NONCOMMUTATIVE BELL POLYNOMIALS AND THE DUAL IMMACULATE BASIS

JEAN-CHRISTOPHE NOVELLI, JEAN-YVES THIBON AND FRÉDÉRIC TOUMAZET

Abstract. We define a new family of noncommutative Bell polynomials in the algebra of free quasi-symmetric functions and relate it to the dual immaculate basis of quasi-symmetric functions. We obtain noncommutative versions of Grinberg's results [Canad. J. Math. 69 (2017), 21–53], and interpret them in terms of the tridendriform structure of **WQSym**. We then present a variant of Rey's self-dual Hopf algebra of set partitions [FPSAC'07, Tianjin] adapted to our noncommutative Bell polynomials and give a complete description of the Bell equivalence classes as linear extensions of explicit posets.

1. Introduction

Many classical combinatorial numbers or polynomials are the dimensions (or graded dimensions) of certain representations of the symmetric groups, and can therefore be regarded as specializations of the symmetric functions encoding the characters of these representations [7]. Classical examples include the Euler numbers [10], or the eulerian polynomials [11]. In both cases, the generating series of the relevant symmetric functions can be obtained as the homomorphic images of multiplicity free series living in some (in general noncommutative) combinatorial Hopf algebra. For example, the generating series $\tan x + \sec x$ of the Euler numbers is the image of the formal sum of all alternating permutations in the Hopf algebra of free quasi-symmetric functions **FQSym** by its canonical character, and its lift by Foulkes to symmetric functions is just the commutative image of the same series [12, 19].

Other examples of this situation include the derangement numbers [15], the numbers of parking functions [26], the Abel polynomials [26] or Arnold's snakes [19].

These combinatorial numbers can also be dimensions of representations of the 0-Hecke algebra, which cannot be lifted to the generic Hecke algebra or to the symmetric group. This is the case, for example, for linear extensions of a poset [8], which can be directly interpreted as free quasi-symmetric functions.

The aim of this paper is to apply this philosophy to the Bell polynomials. Their relation with symmetric functions is well-known, and easily extended to noncommutative symmetric functions. At this level, it is already possible to define a non-trivial q-analogue, which points toward the most promising direction for the next step. There is a natural choice of a representation of set partitions by permutations in **FQSym**

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which is compatible with this q-analogue. The Bell polynomials are then lifted to polynomials in noncommuting variables Y_k, with coefficients in **FQSym**. One can then consider the quasi-symmetric functions C_I which are the commutative images of the coefficients of the monomials Y^I. It turns out that they coincide with the dual immaculate basis of [2], up to mirror image of compositions.

The formal sum of these "free Bell polynomials" satisfies a simple functional equation in terms of the dendriform structure of **FQSym**. This allows us to obtain expressions of the dual immaculate basis similar to (but different from) those of Grinberg [13]. Actually, Grinberg works directly at the level of quasi-symmetric functions, and his formula comes in fact from the tridendriform structure of **WQSym**.

Finally, we discuss briefly the connection with the self-dual Hopf algebra of set partitions introduced by M. Rey in [31].

2. Bell polynomials and noncommutative symmetric functions

2.1. Bell polynomials and symmetric functions. The classical Bell (exponential) polynomials are defined by

$$B_0 = 1,\ B_{n+1}(y_1, \dots, y_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} y_{k+1} \tag{1}$$

or equivalently by the exponential generating series

$$\mathcal{B}(t) := \sum_{n\ge 0} B_n \frac{t^n}{n!} = \exp\left(\sum_{k\ge 1} y_k \frac{t^k}{k!}\right). \tag{2}$$

This is reminescent of the relation between power-sums and complete symmetric functions. The power sums are

$$p_n = \sum_{i\ge 1} x_i^n \tag{3}$$

and the complete symmetric function h_n is the sum of all monomials of degree n, with generating series

$$H(t) = \sum_{n\ge 0} h_n t^n = \prod_{i\ge 1} \frac{1}{1-tx_i} = \exp\left(\sum_{k\ge 1} p_k \frac{t^k}{k}\right) \tag{4}$$

so that if one sets $y_n = (n-1)!p_n$, then $B_n = n!h_n$.

2.2. Noncommutative Bell polynomials. This is easily extended to noncommutative symmetric functions. Recall [12] that the algebra **Sym** of noncommutative symmetric functions is freely generated by noncommuting indeterminates S_n playing the role of the h_n. The noncommutative analogues of the p_n are not canonically determined, and one possibility is to define noncommutative power-sums of the first kind Ψ_n by orienting the Newton recursion as

$$(n+1)S_{n+1} = \sum_{k=0}^{n} S_{n-k}\Psi_{k+1}. \tag{5}$$

These power-sums correspond to the Dynkin elements in the descent algebra [12].

Let $(Y_n)_{n\geq 1}$ be a sequence of noncommuting indeterminates. For a composition $I=(i_1,\dots,i_r)$, let $Y^I = Y_{i_1}\cdots Y_{i_r}$. The recurrence (1) can be similarly oriented as

$$B'_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B'_{n-k} Y_{k+1} \text{ or } B''_{n+1} = \sum_{k=0}^{n} \binom{n}{k} Y_{k+1} B''_{n-k}. \tag{6}$$

Then, $B'_n = n!S_n$ for $Y_n = (n-1)!\Psi_n$ and $B''_n = n!S_n$ for $Y_n = (n-1)!\bar{\Psi}_n$ (where the $\bar{\Psi}_n$ are defined by the opposite Newton recursion

$$(n+1)S_{n+1} = \sum_{k=0}^{n} \bar{\Psi}_{k+1} S_{n-k}, \tag{7}$$

and correspond to the right-sided Dynkin elements). Such noncommutative Bell polynomials have been discussed in [33] (see also [28, 9]).

It will be sufficient to describe one version. Let us choose the second one. For now on, we set $B = B''$.

The first values are

$$\begin{aligned}
B_1 &= Y^1\\
B_2 &= Y^2 + Y^{11}\\
B_3 &= Y^3 + Y^{12} + 2Y^{21} + Y^{111}\\
B_4 &= Y^{1111} + Y^{112} + 2Y^{121} + Y^{13} + 3Y^{22} + 3Y^{211} + 3Y^{31} + Y^4\\
B_5 &= Y^5 + Y^{14} + 4Y^{23} + 6Y^{32} + 4Y^{41} + Y^{113} + 3Y^{122} + 3Y^{131}\\
&\quad + 4Y^{212} + 8Y^{221} + 6Y^{311} + Y^{1112} + 2Y^{1121} + 3Y^{1211} + 4Y^{2111} + Y^{11111}
\end{aligned}$$

The coefficients have a simple combinatorial interpretation. For a set partition π of $[n]$, let $\pi^\sharp$ be the set composition obtained by ordering the blocks of π w.r.t. their minima, and let $K(\pi)$ be the composition recording the sizes of the blocks of $\pi^\sharp$. For example, if $\pi = \{\{3,4,7\},\{2,8\},\{1\},\{5,6\}\}$, then $\pi^\sharp = (\{1\},\{2,8\},\{3,4,7\},\{5,6\})$, and $K(\pi) = (1,2,3,2)$. Then, the coefficient of Y^I in B_n is the number of set partitions π of $[n]$ such that $K(\pi) = I$.

