{\text{even}}_{H,C}$ is*

$$\begin{cases} C_k & \text{if } n = 2k \text{ for some } k \geq 1 \\ C_{k+1} - C_k & \text{if } n = 2k+1 \text{ for some } k \geq 1 \text{ and } C \cap V \text{ itself induces a connected graph,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let V be the set of vertices and $B = \{a_1, a_2\}$ be the bundle of H. First, suppose that $C \cap V = V$. Then $C = V \cup B$ and $|V| = 2k$ for some $k \geq 1$. If $k = 1$, then it is clear. Suppose that $k \geq 2$. Since we have only two multiple edges, from (6.1) the number of falling chains of $\mathcal{P}^{\text{even}}_{H,C}$ is

$$\sum_{q=1}^{2} (\#\text{ falling chains of } \mathcal{P}_{2q} \text{ starting with } 12a_1a_2) \times \sum_{\substack{|I|=2k-2q \\ I \subset \{3,4,\ldots,2k\}}} (\#\text{ falling chains of } [\emptyset, I])$$
$$= C_{k-1} + (\#\text{ falling chains of } \mathcal{P}_4 \text{ starting with } 12a_1a_2) \times \sum_{\substack{|I|=2k-4 \\ I \subset V \setminus \{1,2\}}} (\#\text{ falling chains of } [\emptyset, I]),$$

where $\mathcal{P}_{2q}$ means the poset $\mathcal{P}^{\text{even}}_{H',H'}$ for $H' = \tilde{P}_{2q,2}$, and the second summation is over the vertices I of $\mathcal{P}^{\text{even}}_{H,C}$. Since the number of falling chains of $\mathcal{P}_4$ starting with $12a_1a_2$ is only one (see the second poset of Figure 2), the number of falling chains is $C_{k-1} + s$, where

$$s = \sum_{\substack{|I|=2k-4 \\ I \subset V \setminus \{1,2\}}} (\#\text{ falling chains of } [\emptyset, I]).$$

Let $I \subset \{3, 4, \ldots, 2k\}$ be an element of $\mathcal{P}^{\text{even}}_{H,C}$ with $2k - 4$ vertices. Then $V \setminus I = \{1, 2, v_1, v_2\}$ where $v_1 < v_2$. Since each component of I has an even number of vertices, v_1 is odd and v_2 is even, and so the number of falling chains of $[\emptyset, I]$ is $C_{\frac{v_1-3}{2}} C_{\frac{v_2-v_1-1}{2}} C_{\frac{2n-v_2}{2}}$. By a recursion of the Catalan numbers,

$$s = \sum_{\substack{v_1=3 \\ v_1:\text{ odd}}}^{2k-1} \sum_{\substack{v_2=v_1+1 \\ v_2:\text{ even}}}^{2k} C_{\frac{v_1-3}{2}} C_{\frac{v_2-v_1-1}{2}} C_{\frac{2k-v_2}{2}} = \sum_{\substack{v_1=3 \\ v_1:\text{ odd}}}^{2k-1} C_{\frac{v_1-3}{2}} C_{\frac{2k-v_1+1}{2}} = C_k - C_{k-1}. \tag{6.3}$$

Hence the number of falling chains is $C_{k-1} + s = C_k$ when $n = 2k$ ($k \geq 1$) and C contains V.

Now we suppose that $C \cap V \neq V$. Note that it follows from (6.2) that there is no falling chain of $\mathcal{P}^{\text{even}}_{H,C}$ if $|V|$ is odd and $C \cap V$ does not induce a connected graph. Hence we need to consider the case where $|V|$ is even or $C \cap V$ induces a connected graph. In (6.2), a falling chain of $[I \cup 1a_ic, H]$ for some $a_i \in B$ and $c \in C$ is either $I \lessdot I \cup 1a_2v \lessdot H$ ($v \in V$), or $I \lessdot I \cup 1a_1a_2 = H$. In each of the cases, it is uniquely determined. Hence the number of falling chains is equal to $s_1 + s_2$, where

$$s_1 = (\#\text{ falling chains of } [\emptyset, H \setminus (1 \cup B)]), \qquad s_2 = \sum_{\substack{|I|=|V(H)|-3 \\ I \subset V \setminus \{1\},\ 2 \in I}} (\#\text{ falling chains of } [\emptyset, I]).$$

