

Extension of the product of a post-Lie algebra and application to the SISO feedback transformation group

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Abstract

We describe the both post- and pre-Lie algebra \mathfrak{g}_{SISO} associated to the affine SISO feedback transformation group. We show that it is a member of a family of post-Lie algebras associated to representations of a particular solvable Lie algebra. We first construct the extension of the magmatic product of a post-Lie algebra to its enveloping algebra, which allows to describe free post-Lie algebras and is widely used to obtain the enveloping of \mathfrak{g}_{SISO} and its dual.

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Introduction

The affine SISO feedback transformation group G_{SISO} [4], which appears in Control Theory, can be seen as the character group of a Hopf algebra \mathcal{H}_{SISO} ; let us start by a short presentation of this object (we slightly modify the notations of [4]).

1. First, let us recall some algebraic structures on noncommutative polynomials.

- (a) Let x_1, x_2 be two indeterminates. We consider the algebra of noncommutative polynomials $\mathbb{K}\langle x_1, x_2 \rangle$. As a vector space, it is generated by words in letters x_1, x_2 ; its product is the concatenation of words; its unit, the empty word, is denoted by \emptyset .
- (b) $\mathbb{K}\langle x_1, x_2 \rangle$ is a Hopf algebra with the concatenation product and the deshuffling coproduct Δ_{\sqcup} , defined by $\Delta_{\sqcup}(x_i) = x_i \otimes \emptyset + \emptyset \otimes x_i$, for $i \in \{1, 2\}$.
- (c) $\mathbb{K}\langle x_1, x_2 \rangle$ is also a commutative, associative algebra with the shuffle product \sqcup : for example, if $i, j, k, l \in \{1, 2\}$,

$$\begin{aligned} x_i \sqcup x_j &= x_i x_j + x_j x_i, \\ x_i x_j \sqcup x_k &= x_i x_j x_k + x_i x_k x_j + x_k x_i x_j, \\ x_i \sqcup x_j x_k &= x_i x_j x_k + x_j x_i x_k + x_j x_k x_i, \\ x_i x_j \sqcup x_k x_l &= x_i x_j x_k x_l + x_i x_k x_j x_l + x_i x_k x_l x_j + x_k x_i x_j x_l + x_k x_i x_l x_j + x_k x_l x_i x_j. \end{aligned}$$

2. The vector space $\mathbb{K}\langle x_1, x_2 \rangle^2$ is generated by words $x_{i_1} \dots x_{i_k} \epsilon_j$, where $k \geq 0$, $i_1, \dots, i_k, j \in \{1, 2\}$, and (ϵ_1, ϵ_2) denotes the canonical basis of \mathbb{K}^2 .

3. As an algebra, \mathcal{H}_{SISO} is equal to the symmetric algebra $S(\mathbb{K}\langle x_1, x_2 \rangle^2)$; its product is denoted by μ and its unit by 1. Two coproducts $\overline{\Delta}_*$ and $\overline{\Delta}_\bullet$ are defined on \mathcal{H}_{SISO} . For all $h \in \mathcal{H}_{SISO}$, we put $\overline{\Delta}_*(h) = \Delta_*(h) - 1 \otimes h$ and $\overline{\Delta}_\bullet(h) = \Delta_\bullet(h) - 1 \otimes h$. Then:

- For all $i \in \{1, 2\}$, $\overline{\Delta}_*(\emptyset \epsilon_i) = \emptyset \epsilon_i \otimes 1$.
- For all $g \in \mathbb{K}\langle x_1, x_2 \rangle$, for all $i \in \{1, 2\}$:

$$\begin{aligned} \overline{\Delta}_* \circ \theta_{x_1}(g \epsilon_i) &= (\theta_{x_1} \otimes Id) \circ \overline{\Delta}_*(g \epsilon_i) + (\theta_{x_2} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{\sqcup}(g) \epsilon_i \otimes \epsilon_2), \\ \overline{\Delta}_* \circ \theta_{x_2}(g \epsilon_i) &= (\theta_{x_2} \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{\sqcup}(g) \epsilon_i \otimes \epsilon_1), \end{aligned}$$

where $\theta_x(h \epsilon_i) = x h \epsilon_i$ for all $x \in \{x_1, x_2\}$, $h \in \mathbb{K}\langle x_1, x_2 \rangle$, $i \in \{1, 2\}$. These are formulas of Lemma 4.1 of [4], with the notations $a_w = w \epsilon_2$, $b_w = w \epsilon_1$, $\theta_0 = \theta_{x_1}$, $\theta_1 = \theta_{x_2}$ and $\tilde{\Delta} = \overline{\Delta}_*$.

- for all $g \in \mathbb{K}\langle x_1, x_2 \rangle$:

$$\begin{aligned} \overline{\Delta}_\bullet(g \epsilon_1) &= (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{\sqcup}(g)(\epsilon_1 \otimes \epsilon_1)), \\ \overline{\Delta}_\bullet(g \epsilon_2) &= \overline{\Delta}_*(g \epsilon_2) + (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{\sqcup}(g)(\epsilon_2 \otimes \epsilon_1)). \end{aligned}$$

This coproduct $\overline{\Delta}_\bullet$ makes \mathcal{H}_{SISO} a Hopf algebra, and $\overline{\Delta}_*$ is a right coaction on this coproduct, that is to say:

$$(\overline{\Delta}_\bullet \otimes Id) \circ \overline{\Delta}_* = (Id \otimes \overline{\Delta}_\bullet) \circ \overline{\Delta}_*, \quad (\overline{\Delta}_* \otimes Id) \circ \overline{\Delta}_* = (Id \otimes \overline{\Delta}_\bullet) \circ \overline{\Delta}_*.$$

4. After the identification of $\emptyset \epsilon_1$ with the unit of \mathcal{H}_{SISO} , we obtain a commutative, graded and connected Hopf algebra, in other words the dual of an enveloping algebra $\mathcal{U}(\mathfrak{g}_{SISO})$.

Our aim is to give a description of the underlying Lie algebra \mathfrak{g}_{SISO} . It turns out that it is both a pre-Lie algebra (or a Vinberg algebra [1], see [5] for a survey on these objects) and a post-Lie

algebra [6, 10]: it has a Lie bracket ${}_a[-, -]$ and two nonassociative products $*$ and \bullet , such that for all $x, y, z \in \mathfrak{g}_{SISO}$:

$$\begin{aligned} x * {}_a[y, z] &= (x * y) * z - x * (y * z) - (x * z) * y + x * (z * y), \\ {}_a[x, y] * z &= {}_a[x * z, y] + {}_a[x, y * z]; \end{aligned}$$

$$(x \bullet y) \bullet z - x \bullet (y \bullet z) = (x \bullet z) \bullet y - x \bullet (z \bullet y).$$

The Lie bracket on \mathfrak{g}_{SISO} corresponding to G_{SISO} is ${}_a[-, -]_*$:

$$\forall x, y \in \mathfrak{g}_{SISO}, \quad {}_a[x, y]_* = {}_a[x, y] + x * y - y * x = x \bullet y - y \bullet x.$$

Let us be more precise on these structures. As a vector space, $\mathfrak{g}_{SISO} = \mathbb{K}\langle x_1, x_2 \rangle^2$, and:

$$\forall f, g \in \mathbb{K}\langle x_1, x_2 \rangle, \quad \forall i, j \in \{1, 2\}, \quad {}_a[f\epsilon_i, g\epsilon_j] = \begin{cases} 0 & \text{if } i = j, \\ -f \sqcup g\epsilon_2 & \text{if } i = 2 \text{ and } j = 1, \\ f \sqcup g\epsilon_2 & \text{if } i = 1 \text{ and } j = 2. \end{cases}$$

The magmatic product $*$ is inductively defined. If $f, g \in \mathbb{K}\langle x_1, x_2 \rangle$ and $i, j \in \{1, 2\}$:

$$\begin{aligned} \emptyset\epsilon_i * g\epsilon_j &= 0, & x_2 f\epsilon_i * g\epsilon_1 &= x_2(f\epsilon_i * g\epsilon_1) + x_2(f \sqcup g)\epsilon_i, \\ x_1 f\epsilon_i * g\epsilon_j &= x_1(f\epsilon_i * g\epsilon_j), & x_2 f\epsilon_i * g\epsilon_2 &= x_2(f\epsilon_i * g\epsilon_2) + x_1(f \sqcup g)\epsilon_i. \end{aligned}$$

The pre-Lie product \bullet , first determined in [4], is given by:

$$\forall f, g \in \mathbb{K}\langle x_1, x_2 \rangle, \quad \forall i, j \in \{1, 2\}, \quad f\epsilon_i \bullet g\epsilon_j = (f \sqcup g)\delta_{i,1}\epsilon_j + f\epsilon_i * g\epsilon_j.$$

We shall show here that this is a special case of a family of post-Lie algebras, associated to modules over certain solvable Lie algebras.

We start with general preliminary results on post-Lie algebras. We extend the now classical Oudom-Guin construction on prelie algebras [7, 8] to the post-Lie context in the first section: this is a result of [2] (Proposition 3.1), which we prove here in a different, less direct way; our proof allows also to obtain a description of free post-Lie algebras. Recall that if $(V, *)$ is a pre-Lie algebra, the pre-Lie product $*$ can be extended to $S(V)$ in such a way that the product defined by:

$$\forall f, g \in S(V), \quad f \circledast g = \sum f * g^{(1)}g^{(2)}$$

is associative, and makes $S(V)$ a Hopf algebra, isomorphic to $\mathcal{U}(V)$. For any magmatic algebra $(V, *)$, we construct in a similar way an extension of $*$ to $T(V)$ in Proposition 1. We prove in Theorem 1 that the product \circledast defined by:

$$\forall f, g \in T(V), \quad f \circledast g = \sum f * g^{(1)}g^{(2)}$$

makes $T(V)$ a Hopf algebra. The Lie algebra of its primitive elements, which is the free Lie algebra $\mathcal{L}ie(V)$ generated by V , is stable under $*$ and turns out to be a post-Lie algebra (Proposition 2) satisfying a universal property (Theorem 2). In particular, if V is, as a magmatic algebra, freely generated by a subspace W , $\mathcal{L}ie(V)$ is the free post-Lie algebra generated by W (Corollary 1). Moreover, if $V = ([-, -], *)$ is a post-Lie algebra, this construction goes through the quotient defining $\mathcal{U}(V, [-, -])$, defining a new product \circledast on it, making it isomorphic to the enveloping algebra of V with the Lie bracket defined by:

$$\forall x, y \in V, \quad [x, y]_* = [x, y] + x * y - y * x.$$

For example, if $x_1, x_2, x_3 \in V$:

$$\begin{aligned} x_1 \circledast x_2 &= x_1 x_2 + x_1 * x_2 \\ x_1 \circledast x_2 x_3 &= x_1 x_2 x_3 + (x_1 * x_2) x_3 + (x_1 * x_3) x_2 + (x_1 * x_2) * x_3 - x_1 * (x_2 * x_3) \\ x_1 x_2 \circledast x_3 &= x_1 x_2 x_3 + (x_1 * x_3) x_2 + x_1 (x_2 * x_3). \end{aligned}$$

In the particular case where $[-, -] = 0$, we recover the Oudom-Guin construction.

The second section is devoted to the study of a particular solvable Lie algebra \mathfrak{g}_a associated to an element $a \in \mathbb{K}^N$. As the Lie bracket of \mathfrak{g}_a comes from an associative product, the construction of the first section holds, with many simplifications: we obtain an explicit description of $\mathcal{U}(\mathfrak{g}_a)$ with the help of a product \blacktriangleleft on $S(\mathfrak{g}_a)$ (Proposition 6). A short study of \mathfrak{g}_a -modules when $a = (1, 0, \dots, 0)$ (which is a generic case) is done in Proposition 8, considering \mathfrak{g}_a as an associative algebra, and in Proposition 9, considering it as a Lie algebra. In particular, if \mathbb{K} is algebraically closed, any \mathfrak{g}_a module inherits a natural decomposition in characteristic subspaces.

Our family of post-Lie algebras is introduced in the third section; it is reminiscent of the construction of [3]. Let us fix a vector space V , $(a_1, \dots, a_N) \in \mathbb{K}^N$ and a family F_1, \dots, F_N of endomorphisms of V . We define a product $*$ on $T(V)^N$, such that for all $f, g \in T(V)$, $x \in V$, $i, j \in \{1, \dots, N\}$:

$$\begin{aligned} \emptyset \epsilon_i * g \epsilon_j &= 0, \\ x f \epsilon_i * g \epsilon_j &= x (f \epsilon_i * g \epsilon_j) + F_j(x) (f \sqcup g) \epsilon_i, \end{aligned}$$

where $(\epsilon_1, \dots, \epsilon_N)$ is the canonical basis of \mathbb{K}^N and \sqcup is the shuffle product of $T(V)$. The Lie bracket of $T(V)^N$ that we shall use here is:

$$\forall f, g \in T(V), \forall i, j \in \{1, \dots, N\}, {}_a [f \epsilon_i, g \epsilon_j] = (f \sqcup g) (a_i \epsilon_j - a_j \epsilon_i).$$

This Lie bracket comes from an associative product ${}_a \sqcup$ defined by:

$$\forall f, g \in T(V), \forall i, j \in \{1, \dots, N\}, f \epsilon_i {}_a \sqcup g \epsilon_j = a_i (f \sqcup g) \epsilon_j.$$

We put $\bullet = * + {}_a \sqcup$. We prove in Theorem 3 the equivalence of the three following conditions:

- $(T(V)^N, \bullet)$ is a pre-Lie algebra.
- $(T(V)^N, {}_a [-, -], *)$ is a post-Lie algebra.
- F_1, \dots, F_N defines a structure of \mathfrak{g}_a -module on V .

If this holds, the construction of the first section allows to obtain two descriptions of the enveloping algebra of $\mathcal{U}(T(V)^N)$, respectively coming from the post-Lie product $*$ and from the pre-Lie product \bullet : the extensions of $*$ and of \bullet are respectively described in Propositions 15 and 16. It is shown in Proposition 17 that the two associated descriptions of $\mathcal{U}(T(V)^N)$ are equal. For \mathfrak{g}_{SISO} , we take $a = (1, 0)$, $V = Vect(x_1, x_2)$ and:

$$F_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which indeed define a $\mathfrak{g}_{(1,0)}$ -module. In order to relate this to the Hopf algebra \mathcal{H}_{SISO} of [4], we need to consider the dual of the enveloping of $T(V)^N$. First, if $a = (1, 0, \dots, 0)$, we observe that the decomposition of V as a \mathfrak{g}_a -module of the second section induces a graduation of the post-Lie algebra $T(V)^N$ (Proposition 18), unfortunately not connected: the component of degree 0 is 1-dimensional, generated by $\emptyset \epsilon_1$. Forgetting this element, that is, considering the augmentation ideal of the graded post-Lie algebra $T(V)^N$, we can dualize the product \circledast of $S(T(V)^N)$ in order to obtain the coproduct of the dual Hopf algebra in an inductive way. For \mathfrak{g}_{SISO} , we indeed obtain the inductive formulas of \mathcal{H}_{SISO} , finally proving that the dual Lie algebra of this

Hopf algebra, which in some sense can be exponentiated to G_{SISO} , is indeed post-Lie and pre-Lie.

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Notations.

1. Let \mathbb{K} be a commutative field. The canonical basis of \mathbb{K}^n is denoted by $(\epsilon_1, \dots, \epsilon_n)$.
2. For all $n \geq 1$, we denote by $[n]$ the set $\{1, \dots, n\}$.
3. We shall use Sweeder's notations: if C is a coalgebra and $x \in C$,

$$\begin{aligned}\Delta^{(1)}(x) &= \Delta(x) = \sum x^{(1)} \otimes x^{(2)}, \\ \Delta^{(2)}(x) &= (\Delta \otimes Id) \circ \Delta(x) = \sum x^{(1)} \otimes x^{(2)} \otimes x^{(3)}, \\ \Delta^{(3)}(x) &= (\Delta \otimes Id \otimes Id) \circ (\Delta \otimes Id) \circ \Delta(x) = \sum x^{(1)} \otimes x^{(2)} \otimes x^{(3)} \otimes x^{(4)}, \\ &\vdots\end{aligned}$$

1 Extension of a post-Lie product

We first generalize the Oudom-Guin extension of a pre-Lie product in a post-Lie algebraic context, as done in [2]. Let us first recall what a post-Lie algebra is.

Definition 1 1. A (right) post-Lie algebra is a family $(\mathfrak{g}, \{-, -\}, *)$, where \mathfrak{g} is a vector space, $\{-, -\}$ and $*$ are bilinear products on \mathfrak{g} such that:

- $(\mathfrak{g}, \{-, -\})$ is a Lie algebra.
- For all $x, y, z \in \mathfrak{g}$:

$$x * \{y, z\} = (x * y) * z - x * (y * z) - (x * z) * y + x * (z * y), \quad (1)$$

$$\{x, y\} * z = \{x * z, y\} + \{x, y * z\}. \quad (2)$$

2. If $(\mathfrak{g}, \{-, -\}, *)$ is post-Lie, we define a second Lie bracket on \mathfrak{g} :

$$\forall x, y \in \mathfrak{g}, \{x, y\}_* = \{x, y\} + x * y - y * x.$$

Note that if $\{-, -\}$ is 0, then $(\mathfrak{g}, *)$ is a (right) pre-Lie algebra, that is to say:

$$\forall x, y, z \in \mathfrak{g}, (x * y) * z - x * (y * z) = (x * z) * y - x * (z * y). \quad (3)$$

1.1 Extension of a magmatic product

Let V be a vector space. We here use the tensor Hopf algebra $T(V)$. In this section, we shall denote the unit of $T(V)$ by 1. Its product is the concatenation of words, and its coproduct Δ_{\sqcup} is the cocommutative deshuffling coproduct. For example, if $x_1, x_2, x_3 \in V$:

$$\begin{aligned}\Delta_{\sqcup}(x_1) &= x_1 \otimes 1 + 1 \otimes x_1, \\ \Delta_{\sqcup}(x_1 x_2) &= x_1 x_2 \otimes 1 + x_1 \otimes x_2 + x_2 \otimes x_1 + 1 \otimes x_1 x_2, \\ \Delta_{\sqcup}(x_1 x_2 x_3) &= x_1 x_2 x_3 \otimes 1 + x_1 x_2 \otimes x_3 + x_1 x_3 \otimes x_2 + x_2 x_3 \otimes x_1 \\ &\quad + x_1 \otimes x_2 x_3 + x_2 \otimes x_1 x_3 + x_3 \otimes x_1 x_2 + 1 \otimes x_1 x_2 x_3.\end{aligned}$$

Its counit is denoted by ε : $\varepsilon(1) = 1$ and if $k \geq 1$ and $x_1, \dots, x_k \in V$, $\varepsilon(x_1 \dots x_k) = 0$.

