

A NATURAL PROBABILITY MEASURE DERIVED FROM STERN'S DIATOMIC SEQUENCE

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ABSTRACT. Stern's diatomic sequence with its intrinsic repetition and refinement structure between consecutive powers of 2 gives rise to a rather natural probability measure on the unit interval. We construct this measure and show that it is purely singular continuous, with a strictly increasing, Hölder continuous distribution function. Moreover, we relate this function with the solution of the dilation equation for Stern's diatomic sequence.

1. INTRODUCTION

Stern's diatomic sequence $(s(n))_{n \in \mathbb{N}_0}$, also known as the Stern–Brocot sequence, is defined by $s(0) = 0$, $s(1) = 1$ together with the recursions

$$(1.1) \quad s(2n) = s(n) \quad \text{and} \quad s(2n+1) = s(n) + s(n+1)$$

for $n \in \mathbb{N}$. This well-studied sequence has fascinating properties; see entry A002487 of [12] for a concise summary with many references and links. The initial values and recursions in Eq. (1.1) allow one to determine the value $s(n)$ based on the binary expansion of n . In particular, if

$$(1.2) \quad S_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and if $(n)_2 = b_k b_{k-1} \cdots b_1 b_0$ is the binary expansion of n , one has

$$(1.3) \quad s(n) = v^T S_{b_k} S_{b_{k-1}} \cdots S_{b_1} S_{b_0} v.$$

Sequences with a linear representation as provided by Eqs. (1.2) and (1.3) are called b -regular sequences, where b is the base ($b = 2$ for Stern's diatomic sequence). Regular sequences were introduced by Allouche and Shallit [1] as a mathematical generalisation of sequences that are generated by deterministic finite automata, such as the Thue–Morse sequence.

Here, we reconsider the self-similarity type property of Stern's diatomic sequence, which manifests itself in the fact that the sequence, in the range from 2^n to 2^{n+1} , can be seen as a stretched and interlaced version of what it is between 2^{n-1} and 2^n . In particular, as follows from a simple induction argument, one has the well-known summation relation

$$(1.4) \quad \sum_{m=2^n}^{2^{n+1}-1} s(m) = 3^n,$$

which holds for all $n \in \mathbb{N}_0$. Therefore, if we define

$$(1.5) \quad \mu_n := 3^{-n} \sum_{m=0}^{2^n-1} s(2^n + m) \delta_{m/2^n},$$

where δ_x denotes the unit Dirac measure at x , we can view $(\mu_n)_{n \in \mathbb{N}_0}$ as a sequence of probability measures on the 1-torus, the latter written as $\mathbb{T} = [0, 1)$ with addition modulo 1. Here, we have simply re-interpreted the values of the Stern sequence between 2^n and $2^{n+1} - 1$ as weights of a pure point probability measure on \mathbb{T} with $\text{supp}(\mu_n) = \{\frac{m}{2^n} : 0 \leq m < 2^n\}$.

In the remainder of this article, we will study the sequence $(\mu_n)_{n \in \mathbb{N}_0}$ and its limit, as well as various properties of the latter and how they relate to other known results on Stern's sequence. Our approach is motivated by the similarity of Eq. (1.1) with the recursion relation for the Fourier–Bohr coefficients of the classic Thue–Morse measure and some of its generalisations; see [3, Sec. 10.1] as well as [2] and references therein for background.

2. THE PROBABILITY MEASURE

Each μ_n is a probability measure on \mathbb{T} , which in particular implies that it is Fourier (or Fourier–Stieltjes) transformable, where

$$k \mapsto \widehat{\mu}_n(k) := \int_{\mathbb{T}} e^{-2\pi i k x} d\mu_n(x)$$

defines a continuous function on the dual group $\widehat{\mathbb{T}} = \mathbb{Z}$; see [11, Sec. 4.4] for background.

Remark 2.1. It is sometimes useful to ‘periodise’ the measure μ_n to $\nu_n := \mu_n * \delta_{\mathbb{Z}}$ and interpret it as a translation bounded measure on \mathbb{R} . Its Fourier transform is still well defined, via the Poisson summation formula $\widehat{\delta_{\mathbb{Z}}} = \delta_{\mathbb{Z}}$ and the convolution theorem; compare [3, Prop. 8.5 and Sec. 9.2]. It then reads

$$\widehat{\nu}_n = \widehat{\mu}_n \delta_{\mathbb{Z}} = \sum_{x \in \mathbb{Z}} \widehat{\mu}_n(x) \delta_x,$$

where $\widehat{\mu}_n$ now defines a (continuous) function on \mathbb{R} . The values of $\widehat{\mu}_n$ in the complement of \mathbb{Z} are irrelevant, but still useful; compare [3, Sec. 9.2.4] for a general interpretation of this phenomenon. \diamond