First, suppose $|V(H)| = 2k$ and $k \geq 1$. Then s_1 is equal to C_{k-1}, the number of falling chains of $\mathcal{P}^{\text{even}}_{P_{2k-2}}$. If $k = 1$, then $s_2 = 0$ and so the number of falling chains is C_1 (since $C_0 = C_1 = 1$). Suppose that $k \geq 2$. Let $I \subset \{2, 3, \ldots, 2k\}$ be an element of $\mathcal{P}^{\text{even}}_{H,C}$ with $2k - 3$ vertices containing the vertex 2. Then $V \setminus I = \{1, v_1, v_2\}$ where $2 < v_1 < v_2$. Since each component of I has an even number of vertices and $2 \notin C$, v_1 is odd and v_2 is even. Since s_2 has the same equation in (6.3), $s_1 + s_2 = C_{k-1} + (C_k - C_{k-1}) = C_k$. Hence the number of falling chains is C_k if $n = 2k$.

We suppose $|V(H)| = 2k + 1$ and $k \geq 1$. Then s_1 is equal to C_k, the number of falling chains of $\mathcal{P}^{\text{even}}_{P_{2k}}$. If $k = 1$, then $s_2 = 1$ and so the number of falling chains is $C_2 - C_1$ (note $C_2 = 2$ and $C_1 = 1$). Suppose $k \geq 2$. Let $I \subset \{2, 3, \ldots, 2k+1\}$ be an element of $\mathcal{P}^{\text{even}}_{H,C}$ with $2k - 3$ vertices containing the vertex 2. Then $V \setminus I = \{1, v_1, v_2\}$, where $2 < v_1 < v_2$. Since each component of I has an even number of vertices and $2 \in C$, v_1 is even and v_2 is odd. Thus $s_1 + s_2 = C_k + (C_{k+1} - 2C_k) = C_{k+1} - C_k$, since

$$s_2 = \sum_{\substack{v_1=4 \\ v_1\text{: even}}}^{2k} \sum_{\substack{v_2=v_1+1 \\ v_2\text{: odd}}}^{2k+1} C_{\frac{v_1-2}{2}} C_{\frac{v_2-v_1-1}{2}} C_{\frac{2k+1-v_2}{2}} = \sum_{\substack{v_1=4 \\ v_1\text{: even}}}^{2k} C_{\frac{v_1-2}{2}} C_{\frac{2k-v_1+2}{2}} = C_{k+1} - 2C_k.$$

Thus the number of falling chains is $C_{k+1} - C_k$. It completes the proof. □

Finally, for the homotopy types of $\Delta(\overline{\mathcal{P}^{\text{even}}_{H,C}})$ for $H = \tilde{P}_{n,2}$ and C is an admissible collection of H, we have Table 2 by Corollary 6.1 and Proposition 6.3. One may formulate the number of falling chains of $\mathcal{P}^{\text{even}}_{G,C}$, when $G = \tilde{P}_{n,m}$, in terms of the Catalan numbers (or the secant numbers), and it would be interesting to explain the formula by using other combinatorial objects.