Proposition 1 Let V be a vector space and $*$: $V \otimes V \rightarrow V$ be a magmatic product on V . Then $*$ can be uniquely extended as a map from $T(V) \otimes T(V)$ to $T(V)$ such that for all $f, g, h \in T(V)$, $x, y \in V$:

- $f * 1 = f$.
- $1 * f = \varepsilon(f)1$.
- $x * (fy) = (x * f) * y - x * (f * y)$.
- $(fg) * h = \sum (f * h^{(1)}) (g * h^{(2)})$.

Proof. *Existence.* We first inductively extend $*$ from $V \otimes T(V)$ to V . If $n \geq 0$, $x, y_1, \dots, y_n \in V$, we put:

$$x * y_1 \dots y_n = \begin{cases} x & \text{if } n = 0, \\ x * y_1 & \text{if } n = 1, \\ \underbrace{\underbrace{(x * (y_1 \dots y_{n-1}))}_{\in V} * \underbrace{y_n}_{\in V}}_{\in V} - \sum_{i=1}^{n-1} \underbrace{x * (y_1 \dots \underbrace{(y_i * y_n)}_{\in V} \dots y_{n-1})}_{\in V} & \text{if } n \geq 2. \end{cases}$$

This product is then extended from $T(V) \otimes T(V)$ to $T(V)$ in the following way:

- For all $f \in T(V)$, $1 * f = \varepsilon(f)1$.
- For all $n \geq 1$, for all $x_1, \dots, x_n \in V$, $f \in T(V)$:

$$(x_1 \dots x_n) * f = \sum \underbrace{(x_1 * f^{(1)})}_{\in V} \dots \underbrace{(x_n * f^{(n)})}_{\in V} \in V^{\otimes n}.$$

Note that for all $n \geq 0$, $V^{\otimes n} * T(V) \subseteq V^{\otimes n}$, which induces the second point. Let us prove the first point with $f = x_1 \dots x_n \in V^{\otimes n}$. If $n = 0$, $f * 1 = 1 * 1 = \varepsilon(1)1 = 1 = f$. If $n = 1$, $f \in V$, so $f * 1 = f$ by definition of the extension of $*$ on $V \otimes T(V)$. If $n \geq 2$:

$$f * 1 = (x_1 \dots x_n) * 1 = (x_1 * 1) \dots (x_n * 1) = x_1 \dots x_n = f.$$

Let us prove the third point for $f = y_1 \dots y_n$. Then:

$$x * (fy) = (x * f) * y - \sum x * (y_1 \dots (y_i * y) \dots y_n).$$

Moreover, as $\Delta_{\square}(y) = y \otimes 1 + 1 \otimes y$:

$$f * y = \sum_{i=1}^n (y_1 * 1) \dots (y_i * y) \dots (y_n * 1) = \sum_{i=1}^n y_1 \dots (y_i * y) \dots y_n.$$

So $x * (fy) = (x * f) * y - x * (f * y)$. Let us finally prove the last point for $f = x_1 \dots x_k$ and $g = x_{k+1} \dots x_{k+l}$. Then:

$$\begin{aligned} (fg) * h &= \sum \left(x_1 * h^{(1)} \right) \dots \left(x_{k+l} * h^{(k+l)} \right) \\ &= \sum \left(x_1 * (h^{(1)})^{(1)} \right) \dots \left(x_1 * (h^{(1)})^{(k)} \right) \left(x_{k+1} * (h^{(2)})^{(1)} \right) \dots \left(x_{k+l} * (h^{(2)})^{(l)} \right) \\ &= \sum \left((x_1 \dots x_k) * h^{(1)} \right) \left((x_{k+1} \dots x_{k+l}) * h^{(2)} \right) \\ &= \sum \left(f * h^{(1)} \right) \left(g * h^{(2)} \right). \end{aligned}$$

We used the coassociativity of Δ_{\sqcup} for the second equality.

Unicity. The first and third points uniquely determine $x * (x_1 \dots x_n)$ for $x, x_1, \dots, x_n \in V$, by induction on n ; the second and fourth points then uniquely determine $f * (x_1 \dots x_n)$ for all $f \in T(V)$ by induction on the length of f . \square

Examples. If $x_1, x_2, x_3, x_4 \in V$:

$$\begin{aligned} (x_1 x_2) * x_3 &= (x_1 * x_3) x_2 + x_1 (x_2 * x_3), \\ x_1 * (x_2 x_3) &= (x_1 * x_2) * x_3 - x_1 * (x_2 * x_3), \\ (x_1 x_2 x_3) * x_4 &= (x_1 * x_4) x_2 x_3 + x_1 (x_2 * x_4) x_3 + x_1 x_2 (x_3 * x_4), \\ (x_1 x_2) * (x_3 x_4) &= ((x_1 * x_3) * x_4) x_2 - (x_1 * (x_3 * x_4)) x_2 + x_1 ((x_2 * x_3) * x_4), \\ &\quad - x_1 (x_2 * (x_3 * x_4)) + (x_1 * x_3) (x_2 * x_4) + (x_1 * x_4) (x_2 * x_3), \\ x_1 * (x_2 x_3 x_4) &= ((x_1 * x_2) * x_3) * x_4 - (x_1 * (x_2 * x_3)) * x_4 - (x_1 * (x_2 * x_4)) * x_3 \\ &\quad + x_1 * ((x_2 * x_4) * x_3) - (x_1 * x_2) * (x_3 * x_4) + x_1 * (x_2 * (x_3 * x_4)). \end{aligned}$$

Lemma 1 1. For all $k \in \mathbb{N}$, $V^{\otimes k} * T(V) \subseteq V^{\otimes k}$.

2. For all $f, g \in T(V)$, $\varepsilon(f * g) = \varepsilon(f)\varepsilon(g)$.
3. For all $f, g \in T(V)$, $\Delta_{\sqcup}(f * g) = \Delta_{\sqcup}(f) * \Delta_{\sqcup}(g)$.
4. For all $f, g \in T(V)$, $y \in V$, $f * (gy) = (f * g) * y - f * (g * y)$.
5. For all $f, g, h \in T(V)$, $(f * g) * h = \sum f * ((g * h^{(1)}) h^{(2)})$.

Proof. 1. This was observed in the proof of Proposition 1.

2. From the first point, $\text{Ker}(\varepsilon) * T(V) + T(V) * \text{Ker}(\varepsilon) \subseteq \text{Ker}(\varepsilon)$, so if $\varepsilon(f) = 0$ or $\varepsilon(g) = 0$, then $\varepsilon(f * g) = 0$. As $\varepsilon(1 * 1) = 1$, the second point holds for all f, g .

3. We prove it for $f = x_1 \dots x_n$, by induction on n . If $n = 0$, then $f = 1$. Moreover, $\Delta_{\sqcup}(1 * g) = \varepsilon(g)\Delta_{\sqcup}(1) = \varepsilon(g)1 \otimes 1$, and:

$$\Delta_{\sqcup}(f) * \Delta_{\sqcup}(g) = \sum 1 * g^{(1)} \otimes 1 * g^{(2)} = \varepsilon(g^{(1)}) \varepsilon(g^{(2)}) 1 \otimes 1 = \varepsilon(g)1 \otimes 1.$$

If $n = 1$, then $f \in V$. In this case, from the second point, $f * g \in V$, so $\Delta_{\sqcup}(f * g) = f * g \otimes 1 + 1 \otimes f * g$. Moreover:

$$\begin{aligned} \Delta_{\sqcup}(f) * \Delta_{\sqcup}(g) &= (f \otimes 1 + 1 \otimes f) * \Delta_{\sqcup}(g) \\ &= \sum f * g^{(1)} \otimes 1 * g^{(2)} + \sum 1 * g^{(1)} \otimes f * g^{(2)} \\ &= \sum f * g^{(1)} \otimes \varepsilon(g^{(2)}) 1 + \sum \varepsilon(g^{(1)}) 1 \otimes f * g^{(2)} \\ &= f * g \otimes 1 + 1 \otimes f * g. \end{aligned}$$

If $n \geq 2$, we put $f_1 = x_1 \dots x_{n-1}$ and $f_2 = x_n$. By the induction hypothesis applied to f_1 :

$$\begin{aligned} \Delta_{\sqcup}(f * g) &= \sum \Delta_{\sqcup} \left((f_1 * g^{(1)}) (f_2 * g^{(2)}) \right) \\ &= \Delta_{\sqcup} \left(f_1 * g^{(1)} \right) \Delta_{\sqcup} \left(f_2 * g^{(2)} \right) \\ &= \sum \left(f_1^{(1)} * (g^{(1)})^{(1)} \right) \left(f_2^{(1)} * (g^{(2)})^{(1)} \right) \otimes \left(f_1^{(2)} * (g^{(1)})^{(2)} \right) \left(f_2^{(2)} * (g^{(2)})^{(2)} \right) \\ &= \sum (f_1 f_2)^{(1)} * g^{(1)} \otimes (f_1 f_2)^{(2)} * g^{(2)} \\ &= \Delta_{\sqcup}(f) * \Delta_{\sqcup}(g). \end{aligned}$$

We used the cocommutativity of Δ_{\sqcup} for the fourth equality.

4. We prove it for $f = x_1 \dots x_n$, by induction on n . If $n = 0$, then $f = 1$ and:

$$1 * (gy) = (1 * g) * y - 1 * (g * y) = \varepsilon(g)\varepsilon(y) - \varepsilon(g * y) = 0.$$

For $n = 1$, this comes immediately from Proposition 1-3. If $n \geq 2$, we put $f_1 = x_1 \dots x_{n-1}$ and $f_2 = x_n$. The induction hypothesis holds for f_1 . Moreover:

$$\begin{aligned} f * (gy) &= \sum \left(f_1 * g^{(1)} \right) \left(f_2 * \left(g^{(2)} y \right) \right) + \sum \left(f_1 * \left(g^{(1)} y \right) \right) \left(f_2 * g^{(2)} \right) \\ &= \sum \left(f_1 * g^{(1)} \right) \left(\left(f_2 * g^{(2)} \right) * y \right) - \sum \left(f_1 * g^{(1)} \right) \left(f_2 * \left(g^{(2)} * y \right) \right) \\ &\quad + \sum \left(\left(f_1 * g^{(1)} \right) * y \right) \left(f_2 * g^{(2)} \right) - \sum \left(f_1 * \left(g^{(1)} * y \right) \right) \left(f_2 * g^{(2)} \right), \\ (f * g) * y &= \sum \left(\left(f_1 * g^{(1)} \right) \left(f_2 * g^{(2)} \right) \right) * y \\ &= \sum \left(\left(f_1 * g^{(1)} \right) * y \right) \left(f_2 * g^{(2)} \right) + \sum \left(f_1 * g^{(1)} \right) \left(\left(f_2 * g^{(2)} \right) * y \right), \\ f * (g * y) &= \sum \left(f_1 * \left(g * y \right)^{(1)} \right) \left(f_2 * \left(g * y \right)^{(2)} \right) \\ &= \sum \left(f_1 * \left(g^{(1)} * y \right) \right) \left(f_2 * g^{(2)} \right) + \sum \left(f_1 * g^{(1)} \right) \left(f_2 * \left(g^{(2)} * y \right) \right). \end{aligned}$$

We use the third point for the third computation. So the result holds for all f .

5. We prove this for $h = z_1 \dots z_n$ and we proceed by induction on n . If $n = 0$, then $h = 1$ and $(f * g) * 1 = f * g$. Moreover, $\sum f * \left((g * h^{(1)}) h^{(2)} \right) = f * \left((g * 1) 1 \right) = (f * g) 1 = f * g$. If $n = 1$, then $h \in V$, so $\Delta_{\sqcup}(h) = h \otimes 1 + 1 \otimes h$. So:

$$\begin{aligned} \sum f * \left(\left(g * h^{(1)} \right) h^{(2)} \right) &= f * \left((g * h) 1 \right) + f * \left((g * 1) h \right) \\ &= f * (g * h) + f * (gh) \\ &= f * (g * h) + (f * g) * h - f * (g * h) \\ &= (f * g) * h. \end{aligned}$$

We use Proposition 1-3 for the third equality. If $n \geq 2$, we put $h_1 = z_1 \dots z_{n-1}$ and $h_2 = z_n$. From the fourth point:

$$\begin{aligned} (f * g) * h &= \left((f * g) * h_1 \right) * h_2 - (f * g) * \left(h_1 * h_2 \right) \\ &= \sum \left(f * \left(\left(g * h_1^{(1)} \right) h_1^{(2)} \right) \right) * h_2 - \sum f * \left(\left(g * \left(h_1 * h_2 \right)^{(1)} \right) \left(h_1 * h_2 \right)^{(2)} \right) \\ &= \sum f * \left(\left(\left(g * h_1^{(1)} \right) h_1^{(2)} \right) * h_2 \right) + \sum f * \left(\left(g * h_1^{(1)} \right) h_1^{(2)} h_2 \right) \\ &\quad - \sum f * \left(\left(g * \left(h_1^{(1)} * h_2^{(1)} \right) \right) \left(h_1^{(2)} * h_2^{(2)} \right) \right) \\ &= \sum f * \left(\left(\left(g * h_1^{(1)} \right) * h_2 \right) h_1^{(2)} \right) + \sum f * \left(\left(g * h_1^{(1)} \right) \left(h_1^{(2)} * h_2 \right) \right) \\ &\quad + \sum f * \left(\left(g * h_1^{(1)} \right) h_1^{(2)} h_2 \right) - \sum f * \left(\left(g * \left(h_1^{(1)} * h_2 \right) \right) h_1^{(2)} \right) \\ &\quad - \sum f * \left(\left(g * h_1^{(1)} \right) \left(h_1^{(2)} * h_2 \right) \right) \\ &= \sum f * \left(\left(g * \left(h_1^{(1)} * h_2 \right) \right) h_1^{(2)} \right) + \sum f * \left(\left(g * \left(h_1^{(1)} h_2 \right) \right) h_1^{(2)} \right) \\ &\quad + \sum f * \left(\left(g * h_1^{(1)} \right) \left(h_1^{(2)} * h_2 \right) \right) + \sum f * \left(\left(g * h_1^{(1)} \right) h_1^{(2)} h_2 \right) \\ &\quad - \sum f * \left(\left(g * \left(h_1^{(1)} * h_2 \right) \right) h_1^{(2)} \right) - \sum f * \left(\left(g * h_1^{(1)} \right) \left(h_1^{(2)} * h_2 \right) \right) \\ &= \sum f * \left(\left(g * \left(h_1^{(1)} h_2 \right) \right) h_1^{(2)} \right) + \sum f * \left(\left(g * h_1^{(1)} \right) h_1^{(2)} h_2 \right). \end{aligned}$$

For the second equality, we used the induction hypothesis on h_1 and $h_1 * h_2 \in V^{\otimes(k-1)}$ by the first point; we used the third point for the third equality. As $\Delta_{\sqcup}(h_2) = h_2 \otimes 1 + 1 \otimes h_2$, $\Delta_{\sqcup}(h) = \sum h_1^{(1)} h_2 \otimes h_1^{(2)} + \sum h_1^{(1)} \otimes h_1^{(2)} h_2$, so the result holds for h . \square

1.2 Associated Hopf algebra and post-Lie algebra

Theorem 1 *Let $*$ be a magmatic product on V . This product is extended to $T(V)$ by Proposition 1. We define a product \otimes on $T(V)$ by:*

$$\forall f, g \in T(V), f \otimes g = \sum \left(f * g^{(1)} \right) g^{(2)}.$$

Then $(T(V), \otimes, \Delta_{\sqcup})$ is a Hopf algebra.

Proof. For all $f \in T(V)$:

$$1 \otimes f \sum \left(1 * f^{(1)} \right) f^{(2)} = \sum \varepsilon \left(f^{(1)} \right) f^{(2)} = f; \quad \otimes 1 = (f * 1)1 = f.$$

For all $f, g, h \in T(V)$, by Lemma 1-5:

$$\begin{aligned} (f \otimes g) \otimes h &= \sum \left(\left(f * g^{(1)} \right) g^{(2)} \right) \otimes h \\ &= \sum \left(\left(\left(f * g^{(1)} \right) g^{(2)} \right) * h^{(1)} \right) h^{(2)} \\ &= \sum \left(\left(f * g^{(1)} \right) * h^{(1)} \right) \left(g^{(2)} * h^{(2)} \right) h^{(3)} \\ &= \sum \left(f * \left(\left(g^{(1)} * h^{(1)} \right) h^{(2)} \right) \right) \left(g^{(2)} * h^{(3)} \right) h^{(4)}; \\ f \otimes (g \otimes h) &= \sum f \otimes \left(\left(g * h^{(1)} \right) h^{(2)} \right) \\ &= \sum \left(f * \left(\left(g^{(1)} * h^{(1)} \right) h^{(3)} \right) \right) \left(g^{(2)} * h^{(2)} \right) h^{(4)}. \end{aligned}$$

As Δ_{\sqcup} is cocommutative, $(f \otimes g) \otimes h = f \otimes (g \otimes h)$, so $(T(V), \otimes)$ is a unitary, associative algebra.

For all $f, g \in T(V)$, by Lemma 1-3:

$$\begin{aligned} \Delta_{\sqcup}(f \otimes g) &= \sum \Delta_{\sqcup} \left(\left(f * g^{(1)} \right) g^{(2)} \right) \\ &= \sum \left(f^{(1)} * \left(g^{(1)} \right)^{(1)} \right) \left(g^{(2)} \right)^{(1)} \otimes \left(f^{(2)} * \left(g^{(1)} \right)^{(2)} \right) \left(g^{(2)} \right)^{(2)} \\ &= \sum \left(f^{(1)} * \left(g^{(1)} \right)^{(1)} \right) \left(g^{(1)} \right)^{(2)} \otimes \left(f^{(2)} * \left(g^{(2)} \right)^{(1)} \right) \left(g^{(2)} \right)^{(2)} \\ &= \sum f^{(1)} \otimes g^{(1)} \otimes f^{(2)} \otimes g^{(2)}. \end{aligned}$$

Note that we used the cocommutativity of Δ_{\sqcup} for the third equality. Hence, $(T(V), \otimes, \Delta_{\sqcup})$ is a Hopf algebra. \square

Remark. By Lemma 1:

- For all $f, g, h \in T(V)$, $(f * g) * h = f * (g \otimes h)$: $(T(V), *)$ is a right $(T(V), \otimes)$ -module.
- By restriction, for all $n \geq 0$, $(V^{\otimes n}, *)$ is a right $(T(V), \otimes)$ -module. Moreover, for all $n \geq 0$, $(V^{\otimes n}, *) = (V, *)^{\otimes n}$ as a right module over the Hopf algebra $(T(V), \otimes, \Delta_{\sqcup})$.