For example, the coefficient of Y^{23} counts the partitions

$$12|345,\ 13|245,\ 14|235,\ 15|234. \tag{8}$$

2.3. **q-analogues.** The Dynkin power-sums Ψ_n are limiting cases of the transformed complete functions (see [20] for an explanation of the notation)

$$\Psi_n = \lim_{q\to 1} \frac{S_n((1-q)A)}{1-q} \tag{9}$$

and similarly

$$\bar{\Psi}_n = \lim_{q\to 1} \frac{\Lambda_n((1-q)A)}{1-q} \tag{10}$$

One can then define q-analogues of B'_n and B''_n by

$$B'_n(q) = (q)_n S_n\left(\frac{A}{1-q}\right) \text{ for } Y_n = (q)_{n-1}S_n(A) \tag{11}$$

and

$$B''_n(q) = (q)_n \Lambda_n\left(\frac{A}{1-q}\right) \text{ for } Y_n = (q)_{n-1}\Lambda_n(A) \tag{12}$$

This amounts to the recursion

$$B'_n(q) = \sum_{k=0}^{n-1} q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} B'_k(q) Y_{n-k} \tag{13}$$

(see [20, Eq. (76)]), and to a similar one for $B''_n(q)$.

The first values are

$$\begin{aligned}
B'_1 &= Y^1 \\
B'_2 &= Y_2 + qY^{11} \\
B'_3 &= Y^3 + q^2Y^{21} + \left(q^2+q\right)Y^{12} + q^3Y^{111} \\
B'_4 &= Y^4 + q^3Y^{31} + \left(q^4+q^3+q^2\right)Y^{22} + q^5Y^{211} + \left(q^3+q^2+q\right)Y^{13} \\
&\quad + \left(q^5+q^4\right)Y^{121} + \left(q^5+q^4+q^3\right)Y^{112} + q^6Y^{1111}
\end{aligned}$$

and

$$\begin{aligned}
B''_1 &= Y^1 \\
B''_2 &= Y^2 + qY^{11} \\
B''_3 &= Y^3 + \left(q^2+q\right)Y^{21} + q^2Y^{12} + q^3Y^{111} \\
B''_4 &= Y^4 + \left(q^3+q^2+q\right)Y^{31} + \left(q^4+q^3+q^2\right)Y^{22} + \left(q^5+q^4+q^3\right)Y^{211} + q^3Y^{13} \\
&\quad + \left(q^5+q^4\right)Y^{121} + q^5Y^{112} + q^6Y^{1111}
\end{aligned}$$

Setting $Y_i = 1$ in B'_n or in B''_n, one obtains the triangle A188919 of [34]:

$$\begin{aligned}
B_1(q) &= 1 \\
B_2(q) &= 1+q \\
B_3(q) &= 1+q+2\,q^2+q^3 \\
B_4(q) &= 1+q+2\,q^2+4\,q^3+3\,q^4+3\,q^5+q^6 \\
B_5(q) &= 1+q+2\,q^2+4\,q^3+7\,q^4+8\,q^5+9\,q^6+9\,q^7+6\,q^8+4\,q^9+q^{10}
\end{aligned}$$

The Online Encyclopedia of Integer Sequences [34] suggests that the coefficient of q^k in $B_n(q)$ should be the number of permutations of length n with k inversions that avoid the dashed pattern 1-32 [1]. The proof of this fact will follow from a refined interpretation for the coefficient of Y^I (Proposition 3.3 below).

Applying [20, Prop. 5.5], we have the following q-analogue of [9, Theorem 2.5]:

Proposition 2.1. *$B'_n(q)$ is given by the quasideterminant*

$$B'_n(q) = |\mathbb{B}'_n|_{1n} \tag{14}$$

where $\mathbb{B}'_n$ is the $n \times n$ matrix whose subdiagonal elements are -1, the lower elements are all zero, and the elements on and above the diagonal are

$$(\mathbb{B}'_n)_{ij} = q^{i-1} \begin{bmatrix} n-(n-j+1) \\ j-i \end{bmatrix} Y_{j-i+1} \ (i \le j). \tag{15}$$

Proof – In the quasideterminant (78) of [20], expressing $S_n(A)$ in terms of $\Theta_n(q) = (1-q)^{-1} S_n((1-q)A)$

$$[n]_q\, S_n(A) = \begin{vmatrix} \Theta_1(q) & \Theta_2(q) & \dots & \Theta_{n-1}(q) & \boxed{\Theta_n(q)} \\ -[1]_q & q\,\Theta_1(q) & \dots & q\,\Theta_{n-2}(q) & q\,\Theta_{n-1}(q) \\ 0 & -[2]_q & \dots & q^2\,\Theta_{n-3}(q) & q^2\,\Theta_{n-2}(q) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -[n-1]_q & q^{n-1}\,\Theta_1(q) \end{vmatrix}. \tag{16}$$

multiply column j by $[j-1]_q!$ and divide row i by $[i-1]_q!$. These operations do not change the value of the quasideterminant, except for the last column, which yields a factor $[n-1]_q!$, so that the l.h.s. of (78) is now $[n]_q! S_n$. Replacing each entry $\Theta_k(q)$ by $\frac{1}{[k-1]_q!} Y_k$ yields the desired expression. ■

For example,

$$B'_4(q) = \begin{vmatrix} Y_1 & Y_2 & Y_3 & \boxed{Y_4} \\ -1 & q\begin{bmatrix} 2 \\ 0 \end{bmatrix}_q Y_1 & q\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q Y_2 & q\begin{bmatrix} 2 \\ 2 \end{bmatrix}_q Y_3 \\ 0 & -1 & q^2\begin{bmatrix} 3 \\ 0 \end{bmatrix}_q Y_1 & q^2\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q Y_2 \\ 0 & 0 & -1 & q^3\begin{bmatrix} 4 \\ 0 \end{bmatrix}_q Y_1 \end{vmatrix}. \tag{17}$$

Note that Equation (77) of [20] allows one to express Y_n in terms of the B_k, as in [9, Section 3.2.6].

It follows from the definition that the coefficient of Y^I in $B'(q)$ is[1]

$$\prod_{k=2}^{\ell(I)} \begin{bmatrix} i_1+\cdots+i_k-1 \\ i_k-1 \end{bmatrix}_q q^{i_1+\cdots+i_{k-1}}. \tag{18}$$

This suggests that the counting can be refined according to the descents of these permutations. To do this, we have to replace the binomial coefficients in the recurrence by formal sums of permutations in an appropriate algebra.

[1] This is a q-analogue of the r.h.s. of (78) in [28].