H	$C \in \mathcal{A}(H)$	$\Delta(\overline{\mathcal{P}^{\text{even}}_{H,C}})$	
		Dimension	Homotopy Type
$H = \tilde{P}_{2k,2}$	$V(H) \setminus C = \emptyset$	$\frac{\lvert C\rvert}{2} - 3 = k-2$	$\bigvee_{C_k} S^{k-2}$
	$V(H) \setminus C \neq \emptyset$	$\frac{\lvert C\rvert}{2} - 1 = k-1$	$\bigvee_{C_k} S^{k-1}$
$H = \tilde{P}_{2k+1,2}$	$V(H) \setminus C \neq \emptyset$	$\frac{\lvert C\rvert}{2} - 2 = k-1$	$\bigvee_{C_{k+1}-C_k} S^{k-1}$

TABLE 2. The homotopy types of $\overline{\mathcal{P}^{\text{even}}_{H,C}}$ for $C \in \mathcal{A}^*(H)$ and $H = \tilde{P}_{n,2}$. The last row of the table is true only when $C \cap V$ induces a connected graph.

6.2. **Topology of real toric manifolds arising from graphs.** As it was noticed, the posets $\mathcal{P}^{\text{even}}_{G,C}$ are appeared in [7] to compute the rational Betti numbers of real toric manifolds arising from pseudograph associahedron. First, we summarize a main result in [7], and then discuss how to compute the rational Betti numbers of the real toric manifold associated with the graph $\tilde{P}_{n,2}$.

A polytope coming from a graph G, called the pseudograph associahedron and denoted by P_G, is firstly introduced in [5]; for a connected graph G, the facets of the polytope P_G are bijectively identified with the proper semi-induced connected subgraphs of G, and two facets intersect if and only if the corresponding semi-induced subgraphs are disjoint and cannot be connected by an edge of G, or one contains the other. The dimension of P_G is equal to $|V(G)| - 1 + \sum_{i=1}^{\ell}(|B_i| - 1)$, where B_i's are all the bundles of G. If G has the components $G_1, \ldots, G_k$, then P_G is defined to be the product $P_{G_1} \times \cdots \times P_{G_k}$. Moreover, P_G can be realized as a Delzant polytope[3] canonically. It is well-known in toric geometry that there is a one-to-one correspondence between projective smooth toric varieties and Delzant polytopes. Hence under the canonical Delzant realization, there is the projective smooth toric variety associated with a graph G. Then the real toric manifold M_G is defined as the subset consisting of points with real coordinates of the projective smooth toric variety associated with G. For example, it is known that if G is the simple path graph P_3, then the polytope P_G is a pentagon and M_G is $\#3\mathbb{R}P^2$, the connected sum of three copies of the real projective plane $\mathbb{R}P^2$. See [7, Sections 2 and 3], where the reader may find examples, definitions, and a much more detailed account of results for pseudograph associahedra. For $C \subset \mathcal{C}_G$, let $P^{\text{odd}}_{G,C}$ (respectively, $P^{\text{even}}_{G,C}$) be the set of facets of P_G corresponding semi-induced connected subgraphs are C-odd (respectively, C-even), and $K^{\text{odd}}_{G,C}$ (respectively, $K^{\text{even}}_{G,C}$) its dual simplicial complex. Then the ith rational Betti number $\beta^i(M_G)$ of the real toric manifold M_G is computed as follows. The ith reduced Betti number of a topological space X is denoted by $\tilde{\beta}^i(X)$.

[3]An n-dimensional convex polytope is said to be a *Delzant polytope* if the (outward) normal vectors to the facets (codimension-1 faces) meeting at each vertex (dimension-0 faces) form an integral basis of $\mathbb{Z}^n$.