Examples. Let $x_1, x_2, x_3 \in V$.

$$\begin{aligned} x_1 \otimes x_2 &= x_1 x_2 + x_1 * x_2 \\ x_1 \otimes x_2 x_3 &= x_1 x_2 x_3 + (x_1 * x_2) x_3 + (x_1 * x_3) x_2 + (x_1 * x_2) * x_3 - x_1 * (x_2 * x_3) \\ x_1 x_2 \otimes x_3 &= x_1 x_2 x_3 + (x_1 * x_3) x_2 + x_1 (x_2 * x_3). \end{aligned}$$

The vector space of primitive elements of $(T(V), \otimes, \Delta_{\sqcup})$ is $\mathcal{L}ie(V)$. Let us now describe the Lie bracket induced on $\mathcal{L}ie(V)$ by \otimes .

Proposition 2 1. Let $*$ be a magmatic product on V . The Hopf algebras $(T(V), \otimes, \Delta_{\sqcup})$ and $(T(V), \cdot, \Delta_{\sqcup})$ are isomorphic, via the following algebra morphism:

$$\phi_* : \begin{cases} (T(V), \cdot, \Delta_{\sqcup}) & \longrightarrow & (T(V), \otimes, \Delta_{\sqcup}) \\ x_1 \dots x_k \in V^{\otimes k} & \longrightarrow & x_1 \otimes \dots \otimes x_k. \end{cases}$$

2. $\mathcal{L}ie(V) * T(V) \subseteq \mathcal{L}ie(V)$. Moreover, $(\mathcal{L}ie(V), [-, -], *)$ is a post-Lie algebra. The induced Lie bracket on $\mathcal{L}ie(V)$ is denoted by $\{-, -\}_*$:

$$\forall f, g \in \mathcal{L}ie(V), \{f, g\}_* = [f, g] + f * g - g * f = fg - gf + f * g - g * f.$$

The Lie algebra $(\mathcal{L}ie(V), \{-, -\}_*)$ is isomorphic to $\mathcal{L}ie(V)$.

Proof. 1. There exists a unique algebra morphism $\phi_* : (T(V), \cdot) \longrightarrow (T(V), \otimes)$, sending any $x \in V$ on itself. As the elements of V are primitive in both Hopf algebras, ϕ_* is a Hopf algebra morphism. As $V^{\otimes k} * T(V) \subseteq V^{\otimes k}$ for all $k \geq 0$, we deduce that for all $x_1, \dots, x_{k+l} \in V$:

$$x_1 \dots x_k \otimes x_{k+1} \dots x_{k+l} = x_1 \dots x_{k+l} + \text{a sum of words of length } < k + l.$$

Hence, if $x_1, \dots, x_k \in V$:

$$\phi_*(x_1 \dots x_k) = x_1 \otimes \dots \otimes x_k = x_1 \dots x_k + \text{a sum of words of length } < k.$$

Consequently:

- If $k \geq 0$ and $x_1, \dots, x_k \in V$, an induction on k proves that $x_1 \dots x_k \in \phi_*(T(V))$, so ϕ_* is surjective.
- If f is a nonzero element of $T(V)$, let us write $f = f_0 + \dots + f_k$, with $f_i \in V^{\otimes i}$ for all i and $f_k \neq 0$. Then:

$$\phi_*(f) = f_k + \text{terms in } \mathbb{K} \oplus \dots \oplus V^{\otimes(k-1)},$$

so $\phi_*(f) \neq 0$: ϕ_* is injective.

Hence, ϕ_* is an isomorphism.

2. We consider $A = \{f \in \mathcal{L}ie(V) \mid f * T(V) \subseteq \mathcal{L}ie(V)\}$. By Lemma 1-3, $V \subseteq A$. Let $f, g \in A$. For all $h \in T(V)$:

$$\begin{aligned} [f, g] * h &= (fg) * h - (gf) * h \\ &= \sum (f * h^{(1)}) (g * h^{(2)}) - \sum (g * h^{(1)}) (f * h^{(2)}) \\ &= \sum (f * h^{(1)}) (g * h^{(2)}) - \sum (g * h^{(2)}) (f * h^{(1)}) \\ &= \sum [f * h^{(1)}, g * h^{(2)}]. \end{aligned}$$

We used the cocommutativity for the third equality. By hypothesis, $f * h^{(1)}, g * h^{(2)} \in \mathcal{L}ie(V)$, so $[f, g] \in A$. As A is a Lie subalgebra of $\mathcal{L}ie(V)$ containing V , it is equal to $\mathcal{L}ie(V)$.

Let $f, g, h \in \mathcal{L}ie(V)$. Then $g \circledast h = \sum (g * h^{(1)}) h^{(2)} = gh + g * h$. Similarly, $\sum (h * g^{(1)}) g^{(2)} = hg + h * g$, so, by Lemma 1-5:

$$\begin{aligned} f * [g, h] &= f * (gh) - f * (hg) \\ &= \sum f * \left((g * h^{(1)}) h^{(2)} \right) - f * (g * h) - \sum f * \left((h * g^{(1)}) g^{(2)} \right) + f * (h * g) \\ &= (f * g) * h - f * (g * h) - (f * h) * g + f * (g * h). \end{aligned}$$

Moreover:

$$\begin{aligned} [f, g] * h &= (fg) * h - (gf) * h \\ &= (f * h)g + f(g * h) - (g * h)f - g(f * h) \\ &= [f * h, g] + [f, g * h]. \end{aligned}$$

So $\mathcal{L}ie(V)$ is a post-Lie algebra.

Consequently, $\{-, -\}_*$ is a second Lie bracket on $\mathcal{L}ie(V)$. In $(T(V), \circledast)$, if f and g are primitive:

$$f \circledast g - g \circledast f = fg + f * g - gf - g * f = \{f, g\}_*.$$

So, by the Cartier-Quillen-Milnor-Moore's theorem, $(T(V), \circledast, \Delta_{\sqcup})$ is the enveloping algebra of $(\mathcal{L}ie(V), \{-, -\}_*)$. As it is isomorphic to the enveloping algebra of $\mathcal{L}ie(V)$, namely $(T(V), \cdot, \Delta_{\sqcup})$, these two Lie algebras are isomorphic. \square

Let us give a combinatorial description of ϕ_* .

Proposition 3 *Let $(V, *)$ be a magmatic algebra, and $x_1, \dots, x_k \in V$.*

- *Let $I = \{i_1, \dots, i_p\} \subseteq [k]$, with $i_1 < \dots < i_p$. We put:*

$$x_I^* = (\dots ((x_{i_1} * x_{i_2}) * x_{i_3}) * \dots) * x_{i_p} \in V.$$

- *Let P be a partition of $[p]$. We denote it by $P = \{P_1, \dots, P_p\}$, with the convention $\min(P_1) < \dots < \min(P_p)$. We put:*

$$x_P^* = x_{P_1}^* \dots x_{P_p}^* \in V^{\otimes p}.$$

Then:

$$\phi^*(x_1 \dots x_k) = \sum_{P \text{ partition of } [k]} x_P^*.$$

Proof. By induction on k . As $\phi_*(x) = x$ for all $x \in V$, it is obvious if $k = 1$. Let us assume the result at rank k .

$$\begin{aligned} \phi_*(x_1 \dots x_{k+1}) &= \phi_*(x_1 \dots x_k) \circledast x_{k+1} \\ &= \phi_*(x_1 \dots x_k)x_{k+1} + \phi_*(x_1 \dots x_k) * x_{k+1} \\ &= \sum_{P \text{ partition of } [k]} x_P^* x_{k+1} + \sum_{\substack{P = \{P_1, \dots, P_p\} \\ \text{partition of } [k]}} \sum_{i=1}^p x_{P_1}^* \dots (x_{P_i}^* * x_{k+1}) \dots x_{P_p}^* \\ &= \sum_{\substack{P = \{P_1, \dots, P_p\} \\ \text{partition of } [k]}} x_{\{P_1, \dots, P_p, \{k+1\}\}}^* + \sum_{\substack{P = \{P_1, \dots, P_p\} \\ \text{partition of } [k]}} \sum_{i=1}^p x_{\{P_1, \dots, P_i \cup \{k+1\}, \dots, P_p\}}^* \\ &= \sum_{P \text{ partition of } [k+1]} x_P^*. \end{aligned}$$

So the result holds for all k . □

Examples. Let $x_1, x_2, x_3 \in V$.

$$\begin{aligned}\phi_*(x_1) &= x_1, \\ \phi_*(x_1x_2) &= x_1x_2 + x_1 * x_2, \\ \phi_*(x_1x_2x_3) &= x_1x_2x_3 + (x_1 * x_2)x_3 + (x_1 * x_3)x_2 + x_1(x_2 * x_3) + (x_1 * x_2) * x_3.\end{aligned}$$

Theorem 2 *Let $(V, *)$ be a magmatic algebra and let $(L, \{-, -\}, \star)$ be a post-Lie algebra. Let $\phi : (V, *) \rightarrow (L, \star)$ be a morphism of magmatic algebras. There exists a unique morphism of post-Lie algebras $\bar{\phi} : \mathcal{L}ie(V) \rightarrow L$ extending ϕ .*

Proof. Let $\psi : \mathcal{L}ie(V) \rightarrow L$ be the unique Lie algebra morphism extending ϕ . Let us fix $h \in \mathcal{L}ie(V)$. We consider:

$$A_h = \{h \in \mathcal{L}ie(V) \mid \forall f \in \mathcal{L}ie(V), \psi(f * h) = \psi(f) \star \psi(h)\}.$$

If $f, g \in A_h$, then:

$$\begin{aligned}\psi([f, g] * h) &= \psi([f * h, g] + [f, g * h]) \\ &= \{\psi(f * h), \psi(g)\} + \{\psi(f), \psi(g * h)\} \\ &= \{\psi(f) \star \psi(h), \psi(g)\} + \{\psi(f), \psi(g) \star \psi(h)\} \\ &= \{\psi(f), \psi(g)\} \star \psi(h) \\ &= \psi([f, g]) \star \psi(h).\end{aligned}$$

So $[f, g] \in A_h$: for all $h \in \mathcal{L}ie(V)$, A_h is a Lie subalgebra of $\mathcal{L}ie(V)$. Moreover, if $h \in V$, as $\psi|_V = \phi$ is a morphism of magmatic algebras, $V \subseteq A_h$; as a consequence, if $h \in V$, $A_h = \mathcal{L}ie(V)$.

Let $A = \{h \in \mathcal{L}ie(V) \mid A_h = \mathcal{L}ie(V)\}$. We put $\mathcal{L}ie(V)_n = \mathcal{L}ie(V) \cap V^{\otimes n}$; let us prove inductively that $\mathcal{L}ie(V)_n \subseteq A$ for all n . We already proved that $V \subseteq A$, so this is true for $n = 1$. Let us assume the result at all rank $k < n$. Let $h \in \mathcal{L}ie(V)_n$. We can assume that $h = [h_1, h_2]$, with $h_1 \in \mathcal{L}ie(V)_k$, $h_2 \in \mathcal{L}ie(V)_{n-k}$, $1 \leq k \leq n - 1$. From Lemma 1 and Proposition 2, $1f * h_2 \in \mathcal{L}ie(V)_k$ and $h_2 * h_1 \in \mathcal{L}ie(V)_{n-k}$, so the induction hypothesis holds for $h_1, h_2, h_1 * h_2$ and $h_2 * h_1$. Hence, for all $f \in T(V)$:

$$\begin{aligned}\psi(f * h) &= \psi(f * [h_1, h_2]) \\ &= \psi((f * h_1) * h_2 - f * (h_1 * h_2) - (f * h_2) * h_1 + f * (h_2 * h_1)) \\ &= (\psi(f) \star \psi(h_1)) \star \psi(h_2) - \psi(f) \star (\psi(h_1) \star \psi(h_2)) \\ &\quad - (\psi(f) \star \psi(h_2)) \star \psi(h_1) + \psi(f) \star (\psi(h_2) \star \psi(h_1)) \\ &= \psi(f) \star \{\psi(h_1), \psi(h_2)\} \\ &= \psi(f) \star \psi(h).\end{aligned}$$

As a consequence, $\mathcal{L}ie(V)_n \subseteq A$. Finally, $A = \mathcal{L}ie(V)$, so for all $f, g \in \mathcal{L}ie(V)$, $\psi(f * g) = \psi(f) \star \psi(g)$. □

Corollary 1 *Let V be a vector space. The free magmatic algebra generated by V is denoted by $\mathcal{M}ag(V)$. Then $\mathcal{L}ie(\mathcal{M}ag(V))$ is the free post-Lie algebra generated by V .*

Proof. Let L be a post-Lie algebra and let ϕ be a linear map from V to L . From the universal property of $\mathcal{M}ag(V)$, there exists a unique morphism of magmatic algebras from $\mathcal{M}ag(V)$ to L extending ϕ ; from the universal property of $\mathcal{L}ie(\mathcal{M}ag(V))$, this morphism can be uniquely extended as a morphism of post-Lie algebras from $\mathcal{L}ie(\mathcal{M}ag(V))$ to L . So $\mathcal{L}ie(\mathcal{M}ag(V))$ satisfies the required universal property to be a post-Lie algebra generated by V . □

Remark. Describing the free magmatic algebra generated by V in terms of planar rooted trees with a grafting operation, we get back the construction of free post-Lie algebras of [6].

1.3 Enveloping algebra of a post-Lie algebra

Let $(V, \{-, -\}, *)$ be a post-Lie algebra. We extend $*$ onto $T(V)$ as previously in Proposition 1. The usual bracket of $\mathcal{L}ie(V) \subseteq T(V)$ is denoted by $[f, g] = fg - gf$, and should not be confused with the bracket $\{-, -\}$ of the post-Lie algebra V .

Lemma 2 *Let I be the two-sided ideal of $T(V)$ generated by the elements $xy - yx - \{x, y\}$, $x, y \in V$. Then $I * T(V) \subseteq I$ and $T(V) * I = (0)$.*

Proof. *First step.* Let us prove that for all $x, y \in V$, for all $h \in T(V)$:

$$\{x, y\} * h = \sum \left\{ x * h^{(1)}, y * h^{(2)} \right\}.$$

Note that the second member of this formula makes sense, as $V * T(V) \subseteq V$ by Lemma 1. We assume that $h = z_1 \dots z_n$ and we work by induction on n . If $n = 0$, then $h = 1$ and $\{x, y\} * 1 = \{x, y\} = \{x * 1, y * 1\}$. If $n = 1$, then $h \in V$, so $\Delta_{\square}(h) = h \otimes 1 + 1 \otimes h$.

$$\{x, y\} * h = \{x * h, y\} + \{x, y * h\} = \{x * h, y * 1\} + \{x * 1, y * h\} = \sum \{x * h^{(1)}, y * h^{(2)}\}.$$

If $n \geq 2$, we put $h_1 = z_1 \dots z_{n-1}$ and $h_2 = z_n$. The induction hypothesis holds for h_1, h_2 and $h_1 * h_2$:

$$\begin{aligned} \{x, y\} * h &= (\{x, y\} * h_1) * h_2 - \{x, y\} * (h_1 * h_2) \\ &= \sum \left\{ x * h_1^{(1)}, y * h_1^{(2)} \right\} * h_2 - \sum \left\{ x * (h_1 * h_2)^{(1)}, y * (h_1 * h_2)^{(2)} \right\} \\ &= \sum \left\{ (x * h_1^{(1)}) * h_2^{(1)}, (y * h_1^{(2)}) * h_2^{(2)} \right\} - \sum \left\{ x * (h_1^{(1)} * h_2^{(1)}), y * (h_1^{(2)} * h_2^{(2)}) \right\} \\ &= \sum \left\{ (x * h_1^{(1)}) * h_2, y * h_1^{(2)} \right\} + \sum \left\{ x * h_1^{(1)}, (y * h_1^{(2)}) * h_2 \right\} \\ &\quad - \sum \left\{ x * (h_1^{(1)} * h_2), y * h_1^{(2)} \right\} - \sum \left\{ x * h_1^{(1)}, y * (h_1^{(2)} * h_2) \right\} \\ &= \sum \left\{ (x * h_1^{(1)}) * h_2 - x * (h_1^{(1)} * h_2), y * h_1^{(2)} \right\} \\ &\quad + \sum \left\{ x * h_1^{(1)}, (y * h_1^{(2)}) * h_2 - y * (h_1^{(2)} * h_2) \right\} \\ &= \sum \left\{ x * (h_1^{(1)} h_2), y * h_1^{(2)} \right\} + \sum \left\{ x * h_1^{(1)}, y * (h_1^{(2)} h_2) \right\} \\ &= \sum \left\{ x * h^{(1)}, y * h^{(2)} \right\}. \end{aligned}$$

Consequently, the result holds for all $h \in T(V)$.

Second step. Let $J = Vect(xy - yx - \{x, y\} \mid x, y \in V)$. For all $x, y \in V$, for all $h \in T(V)$, by the first step:

$$(xy - yx - \{x, y\}) * h = \sum \left(x * h^{(1)} \right) \left(y * h^{(2)} \right) - \left(y * h^{(1)} \right) \left(y * h^{(2)} \right) - \left\{ x * h^{(1)}, y * h^{(2)} \right\} \in J.$$

So $J * T(V) \subseteq J$. If $g \in J, f_1, f_2, h \in T(V)$:

$$(f_1 g f_2) * h = \sum \left(f_1 * d^{(1)} \right) \underbrace{\left(g * h^{(2)} \right)}_{\in J} \left(f_2 * h^{(3)} \right) \in I.$$

So $I * T(V) \subseteq I$.