3. Quasi-symmetric analogues of the coefficients

3.1. **Free quasi-symmetric functions and their dendriform structure.** Recall that for a totally ordered alphabet A, $\mathbf{FQSym}(A)$ is the algebra spanned by the noncommutative polynomials

$$\mathbf{G}_\sigma(A) := \sum_{\substack{w\in A^n \\ \mathrm{Std}(w)=\sigma}} w \tag{19}$$

where σ is a permutation in the symmetric group $\mathfrak{S}_n$ and $\mathrm{Std}(w)$ denotes the standardization of the word w, *i.e.*, the unique permutation having the same inversions as w.

The multiplication rule is

$$\mathbf{G}_\alpha \mathbf{G}_\beta = \sum_{\gamma\in\alpha*\beta} \mathbf{G}_\gamma, \tag{20}$$

where the convolution $\alpha * \beta$ of $\alpha \in \mathfrak{S}_k$ and $\beta \in \mathfrak{S}_l$ is the sum in the group algebra of $\mathfrak{S}_{k+l}$ [22]

$$\alpha * \beta = \sum_{\substack{\gamma=uv \\ \mathrm{Std}(u)=\alpha;\, \mathrm{Std}(v)=\beta}} \gamma\,. \tag{21}$$

Thus, the number of terms in $\mathbf{G}_\alpha\mathbf{G}_\beta$ is $\binom{k+l}{k}$. To reduce this number to $\binom{k+l-1}{k}$, we can split the product according to the dendriform structure of $\mathbf{FQSym}$.

The dendriform structure of $\mathbf{FQSym}$ is inherited from that of the algebra of noncommutative formal power series over A, which is [23, 24]

$$u \prec v = \begin{cases} uv & \text{if } \max(v) < \max(u) \\ 0 & \text{otherwise,} \end{cases} \tag{22}$$

$$u \succ v = \begin{cases} uv & \text{if } \max(v) \geq \max(u) \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

This yields

$$\mathbf{G}_\alpha\mathbf{G}_\beta = \mathbf{G}_\alpha \prec \mathbf{G}_\beta + \mathbf{G}_\alpha \succ \mathbf{G}_\beta\,, \tag{24}$$

where

$$\mathbf{G}_\alpha \prec \mathbf{G}_\beta = \sum_{\substack{\gamma=uv\in\alpha*\beta \\ |u|=|\alpha|;\, \max(v)<\max(u)}} \mathbf{G}_\gamma\,, \tag{25}$$

$$\mathbf{G}_\alpha \succ \mathbf{G}_\beta = \sum_{\substack{\gamma=uv\in\alpha*\beta \\ |u|=|\alpha|;\, \max(v)\geq\max(u)}} \mathbf{G}_\gamma\,. \tag{26}$$

Then $x = \mathbf{G}_1$ generates a free dendriform algebra in $\mathbf{FQSym}$, isomorphic to $\mathbf{PBT}$, the Loday-Ronco algebra of planar binary trees [21, Proposition 3.7]. The notations used here are those of [16], where more details can be found.

In terms of the dual basis $\mathbf{F}_\sigma := \mathbf{G}_{\sigma^{-1}}$, these operations read

$$\mathbf{F}_\alpha \mathbf{F}_\beta = \sum_{\gamma \in \alpha \sqcup\!\sqcup \beta[k]} \mathbf{F}_\gamma, \tag{27}$$

$$\mathbf{F}_\alpha \prec \mathbf{F}_\beta = \sum_{\gamma \in \alpha \prec \beta[k]} \mathbf{F}_\gamma, \tag{28}$$

$$\mathbf{F}_\alpha \succ \mathbf{F}_\beta = \sum_{\gamma \in \alpha \succ \beta[k]} \mathbf{F}_\gamma, \tag{29}$$

where $\alpha \in \mathfrak{S}_k$, $\beta[k]$ denotes β with its entries shifted by k, and $\prec$ and $\succ$ are the half-shuffles

$$ua \prec vb = (ua \sqcup\!\sqcup v)b, \;\; ua \succ vb = (u \sqcup\!\sqcup vb)a, \tag{30}$$

the shuffle product $\sqcup\!\sqcup$ being itself recursively defined by

$$ua \sqcup\!\sqcup vb = (ua \sqcup\!\sqcup v)b + (u \sqcup\!\sqcup vb)a, \tag{31}$$

with the scalar 1 (representing the empty word) as neutral element.

There is a Hopf embedding ι : **Sym** $\to$ **PBT** of noncommutative symmetric functions into **PBT** [12, 8, 16], which is given by

$$\iota(S_n) = (\dots((x \succ x) \succ x)\dots) \succ x \quad (n \text{ times}). \tag{32}$$

3.2. **A dendriform lift of the Bell recurrence.** We shall define the free Bell polynomials as elements of $\mathbb{K}\langle Y_1, Y_2, \dots\rangle \otimes$ **FQSym** by

$$\boldsymbol{B}_0 = 1, \quad \boldsymbol{B}_{n+1} = \sum_{k=0}^{n} Y_{k+1} \mathbf{G}_{1\dots k+1} \prec \boldsymbol{B}_{n-k}, \tag{33}$$

the $\otimes$ sign being omitted for notational convenience, and $\prec$ is actually $\cdot \otimes \prec$. Recall also that $\mathbf{G}_{1\dots k+1} = S_{k+1}$.

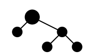
The first $\boldsymbol{B}_n$ are

$$\begin{aligned}
\boldsymbol{B}_0 &= 1\\
\boldsymbol{B}_1 &= Y_1 \mathbf{G}_1\\
\boldsymbol{B}_2 &= Y^{11}\mathbf{G}_{21} + Y_2 \mathbf{G}_{12}\\
\boldsymbol{B}_3 &= Y^{111}\mathbf{G}_{321} + Y^{12}\mathbf{G}_{312} + Y^{21}(\mathbf{G}_{132} + \mathbf{G}_{231}) + Y_3\mathbf{G}_{123}\\
\boldsymbol{B}_4 &= Y^{1111}\mathbf{G}_{4321} + Y^{112}\mathbf{G}_{4312} + Y^{121}(\mathbf{G}_{4132} + \mathbf{G}_{4231}) + Y^{13}\mathbf{G}_{4123}\\
&\quad + Y^{211}(\mathbf{G}_{3421} + \mathbf{G}_{2431} + \mathbf{G}_{1432}) + Y^{22}(\mathbf{G}_{3412} + \mathbf{G}_{2413} + \mathbf{G}_{1423})\\
&\quad + Y^{31}(\mathbf{G}_{1243} + \mathbf{G}_{1342} + \mathbf{G}_{2341}) + Y_4\mathbf{G}_{1234}.
\end{aligned}$$

The coefficient $C_I(A)$ of Y^I in $\boldsymbol{B}_n$ is actually in **PBT**. It is the sum of the $\mathbf{G}_\sigma$ such that the shape of the decreasing tree (*cf.* [21, 16]) of σ is the right-comb tree whose left branches have sizes $i_1, i_2, \dots, i_r$.