Let us analyse the functions $\widehat{\mu}_n$. Clearly, one has $\widehat{\mu}_0 \equiv 1$ and

$$\widehat{\mu}_1(k) = \frac{1}{3}(1 + 2 \cos(\pi k)) = \begin{cases} 1, & k \text{ even,} \\ -\frac{1}{3}, & k \text{ odd,} \end{cases}$$

where $\widehat{\delta}_x(k) = e^{-2\pi i k x}$ was used in the calculation. More generally, by induction on the basis of Eqs. (1.5) and (1.1), one finds

$$(2.1) \quad \widehat{\mu}_n(k) = \prod_{m=1}^n \frac{1}{3} \left(1 + 2 \cos\left(\frac{2\pi k}{2^m}\right) \right)$$

for $n \in \mathbb{N}_0$ and $k \in \mathbb{Z}$, where the empty product is defined to be 1 as usual. Since

$$\frac{1}{3}(1 + 2 \cos(x)) = 1 + \frac{1}{3}x^2 + O(x^4)$$

as $|x| \searrow 0$, one can apply standard arguments to show that, for any fixed k , the sequence $(\widehat{\mu}_n(k))_{n \in \mathbb{N}_0}$ converges. In fact, one has compact convergence, both for $k \in \mathbb{Z}$ and for $k \in \mathbb{R}$. The latter viewpoint is useful in the context of Remark 2.1, and will be vital later.

Let us now formulate some consequences, where the location x of the Dirac measure δ_x is always understood to be an element of \mathbb{T} , hence taken modulo 1. This is important to give the correct meaning to the convolution identity $\delta_x * \delta_y = \delta_{x+y}$ on \mathbb{T} .

Proposition 2.2. *The sequence $(\mu_n)_{n \in \mathbb{N}_0}$ of probability measures on \mathbb{T} converges weakly to a probability measure μ . In particular, one has $\mu_0 = \delta_0$ and $\mu_n = \ast_{m=1}^n \frac{1}{3}(\delta_0 + \delta_{2^{-m}} + \delta_{-2^{-m}})$ for $n \geq 1$. The weak limit as $n \rightarrow \infty$ is given by the convergent infinite convolution product*

$$\mu = \ast_{m \geq 1} \frac{1}{3}(\delta_0 + \delta_{2^{-m}} + \delta_{-2^{-m}}).$$

Its Fourier transform $\widehat{\mu}$ is given by $\widehat{\mu}(k) = \prod_{m \geq 1} \frac{1}{3}(1 + 2 \cos(2\pi k/2^m))$ for $k \in \mathbb{Z}$. Moreover, this infinite product is also well-defined on \mathbb{R} , where it converges compactly.

Proof. Due to the convergence of the sequences $(\widehat{\mu}_n(k))_{n \in \mathbb{N}_0}$, the first claim can be seen as a consequence of Levy's continuity theorem [6, Thm. 3.14]. The explicit formula for μ_n follows from Eq. (2.1) with a simple calculation via the inverse of the convolution theorem.

The representation of μ is clear, with weak convergence, as is the formula for $\widehat{\mu}(k)$ with compact convergence of the infinite product as mentioned above. \square

Since μ is a probability measure on \mathbb{T} , Bochner's theorem [6, Thm. 3.12] implies that $k \mapsto \widehat{\mu}(k)$ defines a (continuous) positive definite function on \mathbb{Z} . In particular, one has

$$(2.2) \quad \widehat{\mu}(-k) = \overline{\widehat{\mu}(k)} = \widehat{\mu}(k)$$

for all $k \in \mathbb{Z}$. Here, $\widehat{\mu}$ is real (which gives the second equality) as a consequence of the invariance of μ on \mathbb{T} under the reflection $x \mapsto -x$, again taken modulo 1, while the normalisation of μ corresponds to $\widehat{\mu}(0) = 1$. The symmetry relation also implies that

$$\widehat{\mu}(k) = \int_{\mathbb{T}} e^{2\pi i k x} d\mu(x) = \int_{\mathbb{T}} \cos(2\pi k x) d\mu(x)$$

holds for all $k \in \mathbb{Z}$.

The representation of μ as an infinite convolution product of pure point measures allows us to use a result by Jessen and Wintner [10, Thm. 35] which tells us that the spectral type of μ is pure. By the general Lebesgue decomposition theorem, this means that μ is either a pure point measure, or purely singular continuous, or purely absolutely continuous — but not a mixture. Its remains to determine the type, for which we need a scaling property of the Fourier–Bohr coefficients $\widehat{\mu}(k)$.

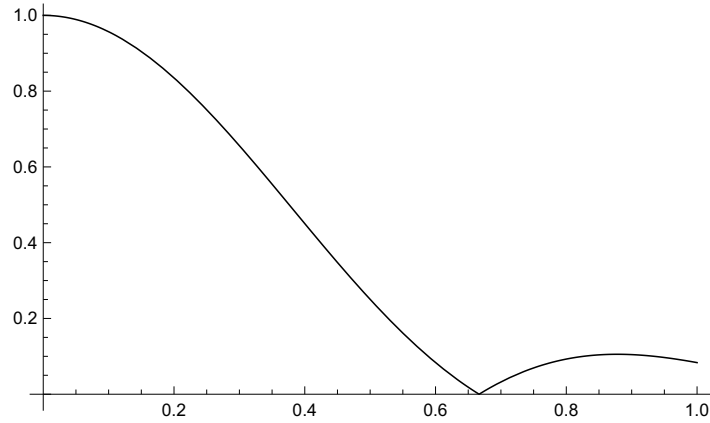


FIGURE 1. The function $|\hat{\mu}(\kappa)|$ for $0 \leq \kappa \leq 1$.