Let us prove that $T(V) * (T(V)JV^{\otimes n}) = (0)$ for all $n \geq 0$. We start with $n = 0$. First, $1 * (T(V)J) = \varepsilon(T(V)J) = (0)$. Let $x, y, z \in V, g \in T(V)$. Then:

$$\begin{aligned}
& x * (gyz - gzy - g\{y, z\}) \\
&= (x * (gy)) * z - x * ((gy) * z) - (x * (gz)) * y + x * ((gz) * y) \\
&\quad - (x * g) * \{y, z\} + x * (g * \{y, z\}) \\
&= ((x * g) * y) * z - (x * (g * y)) * z - x * ((g * z)y) \\
&\quad - x * (g(y * z)) - ((x * g) * z) * y - (x * (g * z)) * y \\
&\quad + x * ((g * y)z) + x * (g(z * y)) - (x * g) * \{y, z\} + x * (g * \{y, z\}) \\
&= ((x * g) * y) * z - (x * (g * y)) * z - (x * (g * z)) * y + x * ((g * z) * y) \\
&\quad - (x * g) * (y * z) + x * (g * (y * z)) - ((x * g) * z) * y + (x * (g * z)) * y \\
&\quad (x * (g * y)) * z - x * ((g * y) * z) + (x * g) * (z * y) - x * (g * (z * y)) \\
&\quad - (x * g) * \{y, z\} + x * (g * \{y, z\}) \\
&= x * ((g * z) * y) + x * (g * (y * z)) - x * ((g * y) * z) - x * (g * (z * y)) + x * (g * \{y, z\}) \\
&\quad + ((x * g) * y) * z - (x * g) * (y * z) - ((x * g) * z) * y + (x * g) * (z * y) - (x * g) * \{y, z\} \\
&= 0 + 0.
\end{aligned}$$

So $V * (T(V)J) = (0)$. As the elements of J are primitive, $T(V)J$ is a coideal. If $n \geq 1$, $x_1, \dots, x_n \in V$ and $g \in T(V)J$, we put $\Delta_{\square}^{(n-1)}(g) = \sum g^{(1)} \otimes \dots \otimes g^{(n)}$, with at least one $g_i \in T(V)J$. Then $(x_1 \dots x_n) * g = \sum (x_1 * g^{(1)}) \dots (x_n * g^{(n)}) = 0$, so $T(V) * (T(V)J) = (0)$.

If $n \geq 1$, we take $f \in T(V)$, $g \in T(V)JV^{\otimes(n-1)}$ and $y \in V$. We put $g = g_1g_2g_3$, with $g_1 \in T(V)$, $g_2 \in J$, $g_3 \in V^{\otimes(n-1)}$. Then:

$$g * y = (g_1 * y)g_2g_3 + g_1 \underbrace{(g_2 * y)}_{\in J * T(V) \subseteq J} g_3 + g_1g_2 \underbrace{(g_3 * y)}_{\in V^{\otimes n}} \in T(V)JV^{\otimes n}.$$

So the induction hypothesis holds for g and for $g * y$. Then $f * (gy) = (f * g) * y - f * (g * y) = 0$. So $T(V) * I = (0)$. \square

As a consequence, the quotient $T(V)/I$ inherits a magmatic product $*$. Moreover, I is a Hopf ideal, and this implies that it is also a two-sided ideal for \otimes . As $T(V)/I$ is the enveloping algebra $\mathcal{U}(V, \{-, -\})$, we obtain Proposition 3.1 of [2]:

Proposition 4 *Let $(\mathfrak{g}, \{-, -\}, *)$ be a post-Lie algebra. Its magmatic product can be uniquely extended to $\mathcal{U}(\mathfrak{g})$ such that for all $f, g, h \in \mathcal{U}(\mathfrak{g})$, $x, y \in \mathfrak{g}$:*

- $f * 1 = f$.
- $1 * f = \varepsilon(f)1$.
- $f * (gy) = (f * g) * y - f * (g * y)$.
- $(fg) * h = \sum (f * h^{(1)}) (g * h^{(2)})$, where $\Delta(h) = \sum h^{(1)} \otimes h^{(2)}$ is the usual coproduct of $\mathcal{U}(\mathfrak{g})$.

We define a product \otimes on $\mathcal{U}(\mathfrak{g})$ by $f * g = \sum (f * g^{(1)}) g^{(2)}$. Then $(\mathcal{U}(\mathfrak{g}), \otimes, \Delta)$ is a Hopf algebra, isomorphic to $\mathcal{U}(\mathfrak{g}, \{-, -\}_*)$.

Proof. By Cartier-Quillen-Milnor-Moore's theorem, $(\mathcal{U}(\mathfrak{g}), \otimes, \Delta)$ is an enveloping algebra; the underlying Lie algebra is $\text{Prim}(\mathcal{U}(\mathfrak{g})) = \mathfrak{g}$, with the Lie bracket defined by:

$$\{x, y\}_{\otimes} = x \otimes y - y \otimes x = xy + x * y - yx - y * x.$$

This is the bracket $\{-, -\}_*$. \square

Remarks.

1. If \mathfrak{g} is a post-Lie algebra with $\{-, -\} = 0$, it is a pre-Lie algebra, and $\mathcal{U}(\mathfrak{g}) = S(\mathfrak{g})$. We obtain again the Oudom-Guin construction [7, 8].
2. By Lemma 1, $(\mathcal{U}(\mathfrak{g}), *)$ is a right $(\mathcal{U}(\mathfrak{g}), \otimes)$ -module. By restriction, $(\mathfrak{g}, *)$ is also a right $(\mathcal{U}(\mathfrak{g}), \otimes)$ -module.

1.4 The particular case of associative algebras

Let (V, \triangleleft) be an associative algebra. The associated Lie bracket is denoted by $[-, -]_{\triangleleft}$. As $(V, 0, \triangleleft)$ is post-Lie, the construction of the enveloping algebra of $(V, [-, -]_{\triangleleft})$ can be done: we obtain a product \triangleleft defined on $S(V)$ and an associative product \blacktriangleleft making $(S(V), \blacktriangleleft, \Delta)$ a Hopf algebra, isomorphic to the enveloping algebra of $(V, [-, -]_{\triangleleft})$.

Lemma 3 *If $x_1, \dots, x_k, y_1, \dots, y_l \in V$:*

$$x_1 \dots x_k \triangleleft y_1 \dots y_l = \sum_{\theta: [l] \rightarrow [k]} \left(\prod_{i \notin \text{Im}(\theta)} x_i \right) \left(\prod_{i=1}^k x_{\theta(i)} \triangleleft y_i \right),$$

$$x_1 \dots x_k \blacktriangleleft y_1 \dots y_l = \sum_{I \subseteq [l]} \sum_{\theta: I \rightarrow [k]} \left(\prod_{i \notin \text{Im}(\theta)} x_i \right) \left(\prod_{j \notin I} y_j \right) \left(\prod_{i \in I} x_{\theta(i)} \triangleleft y_i \right).$$

Proof. We first prove that for all $k \geq 2$, $x, y_1, \dots, y_k \in V$, $x \triangleleft y_1 \dots y_k = 0$. We proceed by induction on k . For $k = 2$, $x \triangleleft y_1 y_2 = (x \triangleleft y_1) \triangleleft y_2 - x \triangleleft (y_1 \triangleleft y_2) = 0$, as \triangleleft is associative. Let us assume the result at rank k . Then:

$$x \triangleleft y_1 \dots y_{k+1} = (x \triangleleft y_1 \dots y_k) \triangleleft y_{k+1} - \sum_{i=1}^k x \triangleleft (y_1 \dots (y_i \triangleleft y_{k+1}) \dots y_k) = 0.$$

Let us now prove the formula for \triangleleft .

$$x_1 \dots x_k \triangleleft y_1 \dots y_l = \sum_{[l] = I_1 \sqcup \dots \sqcup I_k} \left(x_1 \triangleleft \prod_{i \in I_1} y_i \right) \dots \left(x_k \triangleleft \prod_{i \in I_k} y_i \right).$$

Moreover, for all j :

$$x_j \triangleleft \prod_{i \in I_j} y_i = \begin{cases} x_j & \text{if } I_j = \emptyset, \\ x_j \triangleleft y_p & \text{if } I_j = \{p\}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence:

$$\begin{aligned} x_1 \dots x_k \triangleleft y_1 \dots y_l &= \sum_{\substack{[l] = I_1 \sqcup \dots \sqcup I_k \\ \forall p, |I_p| \leq 1}} \left(x_1 \triangleleft \prod_{i \in I_1} y_i \right) \dots \left(x_k \triangleleft \prod_{i \in I_k} y_i \right) \\ &= \sum_{\theta: [l] \rightarrow [k]} \left(\prod_{i \notin \text{Im}(\theta)} x_i \right) \left(\prod_{i=1}^k x_{\theta(i)} \triangleleft y_i \right). \end{aligned}$$

Finally:

$$x_1 \dots x_k \blacktriangleleft y_1 \dots y_l = \sum_{I \subseteq [l]} \left(\prod_{i \notin I} y_i \right) x_1 \dots x_k \triangleleft \left(\prod_{i \in I} y_i \right),$$

as announced. □

Examples. Let $x_1, x_2, y_1, y_2 \in V$.

$$\begin{aligned} x_1 \blacktriangleleft y_1 &= x_1 y_1 + x_1 \triangleleft y_1, \\ x_1 x_2 \blacktriangleleft y_1 &= x_1 x_2 y_1 + (x_1 \triangleleft y_1) x_2 + x_1 (x_2 \triangleleft y_1), \\ x_1 \blacktriangleleft y_1 y_2 &= x_1 y_1 y_2 + (x_1 \triangleleft y_1) y_2 + (x_1 \triangleleft y_2) y_1, \\ x_1 x_2 \blacktriangleleft y_1 y_2 &= x_1 x_2 y_1 y_2 + (x_1 \triangleleft y_1) x_2 y_2 + (x_1 \triangleleft y_2) x_2 y_1 + x_1 (x_2 \triangleleft y_1) y_2 \\ &\quad + x_1 (x_2 \triangleleft y_2) y_1 + (x_1 \triangleleft y_1) (x_2 \triangleleft y_2) + (x_1 \triangleleft y_2) (x_2 \triangleleft y_1). \end{aligned}$$

Remark. The number of terms in $x_1 \dots x_k \triangleleft y_1 \dots y_l$ is:

$$\sum_{i=0}^{\min(k,l)} \binom{l}{i} \binom{k}{i} i!,$$

see sequences A086885 and A176120 of [9].

2 A family of solvable Lie algebras

2.1 Definition

Definition 2 Let us fix $a = (a_1, \dots, a_N) \in \mathbb{K}^N$. We define an associative product \triangleleft on \mathbb{K}^N :

$$\forall i, j \in [N], \epsilon_i \triangleleft \epsilon_j = a_j \epsilon_i.$$

The associated Lie bracket is denoted by $[-, -]_a$:

$$\forall i, j \in [N], [\epsilon_i, \epsilon_j]_a = a_j \epsilon_i - a_i \epsilon_j.$$

This Lie algebra is denoted by \mathfrak{g}_a .

Remarks.

1. Let $A \in M_{N,M}(\mathbb{K})$, and $a \in \mathbb{K}^N$. The following map is a Lie algebra morphism:

$$\begin{cases} \mathfrak{g}_a \cdot {}^t A & \longrightarrow \mathfrak{g}_a \\ x & \longrightarrow Ax. \end{cases}$$

Consequently, if $a \neq (0, \dots, 0)$, \mathfrak{g}_a is isomorphic to $\mathfrak{g}_{(1,0,\dots,0)}$.

2. These Lie algebras \mathfrak{g}_a are characterized by the following property: if \mathfrak{g} is a n -dimensional Lie algebra such that any 2-dimensional subspace of \mathfrak{g} is a Lie subalgebra, there exists $a \in \mathbb{K}^n$ such that \mathfrak{g} and \mathfrak{g}_a are isomorphic.

Definition 3 Let $A = T(V)^N$. The elements of A will be denoted by:

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} = f_1 \epsilon_1 + \dots + f_N \epsilon_N.$$

For all $i, j \in [N]$, we define bilinear products ${}_i \sqcup$ and \sqcup_j :

$$\forall f, g \in T(V)^N, \quad f {}_i \sqcup g = \begin{pmatrix} f_i \sqcup g_1 \\ \vdots \\ f_i \sqcup g_N \end{pmatrix}, \quad f \sqcup_j g = \begin{pmatrix} f_1 \sqcup g_j \\ \vdots \\ f_N \sqcup g_j \end{pmatrix}.$$

In other words, if $f, g \in T(V)$, for all $k, l \in [N]$:

$$f \epsilon_k {}_i \sqcup g \epsilon_l = \delta_{i,k} (f \sqcup g) \epsilon_l, \quad f \epsilon_k \sqcup_j g \epsilon_l = \delta_{j,l} (f \sqcup g) \epsilon_k.$$

If $a = (a_1, \dots, a_N) \in \mathbb{K}^N$, we put ${}_a \sqcup = a_1 {}_1 \sqcup + \dots + a_N {}_N \sqcup$ and $\sqcup_a = a_1 \sqcup_1 + \dots + a_N \sqcup_N$.

Proposition 5 Let $f, g \in \mathbb{K}^N$. For all $f, g, h \in A$:

$$\begin{aligned} (f \sqcup_a g) \sqcup_b h &= f \sqcup_a (g \sqcup_b h), & (f \sqcup_a g)_b \sqcup h &= f \sqcup_a (g_b \sqcup h), \\ (f_a \sqcup g) \sqcup_b h &= f_a \sqcup (g \sqcup_b h), & (f_a \sqcup g)_b \sqcup h &= f_a \sqcup (g_b \sqcup h), \\ f \sqcup_a g &= g_a \sqcup f. \end{aligned}$$

Proof. Direct verifications, using the associativity and the commutativity of \sqcup . \square

Definition 4 Let $a \in \mathbb{K}^N$. We define a Lie bracket on A :

$$\forall f, g \in A, \quad {}_a[f, g] = f_a \sqcup g - g_a \sqcup f = g \sqcup_a f - f \sqcup_a g.$$

This Lie algebra is denoted by \mathfrak{g}'_a .

Remark. If A is an associative commutative algebra and \mathfrak{g} is a Lie algebra, then $A \otimes \mathfrak{g}$ is a Lie algebra, with the following Lie bracket:

$$\forall f, g \in A, x, y \in \mathfrak{g}, \quad [f \otimes x, g \otimes y] = fg \otimes [x, y].$$

Then, as a Lie algebra, \mathfrak{g}'_a is isomorphic to the tensor product of the associative commutative algebra $(T(V), \sqcup)$, and of the Lie algebra \mathfrak{g}_{-a} . Consequently, if $a \neq (0, \dots, 0)$, \mathfrak{g}'_a is isomorphic to $\mathfrak{g}'_{(1,0,\dots,0)}$.

2.2 Enveloping algebra of \mathfrak{g}_a

Let us apply Lemma 3 to the Lie algebra \mathfrak{g}_a :

Proposition 6 The symmetric algebra $S(\mathfrak{g}_a)$ is given an associative product \blacktriangleleft such that for all $i_1, \dots, i_k, j_1, \dots, j_l \in [N]$:

$$\epsilon_{i_1} \dots \epsilon_{i_k} \blacktriangleleft \epsilon_{j_1} \dots \epsilon_{j_l} = \sum_{I \subseteq [l]} k(k-1) \dots (k-|I|+1) \left(\prod_{q \in I} a_{j_q} \right) \left(\prod_{p \notin I} \epsilon_{j_p} \right) \epsilon_{i_1} \dots \epsilon_{i_k}.$$

The Hopf algebra $(S(\mathfrak{g}_a), \blacktriangleleft, \Delta)$ is isomorphic to the enveloping algebra of \mathfrak{g}_a .

The enveloping algebra of \mathfrak{g}_a has two distinguished bases, the Poincaré-Birkhoff-Witt basis and the monomial basis:

$$(\epsilon_{i_1} \blacktriangleleft \dots \blacktriangleleft \epsilon_{i_k})_{k \geq 0, 1 \leq i_1 \leq \dots \leq i_k \leq N}, \quad (\epsilon_{i_1} \dots \epsilon_{i_k})_{k \geq 0, 1 \leq i_1 \leq \dots \leq i_k \leq N}.$$

Here is the passage between them.

Proposition 7 Let us fix $n \geq 1$. For all $I = \{i_1 < \dots < i_k\} \subseteq [n]$, we put:

$$\lambda(I) = (i_1 - 1) \dots (i_k - k), \quad \mu(I) = (-1)^k (i_1 - 1) i_2 (i_3 + 1) \dots (i_k + k - 2).$$

We use the following notation: if $[n] \setminus I = \{q_1 < \dots < q_l\}$, $\overset{\blacktriangleleft}{\prod}_{q \notin I} \epsilon_{i_q} = \epsilon_{i_{q_1}} \blacktriangleleft \dots \blacktriangleleft \epsilon_{i_{q_l}}$. Then:

$$\begin{aligned} \epsilon_{i_1} \blacktriangleleft \dots \blacktriangleleft \epsilon_{i_n} &= \sum_{I \subseteq [n]} \lambda(I) \left(\prod_{p \in I} a_{i_p} \right) \left(\prod_{q \notin I} \epsilon_{i_q} \right), \\ \epsilon_{i_1} \dots \epsilon_{i_n} &= \sum_{I \subseteq [n]} \mu(I) \left(\prod_{p \in I} a_{i_p} \right) \left(\overset{\blacktriangleleft}{\prod}_{q \notin I} \epsilon_{i_q} \right). \end{aligned}$$

Proof. *First step.* Let us prove the first formula by induction on n . It is obvious if $n = 1$, as $\lambda(\emptyset) = 1$ and $\lambda(\{1\}) = 0$. Let us assume the result at rank n .

$$\begin{aligned}
\epsilon_{i_1} \blacktriangleleft \dots \blacktriangleleft \epsilon_{i_{n+1}} &= \sum_{I \subseteq [n]} \lambda(I) \left(\prod_{p \in I} a_{i_p} \right) \left(\prod_{q \notin I} \epsilon_{i_q} \right) \blacktriangleleft \epsilon_{i_{n+1}} \\
&= \sum_{I \subseteq [n]} \lambda(I) \left(\prod_{p \in I} a_{i_p} \right) \left(\prod_{q \notin I} \epsilon_{i_q} \epsilon_{i_{n+1}} + (k - |I|) a_{i_{n+1}} \prod_{q \notin I} \epsilon_{i_q} \right) \\
&= \sum_{\substack{I \subseteq [n+1], \\ n+1 \notin I}} \lambda(I) \left(\prod_{p \in I} a_{i_p} \right) \left(\prod_{q \notin I} \epsilon_{i_p} \right) + \sum_{\substack{I \subseteq [n+1], \\ n+1 \in I}} \lambda(I) \left(\prod_{p \in I} a_{i_p} \right) \left(\prod_{q \notin I} \epsilon_{i_p} \right) \\
&= \sum_{I \subseteq [n+1]} \lambda(I) \left(\prod_{p \in I} a_{i_p} \right) \left(\prod_{q \notin I} \epsilon_{i_p} \right).
\end{aligned}$$

Second step. Let us prove that for all $I \subseteq [n]$, $\sum_{J \subseteq I} \lambda(J) \mu(I \setminus J) = \delta_{I, \emptyset}$.