Example 3.1. The coefficient of Y^{221} in $\boldsymbol{B}_5$ is

$$\begin{aligned}\mathbf{G}_{12} \prec (\mathbf{G}_{132} + \mathbf{G}_{231}) &= \mathbf{G}_{45132} + \mathbf{G}_{35142} + \mathbf{G}_{25143} + \mathbf{G}_{15243} \\ &+ \mathbf{G}_{45231} + \mathbf{G}_{35241} + \mathbf{G}_{25341} + \mathbf{G}_{15342} \\ &= \mathbf{P}\end{aligned}$$



For a set partition π, let $\pi^\flat$ be the set composition obtained by ordering the blocks w.r.t. their maximal values in decreasing order, and let $C(\pi)$ be the composition recording the lengths of these blocks. Let also $\hat{\pi}^\flat$ be the permutation obtained by reading the blocks in this order.

Example 3.2. For $\pi = \{\{3,4,7\},\{2,8\},\{1\},\{5,6\}\}$, $\pi^\flat = (\{2,8\},\{3,4,7\},\{5,6\},\{1\})$, $\hat{\pi}^\flat = 28347561$, and $C(\pi) = (2,3,2,1)$.

Proposition 3.3. *The coefficient of Y^I in $\boldsymbol{B}_n$ is the sum of all $\mathbf{G}_{\hat{\pi}^\flat}$ where π ranges over set partitions such that $C(\pi) = I$:*

$$\boldsymbol{B}_n = \sum_{|\pi|=n} Y^{C(\pi)} \otimes \mathbf{G}_{\hat{\pi}^\flat}. \tag{34}$$

Proof. This follows immediately from the product rule (25). ■

These permutations are not those avoiding the pattern $1-32$ as above, but those avoiding the pattern $21-3$. As observed in [6, Prop. 1], there is a statistic-preserving bijection between both classes (in this case, the Schützenberger involution, which preserves the inversion number). Since the inverse major index and the inversion number are equidistributed on the set of permutations having a decreasing tree of a given shape [18], the generating polynomial of inv on $1-32$ or of imaj on $21-3$-avoiding permutations with descent composition I is

$$c_I(q) = C_I\left(\frac{1}{1-q}\right) \tag{35}$$

where the alphabet $\frac{1}{1-q}$ is defined as usual, so that the specializations of the fundamental quasi-symmetric functions are

$$F_I\left(\frac{1}{1-q}\right) = \frac{q^{\mathrm{maj}(I)}}{(q)_n}. \tag{36}$$

Now that we understand that $c_I(q)$ is the principal specialization of a quasi-symmetric function, we can replace it by the commutative image $C_I(X)$ of $C_I(A)$.

Example 3.4. Recording the recoil compositions of the permutations occuring in C_{221}, we find

$$C_{221}(X) = F_{212} + F_{131} + F_{1211} + F_{311} + F_{122} + F_{1121} + 2F_{221}. \tag{37}$$

This may be compared with the dual immaculate basis of [2]:

$$\mathfrak{S}^*_{221} = F_{1121} + F_{113} + F_{1211} + 2F_{122} + F_{131} + F_{212} + F_{221}. \tag{38}$$

Theorem 3.5. *Define the bar involution on QSym by $\bar{F}_I = F_{\bar{I}}$, where for $I = (i_1, \dots, i_r)$, $\bar{I} = (i_r, \dots, i_1)$ denotes the mirror composition. Then, the dual immaculate basis is given by*

$$\mathfrak{S}^*_I = \overline{C_I(X)}. \tag{39}$$

Proof – This follows from [2, Prop. 3.37]. Indeed, standard immaculate tableaux encode set partitions, and reading them in the appropriate way yields the desired statistics.

Under the involution $\nu : \ \sigma \mapsto \tilde{\sigma}$ replacing each entry i of $\sigma \in \mathfrak{S}_n$ by $n+1-i$, the permutations occuring in $C_I(A)$ become the right-to-left row-readings of the standard immaculate tableaux of shape I. This induces a bijection ϕ between both sets, and this bijection reverts the descent compositions: $C(\tilde{\sigma}) = \overline{C(\sigma)}$. ∎

Note 3.6. Theorem 3.5 provides an expression of the dual immaculate basis which is very similar to the one obtained by Grinberg [13]. However, our dendriform operations are different. The relations between both constructions will be clarified in the forthcoming section.

Example 3.7. $\nu(45132) = 21534$ is the right-to-left row-reading of the first immaculate tableau below. The recoil compositions of the permutations occuring in Example 3.1 are (in this order)

$$212,\ 221,\ 1211,\ 311,\ 122,\ 1121,\ 131,\ 221,$$

and the involution ν maps them to the right-to-left row-readings of the immaculate tableaux

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 1&5\\\hline 2&4\\\hline 3\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 1&4\\\hline 2&3\\\hline 5\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|}\hline 1&5\\\hline 2&3\\\hline 4\\\cline{1-1}\end{array}$$

whose descent compositions (as defined in [2]) are respectively

$$212,\ 122,\ 1121,\ 113,\ 221,\ 1211,\ 131,\ 122.$$

These are also the recoil compositions of the images of the permutations of Example 3.1 under the Schützenberger involution $\sigma \mapsto \overline{\tilde{\sigma}}$

$$43512, 42513, 32514, 32415, 53412, 52413, 52314, 42315,$$

which are the left-to right and bottom-to-top readings of the tableaux.

Corollary 3.8. *The $C_I(X)$ form a basis of QSym, so that the $\mathbf{P}_T$ indexed by right combs form a section of the projection* $\mathbf{PBT} \to QSym$.

Applying the Schützenberger involution to C_I (that is, $\mathbf{G}_\sigma \mapsto \check{\mathbf{G}}_\sigma = \mathbf{G}_{\omega\sigma\omega}$, where $\omega = n \cdots 21$) we have

$$\check{C}_I(X) = \mathfrak{S}^*_I. \tag{40}$$

Corollary 3.9. *$c_I(q)$ is given by the Björner-Wachs q-hook-length formula.*

See [18] for a short proof of the version used here.

Example 3.10. For $I = 221$, the hook-lengths of the right comb are $5, 3, 1, 1, 1$, and the cardinalities of the right subtrees are $3, 1$. Hence,

$$c_{221}(q) = (q)_5 \frac{q^{3+1}}{(1-q^5)(1-q^3)(1-q)^3} = q^4 + 2\,q^5 + 2\,q^6 + 2\,q^7 + q^8. \tag{41}$$

Corollary 3.11. *$\mathfrak{S}_I$ is the characteristic of an indecomposable 0-Hecke algebra module.*

Proof – The immaculate tableaux are the linear extensions of a poset (here a binary tree), hence form the basis of a 0-Hecke module [8]. ∎

4. The dual immaculate basis

4.1. **Half-shuffles and FQSym.** The shuffle product on $\mathbb{K}\langle A\rangle$ can be recursively defined by (31)

$$ua ⧢ vb = (ua ⧢ v)b + (u ⧢ vb)a \tag{42}$$

or symmetrically by

$$au ⧢ bv = a(u ⧢ bv) + b(au ⧢ v) \tag{43}$$

The half-shuffles are known as chronological products. Both versions can be used to define a dendriform structure on **FQSym**. In this section, we shall use the second version and set

$$au \prec' bv = a(u ⧢ bv), \text{ and } au \succ' bv = b(au ⧢ v). \tag{44}$$

If γ is a linear combination of some permutations ρ, we write for short $\mathbf{F}_\gamma$ for the same linear combination of the $\mathbf{F}_\rho$. On **FQSym**, we set, for $\sigma \in \mathfrak{S}_k$ and $\tau \in \mathfrak{S}_l$

$$\mathbf{F}_\sigma \prec' \mathbf{F}_\tau = \mathbf{F}_{\sigma \prec' \tau[k]}, \text{ and } \mathbf{F}_\sigma \succ' \mathbf{F}_\tau = \mathbf{F}_{\sigma \succ' \tau[k]}. \tag{45}$$

Lemma 4.1. *The following identity holds:*

$$u \prec' v := \sum_{v_1 v_2 = v} (-1)^{|v_1|} (\overline{v_1} u) ⧢ v_2, \tag{46}$$

where $\bar{w}$ denotes the mirror image of a word w.

