Egyptian Fractions and Prime Power Divisors

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Abstract

From varying Egyptian fraction equations we obtain generalizations of primary pseudoperfect numbers and Giuga numbers which we call prime power psuedoperfect numbers and prime power Giuga numbers respectively. We show that a sequence of Amarnath Murthy in the OEIS is a subsequence of the sequence of prime power psuedoperfect numbers. Prime factorization conditions sufficient to imply a number is a prime power pseudoperfect number or a prime power Giuga number are given. The conditions on prime factorizations naturally give rise to a generalization of Fermat primes which we call extended Fermat primes.

1 Introduction

We define and study two new types of integers which we call prime power pseudoperfect numbers and prime power Giuga numbers. Each satisfies an Egyptian fraction equation that is a variation of a previously studied Egyptian fraction equation. Throughout we will reference relevant sequences from the OEIS [7]. In Section 1 we review pseudoperfect numbers, primary pseudoperfect numbers, and Giuga numbers. In Section 2 we will define prime power pseudoperfect numbers and show their relation to a sequence in the OEIS contributed by Amarnath Murthy. Prime power Giuga numbers are defined in Section 3. We will give some formulas that can produce more terms of our sequences in Section 4. In Section 5 we discuss some open problems and introduce extended Fermat primes. We have contributed each of our new sequences to the OEIS. The sequences of prime power pseudoperfect numbers, prime power Giuga numbers, and extended Fermat primes are A283423, A286497, and A286499 respectively.

A *pseudoperfect number* is a positive integer n such that there exist $0 < d_1 < \cdots < d_k < n$ where $d_i \mid n$ for each i and $n = d_1 + \cdots + d_k$. For example, the number 20 is a pseudoperfect number since $20 = 1+4+5+10$. Pseudoperfect numbers were first considered in the article [6] and are sequence A005835. Open problems on pseudoperfect numbers can be found in the book [5, B2]. A *primary pseudoperfect number* is a positive integer $n > 1$ which satisfies the

Egyptian fraction equation

$$\sum_{p|n} \frac{1}{p} + \frac{1}{n} = 1$$

where the sum is taken over all prime divisors on n. Primary pseudoperfect numbers were originally defined in the article [3] and are sequence A054377. When $n > 1$ is primary pseudoperfect number it follows that

$$\sum_{p|n} \frac{n}{p} + 1 = n.$$

So, we see that, with the exception of 2, every primary pseudoperfect number is a pseudoperfect number.

A *Giuga number* is a positive composite integer n such that

$$\sum_{p|n} \frac{1}{p} - \frac{1}{n} \in \mathbb{N}$$

where the sum is taken over all prime divisors on n. Giuga numbers were defined in the article [2] and are sequence A007850. All known Giuga numbers satisfy the stronger Egyptian fraction equation

$$\sum_{p|n} \frac{1}{p} - \frac{1}{n} = 1.$$

Giuga numbers are realted to Giuga's conjecture on primality [4]. Open problems relating to Giuga numbers can be found in the book [5, A17]

2 Prime power pseudoperfect numbers

A *prime power pseudoperfect number* is a positive integer $n > 1$ which satisfies the Egyptian fraction equation

$$\sum_{p^k|n} \frac{1}{p^k} + \frac{1}{n} = 1$$

where the sum is taken over all prime power divisors of n. Observe that, with the exception of powers of 2, all prime power pseudoperfect numbers are pseudoperfect. Also note that any primary pseudoperfect number is a prime power pseudoperfect number since primary pseudoperfect numbers must be squarefree. Prime power pseudoperfect numbers are sequence A283423.

