May 19, 2017

# z-CLASSES AND RATIONAL CONJUGACY CLASSES IN ALTERNATING GROUPS 

SUSHIL BHUNIA, DILPREET KAUR, ANUPAM SINGH<br>IISER Pune, Dr. Homi Bhabha Road, Pashan, Pune 411008 INDIA


#### Abstract

In this paper, we compute the number of $z$-classes (conjugacy classes of centralizers of elements) in the symmetric group $S_{n}$, when $n \geq 3$ and alternating group $A_{n}$ when $n \geq 4$. It turns out that the difference between the number of conjugacy classes and the number of $z$-classes for $S_{n}$ is determined by those restricted partitions of $n-2$ in which 1 and 2 do not appear as its part. And, in the case of alternating groups, it is determined by those restricted partitions of $n-3$ which has all its parts distinct, odd and in which 1 (and 2) does not appear as its part, along with an error term. The error term is given by those partitions of $n$ which have each of its part distinct, odd and perfect square. Further, we prove that the number of rational-valued irreducible complex characters for $A_{n}$ is same as the number of conjugacy classes which are rational.


## 1. Introduction

Let $G$ be a group. Two elements $x, y \in G$ are said to be $z$-conjugate if their centralizers $\mathcal{Z}_{G}(x)$ and $\mathcal{Z}_{G}(y)$ are conjugate in $G$. This defines an equivalence relation on $G$ and the equivalence classes are called $z$-classes. Clearly if $x$ and $y$ are conjugate then they are also $z$-conjugate. Thus, in general, $z$-conjugacy is a weaker relation than conjugacy on G. In the theory of groups of Lie type, this is also called "types" (see Gr) and the number of $z$-classes of semisimple elements is called the genus number (see [Ca1, Ca2). This has been studied explicitly for various groups of Lie type in several papers, see for example, $\overline{\mathrm{BS}}, \mathrm{GO}, \mathrm{GK}, \mathrm{Ku}, \mathrm{Si}$. In this work, we want to classify and count the number of $z$-classes for symmetric and alternating groups. For convenience we deal with these groups when they are non-commutative (the commutative cases can be easily calculated), i.e., we assume $n \geq 3$ while dealing with symmetric groups and $n \geq 4$ while dealing with alternating groups.

[^0]Let $\sigma \in S_{n}$. The conjugacy classes of elements in $S_{n}$ are determined by their cycle structure which, in turn, is determined by a partition of $n$. Let $\lambda=\lambda_{1}^{e_{1}} \lambda_{2}^{e_{2}} \cdots \lambda_{r}^{e_{r}}$ be a partition of $n$, i.e., we have $1 \leq \lambda_{1}<\lambda_{2}<\ldots<\lambda_{r} \leq n$, each $e_{i}>0$ and $n=\sum_{i=1}^{r} \lambda_{i} e_{i}$. We may represent an element of $S_{n}$ corresponding to a partition $\lambda$ in cycle notation. We prove the following,

Theorem 1.1. Suppose $n \geq 3$. Let $\nu$ be a restricted partition of $n-2$ in which 1 and 2 do not appear as its part. Let $\lambda=1^{2} \nu$ and $\mu=2^{1} \nu$ be partitions of $n$ obtained by extending $\nu$. Then the conjugacy classes of $\lambda$ and $\mu$ belong to the same $z$-class in $S_{n}$. Further, the converse is also true.

Corollary 1.2. The number of $z$-classes in $S_{n}$ is $p(n)-\tilde{p}(n-2)$, where $p(n)$ is the number of partitions of $n$ and $\tilde{p}(n-2)$ is the number of those restricted partitions of $n-2$ in which 1 and 2 do not appear as its part. Thus, the number of $z$-classes in $S_{n}$ is equal to $p(n)-p(n-2)+p(n-3)+p(n-4)-p(n-5)$.

To prove this theorem, we need to understand the centralizers better which involves the generalised symmetric group. A group $S(a, b)=C_{a} \imath S_{b} \cong C_{a}^{b} \rtimes S_{b}$ is called a generalised symmetric group. We will briefly introduce this group in the following section.

Next we look at the problem of classifying $z$-classes in alternating groups $A_{n}$. Usually the conjugacy classes in $A_{n}$ are studied as a restriction of that of $S_{n}$. First, it is easy to determine for what partitions $\lambda=\lambda_{1}^{e_{1}} \cdots \lambda_{r}^{e_{r}}$ of $n$ the corresponding element $\sigma_{\lambda}$ is in $A_{n}$. This is precisely when $n-\sum e_{i}$ is even. We call such partitions even. Further, when $\sigma_{\lambda} \in A_{n}$, the conjugacy class of $\sigma_{\lambda}$ in $S_{n}$ splits in two conjugacy classes in $A_{n}$ if and only if $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)=\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)$, which is, if and only if the partition $\lambda$ has all its parts distinct and odd, i.e., $e_{i}=1$ and $\lambda_{i}$ odd for all $i$. With this notation we have,

Theorem 1.3. Suppose $n \geq 4$. Let $\lambda=\lambda_{1}^{e_{1}} \cdots \lambda_{r}^{e_{r}}$ be an even partition of $n$. Then the following determines $z$-classes in $A_{n}$.
(1) Suppose $e_{i}=1$ for all $i$ and all $\lambda_{i}$ are odd, i.e., $\lambda$ corresponds to two distinct conjugacy classes in $A_{n}$. Then, $\lambda$ corresponds to two distinct $z$-classes (corresponding to the two distinct conjugacy classes) if and only if all $\lambda_{i}$ are square. Else, the two split conjugacy classes form a single $z$-class.
(2) Suppose either one of the $e_{i} \geq 2$ or at least one of the $\lambda_{i}$ is even, i.e., $\lambda$ corresponds to a unique conjugacy class in $A_{n}$. Then, $\lambda$ is $z$-equivalent to another conjugacy class if and only if $\lambda=1^{3} \nu$, where $\nu$ is a restricted partition of $n-3$, with all its parts distinct and odd, and in which 1 (and 2) does not appear as its part. Further the other equivalent class is $3^{1} \nu$.

We remark that $3^{1} \nu$ could be of the first kind. For example, in $A_{8}$ the partitions $1^{3} 5^{1}$ and $3^{1} 5^{1}$ give same $z$-class. Further, the conjugacy class $3^{1} 5^{1}$ splits into two but both fall in
a single $z$-class. We list few more examples (using GAP) in a table in Section 9 . We also note that $\nu$ could have its first part 3 , in that case while writing $3^{1} \nu$ we appropriately absorb the power of 3 . We denote by $\epsilon(n)$, the number of partitions of $n$ with all of its parts distinct, odd and square. We list the values of $\epsilon(n)$ for small values in a table in Section 9

Corollary 1.4. The number of $z$-classes in $A_{n}$ is

$$
c l\left(A_{n}\right)-(q(n)+\tilde{q}(n-3))+\epsilon(n)
$$

where $\operatorname{cl}\left(A_{n}\right)=\frac{p(n)+3 q(n)}{2}$ is the number of conjugacy classes in $A_{n}, q(n)$ is the number of partitions of $n$ which has all parts distinct and odd, $\tilde{q}(m)$ is the number of restricted partitions of $m$, with all parts distinct, odd and which do not have 1 (and 2) as its part.

Let $G$ be a finite group. An element $g \in G$ is called rational if $g$ is conjugate to $g^{m}$ for all $m$ with property $(m, o(g))=1$ where $o(g)$ is the order of $g$. Clearly if $g$ is rational then all of its conjugates are rational. Thus a conjugacy class of $G$ is said to be rational if it is a conjugacy class of a rational element. It is believed that, for a finite group $G$, the number of conjugacy classes which are rational is related to the number of rational-valued complex irreducible characters of the group $G$ (for example, see Theorem A in [NT]). A group of which all elements are rational (and in that case, all complex irreducible characters are rational-valued) is called a rational group or $\mathbb{Q}$-group (see [K]). The alternating groups $A_{n}$ play an important role in determining simple groups which are rational (see Theorem A [FS]). There is a related notion of rational class in a group which comes from an equivalence relation. For a finite group $G$, a rational class of an element $g$ is a subset containing all elements of $G$ that are conjugate to $g^{m}$, where $(m, o(g))=1$. Thus the rational class of $g$ can be thought of as the conjugacy class of cyclic subgroup $\langle g\rangle$ of $G$. A conjugacy class which is rational is a rational class. However the converse need not be true. It is well known that, for a finite group $G$, the number of isomorphism classes of irreducible representations of $G$ over $\mathbb{Q}$ is equal to the number of rational classes of $G$ (see Corollary 1, Section 13.1 Se]). The symmetric group $S_{n}$ is rational. Alternating groups are not rational (see Corollary B. 1 [FS]). The rationalvalued complex irreducible characters for $A_{n}$ are discussed in Br and $[\mathrm{Pr}$. In this paper we determine conjugacy classes which are rational and the rational classes in alternating group. With notation as above,

Theorem 1.5. Suppose $n \geq 4$. Let $\tilde{C}$ be a conjugacy class in $A_{n}$ and corresponding partition be $\lambda=\lambda_{1}^{e_{1}} \cdots \lambda_{r}^{e_{r}}$ of $n$.
(1) Then the conjugacy class $\tilde{C}$ is rational in $A_{n}$ if and only if one of the following happens:
(a) either one of the $e_{i} \geq 2$ or one of the $\lambda_{i}$ is even, or,
(b) when all $\lambda_{i}$ are distinct (i.e., $e_{i}=1$ for all $i$ ) and odd, the product $\prod_{i=1}^{r} \lambda_{i} i s$ a perfect square. In this case, $\lambda$ corresponds to two conjugacy classes in $A_{n}$ and both are simultaneously rational (or non-rational).
(2) All conjugacy classes which are rational are rational classes. When $\tilde{C}$ is not a rational conjugacy class in $A_{n}$, the conjugacy class $C$ in $S_{n}$ containing $\tilde{C}$ is a rational class in $A_{n}$.

We denote by $\delta(n)$, the number of partitions of $n$ with all parts distinct, odd and the product of parts is a perfect square. We list the values of $\delta(n)$ for small values in a table in Section 9 which is also there in $[\mathrm{Br}$.

Corollary 1.6. For the alternating group $A_{n}$ with $n \geq 4$,
(1) the number of conjugacy classes which are rational is $\operatorname{cl}\left(A_{n}\right)-2 q(n)+2 \delta(n)$, and
(2) the number of rational classes is $\operatorname{cl}\left(A_{n}\right)-q(n)+\delta(n)$.

The character theory of $A_{n}$ is well understood. We use the notation and results from $[\mathrm{Pr}$ and conclude the following,

Theorem 1.7. Suppose $n \geq 4$. Then, the number of conjugacy classes in $A_{n}$ which are rational is same as the number of rational-valued complex irreducible characters.

This theorem is proved in Section 8. We also acknowledge that we have used GAP GAP] on several occasions to verify our computations and results.

Acknowledgement : The authors would like to thank Gerhard Hiss and Alexander Hulpke for wonderful discussion on GAP during the workshop "Group theory and computational methods" held at ICTS Bangalore, India in November 2016.