Proof – By definition

$$au \prec' bv = au ⧢ bv - au \succ' bv, \tag{47}$$

and since

$$au \succ' bv = bau \prec' v, \tag{48}$$

the result follows by induction. ∎

For example,

$$1234 \prec 5671 (234 ⧢ 567) = 1234 ⧢ 567 - 51234 ⧢ 67 + 651234 ⧢ 7 - 7651234. \tag{49}$$

In **FQSym**, this implies

$$\mathbf{F}_\sigma \prec' \mathbf{F}_\tau = \sum_{uv=\tau[k]} (-1)^{|u|} \mathbf{F}_{(\bar{u}\cdot\sigma)\,\sqcup\!\sqcup\, v}. \tag{50}$$

For example,

$$\mathbf{F}_{2143} \prec' \mathbf{F}_{312} = \mathbf{F}_{2143\,\sqcup\!\sqcup\,756} - \mathbf{F}_{72143\,\sqcup\!\sqcup\,56} + \mathbf{F}_{572143\,\sqcup\!\sqcup\,6} - \mathbf{F}_{6572143}. \tag{51}$$

For $w \in A^*$, let $\mathrm{alph}(w) \subseteq A$ be the set of letters occuring in w. The following property appears as Lemma 4.1 in [27]:

Lemma 4.2. *If* $\mathrm{alph}(u) \cap \mathrm{alph}(v) = \emptyset$*, then*

$$\langle u \sqcup\!\sqcup v\rangle = \langle u\rangle\langle v\rangle\,, \tag{52}$$

where the linear map $\langle\,\rangle$ *is defined by* $\langle u\rangle := F_{C(u)}$*. In particular, the descents of the elements of a shuffle on disjoint alphabets depend only on the descents of the initial elements.*

There is a refined statement for the dendriform half-products [27, Theorem 5.2].

Theorem 4.3. *Let* $u = u_1 \cdots u_k$ *and* $v = v_1 \cdots v_\ell$ *of respective lengths* k *and* ℓ*. If* $\mathrm{alph}(u) \cap \mathrm{alph}(v) = \emptyset$*, then*

$$\langle u \prec' v\rangle = \langle \sigma \prec' \tau\rangle \tag{53}$$

where $\sigma = \mathrm{std}(u)$ *and* $\tau = \mathrm{std}(v)[k]$ *if* $u_k < v_\ell$*, and* $\sigma = \mathrm{std}(u)[\ell]$ *and* $\tau = \mathrm{std}(v)$ *if* $u_k > v_\ell$*.*

Applying Theorem 4.3, we can project (46) to $QSym$. Let $\pi :$ **FQSym** $\to QSym$ be the canonical projection. If the descent composition of u is H and that of σ is I, the descent composition of $\bar{u}$ is the conjugate composition $H^\sim$, and if $\min(u) > \max(\sigma)$, the descent composition of $\bar{u}\cdot\sigma$ is $H^\sim \cdot I$. Thus, if $C(\sigma) = I$ and $C(\tau) = J$, we can define

$$F_I \prec' F_J := \pi(\mathbf{F}_\sigma \prec' \mathbf{F}_\tau), \tag{54}$$

and we have therefore

$$F_I \prec' F_J = \sum_{J\in\{HK,\, H\triangleright K\}} (-1)^{|H|} F_{H^\sim\cdot I} F_K. \tag{55}$$

For example,

$$\mathbf{F}_{21} \prec' \mathbf{F}_{132} = \mathbf{F}_{21\,\sqcup\!\sqcup\,354} - \mathbf{F}_{321\,\sqcup\!\sqcup\,54} + \mathbf{F}_{5321\,\sqcup\!\sqcup\,4} - \mathbf{F}_{45321} \tag{56}$$

so that

$$\pi(\mathbf{F}_{21} \prec' \mathbf{F}_{132}) = F_{11}F_{21} - F_{111}F_{11} + F_{1111}F_1 - F_{2111} \tag{57}$$

corresponding to the decompositions $J = \emptyset\cdot 21, 1\triangleright 11, 2\cdot 1$ and $21\cdot\emptyset$. After reduction,

$$\mathbf{F}_{21} \prec' \mathbf{F}_{132} = \mathbf{F}_{21354} + \mathbf{F}_{23154} + \mathbf{F}_{23514} + \mathbf{F}_{23541} \tag{58}$$

$$\text{and } F_{11} \prec' F_{21} = F_{131} + F_{221} + F_{32} + F_{311}. \tag{59}$$

which could have been computed as well from

$$\mathbf{F}_{21} \prec' \mathbf{F}_{231} = \mathbf{F}_{21453} + \mathbf{F}_{25153} + \mathbf{F}_{24513} + \mathbf{F}_{24531}. \tag{60}$$

Since the antipode S of $QSym$ is given by

$$S(F_H) = (-1)^{|H|} F_{H^\sim}, \tag{61}$$

and the coproduct by

$$\Delta(F_J) = \sum_{J \in \{HK, H \triangleright K\}} F_H \otimes F_K \tag{62}$$

we obtain in this way an analogue of [13, Theorem 3.7]:

Theorem 4.4. *Let the $\prec'$ product on $QSym$ be defined by (54). Then, for $f, g \in QSym$,*

$$f \prec' g = \sum_{(g)} \left(S(g_{(1)}) \bullet f \right) g_{(2)} \tag{63}$$

where $F_I \bullet F_J := F_{I \cdot J}$.

