# Integrability properties of renormalization group flow

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**Abstract**

We consider the Polchinski RG equation for a theory of matrix scalar fields interacting with single trace operators and show that it can be written in a Hamiltonian form for a specific choice of the cut-off function. The obtained Hamiltonian equations are a non-linear generalization of the shock-wave equation that is known to be integrable. We present an infinite tower of conserved quantities and recover their relation to Motzkin polynomials.

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# 1 Introduction

Since the discovery of the duality correspondence between gravity theories in a bulk space and gauge theories on its boundary, and in particular the AdS/CFT correspondence, there has been many works putting forward the idea that the renormalization group equations on the field theory side should generate dynamic equations on the gravity side [1, 2, 3, 4, 5, 6]. The most promising approach to address these constructions has happened to be based on the so-called functional (or exact) renormalization group approach (ERG). In this approach one works with the full partition function containing all possible operators appearing during renormalization group evolution – marginal, relevant and irrelevant.

Following the initial idea of the Wilsonian renormalization group one introduces a cut-off parameter $\Lambda$ and splits all fields in the theory into high-momentum and low-momentum modes, then integrating out the former. Imposing the condition that physics driven by the partition function should not change under such manipulations, one gets equations governing change of the operators with respect to the cut-off parameter $\Lambda$, hence renormalization group equations [7, 8] (see [9, 10, 11] for a review). This work is organized around the so-called Polchinski equation that is an exact renormalization group flow equation on operators entering the action of interactions $\mathcal{S}_I$. The next section is devoted to review of some technical details of this construction.

There are two main approaches when constructing the full base of possible operators in a theory: vertex and derivative expansion. In the vertex expansion approach one writes a series of operators each containing a given number of fields but dependence on momentum can be arbitrary. Hence, in a given operator one includes vertices with different powers of momentum constructing an infinite tower of $n$-point functions [12, 13].

The opposite approach would be to expand in powers of momenta while keeping the full field dependence of operators. In this case one ends up with local terms in action with different powers of derivative [14]. It is based on existence of a mass scale in the system which in our case will be hidden inside a source $J_2(x;\Lambda_0) = m^2$. However in principle this scheme behaves well for conformal systems (see also [9]) and hence we do not restrict ourselves in the dimension of the system keeping the discussion general.

The various approaches of exact renormalization group allow to consider flow equations by themselves as dynamical systems and investigate exact solutions, critical points, phase transition etc. The most inspiring are the works [15, 16] where it has been shown that exact renormalization group equations for $O(N)$ vector model can be recast into gravity equation or more general into the Vasiliev's higher spin equations, which are indeed known to be the corresponding holographic duals (see [17, 18]). This is in the context of a more general idea of obtaining dynamical space-time Lorentz (or diffeomorphism) invariant theories from renormalization group flow. For example, one mentions the works [19, 20, 21] where a procedure to arrive to the general relativity equations from the RG equation has been proposed.

The aim of our work is less ambitious. We use the second approach to construct the basis of operators in a theory of $N \times N$ matrix fields with an addition of all single trace operators with arbitrary sources which depend on the cut-off $\Lambda$. We show that it is possible to choose the cut-off function in such a way that ERG equations for such operators can be written as Hamiltonian equations with the Hamiltonian given by

$$H = \int_\gamma d\sigma \int d^D x \left[\Pi^2 J' + \Pi J'^2\right], \tag{1.1}$$

where the fields are complex satisfying the condition $f(\sigma,T)^* = f(-\sigma,T)$. The integration here is performed along the curve $\gamma = \{|\sigma| = 1,\ 0 \leq Arg(\sigma) \leq \frac{\pi}{2}\}$, that is just the first quarter of the unit circle. Physically leaving only single trace operators and the specific dependence of $\mathcal{S}_I$ on momenta in derivative expansion means that we are working in the large $N$ limit and IR approximation. This point is crucial for further study of exact solutions of the obtained equations, and will be commented more in the discussion section.

By studying the Hamiltonian (1.1) we show that there exist an infinite number of integrals of motion, and hence conserved current, parametrized by a single integer. All these stay in involution with respect to the naturally defined Poisson brackets, that might be a good sign of (classical) integrability of the system. Another argument in favor of this conjecture is that taking the limit $J \ll \Pi$ and dropping the second term in the Hamiltonian one is able to rewrite the equations on $J$ as the so-called Burgers-Hopf shock wave equations

$$\partial_T \rho = \rho \partial_s \rho, \quad \rho = \frac{\partial_T J}{\partial_\sigma J}, \tag{1.2}$$

that is known to be integrable. However, after trying to apply the known to us integrability criteria, such as constructing a Lax pair and performing the Painlevé test, we were not able to show this explicitly so far and hence leave this task for a further work.

Irrespective of the possible relations between ERG and the AdS/CFT correspondence exact solutions of renormalization group equations are of great interest by themselves as the flow usually possesses fascinating properties. On the other hand having integrable dynamics governing the flow gives more control over the system and allows more deep investigation. As another example of such correspondence one may recall the work [22] where it was shown that the RG equations of two-dimensional sigma models turn out to be a continual analogue of Toda equations.

Finally, we notice the appearance of the so-called Motzkin numbers in the system, which are usually observed in description of unit paths (see sequence A055151 in [23] and references therein). In the considered system these numbers appear as coefficients in the obtained integrals of motion thus rendering them very similar to Motzkin polynomials and allowing to rewrite them as a very nice expression in terms of hypergeometric ${}_2F_1$ functions. This might be an interesting result from the pure mathematical point of view.

To conclude this section we must set up few remarks concerning our previous works [24] and [25]. The latter can be considered as a particular case of the present consideration for small sources $J$, however with a general cut-off function. Put differently, the present results are derived from a particular case of the system considered in [25] with a fixed form of the cut-off function. This allowed us to go beyond the $J \ll \Pi$ limit and to write the second term in the Hamiltonian in ultra-local form.

The relation with the work [24] is more subtle as the Hamiltonian presented here has already appeared in the previous work. However, as we mention in [25], this derivation contained a non-obvious technical subtlety that under a more detailed consideration actually did not allow to write the Hamiltonian in ultra-local form as it was written in [24]. Here we present a way of avoiding this subtlety by choosing an appropriate cut-off function and hence provide the proper derivation of the Hamiltonian and investigate its properties in more details.

## 2 Exact renormalization group

Renormalization group procedure is known to introduce new types of interactions when the cut-off scale is changed. In the work [7] it was suggested a nice and simple procedure of how to take into account these terms in a consistent way. For further development and review on exact (Wilsonian or functional) renormalization group see [9, 10, 11].