## 2. Restricted partitions

We require certain kind of restricted partitions which we introduce in this section. We denote by $p(m)$, the number of partitions of positive integer $m$. To set the notation clearly, a partition of $m$ is $\lambda=m_{1}^{e_{1}} \cdots m_{r}^{e_{r}}$ where $1 \leq m_{1}<\ldots<m_{r} \leq m, e_{i} \geq 1 \forall i$ and $m=\sum_{i=1}^{r} e_{i} m_{i}$. Sometimes this is also denoted as $\lambda \vdash m$ or $m_{1}^{e_{1}} \cdots m_{r}^{e_{r}} \vdash m$. We clarify that the partition written as $1^{1} 2^{1}$ is same as 1.2 but, in this case, latter notation is confusing if written without a dot. For us the significance of partitions is due to its one-one correspondence with conjugacy classes of the symmetric group $S_{m}$. Let $\tilde{p}(m)$ be the number of those partitions of $m$ in which 1 and 2 do not appear as its part, i.e.,

$$
\tilde{p}(m)=\left|\left\{\lambda=m_{1}^{e_{1}} \cdots m_{r}^{e_{r}} \vdash m \mid m_{1} \geq 3\right\}\right|
$$

Here we list down values of $\tilde{p}(m)$ for some small values.

| $m$ | $\tilde{p}(m)$ | $m$ | $\tilde{p}(m)$ | $m$ | $\tilde{p}(m)$ | $m$ | $\tilde{p}(m)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 0 | 6 | 2 | 11 | 6 | 16 | 21 |
| 2 | 0 | 7 | 2 | 12 | 9 | 17 | 25 |
| 3 | 1 | 8 | 3 | 13 | 10 | 18 | 33 |
| 4 | 1 | 9 | 4 | 14 | 13 | 19 | 39 |
| 5 | 1 | 10 | 5 | 15 | 17 | 20 | 49 |

The generating function for $\tilde{p}(m)$ is

$$
\prod_{i \geq 3} \frac{1}{1-x^{i}}
$$

and a formula to compute $\tilde{p}(m)$ in terms of partition function is

$$
\tilde{p}(m)=p(m)-p(m-1)-p(m-2)+p(m-3) .
$$

This is a well known sequence in OEIS database (see OEIS). This will be used in the study of $z$-classes of symmetric groups later.

Now we introduce the function $q(m)$. For a given integer $m$, the value of $q(m)$ is the number of those partitions of $m$ which have all of its parts distinct and odd, i.e.,

$$
q(m)=\mid\left\{\lambda=m_{1}^{1} \cdots m_{r}^{1} \vdash m \mid m_{i} \text { odd } \forall i\right\} \mid .
$$

This number is same as the number of self-conjugate partitions. For us this would correspond to those partitions which give split conjugacy classes in $A_{n}$. Now we introduce $\tilde{q}(m)$ which is the number of those restricted partitions of $m$ which have all its parts distinct, odd and 1 (and 2) does not appear as its part. The following table gives values of $\tilde{q}(m)$ for some values of $m$.

| $m$ | $q(m)$ | $\tilde{q}(m)$ | $m$ | $q(m)$ | $\tilde{q}(m)$ | $m$ | $q(m)$ | $\tilde{q}(m)$ | $m$ | $q(m)$ | $\tilde{q}(m)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 0 | 6 | 1 | 0 | 11 | 2 | 1 | 16 | 5 | 3 |
| 2 | 0 | 0 | 7 | 1 | 1 | 12 | 3 | 2 | 17 | 5 | 2 |
| 3 | 1 | 1 | 8 | 2 | 1 | 13 | 3 | 1 | 18 | 5 | 3 |
| 4 | 1 | 0 | 9 | 2 | 1 | 14 | 3 | 2 | 19 | 6 | 3 |
| 5 | 1 | 1 | 10 | 2 | 1 | 15 | 4 | 2 | 20 | 7 | 4 |

The generating function for $q(m)$ is $\prod_{i \geq 0}\left(1+x^{2 i+1}\right)$ and the generating function for $\tilde{q}(m)$ is $\prod_{i \geq 1}\left(1+x^{2 i+1}\right)$.

## 3. Symmetric groups

In this section we classify $z$-classes in $S_{n}$. Since the centralizers are a product of generalised symmetric groups, we begin with a brief introduction to them.
3.1. Generalised Symmetric Groups. The group $S(a, b)=C_{a}$ 亿 $S_{b}$, where $C_{a}$ is a cyclic group and $S_{b}$ is a symmetric group, is called a generalised symmetric group. This group is an example of wreath product and has been studied well in literature. Since the centralizer subgroups in the symmetric group are a product of generalised symmetric groups, we need to have more information about this group. For the sake of clarity, let us begin with defining this group. Consider the action of symmetric group $S_{b}$ on the direct product $C_{a}^{b}=C_{a} \times \cdots \times C_{a}$ given by permuting the components:

$$
\sigma\left(x_{1}, \ldots, x_{b}\right)=\left(x_{\sigma(1)}, \ldots, x_{\sigma(b)}\right) .
$$

Then the generalised symmetric group is $S(a, b)=C_{a}$ \} S _ { b } : = C _ { a } ^ { b } \rtimes S _ { b } . Hence the multiplication in this group is given as follows:

$$
\left(x_{1}, \cdots, x_{b}, \sigma\right)\left(y_{1}, \cdots, y_{b}, \tau\right)=\left(x_{1} y_{\sigma^{-1}(1)}, \cdots, x_{b} y_{\sigma^{-1}(b)}, \sigma \tau\right) .
$$

This group has a monomial matrix (each row and each column has exactly one non-zero entry) representation and it can be thought of as a subgroup of $G L_{b}(\mathbb{C})$, in particular as a subgroup of monomial group. Monomial group is well known in the study of $G L_{b}(\mathbb{C})$ as an algebraic group. This gives rise to the Weyl group and Bruhat decomposition. Let $T$ be the diagonal maximal torus (set of all diagonal matrices), then the monomial group is the normaliser $N_{G L_{b}(\mathbb{C})}(T)$. The Weyl group is defined as $W=N_{G L_{b}(\mathbb{C})}(T) / T \cong S_{b}$.

Let $D$ be the set of those diagonal matrices in $G L_{b}(\mathbb{C})$ of which each diagonal entry is an $a$ th roots of unity, i.e., each diagonal entry is from the set $\left\{\zeta^{i} \mid 0 \leq i \leq a-1\right\}$ where $\zeta$ is an $a$ th primitive root of unity. Clearly, $D \cong C_{a}^{b}$ and the group $S(a, b) \cong N_{G L_{b}(\mathbb{C})}(D)$. Thus, $S(a, b)$ is the set of those monomial matrices which have non-zero entries coming from $a$ th roots of unity. The following can be easily verified:
(1) the center $\mathcal{Z}(S(a, b))=\left\{\lambda . I d \mid \lambda^{a}=1\right\} \cong C_{a}$ if $a \geq 2$ or $b \geq 3$.
(2) $N_{G L_{b}(\mathbb{C})}(D) / D \cong S_{b}$.

Representation theory of the generalised symmetric group has been studied by Osima [Os, Can [Ca, Mishra and Srinivasan [MS], just to mention a few.
3.2. $z$-classes in $S_{n}$. In this section we aim to prove Theorem 1.1. For $n=3$ and 4 the conjugacy classes and $z$-classes are same. Thus, if necessary, we may assume $n \geq 5$ in this section. Let $\lambda=\lambda_{1}^{e_{1}} \lambda_{2}^{e_{2}} \cdots \lambda_{r}^{e_{r}}$ be a partition of $n$. Let us denote the partial sums as $n_{i}=\sum_{j=1}^{i} \lambda_{j} e_{j}$ and $n_{0}=0$. We may represent an element of $S_{n}$ corresponding to $\lambda$ as a product of cycles and we choose a representative of class denoted as $\sigma_{\lambda}=\sigma_{\lambda_{1}} \cdots \sigma_{\lambda_{i}} \cdots \sigma_{\lambda_{r}}$ where

$$
\sigma_{\lambda_{i}}=\underbrace{\left(n_{i-1}+1, \cdots, n_{i-1}+\lambda_{i}\right) \cdots\left(n_{i-1}+\left(e_{i}-1\right) \lambda_{i}+1, \cdots, n_{i-1}+e_{i} \lambda_{i}\right)}_{e_{i}}
$$

is a product of $e_{i}$ many disjoint cycles, each of length $\lambda_{i}$. Then the centralizer of this element is (see JK Equation 4.1.19)

$$
\mathcal{Z}_{S_{n}}(\lambda):=\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right) \cong \prod_{i=1}^{r} C_{\lambda_{i}} \backslash S_{e_{i}}
$$

where $C_{\lambda_{i}}$ is a cyclic group of size $\lambda_{i}$ and the size of the centralizer is given by the formula $\left|\mathcal{Z}_{S_{n}}(\lambda)\right|=\prod_{i=1}^{r}\left(\lambda_{i}^{e_{i}} . e_{i}!\right)$. Further, with the above chosen representative element the center of $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)$ is,

$$
Z_{\lambda}=\mathcal{Z}\left(\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)\right)= \begin{cases}\prod_{i=1}^{r}\left\langle\sigma_{\lambda_{i}}\right\rangle & \text { if } \lambda_{1}^{e_{1}} \neq 1^{2} \\ \langle(1,2)\rangle \times \prod_{i=2}^{r}\left\langle\sigma_{\lambda_{i}}\right\rangle & \text { when } \lambda_{1}^{e_{1}}=1^{2}\end{cases}
$$

Note that if $\lambda_{1}=1$ then the element $\sigma_{\lambda_{1}}=1$.
Lemma 3.1. Let $\lambda=\lambda_{1}^{e_{1}} \lambda_{2}^{e_{2}} \cdots \lambda_{r}^{e_{r}}$ be a partition of $n$. Then $\mathcal{Z}_{S_{n}}(\lambda)$ determines $r$ uniquely.

Proof. Consider the natural action of $G=\mathcal{Z}_{S_{n}}(\lambda)$ on the set $\{1,2, \ldots, n\}$ as a subgroup of $S_{n}$. Since $G \cong \prod_{i=1}^{r} C_{\lambda_{i}}\left\{S_{e_{i}}\right.$, the orbits are $\left\{\left\{1, \ldots, n_{1}\right\},\left\{n_{1}+1, \ldots, n_{2}\right\}, \ldots\right\}$. The number of orbits is exactly $r$.

Lemma 3.2. Let $\lambda=\lambda_{1}^{e_{1}} \lambda_{2}^{e_{2}} \cdots \lambda_{r}^{e_{r}}$ be a partition of $n$ and $\lambda_{1}^{e_{1}} \neq 1^{2}$. Let $Z_{\lambda}$ be the center of $\mathcal{Z}_{S_{n}}(\lambda)$. Then $Z_{\lambda}$ determines the partition $\lambda$ uniquely.

Proof. Let us make $Z_{\lambda}$ act on the set $\{1,2, \ldots, n\}$. Then the orbits are of size $\lambda_{i}$ and each of them occur $e_{i}$ many times. This determines the partition $\lambda$.