We put $I = \{i_1 < \dots < i_k\}$ and we proceed by induction on k . As $\lambda(\emptyset) = \mu(\emptyset) = 1$, the result is obvious at rank $k = 0$ and $k = 1$. Let us assume the result at rank $k - 1$, with $k \geq 2$.

$$\begin{aligned}
\sum_{J \subseteq I} \lambda(J) \mu(I \setminus J) &= \sum_{\substack{J \subseteq I, \\ i_k \in J}} \lambda(J) \mu(I \setminus J) + \sum_{\substack{J \subseteq I, \\ i_k \notin J}} \lambda(J) \mu(I \setminus J) \\
&= \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J \cup \{i_k\}) \mu(I \setminus \{i_k\} \setminus J) + \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J) \mu(I \setminus J) \\
&= \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J) (i_k - |J|) \mu(I \setminus \{i_k\} \setminus J) \\
&\quad - \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J) \mu(I \setminus \{i_k\} \setminus J) (i_k + |I \setminus \{i_k\} \setminus J| + 1) \\
&= \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J) \mu(I \setminus \{i_k\} \setminus J) (i_k - |J| - i_k - |I| + 1 + |J| - 1) \\
&= -|I| \sum_{J \subseteq I \setminus \{i_k\}} \lambda(J) \mu(I \setminus \{i_k\} \setminus J) \\
&= 0.
\end{aligned}$$

Therefore:

$$\begin{aligned}
\sum_{I \subseteq [n]} \mu(I) \left(\prod_{p \in I} a_{i_p} \right) \left(\prod_{q \notin I} \epsilon_{i_q} \right) &= \sum_{I \subseteq [n]} \sum_{J \subseteq [n] \setminus I} \mu(I) \lambda(J) \left(\prod_{p \in I} a_{i_p} \right) \left(\prod_{p \in J} a_{i_p} \right) \left(\prod_{q \in [n] \setminus I \setminus J} \epsilon_{i_q} \right) \\
&= \sum_{A \sqcup B \sqcup C = [n]} \mu(A) \lambda(B) \left(\prod_{p \in A \sqcup B} a_{i_p} \right) \left(\prod_{q \in C} \epsilon_{i_q} \right) \\
&= \sum_{I \sqcup J = [n]} \left(\underbrace{\sum_{I' \subseteq I} \lambda(I') \mu(I \setminus I')}_{=\delta_{I, \emptyset}} \right) \left(\prod_{p \in I} a_{i_p} \right) \left(\prod_{q \in J} \epsilon_{i_q} \right) \\
&= \epsilon_{i_1} \dots \epsilon_{i_n},
\end{aligned}$$

which ends the proof. □

2.3 Modules over $\mathfrak{g}_{(1,0,\dots,0)}$

Proposition 8 1. Let V be a module over the associative (non unitary) algebra $(\mathfrak{g}_{(1,0,\dots,0)}, \triangleleft)$. Then $V = V^{(0)} \oplus V^{(1)}$, with:

- $\epsilon_1.v = v$ if $v \in V^{(1)}$ and $\epsilon_1.v = 0$ if $v \in V^{(0)}$.
- For all $i \geq 2$, $\epsilon_i.v \in V^{(0)}$ if $v \in V^{(1)}$ and $\epsilon_i.v = 0$ if $v \in V^{(0)}$.

2. Conversely, let $V = V^{(1)} \oplus V^{(0)}$ be a vector space and let $f_i : V^{(1)} \rightarrow V^{(0)}$ for all $2 \leq i \leq N$. One defines a structure of $(\mathfrak{g}_{(1,0,\dots,0)}, \triangleleft)$ -module over V :

$$\epsilon_1.v = \begin{cases} v & \text{if } v \in V^{(1)}, \\ 0 & \text{if } v \in V^{(0)}; \end{cases} \quad \text{if } i \geq 2, \epsilon_i.v = \begin{cases} f_i(v) & \text{if } v \in V^{(1)}, \\ 0 & \text{if } v \in V^{(0)}. \end{cases}$$

Shortly:

$$\epsilon_1 : \begin{bmatrix} 0 & 0 \\ 0 & Id \end{bmatrix}, \quad \forall i \geq 2, \epsilon_i : \begin{bmatrix} 0 & f_i \\ 0 & 0 \end{bmatrix}.$$

Proof. Note that in $\mathfrak{g}_{(1,0,\dots,0)}$, $\epsilon_i \triangleleft \epsilon_j = \delta_{1,j} \epsilon_i$.

1. In particular, $\epsilon_1 \triangleleft \epsilon_1 = \epsilon_1$. If $F_1 : V \rightarrow V$ is defined by $F_1(v) = \epsilon_1.v$, then:

$$F_1 \circ F_1(v) = \epsilon_1.(\epsilon_1.v) = (\epsilon_1 \triangleleft \epsilon_1).v = \epsilon.v = F_1(v),$$

so F_1 is a projection, which implies the decomposition of V as $V^{(0)} \oplus V^{(1)}$. Let $x \in V^{(1)}$ and $i \geq 2$. Then $F_1(\epsilon_i.v) = \epsilon_i.(\epsilon_i.v) = (\epsilon_1 \triangleleft \epsilon_i).v = 0$, so $\epsilon_i.v \in V^{(0)}$. Let $x \in V^{(0)}$. Then $\epsilon_i.v = (\epsilon_i \triangleleft \epsilon_1).v = \epsilon_i.F_1(v) = 0$, so $\epsilon_i.v = 0$.

2. Let $i \geq 2$ and $j \in [N]$. If $v \in V^{(1)}$:

$$\epsilon_1.(\epsilon_1.v) = v = \epsilon_1.v, \quad \epsilon_i.(\epsilon_1.v) = f_i(v) = \epsilon_i.v, \quad \epsilon_j.(\epsilon_i.v) = \epsilon_j.f_i(v) = 0.v.$$

If $v \in V^{(0)}$:

$$\epsilon_1.(\epsilon_1.v) = 0 = \epsilon_1.v, \quad \epsilon_i.(\epsilon_1.v) = 0 = \epsilon_i.v, \quad \epsilon_j.(\epsilon_i.v) = 0 = 0.v.$$

So V is indeed a $(\mathfrak{g}_{(1,0,\dots,0)}, \triangleleft)$ -module. □

Example. There are, up to an isomorphism, three indecomposable $(\mathfrak{g}_{(1,0)}, \triangleleft)$ -modules:

$$\begin{array}{c|c|c|c} \epsilon_1 & (0) & (1) & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \hline \epsilon_2 & (0) & (0) & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array}$$

Proposition 9 (We assume \mathbb{K} algebraically closed). Let V be an indecomposable finite-dimensional module over the Lie algebra $\mathfrak{g}_{(1,0,\dots,0)}$. There exists a scalar λ and a decomposition:

$$V = V^{(0)} \oplus \dots \oplus V^{(k)}$$

such that, for all $0 \leq p \leq k$:

- $\epsilon_1(V^{(p)}) \subseteq V^{(p)}$ and there exists $n \geq 1$ such that $(\epsilon_1 - (\lambda + p)Id)_{|V^{(p)}}^n = (0)$.
- If $i \geq 2$, $\epsilon_i(V^{(p)}) \subseteq V^{(p-1)}$, with the convention $V^{(-1)} = (0)$.

Proof. First, observe that in the enveloping algebra of $\mathfrak{g}_{(1,0,\dots,0)}$, if $i \geq 2$ and $\lambda \in \mathbb{K}$:

$$\epsilon_i \blacktriangleleft (\epsilon_1 - \lambda) = \epsilon_i \epsilon_1 + \epsilon_i - \lambda \epsilon_i = \epsilon_i \epsilon_1 + (1 - \lambda) \epsilon_i = (\epsilon_1 - \lambda + 1) \blacktriangleleft \epsilon_i.$$

Therefore, for all $i \geq 2$, for all $n \in \mathbb{N}$, for all $\lambda \in \mathbb{K}$:

$$\epsilon_i \blacktriangleleft (\epsilon_1 - \lambda) \blacktriangleleft^n = (\epsilon_1 - \lambda + 1) \blacktriangleleft^n \blacktriangleleft \epsilon_i.$$

Let V be a finite-dimensional module over the Lie algebra $\mathfrak{g}_{(1,0,\dots,0)}$. We denote by E_λ the characteristic subspace of eigenvalue λ for the action of ϵ_1 . Let us prove that for all $\lambda \in \mathbb{K}$, if $i \geq 2$, $\epsilon_i(E_\lambda) \subseteq E_{\lambda-1}$. If $x \in E_\lambda$, there exists $n \geq 1$, such that $(\epsilon_1 - \lambda Id) \blacktriangleleft^n . x = 0$. Hence:

$$0 = \epsilon_i . ((\epsilon_1 - \lambda Id)^n . x) = (\epsilon_1 - (\lambda - 1) Id)^n . (\epsilon_i . x),$$

so $\epsilon_i \in E_{\lambda-1}$.

Let us take now V an indecomposable module, and let Λ be the spectrum of the action of ϵ_1 . The group \mathbb{Z} acts on \mathbb{K} by translation. We consider $\Lambda' = \Lambda + \mathbb{Z}$ and let Λ'' be a system of representants of the orbits of Λ' . Then:

$$V = \bigoplus_{\lambda \in \Lambda''} \underbrace{\left(\bigoplus_{n \in \mathbb{Z}} E_{\lambda+n} \right)}_{V_\lambda}.$$

By the preceding remarks, V_λ is a module. As V is indecomposable, Λ'' is reduced to a single element. As the spectrum of ϵ_1 is finite, it is included in a set of the form $\{\lambda, \lambda + 1, \dots, \lambda + k\}$. We then take $V^{(p)} = E_{\lambda+p}$ for all p . \square

Example. Let us give the indecomposable modules of $\mathfrak{g}_{(1,0)}$ of dimension ≤ 3 . For any $\lambda \in \mathbb{K}$:

ϵ_1	ϵ_2	ϵ_1	ϵ_2
(λ)	(0)	$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda+1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda+1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda+1 & 1 \\ 0 & 0 & \lambda+1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda+1 & 0 \\ 0 & 0 & \lambda+2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Definition 5 Let V be a module over the Lie algebra \mathfrak{g}_a . The associated algebra morphism is:

$$\phi_V : \begin{cases} \mathcal{U}(\mathfrak{g}_a) = (S(\mathfrak{g}_a), \blacktriangleleft) & \longrightarrow & \text{End}(V) \\ \epsilon_i & \longrightarrow & \begin{cases} V & \longrightarrow & V \\ v & \longrightarrow & \epsilon_i . v. \end{cases} \end{cases}$$

For all $i_1, \dots, i_k \in [N]$, we put $F_{i_1, \dots, i_k} = \phi_V(\epsilon_{i_1} \dots \epsilon_{i_k})$; this does not depend on the order on the indices i_p .

By Proposition 7:

Proposition 10 For all $i_1, \dots, i_n \in [N]$:

$$F_{i_1} \circ \dots \circ F_{i_n} = \sum_{\substack{I \subseteq [n], \\ I \setminus J = \{j_1 < \dots < j_l\}}} \lambda(I) \left(\prod_{p \in I} a_{i_p} \right) F_{i_{j_1}, \dots, i_{j_l}},$$

$$F_{i_1, \dots, i_n} = \sum_{\substack{I \subseteq [n], \\ I \setminus J = \{j_1 < \dots < j_l\}}} \mu(I) \left(\prod_{p \in I} a_{i_p} \right) F_{i_{j_1}} \circ \dots \circ F_{i_{j_l}}.$$

When V is a module over the associative algebra $(\mathfrak{g}_A, \triangleleft)$, these morphisms are easy to describe:

Proposition 11 Let V be a module over the associative algebra $(\mathfrak{g}_a, \triangleleft)$; it is also a module over the Lie algebra $(\mathfrak{g}_a, [-, -]_a)$. For all $k \geq 2$, $i_1, \dots, i_k \in [N]$, $F_{i_1, \dots, i_k} = 0$.

Proof. As V is a module over the associative algebra $(\mathfrak{g}_a, \triangleleft)$, for any $i_1, i_2 \in [N]$:

$$F_{i_1} \circ F_{i_2} = a_{i_2} F_{i_1}.$$

We proceed by induction on k . If $k = 2$, $\epsilon_{i_1} \epsilon_{i_2} = \epsilon_{i_1} \triangleleft \epsilon_{i_2} - a_{i_2} \epsilon_{i_1}$, so:

$$F_{i_1, i_2} = F_{i_1} \circ F_{i_2} - a_{i_2} F_{i_1} = a_{i_2} F_{i_1} - a_{i_2} F_{i_1} = 0.$$

Let us assume the result at rank k . Then $\epsilon_1 \dots \epsilon_{i_{k+1}} = \epsilon_{i_1} \dots \epsilon_{i_k} \triangleleft \epsilon_{i_{k+1}} - k a_{i_{k+1}} \epsilon_{i_1} \dots \epsilon_{i_k}$, and $F_{i_1, \dots, i_{k+1}} = F_{i_1, \dots, i_k} \circ F_{i_{k+1}} - k a_{i_{k+1}} F_{i_1, \dots, i_k} = 0$. \square

3 A family of post-Lie algebras

3.1 Reminders

We defined in [3] a family of pre-Lie algebras, associated to endomorphisms. Let us briefly recall this construction.

Proposition 12 Let V be a vector space and $F : V \rightarrow V$ be an endomorphism. We define a product $*$ on $T(V)$: for all $f, g \in T(V)$, for all $x \in V$,

$$\emptyset * g = 0, \quad x f * g = x(f * g) + F(x)(f \sqcup g).$$

This product is pre-Lie. The pre-Lie algebra $(T(V), *)$ is denoted by $T(V, F)$. Moreover, for all $f, g, h \in T(V, F)$:

$$(f \sqcup g) * h = (f * h) \sqcup g + f \sqcup (g * h).$$

We also proved the following result:

Proposition 13 Let $k, l \geq 0$.

- The set $Sh(k, l)$ of (k, l) -shuffles is the set of permutation $\sigma \in \mathfrak{S}_{k+l}$ such that $\sigma(1) < \dots < \sigma(k)$ and $\sigma(k+1) < \dots < \sigma(k+l)$.
- If $\sigma \in Sh(k, l)$, we put $m_k(\sigma) = \max\{i \in [k] \mid \sigma(1) = 1, \dots, \sigma(i) = i\}$. In particular, if $\sigma(1) \neq 1$, $m_k(\sigma) = 0$.

For all $x_1, \dots, x_k, y_1, \dots, y_l \in V$:

$$x_1 \dots x_k * y_1 \dots y_l = \sum_{\sigma \in Sh(k, l)} \sum_{p=1}^{m_k(\sigma)} \left(Id^{\otimes(p)} \otimes F \otimes Id^{\otimes(k+l-p)} \right) \sigma.(x_1 \dots x_k y_1 \dots y_l).$$

3.2 Construction

Let us fix a vector space V , a family of N endomorphisms (F_1, \dots, F_N) of V and $a = (a_1, \dots, a_N) \in \mathbb{K}^N$. We define inductively a product $*$ on $T(V)^N$: for all $f, g \in T(V)^N$, $x \in V$, $i \in [N]$,

$$\emptyset \epsilon_i * g = 0, \quad xf * g = x(f * g) + F_1(x)(f \sqcup_1 g) + \dots + F_N(x)(f \sqcup_N g).$$

We define a second product \bullet on $T(V)^N$:

$$\forall f, g \in T(V)^N, f \bullet g = f * g + f \sqcup_a g.$$

Examples. Let $x, y, z \in V$, $g \in T(V)$, $i, j \in [N]$. Then:

$$\begin{aligned} x\epsilon_i * g\epsilon_j &= F_j(x)g\epsilon_j, \\ xy\epsilon_i * g\epsilon_j &= (xF_j(y)g + F_j(x)(y \sqcup g))\epsilon_i, \\ xyz\epsilon_i * g\epsilon_j &= (xyF_j(z)g + xF_j(y)(z \sqcup g) + F_j(x)(yz \sqcup g))\epsilon_i. \end{aligned}$$

Proposition 14 *Let $x_1, \dots, x_k, y_1, \dots, y_l \in V$, $i, j \in [N]$.*

$$x_1 \dots x_k \epsilon_i * y_1 \dots y_l \epsilon_j = \sum_{\sigma \in Sh(k,l)} \sum_{p=1}^{m_k(\sigma)} \left(Id^{\otimes(p_1)} \otimes F_j \otimes Id^{\otimes(k+l-p)} \right) \sigma.(x_1 \dots x_k y_1 \dots y_l) \epsilon_i.$$

Proof. By induction on k . It is immediate if $k = 0$, as both sides are equal to 0. Let us assume the result at rank $k - 1$.

$$\begin{aligned} x_1 \dots x_k \epsilon_i * y_1 \dots y_l \epsilon_j &= x_1(x_2 \dots x_k \epsilon_i * y_1 \dots y_l \epsilon_j) + F_j(x_1)(x_2 \dots x_k \sqcup y_1 \dots y_l) \epsilon_i \\ &= \sum_{\substack{\sigma \in Sh(k,l), \\ \sigma(1)=1}} \sum_{p=2}^{m_k(\sigma)} (Id^{\otimes(p_1)} \otimes F_j \otimes Id^{\otimes(k+l-p)}) \sigma.(x_1 \dots x_k y_1 \dots y_l) \epsilon_i \\ &\quad + \sum_{\substack{\sigma \in Sh(k,l), \\ \sigma(1)=1}} (F_j \otimes Id^{\otimes(k+l-1)}) \sigma.(x_1 \dots x_k y_1 \dots y_l) \epsilon_i \\ &= \sum_{\substack{\sigma \in Sh(k,l), \\ \sigma(1)=1}} \sum_{p=1}^{m_k(\sigma)} (Id^{\otimes(p_1)} \otimes F_j \otimes Id^{\otimes(k+l-p)}) \sigma.(x_1 \dots x_k y_1 \dots y_l) \epsilon_i \\ &= \sum_{\sigma \in Sh(k,l)} \sum_{p=1}^{m_k(\sigma)} (Id^{\otimes(p_1)} \otimes F_j \otimes Id^{\otimes(k+l-p)}) \sigma.(x_1 \dots x_k y_1 \dots y_l) \epsilon_i, \end{aligned}$$

so the result holds for all k . □

Remark. Let $*_j$ be the pre-Lie product of $T(V, F_j)$, described in [3]. For all $f, g \in T(V)$, for all $i, j \in [N]$:

$$f\epsilon_i * g\epsilon_j = (f *_j g)\epsilon_i.$$

Corollary 2 *For all $f, g, h \in T(V)^N$, for all $i \in [N]$:*

$$\begin{aligned} (f \sqcup_i g) * h &= (f * h) \sqcup_i g + f \sqcup_i (g * h), \\ (f \sqcup_i g) * h &= (f * h) \sqcup_i g + f \sqcup_i (g * h), \\ (f \sqcup g) * h &= (f * h) \sqcup g + f \sqcup (g * h). \end{aligned}$$

Proof. It is enough to prove these assertions for $f = f'\epsilon_k$, $g = g'\epsilon_l$ and $h = h'\epsilon_m$, with $f', g', h' \in T(V)$. For the first assertion:

$$\begin{aligned}
(f \sqcup_i g) * h &= \delta_{i,k}(f' \sqcup g' \epsilon_l) * h' \epsilon_m \\
&= \delta_{i,k}(f' \sqcup g') *_m h' \epsilon_l \\
&= \delta_{i,k}((f' *_m h') \sqcup g' + f' \sqcup (g' *_m h')) \epsilon_l \\
&= (f * h)_i \sqcup g + f \sqcup_i (g * h).
\end{aligned}$$

The second point is deduced from the first one, as $\sqcup_i = \sqcup_i \text{ }^{op}$. Finally:

$$\begin{aligned}
(f \sqcup_i g) * h &= \delta_{k,l}(f' \sqcup g' \epsilon_l) * h' \epsilon_m \\
&= \delta_{k,l}(f' \sqcup g') *_m h' \epsilon_l \\
&= \delta_{k,l}((f' *_m h') \sqcup g' + f' \sqcup (g' *_m h')) \epsilon_l \\
&= (f * h) \sqcup g + f \sqcup (g * h).
\end{aligned}$$

So the last point holds. □

Theorem 3 *The following conditions are equivalent:*

1. $(T(V)^N, \bullet)$ is a pre-Lie algebra.
2. $\mathfrak{g}'_a = (T(V)^N, {}_a[-, -], *)$ is a post-Lie algebra.
3. V is a module over the Lie algebra \mathfrak{g}_a , with the action given by $\epsilon_i.v = F_i(v)$.