Let us consider the case of a scalar field theory for simplicity and for further use in the present work. The corresponding action will include a cut-off function $K(p^2/\Lambda^2)$ that is equal to 1 for low momenta and vanishes for $p \gg \Lambda$

$$S = -\frac{1}{2}\int d^D p \phi(p)\phi(-p)(p^2+m^2)K^{-1}(p^2/\Lambda_0^2) + S_I, \tag{2.1}$$

where $S_I$ contains a finite number of interacting terms at the scale $\Lambda$. Hence, the free propagator has the form $G_0(p) = (p^2+m^2)^{-1}K(p^2/\Lambda^2)$ and the cut-off function

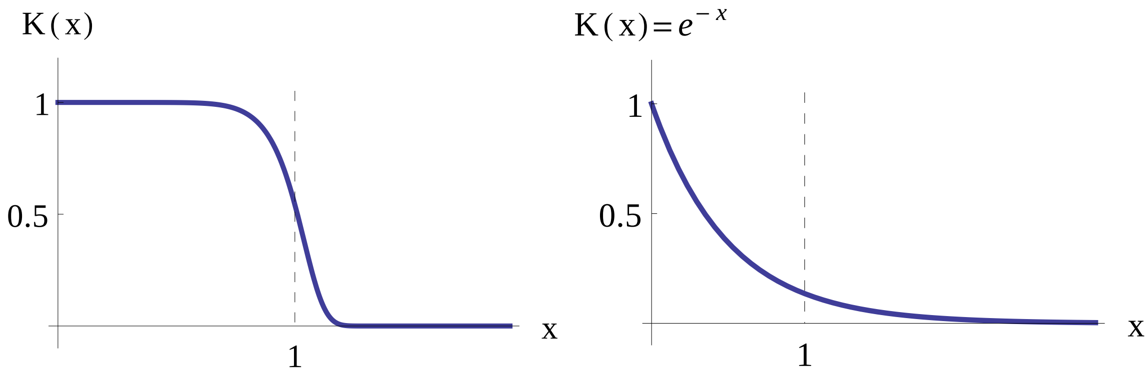


Figure 1: A general possible form for the cut-off function $K(p^2/\Lambda^2)$ (on the left) and the form of the function $K(x) = e^{-x}$ used in the present work.

removes contributions from $p \gg \Lambda$. Another way to understand it is through the partition function where the cut-off function effectively suppresses the integration for modes with higher momentum.

If one now decides to investigate physics on a scale $m^2 \ll \Lambda_R^2$, one has to integrate our modes with momentum higher than this value. During this procedure many new terms are generated and in general one writes the following new effective Lagrangian

$$Z = \int \mathcal{D}\phi \exp\left[ -\frac{1}{2} \int d^D p \phi(p)\phi(-p)(p^2+m^2)K^{-1}(p^2/\Lambda^2) + \mathcal{S}_I \right]. \tag{2.2}$$

Here $\mathcal{S}_I$ now contains all possible terms generated during the procedure. In principle it is an infinite sum of all operators in the theory. Hence, one exchanges shift in the scale for many new terms in the Lagrangian. To keep physics unchanged at the new scale one simply sets

$$\begin{aligned} 0 &= \Lambda \frac{dZ}{d\Lambda} \\ &= \int \mathcal{D}\phi \left[ -\frac{1}{2} \int d^D p \phi(p)\phi(-p)(p^2+m^2)\Lambda \frac{dK^{-1}(p^2/\Lambda^2)}{d\Lambda} + \Lambda \frac{d\mathcal{S}_I}{d\Lambda} \right] e^S. \end{aligned} \tag{2.3}$$

The integrand on the RHS of the above expression becomes full derivative and hence vanishes if

$$\Lambda \frac{d\mathcal{S}_I[\phi]}{d\Lambda} = -\frac{1}{2} \int d^D p \frac{1}{p^2+m^2} \Lambda \frac{dK_\Lambda(p^2)}{d\Lambda} \left[ \frac{\delta^2 \mathcal{S}_I[\phi]}{\delta\phi(-p)\delta\phi(p)} + \frac{\delta \mathcal{S}_I[\phi]}{\delta\phi(p)} \frac{\delta \mathcal{S}_I[\phi]}{\delta\phi(-p)} \right]. \tag{2.4}$$

This is the desired Polchinski renormalization group equation. It has a simple interpretation in terms of vertices. Indeed, as modes are removed from the integration new terms must be placed in order to compensate their contribution. On the level partition function diagrams this looks like a gluing point connected by a propagator carrying momenta higher than the cutoff.

Although the initial effective Lagrangian might be as simple as say an $\phi^6$ theory, the resulting interaction Lagrangian in principle contains an infinite number of terms obtained by gluing vertices in various ways. The Polchinski equation allows to investigate dynamics of these terms with change of the RG scale $\Lambda$ and has been conjectured to be actually a dynamical equation for the corresponding operators and sources. Checking this for the $N \times N$ matrix scalar field theory case is the challenge for the rest of the paper.

# 3 Theory of scalar $N \times N$ matrix fields

## 3.1 Renormalization group flow

In this section we revisit the result of [24] where the Wilsonian renormalization group equations for the matrix scalar field theory

$$\mathcal{S} = -\frac{N}{2}\int d^D x \mathrm{Tr}[\partial_\mu \phi \partial^\mu \phi] + N \int d^D x J_k(x) \mathrm{Tr}[\phi(x)^k] \tag{3.1}$$

was considered. In the momentum representation the action reads:

$$\begin{aligned} \mathcal{S}[\phi] &= -\frac{N}{2}\int_p p^2 K_\Lambda^{-1}(p^2) \mathrm{Tr}\left[\phi(p)\phi(-p)\right] + N\bar{\mathcal{S}}_I, \\ \bar{\mathcal{S}}_I &= \sum_{l=0}^{\infty} \int_{k_1 \dots k_l} \mathrm{Tr}\left[\phi(k_1) \quad \dots \phi(k_l)\right] \, J_l(-k_1 - \dots - k_l), \end{aligned} \tag{3.2}$$

where the notation $\mathcal{S}_I$ was reserved for further use. Here we assume that there is some momentum cut–off imposed, i.e. $K(p^2/\Lambda^2) \sim 1$ as $p^2 << \Lambda^2$, while $K(p^2/\Lambda^2) \sim 0$ as $p^2 >> \Lambda^2$. In what follows the cut-off function will be taken to be of a particular form accompanied by a field redefinition.