For example, applying Lemma 4.2 to (51), we obtain

$$F_{121} \prec' F_{12} = F_{121}F_{12} - F_{1121}F_2 + F_{2121}F_1 - F_{12121}. \tag{64}$$

According to [20, Definition 4.5], for $f \in QSym$,

$$f(X - Y) = \sum_I S(F_I)(Y) \cdot R_I^\perp(f)(X), \tag{65}$$

where the quasi-differential operator $R_I^\perp$ is defined as in [13]. We can therefore rewrite (63) as

$$f \prec' g(X) = [g(X - Y) \bullet_Y f(Y)]_{Y=X}. \tag{66}$$

For example

$$\begin{aligned} F_{12}(X - Y) &= F_{12}((-Y)\hat{+}X) = F_{12}(-Y) + F_{11}(-Y)F_1(X) + F_1(-Y)F_2(X) + F_{12}(X) \\ &= -F_{12}(Y) + F_2(Y)F_1(X) - F_1(Y)F_2(X) + F_{12}(X). \end{aligned} \tag{67}$$

Taking the $\bullet$ product with $F_{121}(Y)$, we obtain

$$F_{12}(X-Y) \bullet F_{121}(Y) = -F_{12121}(Y) + F_{2121}(Y)F_1(X) - F_{1121}(Y)F_2(X) + F_{121}(Y)F_{12}(X), \tag{68}$$

and setting $Y = X$, we recover (64).

Similarly, we have for the right product

$$u \succ' v = \sum_{u_1 u_2 = u} (-1)^{|u_1|} u_2 \,⧢\, \bar{u}_1 v. \tag{69}$$

so that, on $QSym$, if one defines $\blacktriangleright$ by $F_I \blacktriangleright F_J = F_{I \triangleright J}$, with

$$I \triangleright J = (i_1, \ldots, i_r + j_1, j_2, \ldots, j_s), \tag{70}$$

then,

$$f \succ' g = \sum_{(f)} \left(S(f_{(1)}) \blacktriangleright g \right) f_{(2)}. \tag{71}$$

For example,

$$\mathbf{F}_{132} \succ' \mathbf{F}_{21} = \mathbf{F}_{143⧢54-32⧢154+2⧢3154-223154} \tag{72}$$

which projects onto

$$F_{21} \succ' F_{11} = F_{21}F_{11} - F_{11}F_{21} + F_1F_{121} - F_{221} = F_{131} + F_{1121} + F_{122} + F_{1211}. \tag{73}$$

4.2. **Standard dendriform structures.** With the usual definitions

$$ua \prec vb = (u⧢vb)a, \quad ua \succ vb = (ua⧢v)b, \tag{74}$$

the half-shuffle identity becomes

$$u \prec v = \sum_{v_1v_2=v} (-1)^{|v_2|} u\bar{v}_2 ⧢ v_1 \tag{75}$$

which yields on $QSym$

$$F_I \prec F_J = \sum_{J\in\{HK, H\triangleright K\}} (-1)^{|K|} F_{I\triangleright K^\sim} F_H, \tag{76}$$

or equivalently,

$$f \prec g = \sum_{(g)} g_{(1)} \left(f \blacktriangleright S(g_{(2)})\right). \tag{77}$$

For example,

$$\mathbf{F}_{21} \prec \mathbf{F}_{132} = \mathbf{F}_{35421} + \mathbf{F}_{35241} + \mathbf{F}_{32541} + \mathbf{F}_{23541} \tag{78}$$

and

$$F_{11} \prec F_{21} = F_{2111} + F_{221} + F_{1211} + F_{311}. \tag{79}$$

Then,

$$F_I \prec F_J = F_I(Y) \blacktriangleright_Y F_J(X-Y). \tag{80}$$

For example,

$$F_{21}(X-Y) = F_{21}(X) - F_1(Y)F_2(X) + F_2(Y)F_1(X) - F_{21}(Y) \tag{81}$$

and

$$F_{11}(Y) \blacktriangleright_Y F_{21}(X-Y) = F_{11}(Y)F_{21}(X) - F_{12}(Y)F_2(X) + F_{13}(Y)F_1(X) - F_{131}(Y) \tag{82}$$

so that

$$F_{11} \prec F_{21} = F_{11}F_{21} - F_{12}F_2 + F_{13}F_1 - F_{131}. \tag{83}$$

For the right product, we have

$$u \succ v = \sum_{u_1u_2=u} (-1)^{|u_2|} u_1 ⧢ v\bar{u}_2, \tag{84}$$

and on $QSym$,

$$f \succ g = \sum_{(f)} f_{(1)} \left(g \bullet S(f_{(2)})\right). \tag{85}$$

For example,

$$\mathbf{F}_{21} \succ \mathbf{F}_{132} = \mathbf{F}_{21 ⧢ 453 - 2 ⧢ 4531 + 45312} \tag{86}$$

which yields on $QSym$

$$F_{11} \succ F_{21} = F_{11}F_{21} - F_1F_{211} + F_{212} = F_{131} + F_{122} + F_{221} + F_{32} + F_{1121} + F_{212}. \tag{87}$$

4.3. **Grinberg's operations.** In [13], a left product $\prec''$ on $QSym$ is induced from an operation on monomials. This operation is the commutative image of the left tridendriform product on words defined in [25] (with the use of min instead of max, so that we shall consistently denote it by $\prec'$), and amounts to taking the canonical projection $\pi : \mathbf{WQSym} \to QSym$ of the tridendriform product of **WQSym**:

$$M_I \prec'' M_J := \pi(\mathbf{M}_u \prec' \mathbf{M}_v) \tag{88}$$

where u, v are any packed words of evaluation I, J.

For example, to evaluate $F_{11} \prec'' F_{21}$, we compute $\mathbf{M}_{21} \prec' (\mathbf{M}_{132} + \mathbf{M}_{121})$:

$$\mathbf{M}_{21} \prec' \mathbf{M}_{132} = \mathbf{M}_{21243} + \mathbf{M}_{21354} + \mathbf{M}_{31243} + \mathbf{M}_{41243} + \mathbf{M}_{51243} + \mathbf{M}_{41253} + \mathbf{M}_{31254},$$

$$\mathbf{M}_{21} \prec' \mathbf{M}_{121} = \mathbf{M}_{21232} + \mathbf{M}_{31212} + \mathbf{M}_{21343} + \mathbf{M}_{31242} + \mathbf{M}_{41232}$$

so that

$$\begin{aligned} F_{11} \prec'' F_{21} &= 3M_{1211} + 2M_{1121} + M_{1112} + M_{122} + M_{131} + 4M_{1111} \\ &= F_{131} + F_{122} + F_{1121} + F_{1211}, \end{aligned} \tag{89}$$

which is different from all previous examples such as (59).

On **FQSym**, this amounts to the choice

$$\mathbf{F}_\sigma \prec'' \mathbf{F}_\tau := \mathbf{F}_{\sigma[l] \prec' \tau} \tag{90}$$

where $l = |\tau|$, and $au \prec' bv$ is defined as $a(u ⧢ bv)$.