Proposition 3.3. Let $\lambda=\lambda_{1}^{e_{1}} \lambda_{2}^{e_{2}} \cdots \lambda_{r}^{e_{r}}$ and $\mu=\mu_{1}^{f_{1}} \mu_{2}^{f_{2}} \cdots \mu_{s}^{f_{s}}$ be partitions of $n$. Then $\mathcal{Z}_{S_{n}}(\lambda)$ is conjugate to $\mathcal{Z}_{S_{n}}(\mu)$ if and only if
(1) $r=s$,
(2) for all $i \geq 2, \lambda_{i}$ and $\mu_{i}$ are $\geq 3$ and $\lambda_{i}^{e_{i}}=\mu_{i}^{f_{i}}$,
(3) $\lambda_{1}^{e_{1}}=1^{2}$ and $\mu_{1}^{f_{1}}=2^{1}$ or vice versa.

Proof. Clearly if the three conditions are given we have $\lambda=1^{2} \nu$ and $\mu=2^{1} \nu$ where $\nu=\nu_{1}^{l_{1}} \cdots \nu_{k}^{l_{k}}$ is a partition of $n-2$ with $\nu_{1}>2$. Thus the representative elements of the conjugacy classes are $\sigma_{\mu}=(12) \sigma_{\lambda}$ and $\sigma_{\lambda}$, where $\sigma_{\lambda}$ has cycles each of length $>2$. Thus centralizers of these two elements are same.

For the converse, we choose representative elements $\sigma_{\lambda}$ and $\sigma_{\mu}$ and we are given that $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)$ and $\mathcal{Z}_{S_{n}}\left(\sigma_{\mu}\right)$ are conjugate. The Lemma 3.1 implies that $r=s$. Now we take the center of both of these groups $Z_{\lambda}$ and $Z_{\mu}$ and make it act on the set $\{1,2, \ldots, n\}$. If $\lambda_{1}^{e_{1}}$ and $\mu_{1}^{f_{1}}$ both are not $1^{2}$ then from Lemma 3.2 we get the required result.

This proves Theorem 1.1.

## 4. Rational conjugacy classes in $A_{n}$

The group $A_{n}$ is of index 2 in $S_{n}$. Thus, we usually think of conjugacy classes in $A_{n}$ in terms of that of $S_{n}$. We have two kinds of conjugacy classes in $A_{n}$. Let $\sigma_{\lambda}$ be a representative of a conjugacy class, corresponding to a partition $\lambda$, of $S_{n}$. Suppose $\sigma_{\lambda} \in A_{n}$, that is to say, $\lambda$ is an even partition. Then the two kinds of conjugacy classes are,
a. Split: The conjugacy class of $\sigma_{\lambda}$ in $S_{n}$ splits into two conjugacy classes in $A_{n}$ if and only if all parts of $\lambda$ are distinct and odd, which happens, if and only if $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)=\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)$.
b. Non-split: The conjugacy class of $\sigma_{\lambda}$ remains a single conjugacy class in $A_{n}$ if and only if either one of the $e_{i} \geq 2$ for some $i$ or at least one of the $\lambda_{i}$ is even, which is, if and only if $\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right) \subsetneq \mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)$.
While writing proofs in this section and later sections, we consider these two cases separately.

Let $G$ be a finite group and $g \in G$. The Weyl group of an element $g$ in $G$, denoted as, $W_{G}(g):=N_{G}(\langle g\rangle) / \mathcal{Z}_{G}(\langle g\rangle)$ where $\langle g\rangle$ is the subgroup generated by $g$. Using the $\operatorname{map} \iota: N_{G}(\langle g\rangle) \rightarrow \operatorname{Aut}(\langle g\rangle)$ given by $\iota(x)\left(g^{r}\right)=x g^{r} x^{-1}$, one can show that, the element $g$ in $G$ is rational if and only if $W_{G}(g) \cong \operatorname{Aut}(\langle g\rangle)$. We need to understand Weyl group of elements $\sigma$ in $A_{n}$. Since $S_{n}$ is a rational group, we have, the Weyl group $W_{S_{n}}(\sigma) \cong \operatorname{Aut}(\langle\sigma\rangle)$. Thus, to understand if $\sigma$ is rational in $A_{n}$, we need to understand $N_{A_{n}}(\langle\sigma\rangle)$. This is determined by Brison (see Theorem $4.3[\mathrm{Br}]$ ) as follows,

Theorem 4.1. Let $\sigma \in A_{n}$ and corresponding partition be $\lambda=\lambda_{1}^{e_{1}} \cdots \lambda_{r}^{e_{r}}$. Then, $N_{S_{n}}(\langle\sigma\rangle)=N_{A_{n}}(\langle\sigma\rangle)$ if and only if $\lambda$ satisfies the following,
(1) all parts of $\lambda$ are distinct, i.e., $e_{i}=1$ for all $i$,
(2) $\lambda_{i}$ is odd for all $i$, and
(3) the product of parts $\prod_{i=1}^{r} \lambda_{i} \in \mathbb{Z}$ is a perfect square.

Corollary 4.2. Suppose $n$ is odd and $w=(1,2, \ldots, n)$ is in $A_{n}$. Then, $w$ is rational in $A_{n}$ if and only if $n$ is a perfect square (of odd number).

Proof. We know $w$ is rational in $S_{n}$. Thus $W_{S_{n}}(w) \cong \operatorname{Aut}(\langle w\rangle)$. Since $n$ is odd the conjugacy class of $w$ in $S_{n}$ splits in $A_{n}$ and $\mathcal{Z}_{A_{n}}(w)=\mathcal{Z}_{S_{n}}(w)$. Thus $w$ is rational in $A_{n}$ if and only if $N_{S_{n}}(\langle w\rangle)=N_{A_{n}}(\langle w\rangle)$ which is if and only if $n$ is a perfect square (from Theorem 4.1 above).

Now we determine which conjugacy classes are rational in $A_{n}$.
When the conjugacy class does not split, $C=\sigma_{\lambda}^{S_{n}}=\sigma_{\lambda}^{A_{n}}$ and $\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right) \subsetneq \mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)$ is of index 2 . Then,

Proposition 4.3. Let $C$ be a non-split conjugacy class in $A_{n}$. Then, $C$ is rational in $A_{n}$.

Proof. For this, we need to prove $\sigma_{\lambda}$ is conjugate to $\sigma_{\lambda}^{m}$ for all $m$ which is coprime to the order of $\sigma_{\lambda}$. Since $S_{n}$ is rational we have $g \in S_{n}$ such that $g \sigma_{\lambda} g^{-1}=\sigma_{\lambda}^{m}$. If $g$ is in $A_{n}$ we are done. Else take $h \in \mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)$ which is not in $\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)$. Now $g h \in A_{n}$ and $g h \sigma_{\lambda} h^{-1} g^{-1}=\sigma_{\lambda}^{m}$, and we are done.

When the conjugacy class splits, let $C$ be the conjugacy class of $\sigma_{\lambda}$ in $S_{n}$ where $\lambda=$ $\lambda_{1}^{1} \cdots \lambda_{r}^{1}$ with all $\lambda_{i}$ odd (and distinct). Let $C_{1}$ and $C_{2}$ be the conjugacy classes in $A_{n}$, which are obtained by splitting $C$. Then,

Proposition 4.4. With the notation as above, both $A_{n}$ conjugacy classes $C_{1}$ and $C_{2}$ are rational if and only if $\prod_{i=1}^{r} \lambda_{i}$ is a perfect square.

Proof. In this case, we have $\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)=\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)$. Thus $W_{A_{n}}\left(\sigma_{\lambda}\right)=W_{S_{n}}\left(\sigma_{\lambda}\right) \cong \operatorname{Aut}\left(\left\langle\sigma_{\lambda}\right\rangle\right)$ if and only if $N_{A_{n}}\left(\left\langle\sigma_{\lambda}\right\rangle\right)=N_{S_{n}}\left(\left\langle\sigma_{\lambda}\right\rangle\right)$. Which is determined by Theorem 4.1,

We remark that either both conjugacy classes $C_{1}$ and $C_{2}$ are rational or not rational simultaneously.

Proposition 4.5. With the notation as above, suppose both conjugacy classes $C_{1}$ and $C_{2}$ are not rational. Then the subset $C=C_{1} \cup C_{2}$ is a rational class in $A_{n}$.

Proof. This follows easily because $C$ is a rational conjugacy class in $S_{n}$.
Proof of Theorem 1.5. Let $\tilde{C}$ be a conjugacy class in $A_{n}$. Consider the conjugacy class $C$ in $S_{n}$ containing $\tilde{C}$. Let $\lambda=\lambda_{1}^{e_{1}} \cdots \lambda_{r}^{e_{r}}$ be the corresponding partition of $C$. Then either $\tilde{C}=C$ or $C=C_{1} \cup C_{2}$, where $\tilde{C}$ is one of the $C_{1}$ or $C_{2}$. If $\tilde{C}=C$ it follows from Proposition 4.3 that it is always rational and this corresponds to the partitions where either $e_{i} \geq 2$ for some $i$ or one of the $\lambda_{i}$ is even.

Now suppose $C=C_{1} \cup C_{2}$, where $\tilde{C}$ is one of the components. Then from Proposition 4.4, it follows that both $C_{1}$ and $C_{2}$, and hence $\tilde{C}$, are rational if and only if $\prod_{i=1}^{r} \lambda_{i}$ is a square. That is, in this case the partition $\lambda$ has all parts distinct, odd and the product of parts is a square.

When $\tilde{C}$ is not a rational conjugacy class, Proposition 4.5 implies $C$ is a rational class in $A_{n}$. This completes the proof.

## 5. $z$-CLASSES IN $A_{n}$ - WHEN THE CONJUGACY CLASS SPLITS

Since the $z$-equivalence is a relation on conjugacy classes we deal with split and nonsplit classes separately. We begin with a few Lemmas.

Lemma 5.1. Let $x, y$ be elements in $A_{n}$ such that $x$ and $y$ are conjugate in $S_{n}$. If there exists $g \in A_{n}$ such that $g$ is conjugate to $y$ in $A_{n}$ and $\mathcal{Z}_{A_{n}}(g)=\mathcal{Z}_{A_{n}}(x)$ then centralizers $\mathcal{Z}_{A_{n}}(x)$ and $\mathcal{Z}_{A_{n}}(y)$ are conjugate in $A_{n}$.

Proof. Since $g$ and $y$ are conjugate in $A_{n}$, their centralizers $\mathcal{Z}_{A_{n}}(g)$ and $\mathcal{Z}_{A_{n}}(y)$ are conjugate in $A_{n}$. Hence centralizers $\mathcal{Z}_{A_{n}}(x)$ and $\mathcal{Z}_{A_{n}}(y)$ are conjugate in $A_{n}$.