It appears that the Polchinski equation for this theory has closed form for the given choice of canonical variables: sources for the single–trace operators in question and their

VEVs. The quantum average of Polchinski equations for a theory with the interaction action $\mathcal{S}_I$ (the second line in (3.2)) reads:

$$\left\langle \Lambda \frac{d\bar{\mathcal{S}}_I[\phi]}{d\Lambda} \right\rangle = -\frac{1}{2}\int_p \frac{1}{p^2}\,\Lambda\,\frac{dK_\Lambda(p^2)}{d\Lambda}\,\left\langle \left[ N^{-1}\frac{\delta^2 \bar{\mathcal{S}}_I[\phi]}{\delta\phi^{ij}(-p)\delta\phi^{ji}(p)} + \frac{\delta\bar{\mathcal{S}}_I[\phi]}{\delta\phi^{ij}(p)}\frac{\delta\bar{\mathcal{S}}_I[\phi]}{\delta\phi^{ji}(-p)}\right]\right\rangle . \tag{3.3}$$

As it was noted in [25] keeping both the sources and the cut-off function general does not allow to write these equations in a Hamiltonian form. The approach taken in [25] was to consider IR limit of the theory where the sources $J(x)$ become small since all the corresponding operators become suppressed (even the marginal ones). Here we act differently and choose the cut-off function to be

$$K_\Lambda(p^2) = e^{-2\frac{p^2}{\Lambda^2}}, \tag{3.4}$$

that indeed satisfies the necessary conditions. In addition, it proves crucial to perform the following field redefinition

$$\phi(p) \to e^{-\frac{p^2}{\Lambda^2}}\phi(p), \tag{3.5}$$

that can always be done given the existence of the scale $\Lambda$ in the theory. Effectively, this redefinition returns the quadratic part of the action into its canonical form hiding the cut-off into the the fields and from now on we will be talking only about the rescaled fields. Hence, we have for the Polchinski equation

$$\left\langle \Lambda \frac{d\mathcal{S}_I[\phi]}{d\Lambda} \right\rangle = -2\Lambda^{-2}\int_p \left\langle \left[ N^{-1}\frac{\delta^2 \mathcal{S}_I[\phi]}{\delta\phi^{ij}(-p)\delta\phi^{ji}(p)} + \frac{\delta\mathcal{S}_I[\phi]}{\delta\phi^{ij}(p)}\frac{\delta\mathcal{S}_I[\phi]}{\delta\phi^{ji}(-p)}\right]\right\rangle , \tag{3.6}$$

where accordingly redefined action for the interaction terms should be used

$$\mathcal{S}_I[\varphi] = \sum_{l=0}^{\infty}\int_{k_1\ldots k_l} e^{-\frac{k_1^2}{\Lambda^2}-\cdots-\frac{k_l^2}{\Lambda^2}}\,\mathrm{Tr}\left[\phi(k_1)\quad\ldots\,\phi(k_l)\right]\,J_l(-k_1-\ldots-k_l). \tag{3.7}$$

Note, that now the quantum average is performed with the weight $e^{-\mathcal{S}_0}$, where $\mathcal{S}_0$ is the rescaled free action. On the level of the functional integral such rescaling just produces an extra (infinite) prefactor which does not depend on the fields and is cancelled out by normalization.

With this set up let us now briefly repeat the derivation of the equations on sources and the corresponding VEV's of operators $\mathrm{Tr}[\phi(x)^k]$ to introduce notations and to make the narrative self-contained. Throughout the derivation we correct several misprints of [24] and highlight important issues.

Hence, we start with the derivatives of the interaction part $\mathcal{S}_I$ of the action

$$
\begin{aligned}
\mathrm{Tr}\left[\frac{\delta \mathcal{S}_I}{\delta\phi(p)}\frac{\delta \mathcal{S}_I}{\delta\phi(-p)}\right] &= \sum_{k,l=0}^{\infty}(k+1)(l+1)\int_{p_1\ldots p_k q_1\ldots q_l} e^{-\frac{p_1^2}{\Lambda^2}-\cdots-\frac{p_k^2}{\Lambda^2}} e^{-\frac{q_1^2}{\Lambda^2}-\cdots-\frac{q_l^2}{\Lambda^2}}\times \\
&\times \mathrm{Tr}\left[\phi(p_1)\ \ldots\ \phi(p_k)\phi(q_1)\ \ldots\ \phi(q_l)\right]\times \\
&\times J_{k+1}(-p-p_1-\ldots-p_k)\, J_{l+1}(p-q_1-\ldots-q_l), \\
\mathrm{Tr}\left[\frac{\delta^2 \mathcal{S}_I}{\delta\phi(p)\delta\phi(-p)}\right] &= \sum_{n=0}^{\infty}(n+2)\int_{p_1\ldots p_n}\sum_{m=0}^{n} e^{-\frac{p_1^2}{\Lambda^2}-\cdots-\frac{p_n^2}{\Lambda^2}}\times \\
&\times \mathrm{Tr}\Big[\phi(p_1)\ \ \ldots\ \phi(p_m)\Big]\ \mathrm{Tr}\Big[\phi(p_{m+1})\ \ \ldots\ \phi(p_n)\Big]\times \\
&\times J_{n+2}(-p_1-\ldots-p_n).
\end{aligned}
$$

According to the standard Wilsonian renormalization group prescription the quantum averaging in the expression above is performed only with respect to quantum fluctuations $\varphi(p)$ over the classical solution $\phi_0(p)$. Hence, we write $\phi(p)=\phi_0(p)+\varphi(p)$ and integrate out the field $\varphi(p)$. This is a tedious procedure and in what follows it proves useful to introduce shorthand notations

$$
\begin{aligned}
\int_{p_{(n)}} &:= \int_{p_1\ldots p_n}, \\
T_n[\,p_n\,] &:= \mathrm{Tr}\left[\phi_0(p_1)\ldots\ \phi_0(p_n)\right], \\
J_l(-k_{(l)}) &:= J_l(-k_1-\ldots-k_l), \\
e^{-p_{(n)}^2} &:= e^{-\frac{p_1^2}{\Lambda^2}-\cdots-\frac{p_n^2}{\Lambda^2}}.
\end{aligned}
$$

Note that the LHS of the last line does not contain $\Lambda$ explicitly to lighten the notations.

Hence, the quantum average of a single-trace operator $\mathrm{Tr}\left[(\phi_{01}+\varphi_1)\ldots(\phi_{0n}+\varphi_n)\right]$

over the fluctuations can be reduced to action of some operator $\hat{W}$ on $T_l[\,k_l\,]$ as follows

$$
\begin{aligned}
&\left\langle \int_{p_{(n)}} \mathrm{Tr}\left[(\phi_0(p_1)+\varphi(p_1))\dots(\phi_0(p_n)+\varphi(p_n))\right]\right\rangle \\
&=\int_{p_{(n)}}\int \mathcal{D}\varphi\, e^{S_0}\,\mathrm{Tr}\left[(\phi_0(p_1)+\varphi(p_1))\dots(\phi_0(p_n)+\varphi(p_n))\right] \\
&=\int_{p_{(n)}}\int \mathcal{D}\,\varphi\, e^{S_0}\,\exp\left[\int_p \varphi_p \frac{\delta}{\delta\phi_0(p)}\right]\mathrm{Tr}\left[\phi_0(p_1)\dots\phi_0(p_n)\right] \\
&=\hat{W}\left\{\int_{p_{(n)}}\mathrm{Tr}\left[\phi_{01}\dots\phi_{0n}\right]\right\}=\hat{W}\left[\int_{p_{(n)}} T_n[\,p_n\,]\right],
\end{aligned}
\tag{3.8}
$$

where $S_0=-\frac{N}{2}\int_p p^2\mathrm{Tr}[\phi(p)\phi(-p)]$ is the canonical kinetic action and the operator can be written explicitly as

$$
\hat{W}=\exp\left(\frac{1}{2N}\int_p p^{-2}\mathrm{Tr}\left[\frac{\delta}{\delta\phi_{0_p}}\frac{\delta}{\delta\phi_{0_{-p}}}\right]\right). \tag{3.9}
$$