We will now consider the sequences A073932 and A073935 both of which were contributed to the OEIS by Amarnath Murthy. We first define a function d on composite numbers by letting $d(n)$ denote the largest nontrivial divisor of n. For example, $d(15) = 5$. Next we define a function f on positive integers greater than 1 by

$$f(n) := \begin{cases} n - 1 & \text{if } n \text{ is prime;} \\ n - d(n) & \text{otherwise.} \end{cases}$$

As an example, $f(15) = 10$. Given any positive integer $n > 1$ we can iterate the function f until we reach 1. In this way we obtain a triangle with nth row given by $n, f(n), f(f(n)), \ldots, 1$. The sequence A073932 is the sequence consisting of the entries of this triangle read by rows. For any positive integer n let D_n denote the set of divisors of n and we define $F(n) := \{n, f(n), f(f(n)), \ldots, 1\}$. The sequence consisting of all n such that the $F(n) = D_n$ is sequence A073935.

If we consider $n = 20$ we obtain

$$\begin{aligned} f(20) &= 10 \\ f(f(20)) &= 5 \\ f(f(f(20)) &= 4 \\ f(f(f(f(20)))) &= 2 \\ f(f(f(f(f(20))))) &= 1 \end{aligned}$$

which are exactly the divisors of 20. We also notice that 20 is a prime power pseudoperfect number since

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{20} = 1.$$

We will show in Theorem 3 that every number in the sequence A073935, with the exception of 1, is a prime power pseudoperfect number. We first prove two lemmata.

Lemma 1. *Let $n > 1$ be a positive integer with prime factorization $n = p_1 p_2 \cdots p_\ell$ where $p_1 \leq p_2 \leq \cdots \leq p_\ell$. The function f is then given by*

$$f(n) = (p_1 - 1)p_2 \cdots p_\ell.$$

Proof. Take any positive integer $n > 1$ with prime factorization $n = p_1 p_2 \cdots p_\ell$ where $p_1 \leq p_2 \leq \cdots \leq p_\ell$. If n is prime, then $\ell = 1$ and $n = p_1$. In this case $f(n) = n - 1 = p_1 - 1$. When n is composite $\ell > 1$ the largest nontrivial divisor is $p_2 p_3 \cdots p_\ell$. In this case

$$\begin{aligned} f(n) &= n - d(n) \\ &= p_1 p_2 \cdots p_\ell - p_2 p_3 \cdots p_\ell \\ &= (p_1 - 1)p_2 \cdots p_\ell. \end{aligned}$$

We see in any case the lemma holds. □

Lemma 2. *Let $n > 1$ be a positive integer with prime factorization*

$$n = \prod_{i=1}^{\ell} p_i^{a_i}$$

where $p_1 < p_2 < \cdots < p_\ell$. The positive integer n is then in the sequence A073935 if and only if

$$(p_i - 1) = \prod_{j=1}^{i-1} p_j^{a_j}$$

for $1 \leq i \leq \ell$.

Proof. Take any positive integer $n > 1$ with prime factorization

$$n = \prod_{i=1}^{\ell} p_i^{a_i}$$

where $p_1 < p_2 < \cdots < p_\ell$. The divisors of n are

$$D_n = \left\{ \prod_{i=1}^{\ell} p_i^{b_i} : 0 \leq b_i \leq a_i \right\}.$$

We must show $F(n) = D_n$ if and only if

$$(p_i - 1) = \prod_{j=1}^{i-1} p_j^{a_j}$$

for $1 \leq i \leq \ell$.

First assume that

$$(p_i - 1) = \prod_{j=1}^{i-1} p_j^{a_j}$$

for $1 \leq i \leq \ell$. It follows that $F(n) = D_n$ as the divisors of n are obtained in lexicographic order of exponent vectors when we iterate f.