Lemma 2.3. *For all $k \in \mathbb{R}$, the coefficients $\hat{\mu}(k)$ from Proposition 2.2 satisfy*

$$\hat{\mu}(2k) = \frac{1}{3}(1 + 2 \cos(2\pi k)) \hat{\mu}(k).$$

In particular, $\hat{\mu}(2k) = \hat{\mu}(k)$ for all $k \in \mathbb{Z}$.

Proof. Since the infinite product representation of $\hat{\mu}(k)$ is absolutely converging by standard arguments, one may simply calculate

$$\hat{\mu}(2k) = \prod_{m \geq 1} \frac{1}{3}(1 + 2 \cos(2\pi k/2^{m-1})) = \frac{1}{3}(1 + 2 \cos(2\pi k)) \hat{\mu}(k),$$

which obviously implies both relations. \square

The relationship for real k in Lemma 2.3 has some immediate implications. Let us first note that, for each positive integer N , one has

$$(2.3) \quad \hat{\mu}(2^N k) = \hat{\mu}(k) \prod_{m=1}^N \frac{1}{3}(1 + 2 \cos(2^m \pi k)).$$

For the proof of our next result, we will require information about $|\hat{\mu}(k)|^2$. Since the product on the right hand side of Eq. (2.3) is symmetric around $\frac{1}{2}$ in $[0, 1]$ for each $N \in \mathbb{N}$, we can profit from relating values of $\hat{\mu}(k)$ with $0 \leq k \leq 2^N$ to values of $\hat{\mu}(\kappa)$ with $\kappa \in [0, 1]$; see Figure 1 for an illustration of $|\hat{\mu}(\kappa)|$. For larger values of κ , the function values $\hat{\mu}(\kappa)$ are generally small, with (bounded) negative excursions at powers of 2.

Let us first observe that, for $\kappa \in [0, \frac{1}{2}]$, we clearly have

$$(2.4) \quad \frac{\hat{\mu}(2^N(1 - \kappa))}{\hat{\mu}(2^N \kappa)} = \frac{\hat{\mu}(1 - \kappa)}{\hat{\mu}(\kappa)}.$$

Since $|\hat{\mu}(\kappa)| \geq |\hat{\mu}(1 - \kappa)|$ on this interval, we obtain the estimate

$$(2.5) \quad |\hat{\mu}(2^N(1 - \kappa))| \leq |\hat{\mu}(2^N \kappa)|,$$

again for $\kappa \in [0, \frac{1}{2}]$, which implies $|\hat{\mu}(2^N - k)| \leq |\hat{\mu}(k)|$ for $0 \leq k \leq 2^{N-1}$.

Theorem 2.4. *The probability measure μ from Proposition 2.2 is purely singular continuous.*

Proof. Clearly, $\mu \neq 0$, and the spectral type of μ is pure by [10, Thm. 35]. Thus, we can prove the result by showing that μ is neither absolutely continuous nor pure point.

Since $0 \neq \widehat{\mu}(1) = \widehat{\mu}(-1) \approx -0.083432$ and $\widehat{\mu}(2k) = \widehat{\mu}(k)$ for all $k \in \mathbb{Z}$ by Lemma 2.3, the Fourier coefficients cannot decay as $|k| \rightarrow \infty$. Consequently, μ cannot be absolutely continuous by the Riemann–Lebesgue lemma; compare [11, Thm. 4.4.3].

To rule out a pure point nature of μ , we employ Wiener's criterion; see [3, Prop. 8.9]. Because $\widehat{\mu}(k) = \widehat{\mu}(-k)$ for all k , the theorem will follow if $\sum_{k \leq x} |\widehat{\mu}(k)|^2 = o(x)$ as $x \rightarrow \infty$, where here and below the summation for k starts at 0 unless specified otherwise. Moreover, when $x \in [2^N, 2^{N+1}]$, one has the estimate

$$\frac{1}{x} \sum_{k \leq x} |\widehat{\mu}(k)|^2 \leq \frac{1}{2^N} \sum_{k=0}^{2^{N+1}} |\widehat{\mu}(k)|^2 \leq \frac{2}{2^{N+1}} \sum_{k=0}^{2^{N+1}} |\widehat{\mu}(k)|^2,$$

which implies that it suffices to show $\sum_{k \leq 2^{N+1}} |\widehat{\mu}(k)|^2 = o(2^N)$.