Since we are working in the large $N$ approximation it is possible to use the OPE factorization property $\langle\prod_n \mathrm{Tr}O_n\rangle=\prod_n\langle\mathrm{Tr}O_n\rangle$ to write

$$
\hat{W}\Big[T_l[\,k_l\,]T_n[\,p_n\,]\Big]=\hat{W}\Big[T_l[\,k_l\,]\Big]\,\hat{W}\Big[T_n[\,p_n\,]\Big]=\tilde{T}_l[\,k_l\,]\,\tilde{T}_n[\,p_n\,] \tag{3.10}
$$

where the notation $\tilde{T}=\hat{W}\,T$ was used. Hence, the Polchinski equation for the theory reads

$$
\begin{aligned}
&\sum_{l=1}^{\infty}\int_{k_{(l)}} e^{-k^2_{(l)}}\tilde{T}_l[\,k_l\,]\dot{J}_l(-k_{(l)})= \\
&-2\Lambda^{-2}\int_p\Bigg[N^{-1}\sum_{a=1}^{\infty}\sum_{s=0}^{a-1}\int_{k_{(a-1)}}(a+1)e^{-k^2_{(a-s-1)}}e^{-q^2_{(s)}}\tilde{T}_{a-s-1}[\,k_{a-s-1}\,]\,\tilde{T}_s[\,q_s\,] \\
&\times J_{a+1}(-k_{(a-s-1)}-q_{(s)}) \\
&+\sum_{l,k=1}^{\infty} kl\int_{q_{(k-1)}p_{(l-1)}} e^{-p^2_{(l-1)}}e^{-q^2_{(k-1)}}\tilde{T}_{l+k-2}[\,p_{l-1}\,][\,q_{k-1}\,]J_l(-p_{(l-1)}-p)J_j(-q_{(k-1)}+p)\Bigg],
\end{aligned}
\tag{3.11}
$$

where the overdot means differentiation with respect to $d/d\log\Lambda$ and the indices were tuned in such a was as to shift the infinite summations to run from 1. Note that although the operator $\hat{W}$ does not depend on $\Lambda$ explicitly, there is still non-trivial dependence due to the rescaling of the fields, which hides the cut-off inside the field. For the same reason the objects $\tilde{T}_n$ have non-trivial dependence on $\Lambda$ which is governed by the RG flow. In what follows we show, that (for the chosen cut-off) the RG flow actually defines a Hamiltonian dynamics, which has nice properties formulated in terms of the Motzkin polynomials and has some hints for classical integrability.

To do so the equation above suggests to define momenta canonically conjugate to $J_l(k)$ as follows

$$\Pi_l(k) = N^{-1}\Lambda^D \int_{k_{(l)}} e^{-k^2_{(l)}}\delta^{(D)}\Big(k - k_{(l)}\Big)\tilde{T}_l[\,k_l\,]. \tag{3.12}$$

This definition reflects the fact that the sources $J_k$ depend only on a sum of arguments of the corresponding $\tilde{T}_k$. The factor of $N^{-1}$ was included to make the sources and the canonical momentum to be of the same order as $N \to \infty$. Hence we obtain for the RG equation in these variables

$$\begin{aligned} -\frac{1}{2}\Lambda^2 \int_q \sum_{l=0}^{\infty} \Pi_l(q)\dot{J}_l(-q) = \int_{q_1 q_2} \sum_{k,l=0}^{\infty} (k+l+2)\Pi_l(q_1)\Pi_s(q_2)J_{k+l+2}(-q_1-q_2) \\ + \int_{q_1 q_2} \sum_{k,l=0}^{\infty} (k+1)(l+1)\Pi_{k+l}(q_1+q_2)J_{k+1}(-q_1)J_{l+1}(-q_2). \end{aligned} \tag{3.13}$$

It is worth to stop here and discuss relation of the above result to the expressions presented in [24] where the very same equation has already been presented.

As it has been mentioned in [25] the previously obtained Hamiltonian equation [24] was blemished by a subtle technical error at the step when going from (3.11) to (3.13). It was shown, that this procedure can not actually be performed for a general cut-off function and general fields. The result of [25] is based on a general cut-off function and IR limit which suppressed the second term $J^2$. This implies that the Hamiltonian form of the RG equation is governed by that of the Hopf-Burgess equation for shock waves.

In the present work we step out of the IR limit and choose a specific form of the cut-off function to keep the $J^2$ term. As it is shown above this also allows to perform the procedure and turn to Hamiltonian equations. It is not surprising that the result depends on the form of the cut-off function as there has been no the opposite constraint. Indeed, the Polchinski procedure ensures that the physics does not depend on the cut-off itself (hence the equation), but apparently does depend on the form of the function itself. Finally, it is worth mentioning that the present choice of the cut-off function relates the Polchinski and the Wilson equations [10, 26].

## 3.2 Ultra-local Hamiltonian

Following the same idea as in [24, 25] the equation (3.13) can be written in the Hamiltonian form, i.e. as the following system of equations

$$\begin{aligned} \frac{d\,J_l(-q)}{dT} &= \frac{\delta H}{\delta \Pi_l(q)} \\ \frac{d\,\Pi_l(q)}{dT} &= -\frac{\delta H}{\delta J_l(-q)}, \end{aligned} \tag{3.14}$$

where the "time" variable is defined as $T = \Lambda^{-2}$ and the Hamiltonian defined as

$$\begin{aligned} H = \int_{q_1 q_2} \sum_{l,s=0}^{\infty} \Big[ &(l+s+2)\Pi_l(q_1)\Pi_s(q_2)J_{l+s+2}(-q_1-q_2) \\ &+ (l+1)(s+1)\Pi_{l+s}(q_1+q_2)J_{l+1}(-q_1)J_{s+1}(-q_2) \Big]. \end{aligned} \tag{3.15}$$

Indeed, the first equation in (3.14) follows immediately from the Polchinski equation (3.13) and reflects that the path integral does not depend on the choice of the cut-off scale. Similarly the second equation in (3.14) is a direct consequence of the independence of $\Lambda$ of all the VEV's $\langle \mathrm{Tr}\phi^l(x)\rangle$. More straightforward but equivalent derivation of these equations is to perform variation of the Polchinski equation in the form (3.13) with respect to the sources $J_k(-q)$. This follows from the simple fact, that an effective action expressed in terms of sources is related to the one expressed in terms of the corresponding VEV's via the Legendre (functional Fourier) transformation [4].