Proof. By Corollary 2, for all $f, g, h \in \mathfrak{g}'_a$, ${}_a[f, g] * h = {}_a[f * h, g] + {}_a[f, g * h]$.

1. \iff 2. Let $f, g, h \in \mathfrak{g}$.

$$\begin{aligned}
&(f \bullet g) \bullet h - f \bullet (g \bullet h) - (f \bullet h) \bullet g + f \bullet (h \bullet g) \\
&= (f * g) * h - f * (g * h) - (f * h) * g + f * (h * g) \\
&+ (f \sqcup_a g) * h - f \sqcup_a (g * h) - (f \sqcup_a h) * g + f \sqcup_a (h * g) \\
&+ (f * g)_a \sqcup h - f * (g_a \sqcup h) - (f * h)_a \sqcup g + f * (h_a \sqcup g) \\
&+ (f_a \sqcup g)_a \sqcup h - f_a \sqcup (g_a \sqcup h) - (f_a \sqcup g)_a \sqcup h + f_a \sqcup (g_a \sqcup h) \\
&= (f * g) * h - f * (g * h) - (f * h) * g + f * (h * g) \\
&+ f * (g_a \sqcup h) - f * (h_a \sqcup g) \\
&+ [(f_a \sqcup g) * h - f_a \sqcup (g * h) - (f * h)_a \sqcup g] \\
&+ [(f_a \sqcup h) * g - f_a \sqcup (h * g) - (f * g)_a \sqcup h] \\
&- [(f * h)_a \sqcup g - f_a \sqcup (h * g) - (f * g)_a \sqcup h] \\
&+ [(f_a \sqcup g)_a \sqcup h - f_a \sqcup (g_a \sqcup h)] - [(f_a \sqcup g)_a \sqcup h - f_a \sqcup (g_a \sqcup h)] \\
&= (f * g) * h - f * (g * h) - (f * h) * g + f * (h * g) - f * {}_a[g, h].
\end{aligned}$$

So $(\mathfrak{g}'_a, \bullet)$ is pre-Lie if, and only if, $(\mathfrak{g}'_a, {}_a[-, -], *)$ is post-Lie.

2. \implies 3. Let $x, y, v \in V$ and $i, j, k \in [N]$. Then:

$$\begin{aligned}
x\epsilon_i * y\epsilon_j &= F_j(x)y\epsilon_i, & xy\epsilon_i * z\epsilon_k &= xF_k(y)z\epsilon_i + F_k(x)(y \sqcup z)\epsilon_i. \\
x\epsilon_i * yz\epsilon_k &= F_k(x)yz\epsilon_i,
\end{aligned}$$

Hence:

$$\begin{aligned}
(x\epsilon_i * y\epsilon_j) * z\epsilon_k &= F_j(x)y\epsilon_i * z\epsilon_k \\
&= F_j(x)F_k(y)z\epsilon_i + F_k \circ F_j(x)y \sqcup z\epsilon_i, \\
x\epsilon_i * (y\epsilon_j * z\epsilon_k) &= x\epsilon_i * F_k(y)z\epsilon_j \\
&= F_j(x)F_k(y)z\epsilon_i, \\
x\epsilon_i \circ_a [y\epsilon_j, z\epsilon_k] &= a_j x\epsilon_i * (y \sqcup z)\epsilon_k - a_k x\epsilon_i * (y \sqcup z)\epsilon_j \\
&= (a_j F_k(x)(y \sqcup z) - a_k F_j(x)(y \sqcup z))\epsilon_i.
\end{aligned}$$

The post-Lie relation (2) gives:

$$\begin{aligned}
&(a_j F_k(x) - a_k F_j(x))(y \sqcup z) \\
&= F_j(x)F_k(y)z + F_k \circ F_j(x)(y \sqcup z) - F_j(x)F_k(y)z - F_j \circ F_k(x)(y \sqcup z) \\
&= (F_j \circ F_k - F_k \circ F_j)(x)(y \sqcup z).
\end{aligned}$$

Let $y = z$ be a nonzero element of V . Then $y \sqcup z \neq 0$, and we obtain that for all $x \in V$, $a_j F_k(x) - a_k F_j(x) = (F_j \circ F_k - F_k \circ F_j)(x)$: V is a \mathfrak{g}_a -module.

3. \implies 2. Let us prove the post-Lie relation (2) for $f\epsilon_i$, g and h , with $f \in T(V)$, $i \in [N]$, $g, h \in \mathfrak{g}'_a$. We assume that f is a word and we proceed by induction on the length n of f . If $n = 0$, then $f = \emptyset$ and every term is 0 in the relation. Let us assume the result at rank $n - 1$. We put $f = xf'$, with $x \in V$, and f' a word of length $n - 1$.

$$\begin{aligned}
(f * g) * h &= x((f'\epsilon_i * g) * h) + \sum_{p=1}^N F_p(x)((f'\epsilon_i * g) \sqcup_p h) \\
&\quad + \sum_{p=1}^N F_p(x)((f'\epsilon_i \sqcup_p g) * h) + \sum_{p,q=1}^N F_q \circ F_p(x)(f'\epsilon_i \sqcup_p g \sqcup_q h), \\
f * (g * h) &= x(f'\epsilon_i * (g * h)) + \sum_{p=1}^N F_p(x)(f'\epsilon_i \sqcup_p (g * h)), \\
\sum_{p=1}^N a_p f * (g_p \sqcup h) &= \sum_{p=1}^N a_p x(f'\epsilon_i * (g_p \sqcup h)) + \sum_{p,q=1}^N a_p F_q(x)(f'\epsilon_i \sqcup_q (g_p \sqcup h)).
\end{aligned}$$

We put:

$$P(f, g, h) = f * (g * h) - (f * g) * h + \sum_{p=1}^N a_p f * (g_p \sqcup h).$$

In order to prove the post-Lie relation (2), we have to prove that $P(f, g, h) = P(f, h, g)$. First:

$$\begin{aligned}
(f * g) * h &= x((f'\epsilon_i * g) * h) + \sum_{p=1}^N F_p(x)((f'\epsilon_i * g) \sqcup_p h) \\
&\quad + \sum_{p=1}^N F_p(x)((f'\epsilon_i \sqcup_p g) * h) + \sum_{p,q=1}^N F_q \circ F_p(x)(f'\epsilon_i \sqcup_p g \sqcup_q h), \\
f * (g * h) &= x(f'\epsilon_i * (g * h)) + \sum_{p=1}^N F_p(x)(f'\epsilon_i \sqcup_p (g * h)), \\
\sum_{p=1}^N a_p f * (g_p \sqcup h) &= \sum_{p=1}^N a_p x(f'\epsilon_i * (g_p \sqcup h)) + \sum_{p,q=1}^N a_p F_q(x)(f'\epsilon_i \sqcup_q (g_p \sqcup h)).
\end{aligned}$$

Consequently:

$$\begin{aligned}
P(f, g, h) &= xP(f', g, h) \\
&+ \sum_{p=1}^N F_p(x)(-(f'\epsilon_i * g) \sqcup_p h - ((f'\epsilon_i \sqcup_p g) * h + f'\epsilon_i \sqcup_p (g * h))) \\
&+ \sum_{p,q=1}^N a_p F_q(x)(f'\epsilon_i \sqcup_q g \sqcup_p h) - F_q \circ F_p(x)(f'\epsilon_i \sqcup_p g \sqcup_q h) \\
&= xP(f', g, h) - \sum_{p=1}^N F_p(x)((f'\epsilon_i * g) \sqcup_p h + (f'\epsilon_i * h) \sqcup_p g) \\
&+ \sum_{p,q=1}^N a_p F_q(x)(f'\epsilon_i \sqcup_q h \sqcup_p g) - F_q \circ F_p(x)(f'\epsilon_i \sqcup_p g \sqcup_q h) \\
&= xP(f', g, h) - \sum_{p=1}^N F_p(x)(f'\epsilon_i * g) \sqcup_p h + (f'\epsilon_i * h) \sqcup_p g) \\
&+ \sum_{p,q=1}^N (a_p F_q(x) - F_q \circ F_p(x))(f'\epsilon_i \sqcup_p g \sqcup_q h).
\end{aligned}$$

By the induction hypothesis, $P(f', g, h) = P(f', h, g)$, so the first row is symmetric in g, h . As V is a \mathfrak{g}_a -module, $a_p F_q - F_q \circ F_p = a_q F_p - F_p \circ F_q$, so the second row is symmetric in g, h , and finally $P(f, g, h) = P(f, h, g)$: \mathfrak{g}'_a is a post-Lie algebra. \square

Example. The post-Lie algebra \mathfrak{g}_{SISO} is associated to $a = (1, 0)$, $V = Vect(x_1, x_2)$ and:

$$F_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

As F_1 and F_2 define a module over the Lie algebra $\mathfrak{g}_{(1,0)}$, even in fact over the associative algebra $(\mathfrak{g}_{(1,0)}, \triangleleft)$, we obtain indeed a post-Lie algebra. For all $f, g \in T(V)$, for all $i, j \in \{1, 2\}$:

$$\begin{aligned}
\emptyset \epsilon_i * g \epsilon_j &= 0, & x_2 f \epsilon_i * g \epsilon_1 &= x_2 (f \epsilon_i * g \epsilon_1) + x_2 (f \sqcup g) \epsilon_i, \\
x_1 f \epsilon_i * g \epsilon_j &= x_1 (f \epsilon_i * g \epsilon_j), & x_2 f \epsilon_i * g \epsilon_2 &= x_2 (f \epsilon_i * g \epsilon_2) + x_1 (f \sqcup g) \epsilon_i.
\end{aligned}$$

3.3 Extension of the post-Lie product

We now extend the post-Lie product of \mathfrak{g}'_a to the enveloping algebra $\mathcal{U}(\mathfrak{g}'_a)$. As this Lie bracket is obtained from an associative product $\triangleleft = {}_a \sqcup$, we can see $\mathcal{U}(\mathfrak{g}'_a)$ as $(S(\mathfrak{g}'_a), \blacktriangleleft, \Delta)$. The post-Lie product $*$ is extended to $\mathcal{U}(\mathfrak{g}'_a)$, and we obtain a Hopf algebra $(\mathcal{U}(\mathfrak{g}), \otimes, \Delta)$, isomorphic to $\mathcal{U}(\mathfrak{g}'_a, {}_a[-, -]_*)$, with:

$$\forall f, g \in \mathfrak{g}, \quad {}_a[f, g]_* = {}_a[f, g] + f * g - g * f = f {}_a \sqcup g + f * g - g {}_a \sqcup f - g * f.$$

As \bullet is a pre-Lie product, it can also be extended to $S(\mathfrak{g})$ and gives a product \odot , making $S(\mathfrak{g}'_a)$ a Hopf algebra isomorphic to $\mathcal{U}(\mathfrak{g}'_a, [-, -]_\bullet)$.

Remark. Let $f, g \in \mathfrak{g}'_a$.

$$\begin{aligned}
[f, g]_\bullet &= f \bullet g - g \bullet f \\
&= f {}_a \sqcup g + f * g - g {}_a \sqcup f - g * f \\
&= {}_a[f, g] + f * g - g * f \\
&= {}_a[f, g]_*.
\end{aligned}$$

So $[-, -]_\bullet = {}_a[-, -]_*$.

Lemma 4 Let $f_1, \dots, f_k, g \in \mathfrak{g}'_a$, $k \geq 1$.

$$(f_1 \blacktriangleleft \dots \blacktriangleleft f_k) * g = \sum_{p=1}^k f_1 \blacktriangleleft \dots \blacktriangleleft (f_p * g) \blacktriangleleft \dots \blacktriangleleft f_k,$$

$$(f_1 \dots f_k) * g = \sum_{p=1}^k f_1 \dots (f_p * g) \dots f_k.$$

Proof. The first point comes by the very definition of $*$. For the second point, we proceed by induction on k . This is obvious if $k = 1$. Let us assume the result at rank k , $k \geq 1$. Observe that:

$$f_1 \dots f_{k+1} = f_1 \dots f_k \blacktriangleleft f_{k+1} - \sum_{p=1}^k f_1 \dots (f_p \sqcup_a f_{k+1}) \dots f_k,$$

so:

$$\begin{aligned} f_1 \dots f_{k+1} * g &= (f_1 \dots f_k * g) \blacktriangleleft f_{k+1} + f_1 \dots f_k \blacktriangleleft (f_{k+1} * g) - \sum_{p=1}^k f_1 \dots (f_p \sqcup_a f_{k+1}) \dots f_k * g \\ &= \sum_{p=1}^k f_1 \dots (f_p * g) \dots f_k \blacktriangleleft f_{k+1} + f_1 \dots f_k \blacktriangleleft (f_{k+1} * g) \\ &\quad - \sum_{p \neq q} f_1 \dots (f_p \sqcup_a f_{k+1}) \dots (f_q * g) \dots f_k - \sum_{p=1}^k f_1 \dots ((f_p \sqcup_a f_{k+1}) * g) \dots f_k \\ &= \sum_{p=1}^k f_1 \dots (f_p * g) \dots f_k f_{k+1} + \sum_{p \neq q} f_1 \dots (f_p \sqcup_a f_{k+1}) \dots (f_q * g) \dots f_k \\ &\quad + \sum_{p=1}^k f_1 \dots ((f_p * g) \sqcup_a f_{k+1}) \dots f_k - \sum_{p \neq q} f_1 \dots (f_p \sqcup_a f_{k+1}) \dots (f_q * g) \dots f_k \\ &\quad - \sum_{p=1}^k f_1 \dots ((f_p * g) \sqcup_a f_{k+1}) \dots f_k - \sum_{p=1}^k f_1 \dots (f_p \sqcup_a (f_{k+1} * g)) \dots f_k \\ &\quad + f_1 \dots f_k (f_{k+1} * g) + \sum_{p=1}^k f_1 \dots (f_p \sqcup_a (f_{k+1} * g)) \dots f_k \\ &= \sum_{p=1}^{k+1} f_1 \dots (f_p * g) \dots f_{k+1}. \end{aligned}$$

Finally, the result holds for all $k \geq 1$. □

The following result allows to compute $f * g_1 \dots g_k$ by induction on the length of f :

Proposition 15 Let $x \in V$, $k \geq 1$, $f, g_1, \dots, g_k \in T(V)^N$, $i \in [N]$.

$$\emptyset \epsilon_i * (g_1 \blacktriangleleft \dots \blacktriangleleft g_k) = 0,$$

$$x f * (g_1 \blacktriangleleft \dots \blacktriangleleft g_k) = \sum_{\substack{I=\{i_1 < \dots < i_l\} \subseteq [k], \\ j_1, \dots, j_l \in [N]}} F_{j_1} \circ \dots \circ F_{j_l}(x) \left(\left(f * \prod_{i \notin I}^{\blacktriangleleft} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right);$$

$$\emptyset \epsilon_i * (g_1 \dots g_k) = 0,$$

$$x f * (g_1 \dots g_k) = \sum_{\substack{I=\{i_1 < \dots < i_l\} \subseteq [k], \\ j_1, \dots, j_l \in [N]}} F_{j_1, \dots, j_l}(x) \left(\left(f * \prod_{i \notin I} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right).$$

Proof. In order to ease the redaction, we put:

$$I_k = \{(I, j_1, \dots, j_l) \mid I = \{i_1 < \dots < i_l\} \subseteq [k], j_1, \dots, j_l \in [N]\}.$$

We proceed by induction on k . It is immediate if $k = 1$. Let us assume the result at rank k , $k \geq 1$. Then:

$$\begin{aligned} \emptyset \epsilon_i * (g_1 \blacktriangleleft \dots \blacktriangleleft g_{k+1}) &= (\emptyset \epsilon_i * (g_1 \blacktriangleleft \dots \blacktriangleleft g_k)) * g_{k+1} \\ &\quad - \sum_{p=1}^k \emptyset \epsilon_i * (g_1 \blacktriangleleft \dots \blacktriangleleft (g_p * g_{k+1}) \blacktriangleleft \dots \blacktriangleleft g_k) \\ &= 0. \end{aligned}$$