The following Lemma establishes partial converse to the above.
Lemma 5.2. Let $\lambda=\lambda_{1} \cdots \lambda_{r}$ be a partition with all parts (distinct and) odd. Let $x$ and $y$ be elements in $A_{n}$ representing the two distinct conjugacy classes corresponding to $\lambda$. Suppose $x=x_{1} x_{2} \cdots x_{r}$ and $y=y_{1} y_{2} \cdots y_{r} \in A_{n}$, where $x_{i}$ and $y_{i}$ are cycles of length $\lambda_{i}$ and centralizers $\mathcal{Z}_{A_{n}}(x)$ and $\mathcal{Z}_{A_{n}}(y)$ are conjugate in $A_{n}$. Then, $y$ is conjugate to $x_{1}^{i_{1}} x_{2}^{i_{2}} \cdots x_{r}^{i_{r}}$ in $A_{n}$ for some positive integers $i_{1}, \ldots, i_{r}$, where $i_{j}$ is coprime to $\lambda_{j}$ (which is the order of $x_{j}$ ) for all $j$.

Proof. Since $x$ and $y$ are $z$-conjugate in $A_{n}$, i.e., there exist $g \in A_{n}$ such that $\mathcal{Z}_{A_{n}}(x)=$ $g \mathcal{Z}_{A_{n}}(y) g^{-1}=\mathcal{Z}_{A_{n}}\left(g y g^{-1}\right)$. Now, we know that $\mathcal{Z}_{A_{n}}(x)=\mathcal{Z}_{S_{n}}(x)=\left\langle x_{1}, x_{2}, \ldots, x_{r}\right\rangle \cong$ $C_{\lambda_{1}} \times \cdots \times C_{\lambda_{r}}$. Hence $g y g^{-1}=x_{1}^{i_{1}} x_{2}^{i_{2}} \cdots x_{r}^{i_{r}}$ for some $i_{1}, \ldots, i_{r}$. Therefore, $y$ is conjugate to $x_{1}^{i_{1}} x_{2}^{i_{2}} \cdots x_{r}^{i_{r}}$ in $A_{n}$.

Now we prove the main proposition of this section.
Proposition 5.3. Let $\lambda=\lambda_{1} \cdots \lambda_{r}$ be a partition with all parts (distinct and) odd. Let $x$ and $y$ be elements in $A_{n}$ representing the two distinct conjugacy classes corresponding to $\lambda$. Suppose $x=x_{1} x_{2} \cdots x_{r}$ and $y=y_{1} y_{2} \cdots y_{r}$ written as a product of disjoint cycles where $x_{i}$ and $y_{i}$ are of length $\lambda_{i}$. Then, $x$ and $y$ are not $z$-conjugate in $A_{n}$ if and only if each $\lambda_{i}$ is a perfect square (of odd number) $\forall i=1, \ldots, r$.

Proof. First, suppose there exists a $k$ such that $\lambda_{k}$ is not a perfect square of odd number. We define $A_{\lambda_{k}}$ and $S_{\lambda_{k}}$ to be the subgroups of $A_{n}$ and $S_{n}$ respectively, on the symbols involved in the cycle $x_{k}$. Corollary 4.2 implies that the element $x_{k}$ is not a rational element of $A_{\lambda_{k}}$. Hence, there exists $m$ with $\left(m, \lambda_{k}\right)=1$ such that $x_{k}$ is not conjugate to $x_{k}^{m}$ in $A_{\lambda_{k}}$. In any case $x_{k}$ is conjugate to $x_{k}^{m}$ in $S_{\lambda_{k}}$, say, there exists $s \in S_{\lambda_{k}} \backslash A_{\lambda_{k}}$ such that $s x_{k} s^{-1}=x_{k}^{m}$. Thus, $s x s^{-1}=s x_{1} x_{2} \cdots x_{k} \cdots x_{r} s^{-1}=x_{1} x_{2} \cdots x_{k-1} x_{k}^{m} x_{k+1} \cdots x_{r}$. We claim that $x$ is not conjugate to $s x s^{-1}$ in $A_{n}$. Because any two such elements will differ by an element of $\mathcal{Z}_{A_{n}}(x)$ which, in this case, is equal to $\mathcal{Z}_{S_{n}}(x)$ thus all such elements would be even. This implies that $x$ and $s x s^{-1}$ are representatives of the two distinct conjugacy classes obtained by splitting that of $x$ hence $s x s^{-1}$ is conjugate to $y$. But $\mathcal{Z}_{A_{n}}(x)=\left\langle x_{1}, x_{2}, \ldots, x_{r}\right\rangle=\left\langle x_{1}, x_{2}, \ldots, x_{k-1}, x_{k}^{m}, x_{k+1}, \ldots, x_{r}\right\rangle=\mathcal{Z}_{A_{n}}\left(s x s^{-1}\right)$, because of the structure of $s x s^{-1}$. Lemma 5.1 implies that $\mathcal{Z}_{A_{n}}(x)$ and $\mathcal{Z}_{A_{n}}(y)$ are conjugate in $A_{n}$.

Now, assume that each $\lambda_{i}$ is a perfect square (of odd number), for all $i$. We define the subgroups $A_{\lambda_{i}}$ of $A_{n}$ on the symbols appearing in the cycle $x_{i}$ for all $i$. Corollary 4.2 implies that $x_{i}$ is rational in $A_{\lambda_{i}}$, hence $x_{i}$ is conjugate to $x_{i}^{m_{i}}$ in $A_{\lambda_{i}}$ for all $m_{i}$ with $\left(m_{i}, \lambda_{i}\right)=1$. Let $\left(j_{1}, \ldots, j_{r}\right)$ be a tuple where $\left(j_{i}, \lambda_{i}\right)=1$. Then we can find $s_{j_{i}} \in A_{\lambda_{i}}$ such that $s_{j_{i}} x_{i} s_{j_{i}}^{-1}=x_{i}^{j_{i}}$. Thus $s_{j_{1}} s_{j_{2}} \cdots s_{j_{r}}$ is in $A_{n}$ and conjugates $x$ to $x_{1}^{j_{1}} x_{2}^{j_{2}} \cdots x_{r}^{j_{r}}$. Hence, $y$ can not be conjugate to $x_{1}^{j_{1}} x_{2}^{j_{2}} \cdots x_{r}^{j_{r}}$ for any tuple $\left(j_{1}, \ldots, j_{r}\right)$ where $\left(j_{i}, \lambda_{i}\right)=1$. Lemma [5.2, implies that $x$ and $y$ can not be $z$-conjugate in $A_{n}$.

## 6. The center of centralizers in $A_{n}$

In Lemma 3.2 we showed that, for the group $S_{n}$, the center of centralizers $Z_{\lambda}$ determines the partition $\lambda$ uniquely via its action on the set $\{1,2, \ldots, n\}$ except in one case when $\lambda_{1}^{e_{1}}=1^{2}$. For the alternating groups we employ similar strategy.

Let us begin with the case when the partition $\lambda$ has only one part, say, $\lambda=a^{b}$. The representative element can be chosen as follows,

$$
\sigma_{\lambda}=(1,2, \cdots, a)(a+1, a+2, \cdots, 2 a) \cdots((b-1) a+1,(b-1) a+2, \cdots, b a)
$$

which for convenience will be written as $\sigma_{\lambda}=\sigma_{\lambda, 1} \sigma_{\lambda, 2} \cdots \sigma_{\lambda, b}$ where $\sigma_{\lambda, i}$ are cycles of length $a$. And the centralizer is

$$
\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)=(\langle(1,2, \cdots, a)\rangle \times \cdots \times\langle((b-1) a+1,(b-1) a+2, \cdots, b a)\rangle) \rtimes S_{b}
$$

where $S_{b}$ permutes the various cyclic subgroups. To avoid confusion, we write the elements of $S_{b}$ using roman numerals. For example, the element $(I, I I)$ in $S_{b}$ would be actually $(1, a+1)(2, a+2) \cdots(a, 2 a)$ in $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)$, similarly, the element $(I, I I, \cdots, b)$ in $S_{b}$ would be $(1, a+1, \cdots,(b-1) a+1)(2, a+2, \cdots,(b-1) a+2) \cdots(a, 2 a, \cdots, b a)$. In general, the cycle $(I, I I, \cdots, i)$ in $S_{b}$ would be $(1, a+1, \cdots,(i-1) a+1)(2, a+2, \cdots,(i-$ 1) $a+2) \cdots(a, 2 a, \cdots, i a)$ which is a product of $a$ many disjoint cycles, each of length $i$. We can also compute $\operatorname{sgn}((I, I I \cdots, i))=\operatorname{sgn}((1,2, \cdots, i))^{a}$ which will be useful to determine if $(I, I I \cdots, i)$ belongs to $A_{n}$, when needed. We begin with,

Lemma 6.1. If $\lambda=a^{b}$ is a partition of $n$ and $b \geq 2$ then $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)$ contains at least one odd permutation.

Proof. If $a$ is even then the cycle $(1,2, \cdots, a) \in \mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)$ is odd and we are done. Thus we may assume $a$ is odd. From the computation above, $\operatorname{sgn}((I, I I))=(-1)^{a}=-1$ hence $(I, I I)$ is odd.

Lemma 6.2. (1) If $\tau=(I, I I, \cdots, b) \in \mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)$ then $\mathcal{Z}_{\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)}(\tau)=<\tau, \sigma_{\lambda}>$,
(2) If $\tau=(I, I I, \cdots, b-1) \in \mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)$ then $\mathcal{Z}_{\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)}(\tau)=<\tau, \sigma_{\lambda, b}, \prod_{i=1}^{b-1} \sigma_{\lambda, i}>$,

Proof. The proof is simple and follows from the multiplication defined on $S(a, b)$ in Section 3.1 ,

We need to understand the center of centralizers of elements in $A_{n}$. Suppose $\lambda=$ $\lambda_{1}^{e_{1}} \lambda_{2}^{e_{2}} \cdots \lambda_{r}^{e_{r}}$ is a partition of even type, i.e., $\sigma_{\lambda} \in A_{n}$. Recall the notation, $\sigma_{\lambda}=$ $\sigma_{\lambda_{1}} \cdots \sigma_{\lambda_{r}}$, the centralizer is $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right) \cong \prod_{i=1}^{r} \mathcal{Z}_{S_{e_{i} \lambda_{i}}}\left(\sigma_{\lambda_{i}}\right)$ and its center is denoted as $Z_{\lambda}$.
Lemma 6.3. Let $x \in A_{n}$. Then, $Z_{\lambda} \cap A_{n}=\mathcal{Z}\left(\mathcal{Z}_{S_{n}}(x)\right) \cap A_{n} \subseteq \mathcal{Z}\left(\mathcal{Z}_{A_{n}}(x)\right)$.
Proof. Let $g \in \mathcal{Z}\left(\mathcal{Z}_{S_{n}}(x)\right) \cap A_{n}$ then $g \in \mathcal{Z}_{A_{n}}(x)$. Now $\mathcal{Z}_{A_{n}}(x)=\mathcal{Z}_{S_{n}}(x) \cap A_{n}$ thus we get $g \in \mathcal{Z}\left(\mathcal{Z}_{A_{n}}(x)\right)$.

Now we need to decide when $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right) \supsetneq Z_{\lambda} \cap A_{n}$. For the convenience of reader we draw a diagram of the subgroups involved in the proofs. We call the elements of $Z_{\lambda}$ "diagonal elements" and the elements of $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)$ which are not central "non-diagonal elements".