Proposition 6.4 ([7]). *For a connected graph G, the ith rational Betti number of the real toric manifold M_G is*

$$\beta^i(M_G) = \sum_{\substack{H:\ \text{PI-graph} \\ \text{of } G}} \sum_{C \in \mathcal{A}(H)} \tilde{\beta}^{i-1}(K^{\text{odd}}_{H,C}).$$

Note that a real toric manifold M_G is connected, and hence $\beta^0(M_G) = 1$. We can also check it by using Proposition 6.4. For a graph H, if $V(H) = \emptyset$, then $K^{\text{odd}}_{H,C}$ is empty and $\tilde{\beta}^{-1}(K^{\text{odd}}_{H,C}) = 1$; if $V(H) \neq \emptyset$, then $K^{\text{odd}}_{H,C}$ is not empty, and so $\tilde{\beta}^{-1}(K^{\text{odd}}_{H,C}) = 0$. Thus we get $\beta^0(M_G) = 1$. In addition, from [7, Lemma 4.5], for a connected graph G and $(H, C) \in \mathcal{A}^*(G)$, if H_1 is a component of H and $C_1 = C \cap \mathcal{C}_{H_1}$ for $C \in \mathcal{A}(H)$, then $K^{\text{odd}}_{H,C}$ is isomorphic to the join $K^{\text{odd}}_{H_1,C_1} * K^{\text{odd}}_{H_2,C_2}$, where $H_2 = H \setminus H_1$ and $C_2 = C \setminus C_1$. Note that the join $X * Y$ is homotopy equivalent to the reduced suspension of the smash product $X \wedge Y$, and $\Sigma(X \wedge Y) = S^1 \wedge X \wedge Y$. Hence we get the following:

$$\tilde{\beta}^{i-1}(K^{\text{odd}}_{H,C}) = \sum_{\ell} \tilde{\beta}^{\ell}(K^{\text{odd}}_{H_1,C_1}) \times \tilde{\beta}^{i-\ell-2}(K^{\text{odd}}_{H_2,C_2}). \tag{6.4}$$

Remark 6.5. Note that $K^{\text{odd}}_{G,C}$ and $K^{\text{even}}_{G,C}$ have the same homotopy type with $P^{\text{odd}}_{G,C}$ and $P^{\text{even}}_{G,C}$, respectively. For a connected graph H, it was also noted in [7] that $\Delta(\overline{\mathcal{P}^{\text{even}}_{H,C}})$ is a geometric subdivision of $K^{\text{even}}_{H,C}$ for $C \in \mathcal{A}(H)$, and hence $\Delta(\overline{\mathcal{P}^{\text{even}}_{H,C}})$ is homotopy equivalent to $K^{\text{even}}_{H,C}$. Since $P^{\text{odd}}_{H,C} \cup P^{\text{even}}_{H,C} = \partial P_H$, it follows from the Alexander duality that $\tilde{\beta}^i(K^{\text{odd}}_{H,C}) = \tilde{\beta}^{\dim(P_H)-i-2}(\Delta(\overline{\mathcal{P}^{\text{even}}_{H,C}}))$.

We finish the section by explaining how to compute $\beta^i(M_G)$ when $G = \tilde{P}_{n,2}$ in Figure 3. It was shown in [6, Theorem 2.5] that, for the simple path graph P_n with n vertices, $\Delta(\overline{\mathcal{P}^{\text{even}}_{P_n}})$ is homotopy equivalent to $\bigvee_{C_k} S^{k-1}$ for $n = 2k$ and it is contractible for odd n. In addition, for any integer $n \geq 2$,

$$\beta^i(M_{P_n}) = \begin{cases} \binom{n}{i} - \binom{n}{i-1} & \text{if } 0 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

For a non-simple connected graph $H = \tilde{P}_{k,2}$ ($k > 0$) in Table 2, $\dim(P_H) = |V(H)|$ and by Remark 6.5,

$$b^i_k := \sum_{C \in \mathcal{A}(H)} \tilde{\beta}^i(K^{\text{odd}}_{\tilde{P}_{k,2},C}) = \begin{cases} C_{\frac{k}{2}} & \text{if } i = \frac{k}{2} \text{ or } \frac{k}{2} - 1 \text{ for even } k \\ C_{\frac{k+1}{2}} - C_{\frac{k-1}{2}} & \text{if } i = \frac{k-1}{2} \text{ for odd } k \\ 0 & \text{otherwise.} \end{cases} \tag{6.6}$$