To this end, note that using Eq. (2.4) we have

$$\sum_{k \leq 2^N} |\widehat{\mu}(k)|^2 = \sum_{k \leq \frac{3}{5} 2^N} |\widehat{\mu}(k)|^2 + \sum_{\frac{3}{5} 2^N < k \leq 2^N} |\widehat{\mu}(k)|^2 \leq \sum_{k \leq \frac{3}{5} 2^N} |\widehat{\mu}(k)|^2 + \sum_{k \leq \frac{2}{5} 2^N} |r(k) \widehat{\mu}(k)|^2$$

where, due to $0 \leq k \leq \frac{2}{5} 2^N$,

$$\begin{aligned} |r(k)| &= \left| \frac{\widehat{\mu}(2^N - k)}{\widehat{\mu}(k)} \right| = \left| \frac{\widehat{\mu}(1 - k/2^N)}{\widehat{\mu}(k/2^N)} \right| \leq \frac{\max_{0 \leq \ell \leq \frac{2}{5} 2^N} |\widehat{\mu}(1 - \ell/2^N)|}{\min_{0 \leq \ell \leq \frac{2}{5} 2^N} |\widehat{\mu}(\ell/2^N)|} \\ &\leq \frac{\max_{\kappa \in [\frac{3}{5}, 1]} |\widehat{\mu}(\kappa)|}{\min_{\kappa \in [0, \frac{2}{5}]} |\widehat{\mu}(\kappa)|} = \frac{|\widehat{\mu}(t)|}{|\widehat{\mu}(\frac{2}{5})|} = \frac{0.105423890\dots}{0.450342617\dots} < \frac{1}{4}, \end{aligned}$$

with $t = 0.877996139\dots$ being the position of the unique (relative) maximum of $|\widehat{\mu}|$ in the interval $[\frac{3}{5}, 1]$; compare Figure 1. So, we get

$$\begin{aligned} \sum_{k \leq 2^N} |\widehat{\mu}(k)|^2 &< \sum_{k \leq \frac{3}{5} 2^N} |\widehat{\mu}(k)|^2 + \frac{1}{16} \sum_{k \leq \frac{2}{5} 2^N} |\widehat{\mu}(k)|^2 \\ &= \frac{17}{16} \sum_{k \leq 2^{N-1}} |\widehat{\mu}(k)|^2 + \sum_{2^{N-1} < k \leq \frac{3}{5} 2^N} |\widehat{\mu}(k)|^2 - \frac{1}{16} \sum_{\frac{2}{5} 2^N < k \leq 2^{N-1}} |\widehat{\mu}(k)|^2. \end{aligned}$$

To obtain an upper estimate of the last two sums in the previous line, we write $2^N = 2 \cdot 2^{N-1}$ and use Eq. (2.5) to get

$$\begin{aligned}
& \sum_{2^{N-1} < k \leq \frac{6}{5} 2^{N-1}} |\widehat{\mu}(k)|^2 - \frac{1}{16} \sum_{\frac{4}{5} 2^{N-1} < k \leq 2^{N-1}} |\widehat{\mu}(k)|^2 \\
& \leq \sum_{\frac{4}{5} 2^{N-1} < k \leq 2^{N-1}} |\widehat{\mu}(k)|^2 - \frac{1}{16} \sum_{\frac{4}{5} 2^{N-1} < k \leq 2^{N-1}} |\widehat{\mu}(k)|^2 \\
& = \frac{15}{16} \sum_{\frac{4}{5} 2^{N-1} < k \leq 2^{N-1}} |\widehat{\mu}(k)|^2 \\
& \leq \frac{15}{16} \sum_{k \leq 2^{N-1}} |\widehat{\mu}(k)|^2 - \frac{15}{16} \sum_{k \leq 2^{N-2}} |\widehat{\mu}(k)|^2.
\end{aligned}$$

With $\Sigma_N := \frac{1}{2^N} \sum_{k \leq 2^N} |\widehat{\mu}(k)|^2$, inserting the last estimate into the previous one gives

$$(2.6) \quad \Sigma_N < \Sigma_{N-1} - \frac{15}{64} \Sigma_{N-2}.$$

Substituting the recursive inequality (2.6) for Σ_{N-1} with $N \geq 2$ results in

$$\Sigma_N < \frac{49}{64} \Sigma_{N-2} - \frac{15}{64} \Sigma_{N-3} \leq \frac{49}{64} \Sigma_{N-2},$$

where $\Sigma_{-1} := 0$. Consequently, for $N \geq 2$, we obtain

$$0 \leq \Sigma_N < \left(\frac{7}{8}\right)^{N-1} \max\{\Sigma_0, \Sigma_1\} \xrightarrow{N \rightarrow \infty} 0,$$

which proves the absence of pure point components for μ , and completes our argument. \square

3. THE DISTRIBUTION FUNCTION

Here, we are in a situation that is somewhat similar to that of the singular continuous diffraction measures known from the spectral theory of certain substitution systems; compare [4] and references therein. First of all, the distribution function

$$(3.1) \quad F(x) := \mu([0, x])$$

with $x \in \mathbb{T}$, where $F(1) := 1$, defines a continuous function that is monotonically increasing on the unit interval. It is illustrated in Figure 2.

It is clear from Theorem 2.4 that F is a non-decreasing, continuous function. Next, we will show that F is strictly increasing. To do this, we need some specific asymptotics on the summatory function of Stern's sequence. In what follows, for a positive real number y , we use $[y]$ and $\langle y \rangle$ to denote the integer and the fractional part¹ of y , respectively. In particular, $y = [y] + \langle y \rangle$. Also, we write $\log_2(y)$ for the base-2 logarithm of y and let $\tau = (1 + \sqrt{5})/2$ denote the golden ratio.