We can also write

$$u \prec' v = \sum_{v_1v_2 = v} (-1)^{|v_1|} \bar{v}_1 u ⧢ v \tag{91}$$

which yields [13, Theorem 3.7]

$$F_I \prec'' F_J = \sum_{J \in \{HK, H \triangleright K\}} (-1)^{|H|} F_{H^\sim \triangleright I} F_K. \tag{92}$$

For example,

$$\mathbf{F}_{21} \prec'' \mathbf{F}_{132} = \mathbf{F}_{54132} + \mathbf{F}_{51432} + \mathbf{F}_{51342} + \mathbf{F}_{51234}, \tag{93}$$

and one can check that

$$54 \prec' 132 = 54 ⧢ 132 - 154 ⧢ 32 + 3154 ⧢ 2 - 23154, \tag{94}$$

so that

$$F_{11} \prec'' F_{21} = F_{11}F_{21} - F_{21}F_{11} + F_{121}F_1 - F_{221}. \tag{95}$$

Alternatively, we can describe $\prec''$ as

$$\mathbf{F}_\sigma \prec'' \mathbf{F}_\tau = \mathbf{F}_\tau \succ' \mathbf{F}_\sigma. \tag{96}$$

On $QSym$, this translates as

$$f \prec'' g = \sum_{(g)} g_{(2)} \left(S(g_{(1)}) \blacktriangleright f\right) \tag{97}$$

which is now [13, Theorem 3.7] in its original form.

Grinberg's expression of the dual immaculate basis can now be restated as

$$\mathfrak{S}^*_I = (\cdots(F_{i_r} \succ' F_{i_{r-1}}) \succ' \cdots) \succ' F_{i_1}. \tag{98}$$

For example, $\mathfrak{S}^*_{221} = (F_1 \succ' F_2) \succ' F_2 = (F_{12} + F_{21}) \succ' F_2$.

It should be noted that on **FQSym**, the operation $\prec''$ is not a left dendriform product in the usual sense, as it it is fact a flipped right product. The $\prec'$ operation on **WQSym** which induces it does not preserve the standard **FQSym** subalgebra of **WQSym**.

5. Hopf algebras of set partitions

We have seen that the Bell polynomial $\boldsymbol{B}_n$ can be identified with the formal sum of permutations avoiding $21-3$ (up to the Y^I which can be reconstructed from the descent sets). This raises the question of the existence of a Hopf subalgebra or quotient of **FQSym** whose bases are naturally labeled by these permutations.

The most obvious Hopf algebra of set partitions is **WSym**, or symmetric functions in noncommuting variables (not to be confused with noncommutative symmetric functions **Sym**). In [17], a quotient of **WSym** isomorphic to **Sym** and a $QSym$ subalgebra of its dual are related to Bell polynomials. In [3], analogues of the Bell polynomials in various other Hopf algebras are considered.

5.1. **The Bell Hopf algebra.** The Hopf algebra **WSym** is cocommutative. Quite often, combinatorial objects also admit a self-dual Hopf algebra structure. Such an algebra has been constructed by M. Rey [31, 32] for set partitions, from the Burstein-Lankham correspondence, derived from the patience sorting algorithm. More precisely, Rey's Bell classes are indexed by permutations avoiding $23-1$, and these are their minimal elements (for the weak order). We can modify the construction so as to have classes whose maximal elements avoid $21-3$ as follows.

Let A be a totally ordered alphabet. The (modified) Bell congruence on A^* is generated by the relations

$$buca \equiv buac \quad \text{if } a<b<c \text{ and all letters of } u \text{ are smaller than } b. \tag{99}$$

This is a refinement of the reverse sylvester congruence, which is defined by the same relations without restriction on u.

As for the sylvester or reverse sylvester congruences we have:

Theorem 5.1. *The Bell equivalence classes of permutations are intervals of the right weak order on the symmetric group. These intervals consist of the linear extensions of posets which will be explicitly described below, and the maximal elements of these intervals are the* $21-3$*-avoiding permutations.*

Identifying set partitions with their representatives as 21−3-avoiding permutations, we set

$$P_\pi = \sum_{\sigma \equiv \pi} \mathbf{F}_\sigma . \tag{100}$$

Theorem 5.2 (M. Rey [31])**.** *The P_π span a Hopf subalgebra of* **FQSym***.*

5.2. **Proofs.** Since the long version of Rey's paper [31] has never been published, we shall provide detailed proofs of his results (in our modified version), to which we add the explicit description of the posets.

Let us first define an insertion algorithm. This is a rewriting of the Patience Sorting Algorithm defined by Burstein and Lankham adapted to our setting, which amounts to applying some trivial involutions on words. Precisely, if we invert the total order of the alphabet, our algorithm becomes Algorithm 3.1 of [32], our blocks being Rey's piles read downwards.

The first object that we create is a set partition, that we shall regard as ordered by decreasing order on the maximal elements of the blocks.

Let $w = w_1 \dots w_n$ be a word without repeated letters over a totally ordered alphabet. Put $S = \emptyset$. Then, read w from left to right and for each letter w_i do

- Step 1: Let s be the block of S whose maximal element is smaller than (or equal to) w_i and is greater than all other maximal elements smaller than w_i.
- Step 2: If s does not exist, define $s = \{w_i\}$. If s exists, insert w_i into it.

The result of this algorithm will be denoted by $PSA(w)$.

We shall display the blocks of S as columns, increasing from top to bottom (so that an element is inserted at the bottom of its column), the columns being ordered from left to right with their maximal elements in decreasing order.

For example, the insertion of $(3, 1, 2, 6, 4, 5, 7)$ has the following steps:

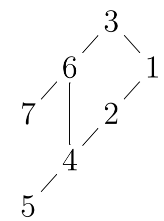
$$S = \emptyset \to S = \begin{array}{|c|}\hline 3\\\hline\end{array} \to S = \begin{array}{|c|c|}\hline 3 & 1\\\hline\end{array} \to S = \begin{array}{cc} 3 & 1\\ & 2\end{array} \to S = \begin{array}{cc} 3 & 1\\ 6 & 2\end{array} \to S = \begin{array}{cc} 3 & 1\\ 6 & 2\\ & 4\end{array} \to S = \begin{array}{cc} 3 & 1\\ 6 & 2\\ & 4\\ & 5\end{array}$$

$$\to S = \begin{array}{cc} 3 & 1\\ 6 & 2\\ 7 & 4\\ & 5\end{array}$$

Now, starting from a set partition S, we shall build a poset P whose relation $>_P$ is generated by the following requirements: order it as above, and, for any element x of a set s of S, write $x >_P x'$ where x' is the greatest element smaller than x in s and write $x >_P x''$ where x'' is the smallest element greater than x in the set just before s in S. This poset will be denoted by $P(S)$ or by $P(w)$ if S is the result of the (modified) patience sorting algorithm applied to w. Note that by removing the edges of $P(S)$ where the element above is greater than the element below, one recovers S itself. Hence it makes sense to refer to the columns of a poset $P(S)$ as the corresponding columns of S. For w in P, we shall write $C(w)$ as its column.

As usual with posets, we shall only represent the covering relations. For example, starting with the set S computed before, we get (orienting the Hasse diagrams upside-down, with minimal elements at the top):

(101)

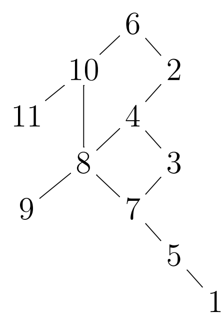


Indeed, the first column of S is $(3, 6, 7)$ and $6 > 3$ and $7 > 6$. Now, on the second column we have $5 > 4 > 2 > 1$ and the relations between these elements and the first column are: $1 > 3$, $2 > 3$, $4 > 6$, and $5 > 6$, whence the Hasse diagram above.