Now we are ready to explain how to compute $\beta^i(M_G)$ from (6.4)$\sim$(6.6), when $G = \tilde{P}_{n,2}$. Assume that $i > 0$. Let $\mathcal{H}_1$ be the set of all simple PI-graphs of G and $\mathcal{H}_2$ the set of all non-simple PI-graphs of G. By Proposition 6.4, $\beta^i(M_G) = s^i_1 + s^i_2$ where

$$s^i_1 = \sum_{H \in \mathcal{H}_1} \sum_{C \in \mathcal{A}(H)} \tilde{\beta}^{i-1}(K^{\text{odd}}_{C,H}), \qquad s^i_2 = \sum_{H \in \mathcal{H}_2} \sum_{C \in \mathcal{A}(H)} \tilde{\beta}^{i-1}(K^{\text{odd}}_{C,H}).$$

As $\mathcal{H}_1$ is the set of PI-graphs of the simple graph P_n, $s^i_1 = \beta^i(M_{P_n})$. By Proposition 6.4 and (6.4),

$$\begin{aligned} s^i_2 &= \sum_{m=2}^{n-2} \sum_{C \in \mathcal{A}(\tilde{P}_{m,2})} \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \tilde{\beta}^{\ell}(K^{\text{odd}}_{\tilde{P}_{m,2},C}) \times \beta^{i-\ell-1}(M_{P_{n-m-1}}) + \sum_{m=n-1}^{n} \sum_{C \in \mathcal{A}(\tilde{P}_{m,2})} \tilde{\beta}^{i-1}(K^{\text{odd}}_{\tilde{P}_{m,2},C}) \\ &= \sum_{\ell=0}^{i-1} \sum_{m=2}^{n-2} b^{\ell}_m \beta^{i-\ell-1}(M_{P_{n-m-1}}) + b^{i-1}_{n-1} + b^{i-1}_{n}, \end{aligned}$$

and we note the second summation above is valid when $n \geq 4$. Hence

$$\beta^i(M_G) \quad = \quad \beta^i(M_{P_n}) + \sum_{\ell=0}^{i-1} \sum_{m=2}^{n-2} b_m^\ell \beta^{i-\ell-1}(M_{P_{n-m-1}}) + b_{n-1}^{i-1} + b_n^{i-1}. \tag{6.7}$$

Combining (6.7) with (6.5) and (6.6), one can completely compute $\beta^i(M_G)$ when $G = \tilde{P}_{n,2}$. Table 3 shows the rational Betti numbers of $M_{\tilde{P}_{n,2}}$ for some small integers n. We observe a more simple formula for $\beta^i(M_G)$ for some i. For example, $\beta^1(M_G) = n$ and $\beta^2(M_G) = \binom{n}{2}$. We also see that $\beta^k(M_{\tilde{P}_{2k,2}}) = \beta^{k+1}(M_{\tilde{P}_{2k+1,2}}) = \frac{6k}{k+2} C_k$, which is known as the total number of nonempty subtrees over all binary trees having $k+1$ internal vertices, see [19, A071721]. It would be interesting if one finds the exact formula of $\beta^i(M_G)$ and figures out that $\beta^i(M_G)$ counts other combinatorial objects for every i.

i \ n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	1	2	6	10	15	21	28	36	45	55	66	78	91	105
3	0	0	0	6	18	33	54	82	118	163	218	284	362	453
4	0	0	0	0	0	18	56	110	192	310	473	691	975	1337
5	0	0	0	0	0	0	0	56	180	372	682	1155	1846	2821
6	0	0	0	0	0	0	0	0	0	180	594	1276	2431	4277
7	0	0	0	0	0	0	0	0	0	0	0	594	2002	4433
8	0	0	0	0	0	0	0	0	0	0	0	0	0	2002

TABLE 3. The rational Betti numbers $\beta^i(M_{\tilde{P}_{n,2}})$ for small n

7. Further Discussions

In this paper, we characterize the family $\mathcal{G}^*$, that is, we find all graphs G such that $\mathcal{P}_{H,C}^{\text{even}}$ is shellable for every $(H, C) \in \mathcal{A}^*(G)$. As the problem was motivated by the topology of a real toric manifold associated with a graph, we could compute the rational Betti numbers of the one associated with $\tilde{P}_{n,2}$.