Next assume that $F(n) = D_n$. Write $D_n = \{1 = d_0 < d_1 < \cdots < d_k = n\}$. Note that the function f is strictly decreasing. Thus $F(n) = D_n$ if and only if $f(d_i) = d_{i-1}$ for $1 \leq i \leq k$. First observe that $(p_1 - 1) \mid f(n)$ and $f(n) \mid n$ by assumption. It follows that $p_1 = 2$. Otherwise any prime divisor of $p_1 - 1$ must divide n, but any prime divisor of $p_1 - 1$ is strictly less than any prime divisor of n. Now assume that for $j < i$

$$(p_j - 1) = \prod_{j'=1}^{j-1} p_{j'}^{a_{j'}}.$$

By iterating f we will initially obtain divisors of n of the form

$$\left(\prod_{j=1}^{i-1} p_j^{b_j}\right)\left(\prod_{j=i}^{\ell} p_j^{a_j}\right)$$

where $0 \le b_j \le a_j$ for $1 \le j < i$.

When we come to the divisor $p_i^{a_i} p_{i+1}^{a_{i+1}} \cdots p_\ell^{a_\ell}$, by Lemma 1

$$f(p_i^{a_i} p_{i+1}^{a_{i+1}} \cdots p_\ell^{a_\ell}) = (p_i - 1) p_i^{a_i - 1} p_{i+1}^{a_{i+1}} \cdots p_\ell^{a_\ell}.$$

By assumption $(p_i - 1) p_i^{a_i - 1} p_{i+1}^{a_{i+1}} \cdots p_\ell^{a_\ell}$ divides n. If

$$(p_i - 1) \ne \prod_{j=1}^{i-1} p_j^{a_j}$$

we see that

$$\begin{aligned} f(p_i^{a_i} p_{i+1}^{a_{i+1}} \cdots p_\ell^{a_\ell}) &= (p_i - 1) p_i^{a_i - 1} p_{i+1}^{a_{i+1}} \cdots p_\ell^{a_\ell} \\ &< p_1^{a_1} p_2^{a_2} \cdots p_{i-1}^{a_{i-1}} p_i^{a_i - 1} p_{i+1}^{a_{i+1}} \cdots p_\ell^{a_\ell} \end{aligned}$$

and the divisor $p_1^{a_1} p_2^{a_2} \cdots p_{i-1}^{a_{i-1}} p_i^{a_i - 1} p_{i+1}^{a_{i+1}} \cdots p_\ell^{a_\ell}$ will not be contained in $F(n)$. Thus when $F(n) = D_n$ we must have

$$(p_i - 1) = \prod_{j=1}^{i-1} p_j^{a_j}$$

for all $1 \le i \le \ell$. □

Algorithm 1 Nondeterministic algorithm to produce terms of sequence A073935.

```
n ← 2
loop
    p ← largest prime divisor of n
    if n + 1 is prime then
        n ← np
        n ← n(n + 1)
    else
        n ← np
    end if
end loop
```

Algorithm 1 is a nondeterministic algorithm which produces the terms of A073935. Lemma 2 implies that Algorithm 1 does indeed produce the sequence. Terms of the sequence coming from various branches of the algorithm are shown in Figure 1.

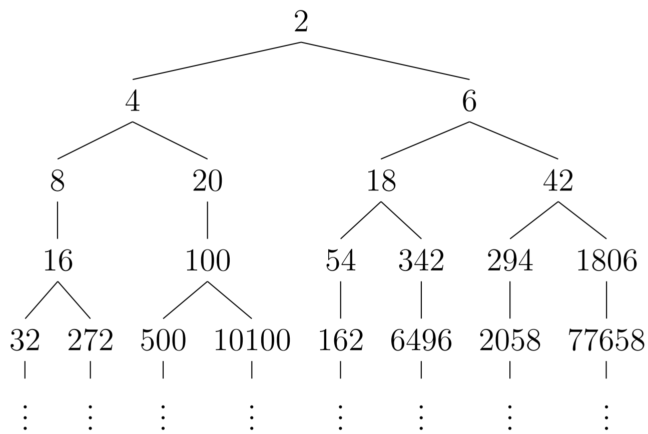

Figure 1: Tree showing terms of sequence A073935 on various branches of Algorithm 1.