To formulate the Hamiltonian in the ultra-local term it is suggestive to perform Fourier transform of the $J_k$ and $\Pi_k$ harmonics

$$\begin{aligned} J(T,\sigma,x) &= \sum_k e^{i\sigma k} J_k(T,x); \\ \Pi(T,s,x) &= \sum_k e^{-i(\sigma+1)k}\Pi_k(T,x), \end{aligned} \tag{3.16}$$

with $\sigma \in [0, 2\pi]$ and periodic $\sigma \sim \sigma + 2\pi$. Since the source fields $J_k(T,x)$ are real the complex field $J(T,\sigma,x)$ satisfies $J(T,\sigma,x)^* = J(T,-\sigma,x)$. This implies that at the point $\sigma = 0 \sim \sigma = 2\pi$ the complex field $J(T, \sigma, x)$ has no imaginary part, i.e.

$$J(T, 0, x)^* = J(T, 0, x). \tag{3.17}$$

The inverse Fourier transformation then reads

$$\begin{aligned} J_k(T, x) &= \frac{1}{4\pi} \int_0^{2\pi} d\sigma \Big( e^{-ik\sigma} J(T, \sigma, x) + e^{ik\sigma} J(T, \sigma, x)^* \Big) \\ &= \frac{1}{4\pi} \int_0^{2\pi} d\sigma \Big( e^{-ik\sigma} J(T, \sigma, x) + e^{ik\sigma} J(T, -\sigma, x) \Big) \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\sigma e^{-ik\sigma} J(T, \sigma, x). \end{aligned} \tag{3.18}$$

Substituting this back to the equation and redefining $\sigma \to e^{-i\sigma}$ the Hamiltonian finally becomes

$$H = \int_{-\pi}^{\pi} d\sigma \int d^D x \left[ \Pi^2 J' + \Pi J'^2 \right]. \tag{3.19}$$

To solve equations of motion, one should observe the constraints on the complex fields $J$ and $\Pi$, that equivalently equates the left-moving modes with to complex conjugate right moving, allowing to perform the Fourier transform properly.

In the next section we investigate properties of integrals of motion corresponding to the obtained Hamiltonian. Since we do not go after solutions, the mentioned constraints do not play role in the following discussion.

# 4 Motzkin Hamiltonian

## 4.1 Conservation laws

In this section we consider dynamics of a theory defined by the following Hamiltonian

$$H = \int_{-\pi}^{\pi} d\sigma \int d^D x \left[ \Pi^2 J' + \Pi J'^2 \right], \tag{4.1}$$

that originates form the renormalization group procedure. Given the direct relation between integrability properties of this system and Motzkin paths and that the system does not look recognizable to the knowledge of the authors, we suggest to call it the Motzkin system.

Let us start with equations of motion

$$\begin{aligned}\dot{J} &= 2\Pi J' + J'^2;\\ \dot{\Pi} &= 2\Pi\Pi' + 2\Pi' J' + 2\Pi J'',\end{aligned} \tag{4.2}$$

which can be easily solved with respect to $\Pi$ to provide the following Lagrangian formulation of the theory

$$\mathcal{L} = \frac{\left(\dot{J} - J'^2\right)^2}{4J'}. \tag{4.3}$$

The EOM for the field $J(t,\sigma)$ can be found either varying the Lagrangian (4.3) with respect to $J(t,\sigma)$ or substituting an expression for $\Pi(t,\sigma)$ into the second equation in (4.2):

$$\begin{aligned}&\dot{\rho} - \frac{1}{2}\partial_\sigma \rho^2 = 5\dot{J}' - 6J''J',\\ &\rho := \frac{\dot{J}}{J'}.\end{aligned} \tag{4.4}$$

One notices, that in the IR limit of the corresponding field theory the sources become infinitesimally small $J \to 0$ , and hence the above equation drops to the Burgess-Hopf equation in agreement with [25].

In general the above equation does not immediately drop into one of the commonly known types of non–linear equations. On the other hand, if one is lucky to find the explicit Lax pair for these equations it would be possible either to compare the system to one of the known systems (e.g. KdV) or to prove that it is a new integrable system. However, so far we were not able to find the corresponding Lax pair and we cannot claim if there exists one. In general the process of finding of the Lax pair is always some kind of art.

However, in the next section we present the infinite set of integrals of motion for the system in question, which is the good arguments that favours the integrability of the Hamiltonian flow in question. It is suggestive to consider the notion of integrability in the Liouvile sence, that means having a maximal set of Poisson-commuting integrals of motion (i.e. function(al)s on the phase space whose Poisson brackets vanish) which are not trivial, i.e. zero or Casimir elements. Certainly one must be subtly when applying the Liouville criterion to infinitely dimensional systems such as field theoretical equations. For this reason to present the required full set of integrals of motion refraining from the claim that the system is indeed integrable.

Starting with some obvious integrals of motion of the type

$$\begin{aligned} I_1 &= \int_\sigma (\Pi + J')\,; \\ I_2 &= \int_\sigma \Pi J'; \\ I_3 &= \int_\sigma \Pi J'(\Pi + J') = H, \end{aligned} \tag{4.5}$$

it is straightforward to see that there exists the following infinite tower of such constructions (see Appendix A)

$$\begin{aligned} I_n &= \int_\sigma \sum_{k=1}^{n} (\Pi J')^k \, (\Pi + J')^{n-2k} \, t_{n,k} + \delta_n^1 (\Pi + J'), \\ t_{n,k} &= \frac{(n-2)!}{(n-2k)!k!(k-1)!}, \forall n > 1 \\ t_{1,1} &= 1. \end{aligned} \tag{4.6}$$

It is worth mentioning here that one shouldn't be confused by the fact that we have discrete set of integrals of motion facing "continuous" variables $J(t, \sigma)$. Since $\sigma$ is compact one actually has a discrete spectrum of variables $J_l(t)$ (3.16).