As a further research, it would be also interesting to see the family $\mathcal{G}_1^*$ of graphs G such that $\mathcal{P}_{G,C}^{\text{even}}$ is shellable for every $C \in \mathcal{A}(G)$. Since G is a PI-graph of itself, it is clear that $\mathcal{G}^* \subset \mathcal{G}_1^*$. Here is an example to show that $\mathcal{G}^*$ is a proper subset of $\mathcal{G}_1^*$. Consider a graph G with five vertices and one bundle $B = \{a, b\}$ of size two in Figure 11. Then $\mathcal{A}(G) = \{1345ab, 2345ab\}$, and both posets $\mathcal{P}_{G,1345ab}^{\text{even}}$ and

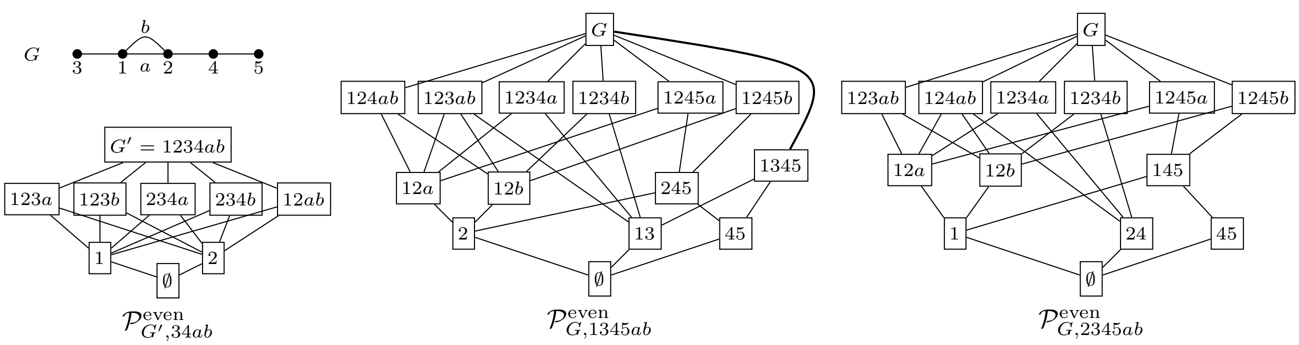


FIGURE 11. A graph $G \in \mathcal{G}_1^*$, and three shellable posets $\mathcal{P}_{G,1345ab}^{\text{even}}$, $\mathcal{P}_{G,2345ab}^{\text{even}}$, and $\mathcal{P}_{1234ab,34ab}^{\text{even}}$

$\mathcal{P}^{\text{even}}_{G,2345ab}$ are shellable, see Figure 11. However, $H \notin \mathcal{G}^*$ by Theorem 1.1 (For a specific reason, refer the proof of Claim 4.4 in Section 4.). One may find infinitely many such graphs.

Going one step further, we ask to completely characterize all pairs (G, C) supporting a shellable poset $\mathcal{P}^{\text{even}}_{G,C}$. It would be the first step to find such pairs (G, C) when G has exactly one bundle. For example, for a subgraph $G' = 1234ab$ of the graph G in Figure 11, $\mathcal{A}(G') = \{34ab, 1234ab\}$, $\mathcal{P}^{\text{even}}_{G',34ab}$ is shellable as in Figure 11 and $\mathcal{P}^{\text{even}}_{G',1234ab}$ is not by the proof of Claim 4.4.

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