Theorem 3. *Every number in the sequence A073935, with the exception of* 1*, is a prime power pseudoperfect number.*

Proof. Let $n > 1$ be in the sequence A073935. Assume n has prime factorization

$$n = \prod_{i=1}^{\ell} p_i^{a_i}.$$

By Lemma 2 we know that

$$(p_i - 1) = \prod_{j=1}^{i-1} p_j^{a_j}$$

for $1 \leq i \leq \ell$. We define

$$n_i := \frac{n}{p_i^{a_i}}$$
$$n'_i := \frac{n}{\prod_{j=1}^{i} p_j^{a_j}}$$

for $1 \le i \le \ell$ and define $n'_0 := n$. We now compute

$$\begin{aligned}
\sum_{p^k|n} \frac{1}{p^k} + \frac{1}{n} &= \sum_{i=1}^{\ell} \sum_{j=1}^{a_i} \frac{1}{p_i^j} + \frac{1}{n} \\
&= \sum_{i=1}^{\ell} \sum_{j=0}^{a_i-1} \frac{p_i^j n_i}{n} + \frac{1}{n} \\
&= \sum_{i=1}^{\ell} \frac{(p_i^{a_i}-1)n_i}{(p_i-1)n} + \frac{1}{n} \\
&= \sum_{i=1}^{\ell} \frac{(p_i^{a_i}-1)n'_i}{n} + \frac{1}{n} \\
&= \sum_{i=1}^{\ell} \frac{p_i^{a_i}n'_i - n'_i}{n} + \frac{1}{n} \\
&= \sum_{i=1}^{\ell} \frac{n'_{i-1} - n'_i}{n} + \frac{1}{n} \\
&= \frac{n'_0 - n'_\ell + 1}{n} \\
&= 1
\end{aligned}$$

Therefore n is prime power pseudoperfect. □

The converse of Theorem 3 is not true. For example, the number $23994 = 2 \cdot 3^2 \cdot 31 \cdot 41$ is a prime power pseudoprime number that is not in the sequence A073935.

3 Prime power Giuga numbers

A *prime power Giuga number* is a positive composite integer $n > 1$ which satisfies the Egyptian fraction condition

$$\sum_{p^k|n} \frac{1}{p^k} - \frac{1}{n} \in \mathbb{N}$$

where the sum is taken over all prime power divisors of n. Since Giuga numbers are squarefree it follows that all Giuga numbers are prime power Giuga numbers. All prime power Giuga numbers we have found obey the stricter Egyptian fraction equation

$$\sum_{p^k|n} \frac{1}{p^k} - \frac{1}{n} = 1.$$

Prime power Giuga numbers are sequence A286497.

We now prove a lemma analogous to Lemma 2.

Lemma 4. *Let $n > 1$ be a positive integer with prime factorization*

$$n = \prod_{i=1}^{\ell} p_i^{a_i}$$

with $p_1 < p_2 < \cdots < p_\ell$ and $a_\ell = 1$. If

$$p_i - 1 = \prod_{j=1}^{i-1} p_j^{a_j}$$

for $1 \leq i < \ell$ and $p_\ell + 1 = \frac{n}{p_\ell}$, then the positive integer n is a prime power Giuga number.

Proof. Assume $n > 1$ is a positive integer satisfying the hypothesis of the lemma. Then by Lemma 2 and Theoerm 3 we know the $\frac{n}{p_\ell}$ is a prime power pseudoperfect number. So,

$$\begin{aligned}
\sum_{p^k | n} \frac{1}{p^k} - \frac{1}{n} &= \sum_{p^k | \frac{n}{p_\ell}} \frac{1}{p^k} + \frac{1}{p_\ell} - \frac{1}{n} \\
&= \frac{\frac{n}{p_\ell} - 1}{\frac{n}{p_\ell}} + \frac{1}{p_\ell} - \frac{1}{n} \\
&= \frac{p_\ell}{p_\ell + 1} + \frac{1}{p_\ell} - \frac{1}{(p_\ell + 1)p_\ell} \\
&= 1.
\end{aligned}$$

□

Lemma 4 gives a sufficient but not necessary condition for being a prime power Giuga number. Table 1 shows prime power Giuga numbers less than 10^7. Notice some numbers in the table, such as $858 = 2 \cdot 3 \cdot 11 \cdot 13$, do not satisfy the condition in Lemma 4.