By making use of the relation between $t_{n,k}$ and the Motzkin polynomial coefficients

$$t_{n,k} = T_{n-2,k-1} \tag{4.7}$$

we can write

$$I_n = \int_\sigma \sum_{k=1}^{n} (\Pi J')^k \, (\Pi + J')^{n-2k} \, T_{n-2,k-1} + (\Pi + J') \, \delta_n^1. \tag{4.8}$$

Interestingly, the sum in the first term above can be performed explicitly and the result for this term can be written in terms of hypergeometric to give (note that $n \neq 1$)

$$\begin{aligned} \bar{I}_n &= \sum_{k=1}^{n} (\Pi J')^k \, (\Pi + J')^{n-2k} \, T_{n-2,k-1} \\ &= (\Pi J') \, (\Pi + J')^{n-2} \int_\sigma {}_2F_2 \Big[ 1 - \frac{n}{2}, \frac{3}{2} - \frac{n}{2}; 2, \frac{4\,(\Pi J')}{(\Pi + J')^2} \Big]. \end{aligned} \tag{4.9}$$

This expression can be further simplified by making use of the following quadratic relation

$${}_2F_1 \Big[ \frac{a}{2}, \frac{a}{2} + \frac{1}{2}; 1 + a - b, \frac{4z}{(1+z)^2} \Big] = (1+z)^a \, {}_2F_1[a, b; 1 + a - b, z] \tag{4.10}$$

to obtain

$$\bar{I}_n = \int_\sigma \Pi J'^{n-1}{}_2F_1\left[2-n, 1-n; 2, \frac{\Pi}{J'}\right]. \tag{4.11}$$

The conserved quantities $I_k$ can be shown to be in involution, i.e they commute with respect to the standard Poissone bracket

$$\begin{aligned}\{f, g\} &= \frac{\partial f}{\partial J}\frac{\partial g}{\partial \Pi} - \frac{\partial f}{\partial \Pi}\frac{\partial g}{\partial J},\\ \{I_m, I_n\} &= 0.\end{aligned} \tag{4.12}$$

Finally it is possible to write the above integrals in terms of conserving currents using the notations $\partial_\mu = (\partial/\partial T, \partial/\partial\sigma)$

$$\dot{I}_m = 0 \rightarrow \partial_\mu {j_m}^\mu = 0. \tag{4.13}$$

hence, the currents read

$$\begin{aligned}{j_n}^0 &= \sum_{k=1}^{n} \alpha^k \beta^{n-2k} t_{n,k},\\ {j_n}^1 &= -2\sum_{k=1}^{n} \alpha^k \beta^{n-2k+1} \frac{n-k}{n-2k+1} t_{n,k},\end{aligned} \tag{4.14}$$

with $\alpha = \Pi J'$ and $\beta = \Pi + J'$. According to the Noether theorem each conserving current is associated to a global symmetry of the system.

## 4.2 Motzkin numbers

Here we review the Motzkin paths and related polynomials which appear in many contexts in the mathematical literature, e.g. [27, 28, 29] and physics, e.g. [30, 31, 32].

First let us define a lattice path. A lattice path $L$ in $\mathbb{Z}^d$ of length $n$ is a sequence $v_0, \ldots, v_n \in \mathbb{Z}^d$ with corresponding steps $s_1, \ldots, s_n \in \mathbb{Z}^d$ defined by consecutive difference $s_i = v_i - v_{i-1}$. A Motzkin path of length $n$ is a lattice path on $\mathbb{N}\times\mathbb{N}$ consisting of up steps $(1, 1)$, down steps $(1, -1)$ and flat steps $(1, 0)$. The number of Motzkin paths from $(0, 0)$ to $(n, 0)$ is given by the Motzkin number[1] (sequence A001006 in [23]) $m_n$

[1]Note that historically the Motzkin numbers appeared in a circle chording setting [33].

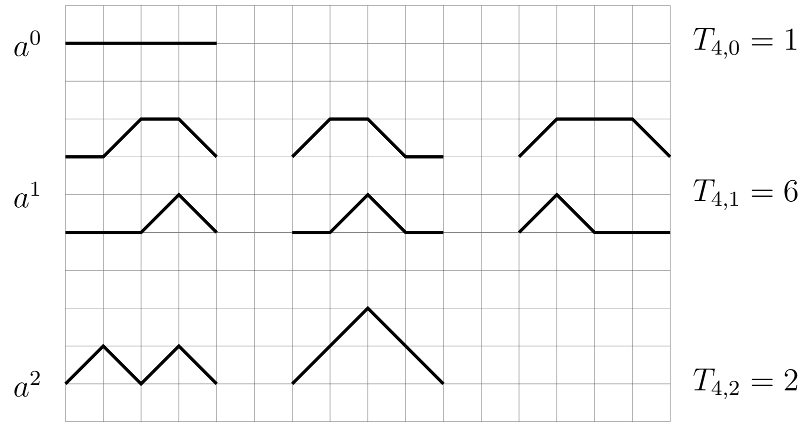


Figure 2: Motzkin paths of length 4 between the points (0,0) and (4,0). The powers of $a$ in the left column denote the number of up steps. The number of paths of length $n$ with $k$ up step is given by the Motzkin polynomial coefficient $T_{n,k}$.

which can be written in the following form

$$m_n = \sum_{k=0}^{[n/2]} \binom{n}{2k} c_k \ , \tag{4.15}$$

where $c_k$ are the Catalan numbers (sequence A000108 in [23]) defined as

$$c_k = \frac{1}{k+1} \binom{2k}{k} \ . \tag{4.16}$$

We are particularly interested in Motzkin polynomial associated to a Motzkin path. In order to define the Motzkin polynomial one needs to assign a weight keeping track of the number of up steps (or flat steps). Then one gets the following polynomial with the corresponding coefficients (sequence A055151 in [23])

$$m_n(a) \ = \ \sum_{k=0}^{[n/2]} T_{n,k} \ a^k \tag{4.17}$$

where $a$ stands for up steps. The coefficient $T_{n,k}$ is number of Motzkin paths of length n with k up steps. Note that the polynomial (4.17) is also known as the Jacobi-Rogers polynomial [29].

It might now be reasonable to expect the occurrence of other lattice path polynomials such that the Riordan polynomials, Dick polynomials etc. in the context of integrable models.

# 5 Conclusion

There are two main results presented in this paper i) an explicitly derived route from the Polchinski equation to the Hamiltonian (4.1) and ii) analysis of integrability properties of the corresponding Hamiltonian system and observation that these are related to Motzkin paths.

We start from the theory of matrix scalar fields and consider renormalization group evolution of single trace operators of the type $\mathrm{Tr}[\varphi^n]$ using the Polchinsky equation. As it was shown in [25] to keep only these operators and to prevent triggering operators with momentum dependence one should working in the IR limit. In this case it is possible to relate the Polchinsky equations of the system to a set of Hamiltonian equations for the operators and the corresponding sources $J_k$. To go beyond of the result of [25] where the Hamiltonian equations were shown to be the Hopf-Burgess shock wave equations, we keep the previously omitted $J_k^2$ terms.

This allows us to recover a non-linear generalization of the shock wave equations which at first glance do not look as a recognizable integrable system. It is interesting that to include these terms in the ultra-local form one has to choose the cut-off function appropriately. The corresponding cut-off function appears to be the one which relates the Polchinsky and the Wilson equations.