Starting with $S = \{\{6, 10, 11\}, \{2, 4, 8, 9\}, \{3, 7\}, \{5\}, \{1\}\}$, one gets $S = \begin{array}{ccccc} 6 & 2 & 3 & 5 & 1 \\ 10 & 4 & 7 & & \\ 11 & 8 & & & \\ & 9 & & & \end{array}$

and

(102)

$P(S) =$ 

Note that in this representation, the columns of $P(S)$ are the straight lines going from south-west to north-east.

We shall now prove Theorem 5.1: all elements in a given Bell class have the same poset $P(w) = P(PSA(w))$, and the linear extensions (taken from top to bottom in our representation) of such a poset give elements of the same Bell class. Hence being in the same Bell class is equivalent to having the same poset.

5.2.1. *All elements in a given Bell class have the same poset.* First, let us prove that all elements in a given Bell class have the same poset, or, equivalently, the same result by the PSA algorithm. We only need to prove this for two elements w and w' obtained from one another by a single rewriting rule. So, let us consider $w = vbuac$ and $w' = vbuca$ where all letters of u are smaller than b. When b is inserted, it is the last element of its column, and it remains so during the insertion of all u, since all its letters are smaller than b. Now, a letter $c > b$ is necessarily inserted into a column to the left of $C(b)$ or into it, independently of the insertion of a. Meanwhile, a letter a smaller than b is necessarily inserted into a column strictly to the right of $C(b)$ (whether or not c has already been inserted). Thus, the insertion of c and a do not interfere with each other. So w and w' satisfy $PSA(w) = PSA(w')$.

Note that the condition that all letters of u are smaller than b is necessary to ensure that c and a cannot both be inserted in the same column, which would prevent w and w' from yielding the same result by the modified patience sorting algorithm.

5.2.2. *All linear extensions of a poset are Bell congruent.* Let us now prove that all linear extensions of a poset $P(\sigma)$ are Bell congruent to σ.[2] First note that the poset obtained from a permutation σ has a unique minimal element m at its top, that σ is a linear extension of it, and that one can rebuild the corresponding set partition $PSA(\sigma)$ since its columns are the maximal increasing sequences of comparable elements in the Hasse diagram of P.

Let us now consider two values $a < c$, both maximal elements of P and let P' be the poset obtained from P by removing a and c. By induction, we can assume that all linear extensions of P' are Bell-congruent with one another. To prove that the same holds for P, we only need to prove that there exists one linear extension v of P' such that $v.ac \equiv v.ca$: indeed, all linear extensions of a poset can be reached from one another by such exchanges.

First, note that $C(a)$ is to the right of $C(c)$ since they are both maximal elements of their column. Let us prove that there exists a linear extension of P' that ends with a letter b such that $a < b < c$, possibly followed by letters smaller than a. We shall use twice a very simple observation: if x is the last letter of a column, then there are linear extensions of the associated poset where x is only followed by letters smaller than itself. Indeed, the letters that must appear after x in all linear extensions of the poset are letters belonging to columns to the right of $C(x)$, hence all smaller than x.

Now, if $C(c)$ is not the column immediately to the left of $C(a)$, define b as the maximal element of this column. It satisfies $a < b < c$ and all elements after it in the linear extensions of P' are smaller than b, and in fact smaller than a. So there are linear extensions of P' where the rewriting $wca \equiv wac$ is possible. Otherwise, since a and c are both maximal elements of P, they cannot be comparable. Define b as the letter immediately above c in its column. It exists since otherwise a would be connected to c (definition of the posets from the sets of columns). Now, b might not be a maximal element of P' but in any case, the values that are below b in P' are strictly to its right in the partition, hence are all smaller than a, again. So there are linear extensions of P' ending with b and then letters smaller than a. So in both cases, we found a value b allowing to rewrite one linear extension w of P' concatenated to ac with $w \cdot ca$, thus proving that all linear extensions of a poset $P(\sigma)$ are Bell congruent to σ.

5.2.3. *Our posets are regular.* Now that we have obtained the equivalence between Bell classes and the linear extensions of our posets, it only remains to prove that the posets are regular to conclude that the classes are intervals of the weak order [4]. We shall make use of a very simple property: if a value u belongs to a column to the left of a value v, and if $u < v$ then $u <_P v$. Indeed, consider the column C' immediately to the left of $C(v)$. Then v is below (in the poset) the smallest value v' greater than

[2] Grinberg [14] has proposed another proof, by induction on the length of σ.

itself in C', hence below all values smaller than itself. Now the same property holds for $v' > v$ on the next column to its left and so on.

Let us now consider two elements x and z such that $x <_P z$, so that $C(x)$ is to the left of $C(z)$. Two cases may appear: $x < z$ or $x > z$. Let us consider the first case and let y be another element of P such that $x < y < z$. In that case, either $C(y)$ is to the left of $C(z)$ or $C(y)$ is to the right of $C(x)$. In both cases, apply our previous result and conclude that either $x <_P y$ or $y <_P z$. Let us now consider the second case where $x > z$ and let y be another element of P such that $x > y > z$. If $C(y)$ is to the left of $C(z)$ or $C(y)$ is to the right of $C(x)$, the same argument as before applies. Otherwise, it means that $C(y)$ is between $C(x)$ and $C(z)$. But in that case, since there is a path of comparable elements from x down to z, all elements in between are either comparable to x (if they are below the path) or to z (if they are above the path). So in all cases, we proved that y satisfies either $x >_P y$ or $y >_P z$, hence proving that our posets are regular.

5.2.4. *The maximal elements of the Bell classes avoid* $21-3$. Let us finally prove that the maximal elements of the Bell classes avoid the pattern $21-3$. There are several ways to prove this. One could say for example that the Bell classes are in bijection with set partitions, hence splitting the permutations of size n into as many classes as the number of permutations avoiding $21-3$. Now, if a permutation σ does not avoid $21-3$, let us consider such a pattern with letters $ba \dots c$ in σ where the distance between a and c is minimal. Then the letter immediately to the left of c has to be smaller than b, hence denoted naturally by a'. All letters between b and a' are necessarily smaller than b, otherwise the rightmost one would create a pattern $21-3$ that would contradict the minimality of the initial one. So we are in the conditions of the Bell congruence, and σ is not a maximal element of its class. So all maximal elements necessarily avoid the pattern $21-3$. Since both sets, the maximal elements of Bell classes and the permutations avoiding $21-3$ have same cardinality, they must be equal.

Another proof consists in applying the greedy algorithm that takes a Bell poset as entry and takes the linear extension where at each step one chooses the maximal available value. It is easy to see that these elements necessarily avoid the pattern $21-3$ and conclude in the same way as before.

Note that the minimal elements are not characterized by pattern avoidance.

5.2.5. *Proof of Theorem 5.2.* As for Theorem 5.2, it follows from a well-known result about algebras defined by congruences, see *e.g.* Theorem 2.1 of [29] or Chap. 2.1 of [30].

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