4 Producing more terms

In this section we give some formulas that can be helpful in finding solutions to our Egyptian fraction equations. Similar results for primary pseudoperfect numbers and Giuga numbers are given in the article [9, Theorem 8]. The article [8, Proposition 1] also contains conditions for primary pseudoperfect numbers. Results to help search for solutions of other related Egyptian fraction equations can be found in the articles [1, Proposition 12, Lemma 17] and [3, Lemma 4.1, Lemma 4.2].

Proposition 5. *Let $n > 1$ be a positive integer.*

(i) If n is in the sequence A073935 and p is largest prime divisor of n, then both $\frac{n}{p}$ and np are in the sequence A073935.

n	prime factorization
12	$2^2 \cdot 3$
30	$2 \cdot 3 \cdot 5$
56	$2^3 \cdot 7$
306	$2 \cdot 3^2 \cdot 17$
380	$2^2 \cdot 5 \cdot 19$
858	$2 \cdot 3 \cdot 11 \cdot 13$
992	$2^5 \cdot 31$
1722	$2 \cdot 3 \cdot 7 \cdot 41$
2552	$2^3 \cdot 11 \cdot 29$
2862	$2 \cdot 3^3 \cdot 53$
16256	$2^7 \cdot 127$
30704	$2^4 \cdot 19 \cdot 101$
66198	$2 \cdot 3 \cdot 11 \cdot 17 \cdot 59$
73712	$2^4 \cdot 17 \cdot 271$
86142	$2 \cdot 3 \cdot 7^2 \cdot 293$
249500	$2^2 \cdot 5^3 \cdot 499$
629802	$2 \cdot 3^3 \cdot 107 \cdot 109$
1703872	$2^6 \cdot 79 \cdot 337$
6127552	$2^6 \cdot 67 \cdot 1429$

Table 1: Prime power Giuga numbers less than 10^7.

(ii) If n is in the sequence A073935 and $n+1$ is prime, then $n(n+1)^k$ is in the sequence A073935 for any nonnegative integer k.

(iii) If n is a prime power pseudoperfect number and $n+1$ is prime, then $n(n+1)^k$ is a prime power pseudoperfect number for any nonnegative integer k.

(iv) If n is a prime power pseudoperfect number and $n-1$ is prime, then $n(n-1)$ is a prime power Giuga number.

Proof. Parts (i) and (ii) follow immediately from Lemma 2.

For part (iii) assume that n is a prime power pseudoperfect number and $n+1$ is prime. So,

$$\begin{aligned}\sum_{p^k|n(n+1)^k}\frac{1}{p^k}+\frac{1}{n(n+1)^k} &= \sum_{p^k|n}\frac{1}{p^k}+\sum_{j=1}^{k}\frac{1}{(n+1)^j}+\frac{1}{n(n+1)^k}\\ &= \frac{n-1}{n}+\frac{(n+1)^k-1}{n(n+1)^k}+\frac{1}{n(n+1)^k}\\ &= \frac{(n-1)(n+1)^k+(n+1)^k}{n(n+1)^k}\\ &= 1.\end{aligned}$$