The Hamiltonian has the following form

$$H = \int_{-\pi}^{\pi} d\sigma \left[ \Pi^2 J' + \Pi {J'}^2 \right], \tag{5.1}$$

with the fundamental fields and their conjugates given by $J(\sigma, T)$ and $\Pi(\sigma, T)$. This system has an infinite number of conserved quantities $I_n$ of the form

$$I_n = \int_\sigma \sum_{k=1}^{n} (\Pi J')^k \, (\Pi + J')^{n-2k} \, T_{n-2,k-1} + (\Pi + J') \, \delta_n^1, \tag{5.2}$$

which Poisson-commute with each other. This allows to conjecture, that the non-linear system in question might be indeed integrable. A way to show that explicitly would be to find the corresponding Lax pair or to satisfy the Painlevé criterion.

The very first step forward from our result would be looking at explicit solutions of the obtained Hamiltonian equation, probably using numerical methods. This will provide RG flows for single-trace operators. For more general approach one would like to go beyond the IR limit and to include momentum dependence into the sources. That can be done in multiple ways, for example along the line of [34].

Finally, from the pure mathematical side the presented results are interesting in the aspect of appearance of the so-called Motzkin number (sequence A055151 in [23]). These enter the theory as the coefficients $T_{n,k}$ in the integrals of motion $I_n$. These numbers and the corresponding polynomials appear in the problem of counting all routes on a lattice with a given number of vertical and horizontal steps (see Figure 2). Hence, each integral of motion $I_n$ is given by a Motzkin polynomial corresponding to a path of length $n-2$ with $k-1$ horizontal steps.

An interesting problem would be to consider other polynomials corresponding to various paths on a lattice, and reversely build a set of expressions understood as integrals of motion for some system. Having such procedure would be a fascinating way of generating dynamical systems.

The presented result is strongly based on the explicit form of the cut-off function chosen. One wonders if it is possible to shift from this dependence and to consider a way to formulate similar or new results using a general cut-off function or using another choices of that. Along the same lines would be the idea to generalize the system itself and to consider more realistic models such as theories with vector gauge bosons, fermions etc. On the mathematical side all these generalizations might lead to new numerical sequences or be related to other known ones.

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# A Derivation of integrals of motion

Let us start by listing few first integrals of motion for which purpose we first introduce the notations

$$\begin{aligned} \alpha &= \Pi J'; \\ \beta &= \Pi + J'. \end{aligned} \tag{A.1}$$

The equations of motion for these variables are rather simple

$$\begin{aligned} \dot{\alpha} &= 2(\alpha\beta)'; \\ \dot{\beta} &= \left(\beta^2 + 2\alpha\right)', \end{aligned} \tag{A.2}$$

using which one easily checks that the following expressions represent conserved charges

$$\begin{aligned} I_1 &= \int_\sigma \beta, \\ I_2 &= \int_\sigma \alpha, \\ I_3 &= \int_\sigma \alpha\beta, \\ I_4 &= \int_\sigma \alpha^2 + \alpha\beta, \\ I_5 &= \int_\sigma 3\alpha^2\beta + \alpha\beta^3, \\ I_6 &= \int_\sigma 2\alpha^3 + 6\alpha^2\beta^2 + \alpha\beta. \end{aligned} \tag{A.3}$$

One immediately notices that all terms in each expression are of the same power in the fields $\Pi$ and $J'$. After observing some other patters above one conjectures the following general expressions for an integral of motion

$$I_n = \int_\sigma \sum_{k=1}^{n} \alpha^k \beta^{n-k} t_{n,k}, \tag{A.4}$$

where the coefficients $t_{n,k}$ are constrained to satisfy certain conditions.

Indeed, let us show that these expressions represent an infinite number of conserving charges and find the coefficients explicitly. Hence, we consider time derivative

$$\begin{aligned}\dot{I}_n &= \int_\sigma \sum_{k=1}^{n} 2k\alpha^{k-1}\alpha'\beta^{n-2k+1}t_{n,k} + \sum_{k=2}^{n+1} 2(n-2k+2)\alpha^{k-1}\beta^{n-2k+1}\alpha' t_{n,k-1} \\ &\quad + \sum_{k=1}^{n} 2(n-2k)\alpha^k\beta^{n-2k}\beta' t_{n,k} \\ &= \int_\sigma \sum_{k=1}^{n} \Big(2kt_{n,k} + 2(n-2k+2)t_{n,k-1}\Big)\alpha^{k-1}\alpha'\beta^{n-2k+1} - 2n\beta^{n-1}\alpha' t_{n,0} \\ &\quad + \sum_{k=1}^{n} 2(n-k)\alpha^k\beta^{n-2k}\beta' t_{n,k} - 2n\alpha^n\alpha'\beta^{-n-1}t_{n,n}\end{aligned} \tag{A.5}$$

Here in the first line we used the equations of motion for $\alpha$ and $\beta$ and shifted the summation index in the second term, while in the second line we added and subtracted the term with $k = 1$ needed to complete the second sum. Now to form a full derivative and to make the additional terms vanish one imposes the following conditions for the coefficients

$$\begin{aligned}\frac{k-1}{n-2k+1}t_{n,k} &= \frac{n-2k+2}{k}t_{n,k-1}, \\ t_{n,0} &= 0, \\ t_{n,n} &= 0\end{aligned} \tag{A.6}$$

The second condition above ensures that all terms in the charges $I_n$ always have at least one power of $\alpha$ as it can be explicitly seen from (A.3) while the last condition removes terms of negative powers from (A.5). In what follows this will be extended to $\{t_{n,k} = 0 \text{ for all } 2k > n\ \}$.

The recurrence relations (A.6) can be used to determine explicit expressions for the coefficients $t_{n,k}$ as follows

$$\begin{aligned}t_{n,k} &= \frac{(n-2k+1)(n-2k+2)}{k(k-1)}t_{n,k-1} \\ &= \frac{(n-2k+1)(n-2k+2)(n-2k+3)(n-2k+4)\cdots(n-3)(n-2)}{k(k-1)(k-1)(k-2)\cdots 2\cdot 1}t_{n,1} \\ &= \frac{(n-2)!}{(n-2k)!k!(k-1)!}t_{n,1}.\end{aligned} \tag{A.7}$$

Finally, using the freedom to choose the overall normalization of each of $I_n$ one is allowed to fix $t_{n,1} = 1$

$$t_{n,k} = \frac{(n-2)!}{(n-2k)!k!(k-1)!}. \tag{A.8}$$

Since this formula does not cover the case $n = 1$ and the cases $2k > n$ we set the additional constraints based on the explicit form of the integrals of motion (A.3)

$$\begin{aligned} t_{1,1} &= 1, \\ t_{n,k} &= 0, \forall k > 2n. \end{aligned} \tag{A.9}$$

# B Poisson brackets

For proper integrable systems one observes all integrals of motion in involution meaning that Poisson brackets $\{I_n, I_m\}$ vanishes for any $I_n$ and $I_m$. Since the fundamental variables for our theory are $\Pi(T,\sigma)$ and $J(T,\sigma)$ the Poisson bracket is written as

$$\{F, G\} = \int_\sigma \frac{\delta F}{\delta J(T,\sigma)} \frac{\delta G}{\delta \Pi(T,\sigma)} - \frac{\delta G}{\delta J(T,\sigma)} \frac{\delta F}{\delta \Pi(T,\sigma)}, \tag{B.1}$$

where $F$ and $G$ are some functionals in $\Pi$ and $J$. In what follows we will not mention dependence on $T$ and $\sigma$ for the sake of space.