Moreover:

$$\begin{aligned} &xf * (g_1 \blacktriangleleft \dots \blacktriangleleft g_{k+1}) \\ &= (xf * (g_1 \blacktriangleleft \dots \blacktriangleleft g_k)) * g_{k+1} - \sum_{p=1}^k xf * (g_1 \blacktriangleleft \dots \blacktriangleleft (g_p * g_{k+1}) \blacktriangleleft \dots \blacktriangleleft g_k) \\ &= \sum_{I_k} F_{j_l} \circ \dots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right) * g_{k+1} \\ &\quad - \sum_{I_k} F_{j_l} \circ \dots \circ F_{j_1}(x) \left(f * \left(\left(\overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \right) * g_{k+1} \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right) \\ &\quad - \sum_{p=1}^k \sum_{I_k} F_{j_l} \circ \dots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_p} (g_{i_p} * g_{k+1}) \dots \sqcup_{j_l} g_{i_l} \right) \\ &= \sum_{I_k} \sum_{j_{l+1} \in [N]} F_{j_l} \circ \dots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \sqcup_{j_{l+1}} g_{k+1} \right) \\ &\quad + \sum_{I_k} F_{j_l} \circ \dots \circ F_{j_1}(x) \\ &\quad \left(\left(\left(f * \overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \right) * g_{k+1} - f * \left(\left(\overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \right) * g_{k+1} \right) \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right) \\ &= \sum_{I_k} \sum_{j_{l+1} \in [N]} F_{j_l} \circ \dots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \sqcup_{j_{l+1}} g_{k+1} \right) \\ &\quad + \sum_{I_k} F_{j_l} \circ \dots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod}_{i \notin J \sqcup \{k+1\}} g_i \blacktriangleleft g_{k+1} \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \sqcup_{j_l} g_{k+1} \right) \\ &= \sum_{I_{k+1}, k+1 \in I} F_{j_l} \circ \dots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod}_{i \notin I} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right) \\ &\quad + \sum_{I_{k+1}, k+1 \notin I} F_{j_l} \circ \dots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod}_{i \notin I} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right) \\ &= \sum_{I_{k+1}} F_{j_l} \circ \dots \circ F_{j_1}(x) \left(\left(f * \overset{\blacktriangleleft}{\prod}_{i \notin I} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right). \end{aligned}$$

So, for all $F \in \mathcal{U}(\mathfrak{g})_+$, $\emptyset \epsilon_i * F = 0$. As $g_1 \dots g_k \in \mathcal{U}(\mathfrak{g})_+$, the first point holds. Let us prove the second point by induction on k . The result is immediate if $k = 1$. Let us assume the result at rank $k \geq 1$.

$$\begin{aligned}
& xf * g_1 \dots g_{k+1} \\
&= xf * (g_1 \dots g_k * g_{k+1}) - \sum_{p=1}^k xf * (g_1 \dots (g_p \sqcup g_{k+1}) \dots g_k) \\
&= (xf * g_1 \dots g_k) * g_{k+1} - \sum_{p=1}^k xf * (g_1 \dots (g_p \sqcup g_{k+1}) \dots g_k) - \sum_{p=1}^k xf * (g_1 \dots (g_p * g_{k+1}) \dots g_k) \\
&= \sum_{I_k} F_{j_1, \dots, j_l}(x) \left(\left(\left(f * \prod_{i \notin J \sqcup \{k+1\}} g_i \right) * g_{k+1} \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right) \\
&+ \sum_{I_k} \sum_{p=1}^k F_{j_1, \dots, j_l}(x) \left(\left(f * \prod_{i \notin J \sqcup \{k+1\}} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_p} (g_{i_p} * g_{k+1}) \dots \sqcup_{j_l} g_{i_l} \right) \\
&+ \sum_{I_k} \sum_{j_{l+1} \in [N]} F_{j_{l+1}} \circ F_{j_1, \dots, j_l}(x) \left(\left(f * \prod_{i \notin J \sqcup \{k+1\}} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right) \\
&- \sum_{I_k} F_{j_1, \dots, j_l}(x) \left(f * \left(\left(\prod_{i \notin J \sqcup \{k+1\}} g_i \right) * g_{k+1} \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right) \\
&- \sum_{I_k} \sum_{p=1}^k F_{j_1, \dots, j_l}(x) \left(\left(f * \prod_{i \notin J \sqcup \{k+1\}} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_p} (g_{i_p} * g_{k+1}) \dots \sqcup_{j_l} g_{i_l} \right) \\
&- \sum_{I_k} \sum_{p=1}^k F_{j_1, \dots, j_l}(x) \left(\left(f * \prod_{i \notin J \sqcup \{k+1\}} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_p} (g_{i_p} \sqcup g_{k+1}) \dots \sqcup_{j_l} g_{i_l} \right) \\
&- \sum_{I_k} F_{j_1, \dots, j_l}(x) \left(f * \left(\left(\prod_{i \notin J \sqcup \{k+1\}} g_i \right) * g_{k+1} - \left(\prod_{i \notin J \sqcup \{k+1\}} g_i \right) g_{k+1} \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right) \\
&= \sum_{I_k} F_{j_1, \dots, j_l}(x) \left(f * \left(\left(\prod_{i \notin J \sqcup \{k+1\}} g_i \right) g_{k+1} \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right) \\
&+ \sum_{I_k} \sum_{j_{l+1} \in [N]} F_{j_{l+1}} \circ F_{j_1, \dots, j_l}(x) \left(\left(f * \left(\prod_{i \notin J \sqcup \{k+1\}} g_i \right) \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \sqcup_{j_{l+1}} g_{i_{l+1}} \right) \\
&- \sum_{I_k} \sum_{p=1}^k \sum_{j \in [N]} a_j F_{j_1, \dots, j_l}(x) \left(\left(f * \left(\prod_{i \notin J \sqcup \{k+1\}} g_i \right) \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_p} (g_{k+1} \sqcup_j g_p) \dots \sqcup_{j_l} g_{i_l} \right) \\
&= \sum_{I_k} F_{j_1, \dots, j_l}(x) \left(f * \left(\left(\prod_{i \notin J \sqcup \{k+1\}} g_i \right) g_{k+1} \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right) \\
&+ \sum_{I_k} \sum_{j_{l+1} \in [N]} \left(F_{j_{l+1}} \circ F_{j_1, \dots, j_l} - \sum_{p=1}^l a_{j_p} F_{j_1, \dots, \widehat{j_p}, \dots, j_{k+1}} \right) (x) \\
&\quad \left(\left(f * \left(\prod_{i \notin J \sqcup \{k+1\}} g_i \right) \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \sqcup_{j_{l+1}} g_{i_{l+1}} \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{I_{k+1}, k+1 \notin J} F_{j_1, \dots, j_l}(x) \left(\left(f * \prod_{i \notin I} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right) \\
&+ \sum_{I_{k+1}, k+1 \in J} F_{j_1, \dots, j_l}(x) \left(\left(f * \prod_{i \notin I} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right) \\
&= \sum_{I_{k+1}} F_{j_1, \dots, j_l}(x) \left(\left(f * \prod_{i \notin I} g_i \right) \sqcup_{j_1} g_{i_1} \dots \sqcup_{j_l} g_{i_l} \right).
\end{aligned}$$

Note that we used $F_{j_{i+1}} \circ F_{j_1, \dots, j_i} = F_{j_1, \dots, j_{i+1}} + \sum_{p=1}^l a_{j_p} F_{j_1, \dots, \widehat{j_p}, \dots, j_{k+1}}$. □

Proposition 16 *Let $k \geq 1$, $f, g_1, \dots, g_k \in T(V)^N$. Then:*

$$f \bullet g_1 \dots g_k = f * g_1 \dots g_k + \sum_{p=1}^k (f * g_1 \dots g_{p-1} g_{p+1} \dots g_k) \mathbin{a} \sqcup g_p.$$

Proof. We proceed by induction on k . This is obvious if $k = 1$. Let us assume the result at rank k , $k \geq 1$.

$$\begin{aligned}
&f \bullet g_1 \dots g_{k+1} \\
&= (f \bullet g_1 \dots g_k) \bullet g_{k+1} - \sum_{p=1}^k f \bullet (g_1 \dots (g_p \bullet g_{k+1}) \dots g_k) \\
&= (f * g_1 \dots g_k) * g_{k+1} + (f * g_1 \dots g_k) \mathbin{a} \sqcup g_{k+1} \\
&+ \sum_{p=1}^k ((f * g_1 \dots g_{p-1} g_{p+1} \dots g_k) \mathbin{a} \sqcup g_p) * g_{k+1} + \sum_{p=1}^k (f * g_1 \dots g_{p-1} g_{p+1} \dots g_k) \mathbin{a} \sqcup g_p \mathbin{a} \sqcup g_{k+1} \\
&- \sum_{p=1}^k f * (g_1 \dots (g_p \bullet g_{k+1}) \dots g_k) - \sum_{p=1}^k (f * g_1 \dots g_{p-1} g_{p+1} \dots g_k) \sqcup (g_p \bullet g_{k+1}) \\
&- \sum_{p \neq q} f * (g_1 \dots (g_p \bullet g_{k+1}) \dots \widehat{g}_q \dots g_k) \mathbin{a} \sqcup g_q \\
&= (f * g_1 \dots g_k) * g_{k+1} + (f * g_1 \dots g_k) \mathbin{a} \sqcup g_{k+1} + \sum_{p=1}^k ((f * g_1 \dots g_{p-1} g_{p+1} \dots g_k) * g_{k+1}) \mathbin{a} \sqcup g_p \\
&+ \sum_{p=1}^k (f * g_1 \dots g_{p-1} g_{p+1} \dots g_k) \mathbin{a} \sqcup (g_p * g_{k+1} - g_p \bullet g_{k+1} + g_p \mathbin{a} \sqcup g_{k+1}) \\
&- \sum_{p \neq q} f * (g_1 \dots (g_p \bullet g_{k+1}) \dots \widehat{g}_q \dots g_k) \mathbin{a} \sqcup g_q - \sum_{p=1}^k f * (g_1 \dots (g_p \bullet g_{k+1}) \dots g_k) \\
&= (f * g_1 \dots g_k) * g_{k+1} + (f * g_1 \dots g_k) \mathbin{a} \sqcup g_{k+1} \\
&+ \sum_{p=1}^k \left((f * g_1 \dots g_{p-1} g_{p+1} \dots g_k) * g_{k+1} - \sum_{q \neq p} f * g_1 \dots g_{p-1} g_{p+1} \dots (g_q \bullet g_{k+1}) \dots g_k \right) \mathbin{a} \sqcup g_p \\
&- \sum_{p=1}^k f * g_1 \dots (g_p * g_{k+1}) \dots g_k - \sum_{p=1}^k f * g_1 \dots (g_p \mathbin{a} \sqcup g_{k+1}) \dots g_k \\
&= f * \left(g_1 \dots g_k \blacktriangleleft g_{k+1} - \sum_{p=1}^k g_1 \dots (g_p \mathbin{a} \sqcup g_{k+1}) \dots g_k \right) + \sum_{p=1}^{k+1} (f * g_1 \dots g_{p-1} g_{p+1} \dots g_{k+1}) \mathbin{a} \sqcup g_p
\end{aligned}$$

$$= f * g_1 \dots g_{k+1} + \sum_{p=1}^{k+1} (f * g_1 \dots g_{p-1} g_{p+1} \dots g_{k+1}) \mathbin{a} \sqcup g_p.$$

So the result holds for all $k \geq 1$. □

Proposition 17 *On $S(\mathfrak{g}'_a)$, $\otimes = \odot$.*

Proof. Let $f, g \in S(\mathfrak{g}'_a)$; let us prove that $f \otimes g = f \odot g$. We assume that $f = f_1 \dots f_k$, $g = g_1, \dots, g_l$, with $f_1, \dots, f_k, g_1, \dots, g_l \in \mathfrak{g}'_a$, and we proceed by induction on k . If $k = 0$, then $f = 1$ and $f \otimes g = f \odot g = g$. Let us assume the result at all ranks $< k$. We proceed by induction on l . If $l = 0$, then $g = 1$ and $f \otimes g = f \odot g = f$. Let us assume the result at all ranks $< l$. We put:

$$\Delta(f) = f \otimes 1 + 1 \otimes f + f' \otimes f'', \quad \Delta(g) = g \otimes 1 + 1 \otimes g + g' \otimes g''.$$

The induction hypothesis on k holds for f' and f'' and the induction hypothesis on l holds for g' and g'' . From:

$$\Delta(f \otimes g - f \odot g) = f^{(1)} \otimes g^{(1)} \otimes f^{(2)} \otimes g^{(2)} - f^{(1)} \odot g^{(1)} \otimes f^{(2)} \odot g^{(2)},$$

these two induction hypotheses give:

$$\Delta(f \otimes g - f \odot g) = (f \otimes g - f \odot g) \otimes 1 + 1 \otimes (f \otimes g - f \odot g).$$

So $f \otimes g - f \odot g \in \text{Prim}(S(\mathfrak{g}'_a)) = \mathfrak{g}'_a$. Let π be the canonical projection on \mathfrak{g}'_a in $S(\mathfrak{g}'_a)$. We obtain:

$$\begin{aligned} \pi(f \otimes g) &= \pi \left(\sum_{I \subseteq [l]} \left(f * \prod_{i \in I} g_i \right) \blacktriangleleft \prod_{j \notin I} g_j \right) \\ &= \pi \left(\sum_{[l]=I_0 \sqcup \dots \sqcup I_k} \left(f_1 * \prod_{i \in I_1} g_i \right) \dots \left(f_k * \prod_{i \in I_k} g_i \right) \blacktriangleleft \prod_{i \in I_0} g_i \right) \\ &= \pi \left(\sum_{[l]=J_1 \sqcup \dots \sqcup J_k} \prod_{p=1}^k \left(f_p * \prod_{i \in J_k} g_i + \sum_{j_p \in J_p} \left(f_p * \prod_{i \in J_p \setminus \{j_p\}} g_i \right) \mathbin{a} \sqcup g_{j_p} \right) \right) \\ &= \pi \left(\sum_{[l]=J_1 \sqcup \dots \sqcup J_k} \left(\prod_{p=1}^k f_p \bullet \prod_{i \in J_p} g_i \right) \right) \\ &= \pi \left(\left(f_1 \bullet g^{(1)} \right) \dots \left(f_k \bullet g^{(k)} \right) \right) \\ &= \pi(f \bullet g) \\ &= \pi(f \odot g). \end{aligned}$$

As $f \otimes g - f \odot g \in \mathfrak{g}'_a$, $f \otimes g = f \odot g$. □

3.4 Graduation

We assume in this whole paragraph that $a = (1, 0, \dots, 0)$ and V is finite-dimensional. We decompose the \mathfrak{g}_a -module V as a direct sum of indecomposables. By Proposition 9, decomposing each indecomposables, we obtain a decomposition of V of the form:

$$V = V^{(0)} \oplus \dots \oplus V^{(k)},$$

with $F_1(V^{(p)}) \subseteq V^{(p)}$ and $F_i(V^{(p)}) \subseteq V^{(p-1)}$ for all $i \geq 2$, for all $p \in [k]$. We put $V_p = V^{(k+1-p)}$ for all $p \in [k+1]$. This defines a graduation of V , which induces a connected graduation of $T(V)$. For this graduation of V , F_1 is homogeneous of degree 0 and F_i is homogeneous of degree 1 for all $i \geq 2$. We define a graduation of $\mathfrak{g}'_a = T(V)^N$:

$$\forall n \geq 0, (\mathfrak{g}'_a)_n = T(V)_n \epsilon_1 \oplus \bigoplus_{i=2}^N T(V)_{n-1} \epsilon_i.$$

Let $v, w \in T(V)$, homogeneous of respective degree k and l . Let $i, j \geq 2$. Then:

- $v\epsilon_1$ is homogeneous of degree k .
- $v\epsilon_i$ is homogeneous of degree $k+1$.
- $w\epsilon_1$ is homogeneous of degree l .
- $w\epsilon_j$ is homogeneous of degree $l+1$.

As $v \sqcup w$ is homogeneous of degree $k+l$:

- $v\epsilon_1 \sqcup w\epsilon_1 = v \sqcup w\epsilon_1$ is homogeneous of degree $k+l$.
- $v\epsilon_1 \sqcup w\epsilon_j = v \sqcup w\epsilon_j$ is homogeneous of degree $k+l+1$.
- $v\epsilon_i \sqcup w\epsilon_1 = 0$ is homogeneous of degree $k+l+1$.
- $v\epsilon_i \sqcup w\epsilon_j = 0$ is homogeneous of degree $k+l+2$.

Consequently, the product $\sqcup_{(1,0,\dots,0)}$ is homogeneous of degree 0. Proposition 14 implies that $*$ is homogeneous of degree 0; summing, \bullet is also homogeneous of degree 0. Hence:

Proposition 18 *The decomposition of V in indecomposable $\mathfrak{g}_{(1,0,\dots,0)}$ -modules induces a graduation of the post-Lie algebra $\mathfrak{g}'_{(1,0,\dots,0)}$.*

We put:

$$P(X) = \sum_{i=1}^{k+1} \dim(V_p) X^p \in \mathbb{K}[X].$$

the formal series of $\mathfrak{g}'_{(1,0,\dots,0)}$ is:

$$\begin{aligned} R(X) &= \sum_{p=1}^{\infty} \dim((\mathfrak{g}'_{(1,0,\dots,0)})_p) X^p \\ &= \frac{1}{1-P(X)} + (N-1) \frac{X}{1-P(X)} = \frac{1+(N-1)X}{1-P(X)}. \end{aligned}$$

Note that $R(0) = 1$: indeed, $(\mathfrak{g}'_{(1,0,\dots,0)})_0 = \text{Vect}(\emptyset\epsilon_1)$. The augmentation ideal of $\mathfrak{g}'_{(1,0,\dots,0)}$ is:

$$(\mathfrak{g}'_{(1,0,\dots,0)})_+ = T(V)_+ \times T(V)^{N-1}.$$

This is a graded, connected post-Lie algebra.

Example. For the SISO case, $V_1 = \text{Vect}(x_2)$ and $V_2 = \text{Vect}(x_1)$. The formal series of \mathfrak{g}_{SISO} is:

$$R_{SISO}(X) = \frac{1+X}{1-X-X^2} = 1 + 2X + 3X^2 + 5X^3 + 8X^4 + 13X^5 + \dots$$

Hence, $(\dim(\mathfrak{g}_{SISO})_n)_{n \geq 0}$ is the Fibonacci sequence A000045 [9]. For example:

$$\begin{aligned} (\mathfrak{g}_{SISO})_0 &= \text{Vect}(\emptyset\epsilon_1), \\ (\mathfrak{g}_{SISO})_1 &= \text{Vect}(x_2\epsilon_1, \emptyset\epsilon_2), \\ (\mathfrak{g}_{SISO})_2 &= \text{Vect}(x_1\epsilon_1, x_2x_2\epsilon_1, x_2\epsilon_2), \\ (\mathfrak{g}_{SISO})_3 &= \text{Vect}(x_1x_2\epsilon_1, x_2x_1\epsilon_1, x_2x_2x_2\epsilon_1, x_1\epsilon_2, x_2x_2\epsilon_2). \end{aligned}$$

4 Graded dual

We assume in this section that $a = (1, 0, \dots, 0)$. The augmentation ideal of \mathfrak{g}'_a is denoted by $(\mathfrak{g}'_a)_+$; recall that $(\mathfrak{g}'_a)_0 = Vect(\emptyset\epsilon_1)$.