The main theorem is as follows,
Theorem 6.4. Let $\lambda$ be a partition of $n$ and $\sigma_{\lambda} \in A_{n}$. Then $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right) \supsetneq Z_{\lambda} \cap A_{n}$ if and only if $\lambda$ is one of the following:
(1) $1^{3} \nu ; 2^{2} \nu ; 1^{1} 2^{2} \nu$ where $\nu=\lambda_{3} \cdots \lambda_{r}$ with all $\lambda_{i} \geq 3$ and odd.
(2) $1^{1} \nu ; \nu$ where $\nu=\lambda_{3} \cdots \lambda_{j-1} \lambda_{j}^{2} \lambda_{j+1} \cdots \lambda_{r}$ where $\lambda_{i} \geq 3$ and odd for all $i$.

The rest of this section is devoted to the proof of this theorem.
Lemma 6.5. Let $\lambda=\lambda_{1}^{e_{1}} \cdots \lambda_{r}^{e_{r}}$ where at least two distinct $e_{i}$ and $e_{j}$ are $\geq 2$. Then, $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)=Z_{\lambda} \cap A_{n}$.

Proof. Let us first take the case when $\lambda_{1}=1, e_{1} \geq 2$ and some other $e_{i}$ is $\geq 2$. We have $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)=S_{e_{1}} \times \mathcal{Z}_{S_{e_{2} \lambda_{2}}}\left(\sigma_{\lambda_{2}}\right) \times \cdots \times \mathcal{Z}_{S_{e_{r} \lambda_{r}}}\left(\sigma_{\lambda_{r}}\right)$. We note that when $e_{1}=2$ the subgroup $S_{2}$ is central. Let $g=\left(g_{1}, \ldots, g_{r}\right) \in \mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)$ but $g \notin Z_{\lambda}$. That is, there exists some $j$ such that $g_{j}$ is non-diagonal element in $\mathcal{Z}_{S_{e_{j} \lambda_{j}}}\left(\sigma_{\lambda_{j}}\right)$.

Suppose $j \neq 1$. Since $g_{j}$ is non-diagonal there exists $h_{j} \in \mathcal{Z}_{S_{e_{j} \lambda_{j}}}\left(\sigma_{\lambda_{j}}\right)$ such that $h_{j} g_{j} \neq$ $g_{j} h_{j}$. Now define $h=\left(1, \ldots, 1, h_{j}, 1, \ldots, 1\right)$ if $h_{j}$ is even else $h=\left((1,2), 1, \ldots, 1, h_{j}, 1, \ldots, 1\right)$. Then $h \in A_{n} \cap \mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)=\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)$ but $g h \neq h g$, a contradiction.

Now if $j=1$ the element $g_{1}$ is non-diagonal in $S_{e_{1}}$, that is, $g_{1} \neq 1$. We may also assume that all other $g_{i}$, other than the first one, are diagonal. However if $e_{1}=2$ the element $g=\left((1,2), g_{2}, \ldots g_{r}\right)$ is already in $Z_{\lambda}$, so we couldn't have assumed otherwise. Now if $e_{1} \geq 3$, pick $h_{1} \in S_{e_{1}}$ which does not commute with $g_{1}$. Now define $h=\left(h_{1}, 1, \ldots, 1\right)$
if $h_{1}$ is even. Else define $h=\left(h_{1}, 1, \ldots, 1, w, 1, \ldots, 1\right)$ where $w \in \mathcal{Z}_{S_{e_{i} \lambda_{i}}}\left(\sigma_{\lambda_{i}}\right)$ is an odd permutation guaranteed by Lemma 6.1. Then $h \in A_{n}$ but $g h \neq h g$, a contradiction.

The proof when $e_{1}=1$ and $\lambda_{1}=1$ or $\lambda_{1}>1$ follows similarly. Now two components $i$ and $j$ will have odd elements because of Lemma 6.1 which can be used to change the sign to get an appropriate $h$.

This reduces drastically the number of cases we need to look at. Thus we may assume that at most one $e_{i}$ is greater than 2 or none, i.e., $\lambda=\lambda_{1} \cdots \lambda_{i-1} \lambda_{i}^{e_{i}} \lambda_{i+1} \cdots \lambda_{r}$ with $e_{i} \geq 1$. Let us deal with the case when $i=1$ and $\lambda_{1}=1$.

Lemma 6.6. Let $\lambda=1^{e_{1}} \lambda_{2} \cdots \lambda_{r}$ be a partition of $n$. Then, $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right) \supsetneq Z_{\lambda} \cap A_{n}$ if and only if $\lambda=1^{3} \lambda_{2} \cdots \lambda_{r}$ where $\lambda_{i}>1$ and odd for all $i$.

Proof. Suppose $\lambda$ is not of the form $1^{3} \lambda_{2} \cdots \lambda_{r}$ where $\lambda_{i}>1$ and odd for all $i$. So, if $e_{1}=0,1$ or 2 then $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)$ is Abelian and its $\operatorname{subgroup} \mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)$ is also Abelian. Therefore,

$$
\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)=\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)=\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right) \cap A_{n}=Z_{\lambda} \cap A_{n}
$$

Thus, we assume $e_{1} \geq 3$. Suppose at least one $\lambda_{j}$ is even. Now $\sigma_{\lambda}$ has $e_{1}$ fixed points. Hence, $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)=S_{e_{1}} \times\left\langle\sigma_{\lambda_{2}}\right\rangle \times \cdots \times\left\langle\sigma_{\lambda_{r}}\right\rangle$ and $Z_{\lambda}=\mathcal{Z}\left(\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)\right)=\left\langle\sigma_{\lambda_{2}}\right\rangle \times \cdots \times\left\langle\sigma_{\lambda_{r}}\right\rangle$ since $\lambda_{i}$ are distinct. Now let $g \in \mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right) \subset \mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)$. Write $g=\left(g_{1}, g_{2}, \ldots, g_{r}\right)$. If $g \notin Z_{\lambda} \cap A_{n}$ then $g_{1} \neq 1$. But we can find $h_{1} \in S_{e_{1}}$ such that $g_{1} h_{1} \neq h_{1} g_{1}$. Define $h=$ $\left(h_{1}, 1, \ldots, 1\right)$ if $h_{1}$ is even else $h=\left(h_{1}, 1, \ldots, 1, \sigma_{\lambda_{j}}, 1, \ldots, 1\right)$. Clearly $h \in A_{n} \cap \mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)=$ $\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)$ and $g h \neq h g$. This contradicts that $g \in \mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)$, thus $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)=$ $Z_{\lambda} \cap A_{n}$.

Now suppose $e_{1} \geq 4$ and all $\lambda_{i}$ are odd. In this case, $\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)=A_{e_{1}} \times\left\langle\sigma_{\lambda_{2}}\right\rangle \times \cdots \times\left\langle\sigma_{\lambda_{r}}\right\rangle$ since all $\sigma_{\lambda_{i}}$ are even. And $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)=\left\langle\sigma_{\lambda_{2}}\right\rangle \times \cdots \times\left\langle\sigma_{\lambda_{r}}\right\rangle$ which is equal to $Z_{\lambda} \cap A_{n}$.

For the converse, $\lambda=1^{3} \lambda_{2} \cdots \lambda_{r}$ with all $\lambda_{i}$ odd. Then $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)=S_{3} \times\left\langle\sigma_{\lambda_{2}}\right\rangle \times \cdots \times$ $\left\langle\sigma_{\lambda_{r}}\right\rangle$ and $Z_{\lambda}=\mathcal{Z}\left(\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)\right)=\left\langle\sigma_{\lambda_{2}}\right\rangle \times \cdots \times\left\langle\sigma_{\lambda_{r}}\right\rangle \subset A_{n}$. Also $\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)=A_{3} \times\left\langle\sigma_{\lambda_{2}}\right\rangle \times$ $\cdots \times\left\langle\sigma_{\lambda_{r}}\right\rangle=\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)$. Hence $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right) \supsetneq Z_{\lambda} \cap A_{n}$.

This also takes care of the case when all parts are distinct so we may assume $e_{i} \geq 2$. Thus, assume either $i \geq 2$ or $\lambda_{1} \geq 2$. That is we can have at most one fixed point, if at all. If $\sigma_{\lambda}$ has one fixed point, say, $\sigma_{\lambda}(n)=n$ then we may consider $\sigma_{\lambda} \in A_{n-1}$ with no fixed points. Further $\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)=\mathcal{Z}_{A_{n-1}}\left(\sigma_{\lambda}\right)$. Therefore, it is enough to study the partitions which do not have 1 as its part, i.e., we have $\lambda_{1}>1$.

Lemma 6.7. Let $\lambda=\lambda_{1} \cdots \lambda_{i-1} \lambda_{i}^{e_{i}} \lambda_{i+1} \cdots \lambda_{r}$ be a partition with $\lambda_{1}>1$. Further suppose $\lambda$ satisfies one of the followings,
(1) $e_{i} \geq 3$, or,
(2) $\lambda_{i}>2$ and is even.

Then, $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)=Z_{\lambda} \cap A_{n}$.

Proof. We prove (1) first. Let $g=\left(g_{1}, \ldots, g_{r}\right) \in \mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)$ but $g \notin Z_{\lambda}$. All $g_{j}$ are diagonal except $g_{i}$ which is non-diagonal. The element $g_{i} \in \mathcal{Z}_{S_{e_{i} \lambda_{i}}}\left(\sigma_{\lambda_{i}}\right)$ where $\sigma_{\lambda_{i}}=$ $\sigma_{\lambda_{i, 1}} \ldots \sigma_{\lambda_{i, e_{i}}}$. Recall the notation that $\sigma_{\lambda_{i, j}}$ are cycles of length $\lambda_{i}$ as introduced in the beginning of Section 6. Now consider $\tau=\left(I, I I, \ldots, e_{i}\right)$ as in Lemma 6.2, Then $\mathcal{Z}_{\mathcal{Z}_{S_{e_{i} \lambda_{i}}}\left(\sigma_{\lambda_{i}}\right)}(\tau)=\left\langle\sigma_{\lambda_{i}}, \tau\right\rangle$. If $g_{i}=\tau$ then it does not commute with $h_{i}=\sigma_{\lambda_{i, 1}} \sigma_{\lambda_{i, 2}}$ (remember that $e_{i} \geq 3$ ). Since all $\sigma_{\lambda_{i, j}}$ are of same length $\lambda_{i}$ this is an element in $A_{n}$. Thus we get $h=\left(1, \ldots, 1, h_{i}, 1 \ldots, 1\right)$ in $\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)$ which does not commute with $g$, a contradiction.

On other hand if $g_{i} \neq \tau$ then $g_{i}$ does not commute with $\tau$. We observe that $\tau$ is a product of $\lambda_{i}$ many cycles, each of length $e_{i}$. If $e_{i}$ is odd then $\tau$ is an even permutation. Further, if both $e_{i}$ and $\lambda_{i}$ are even then, also, $\tau$ is an even permutation. And in these cases we may take $h_{i}=\tau$ and get a contradiction as above.