¹We use this version for the fractional part because the more common notation, $\{y\}$, represents a singleton set in our measure-theoretic arguments.

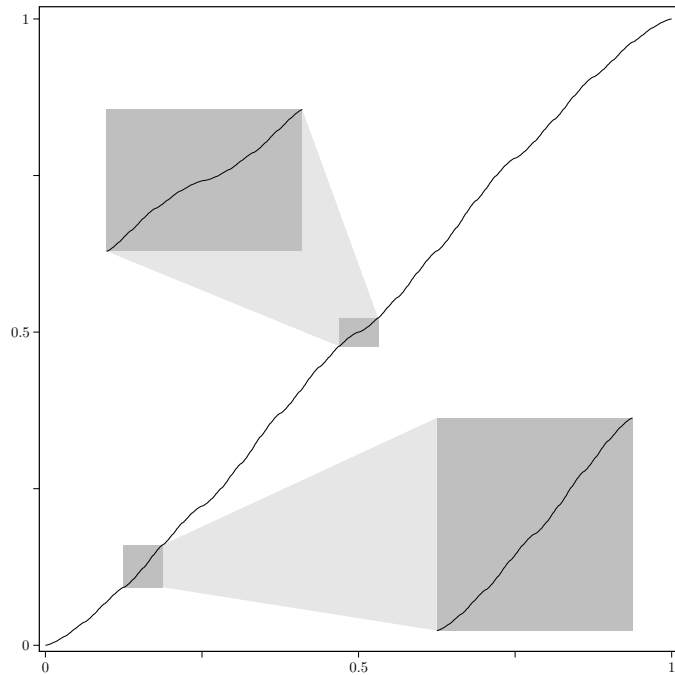


FIGURE 2. The distribution function F of the purely singular continuous probability measure μ derived from Stern's diatomic sequence.

Proposition 3.1. *If $(s(n))_{n \geq 0}$ is Stern's diatomic sequence, its summatory function satisfies*

$$\sum_{n \leq x} s(n) = 3^{\lfloor \log_2(x) \rfloor + 1} f_0(2^{(\log_2(x)) - 1}) + O(x^{\log_2(3/\tau) + \varepsilon}),$$

where the function f_0 is Hölder continuous with exponent $\log_2(3/\tau)$. Moreover, $f_0(t)$ is the first coordinate of the column vector $f(t) = (f_0(t), f_1(t))^T$ that is the unique solution of the dilation equation

$$f(t) = \frac{1}{3}(S_0 f(2t) + S_1 f(2t - 1)),$$

with the conditions $f_0(t) = f_1(t) = 0$ for $t \leq 0$ and $f_0(t) = f_1(t) = \frac{1}{2}$ for $t \geq 1$.

Sketch of proof. This result follows from a method of Dumas. In particular, it follows from an application of [9, Thm. 3], when using the linear representation of the Stern sequence (1.2) along with the facts that the set $\{S_0, S_1\}$ satisfies the finiteness property with joint spectral radius equal to the golden ratio (see [7, 8]), that $Q := S_0 + S_1$ has eigenvalues 3 and 1 with Jordan basis $v_3 = (\frac{1}{2}, \frac{1}{2})^T$ and $v_1 = (\frac{1}{2}, -\frac{1}{2})^T$, and that $v = v_3 + v_1$. \square

Theorem 3.2. *The distribution function F from Eq. (3.1) is strictly increasing.*

Proof. Let $x \in [0, 1)$ and $\varepsilon > 0$ with $x + \varepsilon \leq 1$ be arbitrary. Since F is continuous and non-decreasing, we have that

$$F(x + \varepsilon) - F(x) = \mu([0, x + \varepsilon]) - \mu([0, x]) = \mu([x, x + \varepsilon]) \geq 0.$$

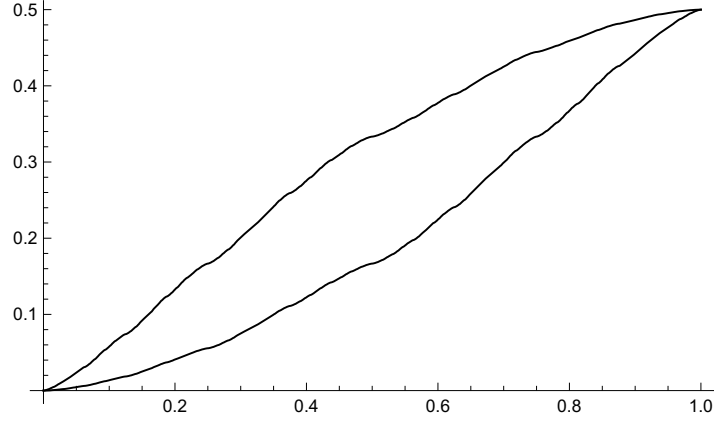


FIGURE 3. The functions f_0 (lower curve) and f_1 (upper curve).

Thus, to prove the theorem, it is enough to show that $\mu([x, x + \varepsilon]) > 0$. Note also that, since the dyadic rationals are dense in $[0, 1]$, there exist positive integers m and k so that $[2m/2^k, (2m + 1)/2^k] \subseteq [x, x + \varepsilon]$. Thus, to show that $\mu([x, x + \varepsilon]) > 0$, it suffices to show that $\mu([2m/2^k, (2m + 1)/2^k]) > 0$.