- As $(\mathfrak{g}'_a)_+$ is a graded, connected Lie algebra, its enveloping algebra $\mathcal{U}((\mathfrak{g}'_a)_+)$ is a graded, connected Hopf algebra, and its graded dual also is. We denote it by \mathcal{H}_V .
- As an algebra, \mathcal{H}_V is identified with $S((\mathfrak{g}'_a)^*)/\langle\emptyset\epsilon_1\rangle$. We identify $(\mathfrak{g}'_a)^*$ with $T(V^*)^N$ via the pairing:

$$\langle f_1 \dots f_k \epsilon_i, x_1 \dots x_l \epsilon_j \rangle = \delta_{i,j} \delta_{k,l} f_1(x_1) \dots f_k(x_k).$$

- The coproduct dual of $\odot = \otimes$ is denoted by Δ_\bullet .
- The dual of the product \sqcup_j defined on \mathfrak{g}'_a is denoted by Δ_{\sqcup_j} , defined on $(\mathfrak{g}'_a)^* = T(V^*)^N$.
- We define a coproduct Δ_* on $S((\mathfrak{g}'_a)_+)^*$, dual of the right action $*$. Therefore, this is right coaction of $(\mathcal{H}_V, \Delta_\bullet)$ on itself:

$$(\Delta_* \otimes Id) \circ \Delta_* = (Id \otimes \Delta_\bullet) \circ \Delta_*.$$

Notations.

1. For all $y \in V^*$, we define $\theta_y : (\mathfrak{g}'_a)^* \rightarrow (\mathfrak{g}'_a)^*$ by $\theta_y(f) = yf$.
2. For all $x \in (\mathcal{H}_V)_+$, we put $\overline{\Delta}_\bullet(x) = \Delta_\bullet(x) - 1 \otimes x$ and $\overline{\Delta}_*(x) = \Delta_*(x) - 1 \otimes x$. For all $g, f, f_1, \dots, f_k \in (\mathfrak{g}'_a)_+^*$:

$$\langle \overline{\Delta}_*(g), f \otimes f_1 \dots f_k \rangle = \langle g, f * f_1 \dots f_k \rangle.$$

4.1 Deshuffling coproducts

Proposition 19 For all $g \in T(V)$, for all $i \in [N]$, $\Delta_{\sqcup_j}(g\epsilon_k) = \Delta_{\sqcup}(g)(\epsilon_k \otimes \epsilon_j)$.

Proof. Let $f_1, f_2 \in T(V)$, $i_1, i_2 \in [N]$.

$$\begin{aligned} \langle \Delta_{\sqcup_j}(g\epsilon_k), f_1 \epsilon_{i_1} \otimes f_2 \epsilon_{i_2} \rangle &= \langle g\epsilon_k, f_1 \epsilon_{i_1} \sqcup_j f_2 \epsilon_{i_2} \rangle \\ &= \delta_{i_2, j} \langle g\epsilon_k, f_1 \sqcup f_2 \epsilon_{i_1} \rangle \\ &= \delta_{i_2, j} \delta_{i_1, k} \langle g, f_1 \sqcup f_2 \rangle \\ &= \delta_{i_2, j} \delta_{i_1, k} \langle \Delta_{\sqcup}(g), f_1 \otimes f_2 \rangle \\ &= \langle \Delta_{\sqcup}(g)(\epsilon_k \otimes \epsilon_j), f_1 \epsilon_{i_1} \otimes f_2 \epsilon_{i_2} \rangle. \end{aligned}$$

As the pairing is nondegenerate, we obtain the result. \square

Notations. We define inductively, for $l \geq 0$, $j_1, \dots, j_l \in [N]$:

$$\begin{cases} \Delta_{\sqcup_\emptyset} = Id, \\ \Delta_{\sqcup_{j_1, \dots, j_l}} = \left(\Delta_{\sqcup_{j_1}} \otimes Id^{\otimes(l-1)} \right) \circ \Delta_{\sqcup_{j_2, \dots, j_l}}. \end{cases}$$

For all $g \in T(V^*)$, for all $i \in [N]$:

$$\Delta_{\sqcup_{j_1, \dots, j_l}}(g\epsilon_k) = \Delta_{\sqcup}^{(l)}(g)(\epsilon_k \otimes \epsilon_{j_1} \otimes \dots \otimes \epsilon_{j_l});$$

for all $f_1, \dots, f_l \in T(V)$:

$$\langle \Delta_{\sqcup_{j_1, \dots, j_l}}(g), f_1 \otimes \dots \otimes f_{l+1} \rangle = \langle g, f_1 \sqcup_{j_1} \dots \sqcup_{j_l} f_{l+1} \rangle.$$

4.2 Dual of the post-Lie product

Proposition 20 In $\mathcal{H}_V = S((\mathfrak{g}'_a)^*)/\langle \emptyset\epsilon_1 \rangle$:

- For all $i \in [N]$, $\Delta_*(\emptyset\epsilon_i) = \emptyset\epsilon_i \otimes 1 + 1 \otimes \emptyset\epsilon_i$.
- For all $y \in V^*$, $g \in (\mathfrak{g}'_a)^*$:

$$\bar{\Delta}_* \circ \theta_y(g) = \sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{F_{j_1, \dots, j_l}^*(y)} \otimes \mu) \circ (\bar{\Delta}_* \otimes Id) \circ \Delta_{\sqcup_{j_1, \dots, j_l}}(g),$$

where we denote by μ the sum of the iterated products of \mathcal{H}_V :

$$\mu : \begin{cases} T(\mathcal{H}_V) & \longrightarrow \mathcal{H}_V \\ g_1 \otimes \dots \otimes g_k & \longrightarrow g_1 \dots g_k. \end{cases}$$

Proof. The first point comes from $\emptyset\epsilon_i * \mathcal{U}(\mathfrak{g}'_a)_+ = (0)$.

In order to prove the formula, it is enough to prove that, for $f, f_1, \dots, f_k \in \mathfrak{g}$:

$$\langle \bar{\Delta}_* \circ \theta_y(g), f \otimes f_1 \dots f_k \rangle = \left\langle \sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{F_{j_1, \dots, j_l}^*(y)} \otimes \mu) \circ (\bar{\Delta}_* \otimes Id) \circ \Delta_{\sqcup_{j_1, \dots, j_l}}(g), f \otimes f_1 \dots f_k \right\rangle,$$

or equivalently:

$$\langle \theta_y(g), f * f_1 \dots f_k \rangle = \left\langle \sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{F_{j_1, \dots, j_l}^*(y)} \otimes \mu) \circ (\bar{\Delta}_* \otimes Id) \circ \Delta_{\sqcup_{j_1, \dots, j_l}}(g), f \otimes f_1 \dots f_k \right\rangle,$$

If $f = \emptyset\epsilon_i$, both sides are equal to 0. Otherwise, we can assume that $f = x f'$, with $x \in V$ and $f' \in \mathfrak{g}$.

$$\begin{aligned} & \langle \theta_y(g), f * f_1 \dots f_k \rangle \\ &= \left\langle yg, \sum_{I=\{i_1 < \dots < i_l\} \subseteq [k]} \sum_{j_1, \dots, j_l \in [N]} F_{j_1, \dots, j_l}(x) \left(f' * \left(\prod_{i \notin I} f_i \right) \sqcup_{j_1} f_{i_1} \dots \sqcup_{j_l} f_{i_l} \right) \right\rangle \\ &= \sum_{I=\{i_1 < \dots < i_l\} \subseteq [k]} \sum_{j_1, \dots, j_l \in [N]} \langle y, F_{j_1, \dots, j_l}(x) \rangle \langle \Delta_{\sqcup_{j_1, \dots, j_l}}(g), f' * \left(\prod_{i \notin I} f_i \right) \otimes f_{i_1} \dots \otimes f_{i_l} \rangle \\ &= \sum_{j_1, \dots, j_l \in [N]} \langle F_{j_1, \dots, j_l}^*(y), x \rangle \langle (Id \otimes \mu) \circ (\bar{\Delta}_* \otimes Id) \circ \Delta_{\sqcup_{j_1, \dots, j_l}}(g), f' \otimes f_1 \dots f_k \rangle \\ &= \sum_{j_1, \dots, j_l \in [N]} \langle (\theta_{F_{j_1, \dots, j_l}^*(y)} \otimes Id) \circ (Id \otimes \mu) \circ (\bar{\Delta}_* \otimes Id) \circ \Delta_{\sqcup_{j_1, \dots, j_l}}(g), x f' \otimes f_1 \dots f_k \rangle, \end{aligned}$$

which ends the proof. \square

In order to obtain a better description of the coproduct $\bar{\Delta}_*$, we are going to identify the following three objects:

$$\begin{array}{ccc} & S((\mathfrak{g}'_a)_+^*) & \\ \sim \swarrow & & \searrow \sim \\ S((\mathfrak{g}'_a)^*)/\langle \emptyset\epsilon_1 \rangle & & S((\mathfrak{g}'_a)^*)/\langle \emptyset\epsilon_1 - 1 \rangle \end{array}$$

Both identification sends $x \in (\mathfrak{g}'_a)_+^*$ to its class. Let us reformulate Proposition 20 in the vector space $S((\mathfrak{g}'_a)^*)/\langle \emptyset\epsilon_1 - 1 \rangle$:

$$\begin{aligned} \bar{\Delta}_* \circ \theta_y(g\epsilon_k) &= \sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{F_{j_1, \dots, j_l}^*}(y) \otimes \mu) \circ (\bar{\Delta}_* \otimes Id)(\Delta_{\sqcup}^{(l)}(g)\epsilon_k \otimes \epsilon_{j_1} \otimes \dots \otimes \epsilon_{j_l}) \\ &\quad - \sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{F_{j_1, \dots, j_l, 1}^*}(y) \otimes \mu) \circ (\bar{\Delta}_* \otimes Id)(\Delta_{\sqcup}^{(l+1)}(g)\epsilon_k \otimes \epsilon_{j_1} \otimes \dots \otimes \epsilon_{j_l} \otimes \epsilon_1) \\ &= \sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{F_{j_1, \dots, j_l}^*}(y) \otimes \mu) \circ (\bar{\Delta}_* \otimes Id)(\Delta_{\sqcup}^{(l)}(g)\epsilon_k \otimes \epsilon_{j_1} \otimes \dots \otimes \epsilon_{j_l}) \\ &\quad - \left(\sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{F_{j_1, \dots, j_l, 1}^*}(y) \otimes \mu) \circ (\bar{\Delta}_* \otimes Id)(\Delta_{\sqcup}^{(l)}(g)\epsilon_k \otimes \epsilon_{j_1} \otimes \dots \otimes \epsilon_{j_l}) \right) (1 \otimes \emptyset\epsilon_1). \end{aligned}$$

Finally, identifying in $S((\mathfrak{g}'_a)_+^*)$:

Proposition 21 For all $j_1, \dots, j_l \in [N]$, we put:

$$G_{j_1, \dots, j_l} = F_{j_1, \dots, j_l} - F_{j_1, \dots, j_l, 1}.$$

In $S((\mathfrak{g}'_a)_+^*)/\langle \emptyset\epsilon_1 - 1 \rangle$:

- For all $i \in [N]$, $\bar{\Delta}_*(\emptyset\epsilon_i) = \emptyset\epsilon_i \otimes 1$.
- For all $y \in V^*$, for all $g \in (\mathfrak{g}'_a)_+^*$:

$$\bar{\Delta}_* \circ \theta_y(g) = \sum_{l \geq 0} \sum_{j_1, \dots, j_l \in [N]} (\theta_{G_{j_1, \dots, j_l}^*}(y) \otimes \mu) \circ (\bar{\Delta}_* \otimes Id) \circ \Delta_{\sqcup_{j_1, \dots, j_l}}(g).$$

Example. For \mathfrak{g}_{SISO} , as V is a module over the associative algebra $(\mathfrak{g}_{(1,0)}, \triangleleft)$, if $l \geq 2$, $F_{j_1, \dots, j_l} = 0$ by Proposition 11, so $G_{j_1, \dots, j_l} = 0$. Moreover:

$$\begin{aligned} F_{\emptyset} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & F_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & F_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \\ G_{\emptyset} = F_{\emptyset} - F_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & G_1 = F_1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & G_2 = F_2 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \\ G_{\emptyset}^* &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & G_1^* &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & G_2^* &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

The coproduct $\bar{\Delta}_*$ on $S((\mathfrak{g}_{SISO})_+^*)$ is given by:

- For all $i \in [2]$, $\bar{\Delta}_*(\emptyset\epsilon_i) = \emptyset\epsilon_i \otimes 1$.
- For all $g \in \mathbb{K}\langle x_1, x_2 \rangle$, for all $i \in [2]$:

$$\begin{aligned} \bar{\Delta}_* \circ \theta_{x_1}(g\epsilon_i) &= (\theta_{x_1} \otimes Id) \circ \bar{\Delta}_*(g\epsilon_i) + (\theta_{x_2} \otimes \mu) \circ (\bar{\Delta}_* \otimes Id)(\Delta_{\sqcup}(g)\epsilon_i \otimes \epsilon_2), \\ \bar{\Delta}_* \circ \theta_{x_2}(g\epsilon_i) &= (\theta_{x_2} \otimes \mu) \circ (\bar{\Delta}_* \otimes Id)(\Delta_{\sqcup}(g)\epsilon_i \otimes \epsilon_1). \end{aligned}$$

These are formulas of Lemma 4.1 of [4], where $a_w = w\epsilon_2$, $b_w = w\epsilon_1$, $\theta_0 = \theta_{x_1}$, $\theta_1 = \theta_{x_2}$ and $\tilde{\Delta} = \bar{\Delta}_*$.

4.3 Dual of the pre-Lie product

Notations. We denote by Δ_{\sqcup} the coproduct on $T_+(V^*) \otimes (V)^{N-1}$ dual to the product \sqcup . As $\sqcup = \sqcup_1^{op}$, $\Delta_{\sqcup} = \Delta_{\sqcup_1}^{cop}$, and for all $g \in T(V)$, for all $i \in [N]$:

$$\Delta_{\sqcup}(g\epsilon_i) = \Delta_{\sqcup}(g)(\epsilon_1 \otimes \epsilon_k).$$

Proposition 22 *In $S((\mathfrak{g}'_a)_+^*)/\langle \emptyset\epsilon_1 \rangle$, for all $g \in (\mathfrak{g}'_a)_+^*$:*

$$\overline{\Delta}_\bullet(g) = \overline{\Delta}_*(g) + (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id) \circ \Delta_{\sqcup}(g).$$

Proof. Let $f, f_1, \dots, f_k \in (\mathfrak{g}'_a)_+$.

$$\begin{aligned} \langle \overline{\Delta}_\bullet(g), f \otimes f_1 \dots f_k \rangle &= \langle g, f \bullet f_1 \dots f_k \rangle \\ &= \langle g, f * f_1 \dots f_k + \sum_{p=1}^k (f * f_1 \dots \widehat{f}_p \dots f_k) \sqcup f_p \rangle \\ &= \langle \overline{\Delta}_*(g), f \otimes f_1 \dots f_k \rangle + \langle \Delta_{\sqcup}(g), \sum_{p=1}^k f * f_1 \dots \widehat{f}_p \dots f_k \otimes f_p \rangle \\ &= \langle \overline{\Delta}_*(g), f \otimes f_1 \dots f_k \rangle + \langle (\Delta_* \otimes Id) \circ \Delta_{\sqcup}(g), \sum_{p=1}^k f \otimes f_1 \dots \widehat{f}_p \dots f_k \otimes f_p \rangle \\ &= \langle \overline{\Delta}_*(g), f \otimes f_1 \dots f_k \rangle + \langle (Id \otimes \mu) \circ (\Delta_* \otimes Id) \circ \Delta_{\sqcup}(g), f \otimes f_1 \dots f_k \rangle. \end{aligned}$$

As $(\mathfrak{g}'_a, *)$ is pre-Lie, $\overline{\Delta}_\bullet(g) \in (\mathfrak{g}'_a)_+^* \otimes S((\mathfrak{g}'_a)_+^*)$ and the nondegeneracy of the pairing implies the formula. \square

Rewriting this formula in $S((\mathfrak{g}'_a)_+^*)/\langle \emptyset\epsilon_1 - 1 \rangle$:

$$\begin{aligned} \overline{\Delta}_\bullet(g\epsilon_1) &= \overline{\Delta}_*(g\epsilon_1) + (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{\sqcup}(g)(\epsilon_1 \otimes \epsilon_1)) \\ &= \overline{\Delta}_*(g\epsilon_1) + (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)((\Delta_{\sqcup}(g) - g \otimes \emptyset)(\epsilon_1 \otimes \epsilon_1)) \\ &= \overline{\Delta}_*(g\epsilon_1)(1 \otimes (1 - \emptyset\epsilon_1)) + (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{\sqcup}(g)(\epsilon_1 \otimes \epsilon_1)) \\ &= (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{\sqcup}(g)(\epsilon_1 \otimes \epsilon_1)). \end{aligned}$$

Identifying in $S((\mathfrak{g}'_a)_+^*)$:

Proposition 23 *In $S((\mathfrak{g}'_a)_+^*)/\langle \emptyset\epsilon_1 - 1 \rangle$, if $g \in T(V^*)$:*

$$\begin{aligned} \overline{\Delta}_\bullet(g\epsilon_1) &= (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{\sqcup}(g)(\epsilon_1 \otimes \epsilon_1)), \\ \text{if } i \geq 2, \overline{\Delta}_\bullet(g\epsilon_i) &= \overline{\Delta}_*(g\epsilon_i) + (Id \otimes \mu) \circ (\overline{\Delta}_* \otimes Id)(\Delta_{\sqcup}(g)(\epsilon_i \otimes \epsilon_1)), \end{aligned}$$

with the convention $\emptyset\epsilon_1 = 1$. We put $\Delta_\bullet(g) = \overline{\Delta}_\bullet(g) + 1 \otimes g$ for all $g \in (\mathfrak{g}'_a)_+^*$ and extend Δ_\bullet to $S((\mathfrak{g}'_a)_+^*)$ as an algebra morphism. This coproduct makes $S((\mathfrak{g}'_a)_+^*)$ a Hopf algebra, isomorphic to the graded dual of the enveloping algebra of $((\mathfrak{g}'_a)_+, [-, -]_*)$.

Remark. These are *mutatis mutandis* the formulas of Lemma 4.3 in [4].

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