Now let us assume that $\lambda_{i}$ is odd and $e_{i}$ is even and thus $\tau$ is an odd permutation. In this case instead of $\tau$ we make use of two elements $\tau_{1}, \tau_{2} \in \mathcal{Z}_{S_{e_{i} \lambda_{i}}}\left(\sigma_{\lambda_{i}}\right)$ as follows. The element $\tau_{1}=\left(I I, I I I, \ldots, e_{i}\right)$ and $\tau_{2}=\left(I, I I, \ldots, e_{i}-1\right)$. Each of the $\tau_{1}$ and $\tau_{2}$ are product of $\lambda_{i}$ many cycles, each of length $e_{i}-1$ and hence even. Now we note that $\mathcal{Z}_{\mathcal{Z}_{S_{e_{i} \lambda_{i}}}\left(\sigma_{\lambda_{i}}\right)}\left(\tau_{1}\right)=\left\langle\sigma_{\lambda_{i, 1}}, \prod_{j=2}^{e_{i}} \sigma_{\lambda_{i, j}}, \tau_{1}\right\rangle$ and $\mathcal{Z}_{\mathcal{Z}_{S_{e_{i} \lambda_{i}}}\left(\sigma_{\lambda_{i}}\right)}\left(\tau_{2}\right)=\left\langle\sigma_{\lambda_{i, e_{i}}}, \prod_{j=1}^{e_{i}-1} \sigma_{\lambda_{i, j}}, \tau_{2}\right\rangle$ (see Lemma (6.2). This gives us that $\mathcal{Z}_{\mathcal{Z}_{S_{e_{i} \lambda_{i}}}\left(\sigma_{\lambda_{i}}\right)}\left(\tau_{1}\right) \cap \mathcal{Z}_{\mathcal{Z}_{S_{e_{i} \lambda_{i}}}\left(\sigma_{\lambda_{i}}\right)}\left(\tau_{2}\right)=\left\langle\sigma_{\lambda_{i}}\right\rangle$. Since $g_{i}$ is non-diagonal it does not commute with either $\tau_{1}$ or $\tau_{2}$ else it would be in the intersection of centralizers which is diagonal. Thus we may take $h_{i}$ to be $\tau_{1}$ or $\tau_{2}$ as required, and get a contradiction.

For the proof of (2), let $g=\left(g_{1}, \ldots, g_{r}\right) \in \mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)$ but $g \notin Z_{\lambda}$. The component $g_{i}$ is non-diagonal element in $\mathcal{Z}_{S_{e_{i} \lambda_{i}}}\left(\sigma_{\lambda_{i}}\right)$. In this case $\sigma_{\lambda_{i}}=\sigma_{\lambda_{i, 1}} \sigma_{\lambda_{i, 2}}$. Take $\tau=(I, I I)$ then $\tau$ is an even permutation as $\lambda_{i}$ is even. If $\tau \neq g_{i}$ take $h_{i}=\tau$ and we are done. Else if $g_{i}=\tau$ then we take $\sigma_{\lambda_{i, 1}}^{2}$. Since $\lambda_{i}>2, \sigma_{\lambda_{i, 1}}^{2} \neq 1$ and it is even permutation. And now taking $h_{i}=\sigma_{\lambda_{i}, 1}^{2}$ would lead to a contradiction.

This leaves us with the following case now. The partition is $\lambda=\lambda_{1} \cdots \lambda_{i-1} \lambda_{i}^{2} \lambda_{i+1} \cdots \lambda_{r}$ with $\lambda_{1}>1$ and either $\lambda_{i}=2$ or $\lambda_{i}$ is odd. And this is where all complication lies.

Lemma 6.8. Let $\lambda$ with $\lambda_{1}>1$ be one of the followings,
(1) $\lambda_{1} \cdots \lambda_{i-1} \lambda_{i}^{2} \lambda_{i+1} \cdots \lambda_{r}$, and suppose, $\lambda_{i}$ is odd and $\lambda_{m}$ even for some $m \neq i$, or,
(2) $2^{2} \lambda_{2} \cdots \lambda_{r}$ with some $\lambda_{m}$ even.

Then, $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)=Z_{\lambda} \cap A_{n}$.
Proof. For the proof of (1), let $g=\left(g_{1}, \ldots, g_{r}\right) \in \mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)$ but $g \notin Z_{\lambda}$. Then $g_{i}$ is non-diagonal. Pick $h_{i} \in \mathcal{Z}_{S_{e_{i} \lambda_{i}}}\left(\sigma_{\lambda_{i}}\right)$ such that $h_{i} g_{i} \neq g_{i} h_{i}$. If $h_{i}$ is even then $h=\left(1, \ldots, h_{i}, \ldots, 1\right)$ would do the job. Else take $h=\left(1, \ldots, h_{i}, 1, \ldots, \sigma_{\lambda_{m}}, 1 \ldots, 1\right)$ which is an even permutation, and does the job.

In the second case, we have $\sigma_{\lambda}=(1,2)(3,4) \sigma_{\lambda_{2}} \cdots \sigma_{\lambda_{r}}$ and $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)=\mathcal{Z}_{S_{4}}((1,2)(3,4)) \times$ $\left\langle\sigma_{\lambda_{2}}\right\rangle \times \cdots \times\left\langle\sigma_{\lambda_{r}}\right\rangle$. Let $g=\left(g_{1}, \ldots, g_{r}\right) \in \mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)$ but $g \notin Z_{\lambda}$. In this case $g_{1}$ has to be non-diagonal. Now we can do the same thing as above to get a contradiction.

At this step we are left with the $\lambda$ of following kinds, and its variant (see the discussion following Lemma (6.6) with exactly one fixed point,
(1) $\lambda_{1} \cdots \lambda_{i-1} \lambda_{i}^{2} \lambda_{i+1} \cdots \lambda_{r}$, where all $\lambda_{j}$ are odd, and,
(2) $2^{2} \lambda_{2} \cdots \lambda_{r}$, where all $\lambda_{j}$ are odd.

Now we are ready to prove the main theorem of this section,
Proof of Theorem 6.4. Lemma 6.5, 6.6, 6.7 and 6.8 prove that if the partition $\lambda$ is not of the type listed in the theorem then $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)=Z_{\lambda} \cap A_{n}$. Thus it remains to prove if $\lambda$ is of one the kinds listed in the theorem then we do not get equality. Which we prove now case-by-case.

In case $\lambda=1^{3} \lambda_{3} \cdots \lambda_{r}$ and $\lambda_{i}$ are odd for all $i$ then the result follows from Lemma 6.6. Now, take $\lambda=2^{2} \lambda_{3} \lambda_{4} \cdots \lambda_{r}$ and $\lambda_{3} \geq 2$ and odd for all $i$. Write $\sigma_{\lambda}=(1,2)(3,4) \sigma_{\lambda_{3}} \ldots \sigma_{\lambda_{r}}$ then $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)=\{1,(1,2),(3,4),(1,2)(3,4),(1,3)(2,4),(1,3,2,4),(1,4,2,3),(1,4)(2,3)\} \times$ $\left\langle\sigma_{\lambda_{3}}\right\rangle \cdots \times\left\langle\sigma_{\lambda_{r}}\right\rangle$. And $\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)=\{1,(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)\} \times\left\langle\sigma_{\lambda_{3}}\right\rangle \cdots \times\left\langle\sigma_{\lambda_{r}}\right\rangle$ which is equal to its own center, being commutative. However the element $(1,3)(2,4) \notin$ $Z_{\lambda}$. Thus we get strict inequality in this case. The argument is similar when $\lambda=$ $1^{1} 2^{2} \lambda_{3} \cdots \lambda_{r}$.

Now suppose $\lambda=\lambda_{3} \cdots \lambda_{i-1} \lambda_{i}^{2} \lambda_{i+1} \cdots \lambda_{r}$ with $\lambda_{3} \geq 3$ and all odd. In this case, $\sigma_{\lambda}=\sigma_{\lambda_{3}} \cdots \sigma_{\lambda_{i}} \cdots \sigma_{\lambda_{r}}$ where $\sigma_{\lambda_{j}}$ is a cycle of length $\lambda_{j}$ for $j \neq i$ and $\sigma_{\lambda_{i}}=\sigma_{\lambda_{i, 1}} \sigma_{\lambda_{i, 2}}$ is a product of two cycles, each of length $\lambda_{i}$. Then $\sigma_{\lambda_{i}, 1}$ and $\sigma_{\lambda_{i}, 2}$ both belong to $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)$ but none of them belong to $Z_{\lambda}$ instead their product belongs. A similar argument works for the case when $\lambda=1^{1} \lambda_{3} \cdots \lambda_{i-1} \lambda_{i}^{2} \lambda_{i+1} \cdots \lambda_{r}$.
7. $z$-Classes in $A_{n}$ - When the conjugacy class does not split

Our strategy for the proof is similar to that of $S_{n}$ case. That is, we look at the action of $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)$ on $\{1,2, \ldots, n\}$ and decide when it determines the partition. This works in almost all cases. We continue to use notation from previous sections.

Proposition 7.1. The action of $Z_{\lambda} \cap A_{n}$ on the set $\{1,2, \ldots, n\}$ determines the partition $\lambda$ uniquely except when $\lambda_{1}^{e_{1}}=1^{2}$.

Proof. We know that the action of $Z_{\lambda}$ on the set $\{1,2, \ldots, n\}$ determines the partition uniquely except when $\lambda_{1}^{e_{1}}=1^{2}$ (see Lemma 3.2). We need to prove that if two points in $\{1,2, \ldots, n\}$ are related under the action of $Z_{\lambda}$ then they are so under the action of $Z_{\lambda} \cap A_{n}$.

Since $\sigma_{\lambda}=\sigma_{\lambda_{1}} \cdots \sigma_{\lambda_{r}}$, we reorder $\sigma_{\lambda_{k}}$ 's, if required, so that $\sigma_{\lambda_{k}}$ for $1 \leq k \leq l$ are even permutations and $\sigma_{\lambda_{k}}$ for $l<k \leq r$ are odd permutations. Since $\sigma_{\lambda}$ is an even
permutation, the number of odd permutations $r-l$ is even (including 0 ). If $r=l$ then $Z_{\lambda}=Z_{\lambda} \cap A_{n}$ and we are done. Else suppose $i \neq j$ are related under $Z_{\lambda}$. That is, there exists $t$ such that $\sigma_{\lambda_{t}}^{m}(i)=j$ for some power $m$. If $\sigma_{\lambda_{t}}^{m}$ is even, we are done. So we may assume $\sigma_{\lambda_{t}}^{m}$ is odd. But since the number of odd permutations is assumed to be even we have another odd permutation $\sigma_{\lambda_{s}}$ disjoint from this one. Thus, $\sigma_{\lambda_{s}} \sigma_{\lambda_{t}}^{m}$ will do the job.