To this end, observe that, for any real number $y \in [2^n, 2^{n+1})$, we have $\lfloor \log_2(y) \rfloor = n$. Thus, combining Eq. (1.5) with Proposition 3.1 gives

$$\begin{aligned} \mu_n([2m/2^k, (2m + 1)/2^k]) &= 3^{-n} \sum_{\ell=2^n+2^n(2m/2^k)}^{2^n+2^n((2m+1)/2^k)} s(\ell) \\ &= 3f_0(2^{\langle \log_2(2^n+2^n((2m+1)/2^k)) \rangle - 1}) - 3f_0(2^{\langle \log_2(2^n+2^n(2m/2^k)) \rangle - 1}) \\ &\quad + O_n\left(3^{-n}(2^n + 2^n((2m + 1)/2^k))^{\log_2(\tau) + \varepsilon}\right), \end{aligned}$$

where we have used the notation O_n to indicate the dependence on n . Since m and k are fixed, one has

$$O_n((2^n + 2^n((2m + 1)/2^k))^{\log_2(\tau) + \varepsilon} 3^{-n}) = O_n(2^{n(\log_2(\tau) + \varepsilon)} 3^{-n}) = o_n(1), \quad \text{as } n \rightarrow \infty,$$

and, for any $y \in (0, 1)$,

$$\langle \log_2(2^n + 2^n y) \rangle = \langle n + \log_2(1 + y) \rangle = \log_2(1 + y).$$

Continuing the above then gives

$$\mu_n([2m/2^k, (2m + 1)/2^k]) = 3f_0\left(\frac{((2m + 1)/2^k) + 1}{2}\right) - 3f_0\left(\frac{(2m/2^k) + 1}{2}\right) + o_n(1).$$

Taking the limit as n goes to infinity gives

$$(3.2) \quad \mu([2m/2^k, (2m + 1)/2^k]) = 3f_0\left(\frac{2^k + 2m + 1}{2^{k+1}}\right) - 3f_0\left(\frac{2^k + 2m}{2^{k+1}}\right).$$

We can now use the dilation equation for f to determine the value of the right-hand side of (3.2) based on the binary expansion of the numerators of the dyadic rationals involved. In particular, if $b_i \in \{0, 1\}$ for $i \in \{0, 1, \dots, k\}$, we have

$$\begin{aligned}
& f\left(\frac{b_k 2^k + b_{k-1} 2^{k-1} + \dots + b_1 2 + b_0}{2^{k+1}}\right) \\
&= \begin{cases} \frac{1}{6} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} f\left(\frac{b_{k-1} 2^{k-1} + \dots + b_1 2 + b_0}{2^k}\right), & \text{if } b_k = 1 \\ \frac{1}{3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} f\left(\frac{b_{k-1} 2^{k-1} + \dots + b_1 2 + b_0}{2^k}\right), & \text{if } b_k = 0 \end{cases} \\
&= \frac{b_k}{6} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left[\frac{b_k}{3} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \frac{1-b_k}{3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right] f\left(\frac{b_{k-1} 2^{k-1} + \dots + b_1 2 + b_0}{2^k}\right) \\
&= \frac{b_k}{6} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{b_k} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{1-b_k} f\left(\frac{b_{k-1} 2^{k-1} + \dots + b_1 2 + b_0}{2^k}\right).
\end{aligned}$$

Note next that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{b_k} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{1-b_k} = S_{b_k},$$

where S_{b_k} is as given in the linear representation of $s(n)$. Thus setting

$$u_{b_k} := \frac{b_k}{2} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{b_k}{2} \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

we have

$$(3.3) \quad \begin{pmatrix} f\left(\frac{b_k 2^k + \dots + b_1 2 + b_0}{2^{k+1}}\right) \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} S_{b_k} & u_{b_k} \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} f\left(\frac{b_{k-1} 2^{k-1} + \dots + b_1 2 + b_0}{2^k}\right) \\ 1 \end{pmatrix}.$$

Iterating Eq. (3.3) we get

$$(3.4) \quad \begin{pmatrix} f\left(\frac{b_k 2^k + \dots + b_1 2 + b_0}{2^{k+1}}\right) \\ 1 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} S_{b_k} & u_{b_k} \\ 0 & 0 & 3 \end{pmatrix} \dots \begin{pmatrix} S_{b_1} & u_{b_1} \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} f\left(\frac{b_0}{2}\right) \\ 1 \end{pmatrix}.$$