Let us now show that $\{I_n, I_m\} = 0$ for any $n, m$, and start with derivatives of $I_n$ with respect to the fundamental variables. Denoting variation with respect to a function $f$ by $\delta_f$ we note the following

$$\begin{aligned} \delta_J I_m &= -\partial_\sigma \delta_{J'} I_m, \\ \delta_{J'} I_m &= \delta_\alpha I_m \Pi + \delta_\beta I_m, \\ \delta_\Pi I_m &= \delta_\alpha I_m J' + \delta_\beta I_m. \end{aligned} \tag{B.2}$$

Given these we can write for the Poisson bracket

$$\begin{aligned}\{I_m, I_n\} &= 2\int_\sigma \delta_J I_{[m}\delta_\Pi I_{n]} = -2\int_\sigma \partial_\sigma\Big(\delta_{J'} I_{[m}\Big)\delta_\Pi I_{n]} \\ &= -2\int_\sigma \partial_\sigma\Big(\delta_\alpha I_m \Pi + \delta_\beta I_m\Big)\Big(\delta_\alpha I_n J' + \delta_\beta I_n\Big) \\ &= -2\int_\sigma \alpha(\delta_\alpha I_m)'\delta_\alpha I_n + \Pi'\delta_\alpha I_m \delta_\beta I_n + J'(\delta_\beta I_m)'\delta_\alpha I_n \\ &+ (\delta_\beta I_m)'\delta_\beta I_n + \Pi(\delta_\alpha I_m)'\delta_\beta I_n \\ &= -2\int_\sigma \Big[\alpha(\delta_\alpha I_m)'\delta_\alpha I_n + \beta'\delta_\alpha I_m\delta_\beta I_n + \beta(\delta_\alpha I_m)'\delta_\beta I_n + (\delta_\beta I_m)'\delta_\beta I_n\Big],\end{aligned} \tag{B.3}$$

where antisymmetrization in $\{m, n\}$ is always undermined. Here we used integration by parts and the antisymmetry properties to recollect terms with $J, \Pi$ and their derivatives back into $\alpha$ and $\beta$.

Substituting the explicit form of the integrals $I_m$ and introducing a new variable $\gamma = \alpha\beta^{-2}$ for convenience the integrand of the above expression can be written as (all terms are antisymmetric in $\{m, n\}$)

$$\begin{aligned}&\sum_{k,l} t_{n,k}t_{m,l}\Big[kl\gamma^k\beta^n(\gamma^{l-1}\beta^{m-2})' + (m-2l)k\gamma^{k-1}\beta^{n-1}(\gamma^l\beta^{m-1})' \\ &\qquad + (n-2k)(m-2l)(\gamma^l\beta^{m-1})'\gamma^k\beta^{n-1}\Big] - (n\leftrightarrow m) \\ &= \sum_{k,l} t_{n,k}t_{m,l}\Big[kl\gamma^{k-1}\beta^{n-1}(\gamma^l\beta^{m-1})' + (m-2l)k\gamma^{k-1}\beta^{n-1}(\gamma^l\beta^{m-1})' \\ &\qquad + (n-2k)(m-2l)(\gamma^l\beta^{m-1})'\gamma^k\beta^{n-1}\Big] - (n\leftrightarrow m) \\ &= \sum_{k,l} t_{n,k}t_{m,l}\Big[(m-l)(m-1)k\gamma^{k+l-1} + (n-2k)(m-2l)(m-1)\gamma^{k+l}\Big]\beta^{n+m-3}\beta' \\ &\quad + t_{n,k}t_{m,l}\Big[(m-l)kl\gamma^{k+l-2} + l(n-2k)(m-2l)\gamma^{k+l-1}\Big]\gamma'\beta^{n+m-2} - (n\leftrightarrow m)\end{aligned} \tag{B.4}$$

where in the second line we the antisymmetry to shift powers of $\beta$ and $\gamma$ out of the derivative in the first term. Noticing that the power of $\gamma$ in the second term in each line is just that of the first term shifted as $k \to k+1$ we can use the property of the Motzkin coefficients $k(k-1)t_{n,k} = (n-2k+1)(n-2k+2)t_{n,k-1}$ to write the above expression as

$$\{I_n, I_m\} = \sum_{k,l} t_{n,k} t_{m,l} \Big[A_{n,m,k,l}\gamma^{k+l-2}\gamma'\beta^{n+m-2} + B_{n,m,k,l}\gamma^{k+l-1}\beta^{n+m-3}\beta'\Big] - (n \leftrightarrow m),$$

where

$$\begin{aligned} A_{n,m,k,l} &= \frac{kl(m-l)(n+2k+1) + kl(k-1)(m-2l)}{n-2k+1}, \\ B_{n,m,k,l} &= \frac{k(m-l)(m-1)(n+2k+1) + k(k-1)(m-2l)(m-1)}{n-2k+1}. \end{aligned} \tag{B.5}$$

This sum does not form a full derivative term by term and one must turn to summation over $p = k + l$ to actually get cancellation. Hence, we write

$$\begin{aligned} \{I_n, I_m\} &= \sum_{p=2}^{N}\sum_{k=1}^{p-1}\Big[\hat{A}_{n,N,k,p}\gamma^{p-2}\gamma'\beta^{N-2} + \hat{B}_{n,N,k,p}\gamma^{p-1}\beta^{N-3}\beta'\Big], \\ &\text{with} \\ \hat{A}_{n,N,k,p} &= A_{n,N-n,k,p-k} - A_{N-n,n,p-k,k}, \\ \hat{B}_{n,N,k,p} &= B_{n,N-n,k,p-k} - B_{N-n,n,p-k,k}, \end{aligned} \tag{B.6}$$

and $N = n + m$. Although each term in the sum has now the same power of the variables $\gamma$ and $\beta$ the full derivative can be obtained only after taking the summation along $k$ explicitly. This is a tough calculational task and it is much easier to check

$$\sum_{k=1}^{p-1}\frac{\hat{A}_{n,N,k,p}}{p-1} - \sum_{k=1}^{p-1}\frac{\hat{B}_{n,N,k,p}}{N-2} = 0. \tag{B.7}$$

Indeed, using Wolfram Mathematica and performing the calculation explicitly one gets the desired cancellation for any $p, N$ and $n$. Obviously, this ensures that the expression (B.6) is indeed a full derivative and hence all the integrals of motion are in involution.

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