We record the following example of the exception case. Take $\lambda=1^{2} 4^{1} \vdash 6$ then $\mathcal{Z}_{S_{6}}(3,4,5,6)=\langle(1,2)\rangle \times\langle(3,4,5,6)\rangle=Z_{1^{2} 4^{1}}$. And $Z_{1^{2} 4^{1}} \cap A_{5}=\langle(1,2)(3,4,5,6)\rangle$ which would determine the partition $2^{1} 4^{1}$. Now let us look at the case when $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right) \neq$ $Z_{\lambda} \cap A_{n}$. In this case we have the following,

Proposition 7.2. If $\lambda$ is one of the following with $\sigma_{\lambda}$ in $A_{n}$,
(1) $1^{2} \lambda_{2}^{e_{2}} \cdots \lambda_{r}^{e_{r}}$, where $\lambda_{2} \geq 2$, or,
(2) $1^{1} \nu, \nu$ where $\nu=\lambda_{2} \cdots \lambda_{i-1} \lambda_{i}^{2} \lambda_{i+1} \cdots \lambda_{r}$, where $\lambda_{j} \geq 3$ and odd for all $j$,
then, the action of $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)$ on the set $\{1,2, \ldots, n\}$ determines the partition $\lambda$ uniquely.
Proof. The first case appears in $S_{n}$, where $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)$ determines all $\lambda_{i}>2$ except for the first orbit which is $\{1,2\}$. Thus there are two possibilities either $1^{2}$ or $2^{1}$. Since $\sigma_{\lambda}=\sigma_{\lambda_{2}} \cdots \sigma_{\lambda_{r}} \in A_{n}$ we note that the partition $2^{1} \lambda_{2}^{e_{2}} \cdots \lambda_{r}^{e_{r}}$ is not even because this would correspond to the element $(1,2) \sigma_{\lambda}=(1,2) \sigma_{\lambda_{2}} \cdots \sigma_{\lambda_{r}}$ which is odd. Thus this leaves a unique choice for $\lambda$ where the first part must be $1^{2}$.

For the part (2), from the proof of Theorem 6.4, we see that

$$
\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)=\left\langle\sigma_{\lambda_{1}}, \ldots, \sigma_{\lambda_{i-1}}, \sigma_{\lambda_{i, 1}}, \sigma_{\lambda_{i, 2}}, \sigma_{\lambda_{i+1}}, \ldots, \sigma_{\lambda_{r}}\right\rangle .
$$

Clearly this determines the partition $\lambda$ uniquely.
Now, we prove the main proposition as follows.
Proposition 7.3. Let $n \geq 4$. Let $\nu$ be a restricted partition of $n-3$, with distinct and odd parts, in which 1 (and 2) does not appear as its part. Let $\lambda=1^{3} \nu$ and $\mu=3^{1} \nu$ be partitions of $n$ obtained by extending $\nu$. Then $\lambda$ and $\mu$ belong to the same $z$-class in $A_{n}$. Conversely, if $\lambda$ corresponds to a non-split class in $A_{n}$ then it can be $z$-equivalent to at most one more class (possibly split), provided $\lambda$ is of the form $1^{3} \nu$.

Proof. When the partition $\lambda=1^{3} \nu$ then $\mathcal{Z}_{S_{n}}\left(\sigma_{\lambda}\right)=S_{3} \times \mathcal{Z}_{S_{n-3}}\left(\sigma_{\nu}\right)$ and its center is $Z_{\lambda}=\{1\} \times Z_{\nu}$. However $\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)=A_{3} \times \mathcal{Z}_{A_{n-3}}\left(\sigma_{\nu}\right)$ is Abelian and its action would give the partition $3^{1} \nu$. In this case, if we take partition $\lambda^{\prime}=3^{1} \nu$ then $\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda^{\prime}}\right)=$ $\langle(1,2,3)\rangle \times \mathcal{Z}_{A_{n-3}}\left(\sigma_{\nu}\right)$ and $\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda^{\prime}}\right)=\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)$ (in case $\lambda^{\prime}$ corresponds to a split class they are $z$-conjugate thus we may choose this representative). And thus $\sigma_{\lambda}$ and $\sigma_{\lambda^{\prime}}$ would be $z$-conjugate.

For the converse, if $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)=Z_{\lambda} \cap A_{n}$, then from Proposition 7.1, the action of $\mathcal{Z}\left(\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)\right)$ determines the partition $\lambda$ of $n$ uniquely, and we are done. Otherwise, we use Proposition 7.2 which implies that $\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)$ determines the partition $\lambda$ uniquely except in two cases. One of the cases is $1^{3} \nu$ where the centralizer is conjugate to that of $3^{1} \nu$ as required in the proposition. Thus we need to rule out the possibility when $\lambda=2^{2} \nu$ and $1^{1} 2^{2} \nu$ where $\nu=\lambda_{3} \lambda_{4} \cdots \lambda_{r}, \lambda_{i} \geq 3$ and are odd for all $i$.

Let us deal with the case when $\lambda=2^{2} \nu$, the other case is similar. The element $\sigma_{\lambda}=(1,2)(3,4) \sigma_{\lambda_{2}} \cdots \sigma_{\lambda_{r}}$ and

$$
\mathcal{Z}_{A_{n}}\left(\sigma_{\lambda}\right)=\langle(1,2)(3,4),(1,3)(2,4)\rangle \times\left\langle\sigma_{\lambda_{2}}\right\rangle \times \cdots \times\left\langle\sigma_{\lambda_{r}}\right\rangle
$$

which has size $4 . \lambda_{2} \cdots . \lambda_{r}$. Since this is Abelian its center is itself which determines the partition $\lambda$ except for the first orbit which is $\{1,2,3,4\}$. Considering that $\lambda$ is even, we have the possibilities of the first part being $1^{4}, 1^{1} 3^{1}, 2^{2}$. We claim that if $\lambda=1^{4} \nu, 1^{1} 3^{1} \nu$ or $2^{2} \nu$ the size of centralizers is different and hence they can not be $z$-equivalent. We note that, $\mathcal{Z}_{A_{n}}\left(\sigma_{1^{4} \nu}\right)=\left\langle A_{4}\right\rangle \times\left\langle\sigma_{\lambda_{2}}\right\rangle \times \cdots \times\left\langle\sigma_{\lambda_{r}}\right\rangle$ which has size $12 . \lambda_{2} \cdots . \lambda_{r}$. And if $\lambda_{2}>3, \mathcal{Z}_{A_{n}}\left(\sigma_{1^{1} 3^{1} \nu}\right)=\langle(1,2,3)\rangle \times\left\langle\sigma_{\lambda_{2}}\right\rangle \times \cdots \times\left\langle\sigma_{\lambda_{r}}\right\rangle$ of size $3 . \lambda_{2} \cdots . \lambda_{r}$ and if $\lambda_{2}=3$, $\mathcal{Z}_{A_{n}}\left(\sigma_{1^{1} 3^{2} \lambda_{3} \cdots \lambda_{r}}\right)=\langle(2,3,4),(5,6,7)\rangle \times\left\langle\sigma_{\lambda_{3}}\right\rangle \times \cdots \times\left\langle\sigma_{\lambda_{r}}\right\rangle$ of size $3^{2} . \lambda_{3} \cdots . \lambda_{r}$.
7.1. Proof of Theorem 1.3. Let $C$ be a conjugacy class of $S_{n}$ corresponding to the partition $\lambda_{1} \lambda_{2} \cdots \lambda_{r}$ of $n$ with all $\lambda_{i}$ distinct and odd. Then the conjugacy class $C$ splits in two conjugacy classes, say, $C_{1}$ and $C_{2}$ in $A_{n}$. From Proposition 5.3 if each $\lambda_{i}$ is a perfect square for all $1 \leq i \leq r$, then both the conjugacy classes $C_{1}$ and $C_{2}$ are distinct $z$-classes in $A_{n}$. Else $C_{1} \cup C_{2}$ form a single $z$-class in $A_{n}$.

Now, when $C$ does not split, it follows from Proposition 7.3, that except the partition $1^{3} \nu$ where $\nu$ is a partition of $n-3$, with all parts odd and distinct without 1 as its part, all conjugacy classes remain distinct $z$-classes. And in the case when $\lambda=1^{3} \nu$ its $z$-class can coincide with that of $3^{1} \nu$.

## 8. Rational-valued Characters of $A_{n}$

We begin with recalling characters of the alternating group from Pr . First we note that, the number of partitions of $n$ with distinct and odd parts is equal to the number of self-conjugate partitions of $n$ (see Lemma 4.6.16 in $[\mathrm{Pr}$ ). In fact, these are in oneone correspondence via folding. This corresponds to the split conjugacy classes. The complex irreducible characters of $A_{n}$ are given as follows (see Theorem 4.6.7 and 5.12.5 in Pr ). For every partition $\mu$ of $n$ which is not self-conjugate (this corresponds to non-split conjugacy classes), the irreducible character $\chi_{\mu}$ of $S_{n}$ restricts to an irreducible character of $A_{n}$. Since all characters of $S_{n}$ are integer-valued, these characters of $A_{n}$ are rational-valued too. Now, for all partitions $\mu$ of $n$ which are self-conjugate (these correspond to split conjugacy classes), there exists a pair of irreducible characters $\chi_{\mu}^{+}$
and $\chi_{\mu}^{-}$. The character values are given by the following formula. When $g \in A_{n}$ of cycle type $\lambda$ with all parts distinct and odd, say, $\lambda=\left(2 m_{1}+1, \cdots, 2 m_{l}+1\right)$, and the folding corresponding to $\lambda$ is the partition $\mu$ then

$$
\chi_{\mu}^{ \pm}\left(g_{\lambda}^{+}\right)=\frac{1}{2}\left(e_{\lambda} \pm \sqrt{e_{\lambda}\left|Z_{\lambda}\right|}\right)
$$

and $\chi_{\mu}^{ \pm}\left(g_{\lambda}^{-}\right)=\chi_{\mu}^{\mp}\left(g_{\lambda}^{+}\right)$. Here $g_{\lambda}^{+}$and $g_{\lambda}^{-}$denote the two split conjugacy classes in $A_{n}$ and $e_{\lambda}=(-1)^{\sum_{i=1}^{l} m_{i}}$. Else $\chi_{\mu}^{+}(g)=\chi_{\mu}^{-}(g)=\frac{\chi_{\mu}(g)}{2}$. Clearly the characters $\chi_{\mu}^{\mp}$ are rational valued if and only if $e_{\lambda}=1$ and $\left|Z_{\lambda}\right|$ is a perfect square.

Lemma 8.1. For $\lambda=\left(2 m_{1}+1, \cdots, 2 m_{l}+1\right)$, if $\left|Z_{\lambda}\right|$ is a square then $e_{\lambda}=1$.
Proof. In this case, $\left|Z_{\lambda}\right|=\prod_{i=1}^{l}\left(2 m_{i}+1\right)=(2 a+1)^{2}$ for some $a$. Then $\sum_{i=1}^{l} m_{i}$ must be even.

Proof of Theorem 1.7. From the discussion above, all characters of $A_{n}$ corresponding to non-split conjugacy classes are rational-valued. And, both characters corresponding to split conjugacy classes are simultaneously rational-valued if and only if the partition $\lambda$ has all its parts distinct and odd and the product of parts is a perfect square. Clearly this is same as the criteria determining conjugacy classes which are rational.