For the integer $2m \in [0, 2^k)$, consider the binary expansions $2^k + 2m + 1 = 1b_{k-1} \dots b_1 1$ and $2^k + 2m = 1b_{k-1} \dots b_1 0$ in obvious notation. Then, using Eqs. (3.2) and (3.4) and observing

that $f(\frac{1}{2}) = (\frac{1}{6}, \frac{1}{3})^T$, we have

$$\begin{aligned} \mu([2m/2^k, (2m+1)/2^k]) &= (3^{1-k}, 0, 0) \begin{pmatrix} S_1 & u_1 \\ 0 & 0 & 3 \end{pmatrix} \cdots \begin{pmatrix} S_{b_1} & u_{b_1} \\ 0 & 0 & 3 \end{pmatrix} \left[\begin{pmatrix} f(\frac{1}{2}) \\ 1 \end{pmatrix} - \begin{pmatrix} f(0) \\ 1 \end{pmatrix} \right] \\ &= \frac{1}{6} (3^{1-k}, 0, 0) \begin{pmatrix} S_1 & u_1 \\ 0 & 0 & 3 \end{pmatrix} \cdots \begin{pmatrix} S_{b_1} & u_{b_1} \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \\ &= \frac{1}{6} (3^{1-k}, 0) S_1 S_{b_{k-1}} \cdots S_{b_1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \geq \frac{3}{6 \cdot 3^k} > 0, \end{aligned}$$

which proves the theorem. \square

Unlike other distribution functions of singular continuous measures, the distribution function F from Eq. (3.1) looks relatively ‘smooth’; this is quantified in the following corollary.

Corollary 3.3. *The distribution function F from Eq. (3.1) is Hölder continuous with exponent $\log_2(3/\tau)$.*

Proof. The distribution function F inherits this exponent from the dilation equation for the associated function f . One can see this by following through the proof of Theorem 3.2 through Eq. (3.2) using the interval $[x, y] \subseteq [0, 1]$. \square

Further, it will be an interesting question to analyse some of the scaling properties of μ , for instance in analogy to the treatment of the Thue–Morse measure in [5]. For this, it will be helpful to understand the precise relation between the measure μ and the dilation equation from Proposition 3.1. One such relation can be stated as follows.

Corollary 3.4. *For $x \in [0, 1]$, the distribution function F from Eq. (3.1) satisfies*

$$F(x) = 3(f_0(\frac{1+x}{2}) - f_0(\frac{1}{2})) = f_0(x) + f_1(x),$$

where f_0 and f_1 are the functions from Proposition 3.1.

Proof. The first identity is a rather direct consequence of Eq. (3.2) in the proof of Proposition 3.1. While $f_0(\frac{1}{2}) = \frac{1}{6}$, the dilation equation for the functions f_i gives

$$f_0(\frac{1+x}{2}) = \frac{1}{6} + \frac{1}{3}(f_0(x) + f_1(x)).$$

This implies the second identity. \square

APPENDIX

The purpose of this appendix is to provide two other proofs of the continuity of the measure μ from Proposition 2.2, in view of their potential usefulness in other applications to recursive sequences of a similar kind.

The first one starts from the observation that $\mu = \star_{m \geq 1} \nu_m$, with the probability measures $\nu_m = \frac{1}{3}(\delta_0 + \delta_{2^{-m}} + \delta_{-2^{-m}})$ on \mathbb{T} as building blocks, is absolutely convergent in the weak

topology, which is to say that that it is weakly convergent to the same limit in any order of its terms. This follows from [10, Thm. 6], where one has to notice that

$$M_r(\nu_m) := \int_{\mathbb{T}} x^r d\nu_m(x) = \frac{1}{3}(0^r + 2^{-rm} + (-2)^{-rm})$$

for $r \geq 0$. In particular, $M_1(\nu_m) = 0$ and $M_2(\nu_m) = \frac{2}{3}4^{-m}$, so both $\sum_{m=1}^{\infty} |M_1(\nu_m)|$ and $\sum_{m=1}^{\infty} M_2(\nu_m)$ are finite, which ensures absolute convergence.

If we now assume, contrary to the claim, that μ fails to be continuous, there must be an $x \in \mathbb{T}$ with $\mu(\{x\}) > 0$. Now, rewrite μ as $\mu = \nu_n * \rho_n$ with $\rho_n := \ast_{m \neq n} \nu_m$, which is possible for any $n \in \mathbb{N}$. This implies

$$\mu(\{x\}) = (\nu_n * \rho_n)(\{x\}) = \frac{1}{3}(\rho_n(\{x\}) + \rho_n(\{x + 2^{-n}\}) + \rho_n(\{x - 2^{-n}\})).$$

Analogously, one obtains

$$\mu(\{x \pm 2^{-n}\}) = \frac{1}{3}(\rho_n(\{x\}) + \rho_n(\{x \pm 2^{-n}\}) + \rho_n(\{x \pm 2 \cdot 2^{-n}\})).$$

Together with the previous relation, this implies the estimate²

$$(3.5) \quad \mu(\{x\}) \leq \mu(\{x + 2^{-n}\}) + \mu(\{x - 2^{-n}\}).$$

Now, choose $r \in \mathbb{N}$ with $\mu(\{x\}) > \frac{1}{r}$, and select r integers $j_1 < j_2 < \dots < j_r$ with $j_1 \geq 2$. Since μ is a probability measure on \mathbb{T} , we then get

$$\begin{aligned} 1 &\geq \mu\left(\bigcup_{1 \leq q \leq r} (\{x - 2^{-j_q}\} \cup \{x + 2^{-j_q}\})\right) \\ &= \sum_{q=1}^r (\mu(\{x - 2^{-j_q}\}) + \mu(\{x + 2^{-j_q}\})) > r\mu(\{x\}) \geq r \frac{1}{r} = 1. \end{aligned}$$

This contradiction shows that μ is continuous.

The second alternative proof employs Wiener's criterion again. Observing (without proof) that the inequalities

$$(3.6) \quad |\widehat{\mu}(2k+1)| \leq \frac{1}{2}|\widehat{\mu}(k) + \widehat{\mu}(k+1)| \quad \text{and} \quad \widehat{\mu}(2k+1)(\widehat{\mu}(2k) + \widehat{\mu}(2k+2)) \leq 0$$

hold for all $k \in \mathbb{Z}$, one can proceed as follows. With $\Sigma(N) := \sum_{k=-N}^N \widehat{\mu}(k)^2$, one has

$$\begin{aligned} \Sigma(4N) &= \sum_{k=-2N}^{2N} \widehat{\mu}(2k)^2 + \sum_{k=-2N}^{2N-1} \widehat{\mu}(2k+1)^2 \leq \Sigma(2N) + \frac{1}{4} \sum_{k=-2N}^{2N-1} (\widehat{\mu}(k) + \widehat{\mu}(k+1))^2 \\ &= \frac{3}{2}\Sigma(2N) - \frac{\widehat{\mu}(2N)^2}{2} + \frac{1}{2} \sum_{k=-2N}^{2N-1} \widehat{\mu}(k) \widehat{\mu}(k+1) \\ &\leq \frac{3}{2}\Sigma(2N) + \frac{1}{2} \sum_{k=-N}^{N-1} \widehat{\mu}(2k+1)(\widehat{\mu}(2k) + \widehat{\mu}(2k+2)) \leq \frac{3}{2}\Sigma(2N), \end{aligned}$$

²More generally, one has the relation $\nu_n \leq (\delta_{2^{-n}} + \delta_{-2^{-n}}) * \nu_n$ as an inequality between positive measures, and hence — by convolution with ρ_n — also $\mu \leq (\delta_{2^{-n}} + \delta_{-2^{-n}}) * \mu$, which implies Eq. (3.5).

where Lemma 2.3 was used several times, while Eq. (3.6) was needed for the two inequalities.

This estimate implies $\Sigma(2^{k+1}) \leq (\frac{3}{2})^k \Sigma(2)$ and thus also $\Sigma(N) \leq C(\frac{3}{2})^{\log_2(N)}$ for some positive constant C . With $\alpha = \log_2(3/2) < 1$, one then obtains the asymptotic behaviour $\frac{1}{N}\Sigma(N) = O(1/N^{1-\alpha})$ as $N \rightarrow \infty$, which implies the absence of pure point components in μ by Wiener's criterion. This approach has the advantage (over the Jessen–Wintner argument) that one also gets a lower bound on the Hölder exponent from Corollary 3.3

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REFERENCES

- [1] Allouche J-P and Shallit J, The ring of k -regular sequences, *Theor. Comput. Sci.* **98** (1992) 163–197.
- [2] Baake M, Gähler F and Grimm U, Spectral and topological properties of a family of generalised Thue–Morse sequences, *J. Math. Phys.* **53** (2012) 032701 (24pp); [arXiv:1201.1423](https://arxiv.org/abs/1201.1423).
- [3] Baake M and Grimm U, *Aperiodic Order. Vol. 1: A Mathematical Invitation*, Cambridge University Press, Cambridge (2013).
- [4] Baake M and Grimm U, Squirals and beyond: Substitution tilings with singular continuous spectrum, *Ergodic Th. & Dynam. Syst.* **34** (2014) 1077–1102; [arXiv:1205.1384](https://arxiv.org/abs/1205.1384).
- [5] Baake M, Grimm U and Nilsson J, Scaling of the Thue–Morse diffraction measure, *Acta Phys. Pol. A* **126** (2014) 431–434; [arXiv:1311.4371](https://arxiv.org/abs/1311.4371).
- [6] Berg C and Forst G, *Potential Theory on Locally Compact Abelian Groups*, Springer, Berlin (1975).
- [7] Coons M, Regular sequences and the joint spectral radius, *Intern. J. Found. Comput. Sci.* **28** (2017) 135–140; [arXiv:1511.07535](https://arxiv.org/abs/1511.07535).
- [8] Coons M and Tyler J, The maximal order of Stern's diatomic sequence, *Moscow J. Comb. Number Th.* **4** (2014) 3–14; [arXiv:1307.1521](https://arxiv.org/abs/1307.1521).
- [9] Dumas P, Joint spectral radius, dilation equations, and asymptotic behavior of radix-rational sequences, *Lin. Alg. Appl.* **438** (2013) 2107–2126.
- [10] Jessen B and Wintner A, Distribution functions and the Riemann zeta function, *Trans. Amer. Math. Soc.* **38** (1935) 48–88.
- [11] Reiter H and Stegeman J D, *Classical Harmonic Analysis and Locally Compact Groups*, Clarendon Press, Oxford (2000).
- [12] Sloane N J A (ed.), The On-Line Encyclopedia of Integer Sequences, available at <https://oeis.org/>

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