## 9. Some GAP calculations

In this work, we have come across two functions on natural numbers. The first one is $\epsilon$ defined as

$$
\epsilon(n)=\mid\left\{n=m_{1}^{2}+m_{2}^{2}+\cdots+m_{r}^{2} \mid 1 \leq m_{1}<m_{2}<\ldots<m_{r} \leq n, m_{i} \text { odd } \forall i\right\} \mid
$$

and its generating function is $\prod_{i=0}^{\infty}\left(1+x^{(2 i+1)^{2}}\right)$. And another one is $\delta$ defined as,

$$
\delta(n)=\mid\left\{n=n_{1}+\cdots+n_{r} \mid 1 \leq n_{1}<\ldots<n_{r} \leq n, n_{i} \text { odd } \forall i, \prod_{i=1}^{r} n_{i} \in \mathbb{N}^{2}\right\} \mid .
$$

Writing a natural number as a sum of squares is well studied problem in number theory. However, we could not find references to these functions. Clearly $\epsilon(n) \leq \delta(n)$. The inequality could be strict, for example, $n=78=3+75$ where $3.75=15^{2}$ but none of the components are square. This happens infinitely often. For example, let $p_{1}$ and $p_{2}$ be odd and distinct primes. Consider, $n=p_{1}+p_{2}+p_{1} p_{2}$ and the partition of $n$ given by $p_{1}^{1} p_{2}^{1}\left(p_{1} p_{2}\right)^{1}$. Then $\epsilon(n)<\delta(n)$. We may also consider, for example, $m=p_{1}+p_{1} p_{2}^{2}$, i.e., we have the partition of $m$ given as $p_{1}^{1}\left(p_{1} p_{2}^{2}\right)^{1}$. Then $\epsilon(m)<\delta(m)$. We make a table for the values of $\epsilon$ and $\delta$ for small values of $n$ and also note down the partitions giving rise to the function $\delta$. Some values of $\delta(n)$ are also given in Br .

| $n$ | $\epsilon(n)$ | $\delta(n)$ | partitions | $n$ | $\epsilon(n)$ | $\delta(n)$ | partitions |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 9 | 1 | 1 | $9^{1}$ | 34 | 1 | 1 | $9^{1} 25^{1}$ |
| 10 | 1 | 1 | $1^{1} 9^{1}$ | 35 | 1 | 1 | $1^{1} 9^{1} 25^{1}$ |
| 23 | 0 | 1 | $3^{1} 5^{1} 15^{1}$ | 39 | 0 | 1 | $3^{1} 9^{1} 27^{1}$ |
| 24 | 0 | 1 | $1^{1} 3^{1} 5^{1} 15^{1}$ | 40 | 0 | 2 | $1^{1} 3^{1} 9^{1} 27^{1}, 3^{1} 7^{1} 9^{1} 21^{1}$ |
| 25 | 1 | 1 | $25^{1}$ | 41 | 0 | 1 | $1^{1} 3^{1} 7^{1} 9^{1} 21^{1}$ |
| 26 | 1 | 1 | $1^{1} 25^{1}$ | 47 | 0 | 3 | $3^{1} 11^{1} 33^{1}, 5^{1} 7^{1} 35^{1}, 5^{1} 15^{1} 27^{1}$ |
| 30 | 0 | 1 | $3^{1} 27^{1}$ | 48 | 0 | 5 | $1^{1} 3^{1} 11^{1} 33^{1}, 1^{1} 5^{1} 7^{1} 35^{1}$, |
| $1^{1} 5^{1} 15^{1} 27^{1}, 5^{1} 7^{1} 15^{1} 21^{1}$, |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  | $3^{1} 5^{1} 15^{1} 25^{1}$ |
| 31 | 0 | 2 | $1^{1} 3^{1} 27^{1}, 3^{1} 7^{1} 21^{1}$ | 49 | 1 | 3 | $49^{1}, 1^{1} 3^{1} 5^{1} 15^{1} 25^{1}$, |
| 32 | 0 | 2 | $1^{1} 3^{1} 7^{1} 21^{1}, 3^{1} 5^{1} 9^{1} 15^{1}$ | 50 | 1 | 2 | $1^{1} 5^{1} 7^{1} 15^{1} 21^{1}$ |
| 33 | 0 | 1 | $1^{1} 3^{1} 5^{1} 9^{1} 15^{1}$ | 51 | 0 | 1 | $1^{1} 49^{1}, 5^{1} 45^{1}$ |

Next, we used GAP [GAP] to compute $z$-classes, rational conjugacy classes etc. Here we have some examples for $A_{n}$ which verifies our theorem.

| $n$ | number of <br> conj. classes | number of <br> $z$-classes | partitions |
| :--- | :---: | :---: | :---: |
| 20 | 324 | 315 | $\left\{1^{3} 3^{1} 5^{1} 9^{1}, 3^{2} 5^{1} 9^{1}\right\},\left\{1^{1} 3^{1} 7^{1} 9^{1}, 1^{1} 3^{1} 7^{1} 9^{1}\right\}$ |
|  |  |  | $\left\{1^{1} 3^{1} 5^{1} 11^{1}, 1^{1} 3^{1} 5^{1} 11^{1}\right\},\left\{9^{1} 11^{1}, 9^{1} 11^{1}\right\},\left\{1^{1} 19^{1}, 1^{1} 19^{1}\right\}$ |
| $\{27$ | 1526 | 1506 | $\left\{7^{1} 13^{1}, 7^{1} 13^{1}\right\},\left\{5^{1} 15^{1}, 5^{1} 15^{1}\right\},\left\{1^{3} 17^{1}, 3^{1} 17^{1}, 7^{1} 9^{1}, 3^{2} 5^{1} 7^{1} 9^{1}\right\},\left\{1^{1} 3^{1} 5^{1} 7^{1} 11^{1}, 1^{1} 3^{1} 5^{1} 7^{1} 11^{1}\right\}$ |
|  |  |  | $\left\{7^{1} 9^{1} 11^{1}, 7^{1} 9^{1} 11^{1}\right\},\left\{5^{1} 9^{1} 13^{1}, 5^{1} 9^{1} 13^{1}\right\}$ |
|  |  |  | $\left\{1^{3} 11^{1} 13^{1}, 3^{1} 11^{1} 13^{1}, 3^{1} 11^{1} 13^{1}\right\},\left\{5^{1} 7^{1} 15^{1}, 5^{1} 7^{1} 15^{1}\right\}$ |
|  |  |  | $\left\{1^{3} 9^{1} 15^{1}, 3^{1} 9^{1} 15^{1}, 3^{1} 9^{1} 15^{1}\right\},\left\{1^{1} 11^{1} 15^{1}, 1^{1} 11^{1} 15^{1}\right\}$ |
|  |  |  | $\left\{1^{3} 7^{1} 17^{1}, 3^{1} 7^{1} 17^{1}, 3^{1} 7^{1} 17^{1}\right\},\left\{1^{3} 5^{1} 19^{1}, 3^{1} 5^{1} 19^{1}, 3^{1} 5^{1} 19^{1}\right\}$ |
|  |  |  | $\left\{1^{1} 7^{1} 19^{1}, 1^{1} 7^{1} 19^{1}\right\},\left\{1^{1} 9^{1} 17^{1}, 1^{1} 9^{1} 17^{1}\right\}$ |
|  |  |  | $\left\{1^{3} 3^{1} 21^{1}, 3^{2} 21^{1}\right\},\left\{1^{1} 5^{1} 21^{1}, 1^{1} 5^{1} 21^{1}\right\}$ |
|  |  | $\left\{1^{1} 3^{1} 23^{1}, 1^{1} 3^{1} 23^{1}\right\},\left\{27^{1}, 27^{1}\right\}$ |  |

The last column combines together the partitions which give same $z$-class and the repetition of a partition indicates a split conjugacy class.

## References

[BS] Bhunia, Sushil; Singh, Anupam, "Conjugacy classes of centralizers in unitary groups", 2016,

[Ca1] Carter, R. W., "Centralizers of semisimple elements in finite groups of Lie type", Proc. London Math. Soc. (3) 37 (1978), no. 3, 491-507.
[Ca2] Carter, R. W., "Centralizers of semisimple elements in the finite classical groups", Proc. London Math. Soc. (3) 42 (1981), no. 1, 1-41.
[Pr] Prasad, Amritanshu, "Representation theory, A combinatorial viewpoint", Cambridge Studies in Advanced Mathematics, 147. Cambridge University Press, Delhi, 2015.
[Br] Brison, Owen J., "Alternating groups and a conjecture about rational valued characters", Linear and Multilinear Algebra 24 (1989), no. 3, 199-207.
[Ca] Can, Himmet, "Representations of the generalized symmetric groups", Beitrge Algebra Geom. 37 (1996), no. 2, 289-307.
[FS] Feit, Walter; Seitz, Gary M., "On finite rational groups and related topics", Illinois J. Math. 33 (1989), no. 1, 103-131.
[Go] Gongopadhyay, Krishnendu, "The z-classes of quaternionic hyperbolic isometries", J. Group Theory 16 (2013), no. 6, 941-964.
[GAP] The GAP Group, "GAP - Groups, Algorithms, and Programming, Version 4.8.7", 2017,

[GK] Gongopadhyay, Krishnendu; Kulkarni, Ravi S., "z-classes of isometries of the hyperbolic space", Conform. Geom. Dyn. 13 (2009), 91-109.
[Gr] Green, J. A., "The characters of the finite general linear groups", Trans. Amer. Math. Soc. 80 (1955), 402-447.
[JK] James, Gordon; Kerber, Adalbert, "The representation theory of the symmetric group", With a foreword by P. M. Cohn. With an introduction by Gilbert de B. Robinson. Encyclopedia of Mathematics and its Applications, 16. Addison-Wesley Publishing Co., Reading, Mass., 1981.
[Kl] Kletzing, Dennis, "Structure and representations of Q-groups", Lecture Notes in Mathematics, 1084. Springer-Verlag, Berlin, 1984. vi+290 pp.
[Ku] Kulkarni, Ravi S., "Dynamical types and conjugacy classes of centralizers in groups", J. Ramanujan Math. Soc. 22 (2007), no. 1, 35-56.
[MS] Mishra, Ashish; Srinivasan, Murali K., "The Okounkov-Vershik approach to the representation theory of $G \sim S_{n} "$, J. Algebraic Combin. 44 (2016), no. 3, 519-560.
[NT] Navarro, Gabriel; Tiep, Pham Huu, "Rational irreducible characters and rational conjugacy classes in finite groups", Trans. Amer. Math. Soc. 360 (2008), no. 5, 2443-2465.
[OEIS] The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org.
[Os] Osima, Masaru, "On the representations of the generalized symmetric group", Math. J. Okayama Univ. 4, (1954). 39-56.
[Se] Serre, Jean-Pierre, "Linear representations of finite groups, Translated from the second French edition by Leonard L. Scott", Graduate Texts in Mathematics, Vol. 42. Springer-Verlag, New YorkHeidelberg, 1977.
[Si] Singh, Anupam, "Conjugacy classes of centralizers in $G_{2} "$, J. Ramanujan Math. Soc. 23 (2008), no. 4, 327-336.


[^0]:    E-mail address: sushilbhunia@gmail.com, dilpreetmaths@gmail.com, anupamk18@gmail.com.
    2010 Mathematics Subject Classification. 20B30, 20B35.
    Key words and phrases. Centralizers, alternating group, rational conjugacy classes, rational-valued characters.

    During this work, the first named author was supported by CSIR Ph.D. fellowship, and second named author was supported by NBHM Post-Doctoral fellowship.