For part (iv) assume that n is a prime power pseudoperfect number and $n-1$ is prime. So,

$$\begin{aligned}\sum_{p^k|n(n+1)}\frac{1}{p^k}-\frac{1}{n(n-1)} &= \sum_{p^k|n}\frac{1}{p^k}+\frac{1}{(n+1)}-\frac{1}{n(n-1)}\\ &= \frac{n-1}{n}+\frac{1}{(n+1)}-\frac{1}{n(n-1)}\\ &= \frac{(n-1)(n-1)+n-1}{n(n-1)}\\ &= 1.\end{aligned}$$

□

Consider the number

$$n = 23994 = 2\cdot 3^2\cdot 31\cdot 43$$

which is a prime power pseudoperfect number. However, neither

$$\frac{n}{43} = 558 = 2\cdot 3^2\cdot 31$$

nor

$$43n = 1031742 = 2\cdot 3^2\cdot 31\cdot 43^2$$

is a prime power pseudoperfect number. Hence, a version of Proposition 5 (i) does not hold from prime power pseudoperfect numbers. Also consider the number $n = 18$ which is a prime power pseudoperfect number, and the number $n(n-1) = 306$ is a prime power Giuga number since $n - 1 = 17$ is prime. However, the number $n(n-1)^2 = 5202$ is not a prime power Giuga number. Thus a version of Proposition 5 (ii) or (iii) does not hold for prime power Giuga numbers.

5 Open questions

Proposition 5 immediately shows that there are infinitely many terms in both the sequence A073935 and the sequence of prime power pseudoperfect numbers A283423. Proposition 5 does not give a way to produce infinitely many prime power Giuga numbers, but we conjecture there are infinitely many such numbers.

Conjecture 6. There are infinitely many prime power Giuga numbers.

A *Mersenne prime* is prime number p such that $p = 2^k - 1$ for some integer k. Mersenne primes are sequence A000668. By Lemma 4, the number $n = 2^k(2^k - 1)$ is a prime power Giuga number whenever $2^k - 1$ is a Mersenne prime. Hence, Conjecture 6 would follow from an infinitude of Mersenne primes, and it is believed that there are infinitely many Mersenne primes.

A *Fermat prime* is prime number p such that $p = 2^k + 1$ for some positive integer k. Fermat primes are sequence A019434. By Lemma 2 the number 2^k is in the sequence A073935 for any positive integer k, and the powers of 2 are the only numbers in the sequence that have a unique prime divisor. If a number with two distinct prime divisors is in sequence A073935 it must be of the form $2^k(2^k + 1)^j$ where $2^k + 1$ is a Fermat prime and j is a positive integer.

The primes which occur as divisors of terms of the sequence A073935 are primes p such that

$$p - 1 = \prod_{i=1}^{\ell} p_i^{a_i}$$

where for $1 \le i \le \ell$

$$p_i - 1 = \prod_{j=1}^{i-1} p_i^{a_i}.$$

A table of such primes is included in Table 2. Let us call such primes *extended Fermat primes* and if p is an extended Fermat prime such that

$$p - 1 = \prod_{i=1}^{\ell} p_i^{a_i}$$

we say p is a *level-ℓ extended Fermat prime*. By convention the prime 2 is the only level-0 extended Fermat prime. With this new definition usual Fermat primes are now level-1

p	$p-1$
2	1
3	2
5	2^2
7	$2 \cdot 3$
17	2^4
19	$2 \cdot 3^2$
43	$2 \cdot 3 \cdot 7$
101	$2^2 \cdot 5^2$
163	$2 \cdot 3^4$
257	2^8
487	$2 \cdot 3^5$
1459	$2 \cdot 3^6$
14407	$2 \cdot 3 \cdot 7^4$
26407	$2 \cdot 3^4 \cdot 163$
39367	$2 \cdot 3^9$
62501	$2^2 \cdot 5^6$
65537	2^{16}
77659	$2 \cdot 3 \cdot 7 \cdot 43^2$
1020101	$2^2 \cdot 5^2 \cdot 101^2$

Table 2: Table of extended Fermat primes p along with factorizations of $p-1$.

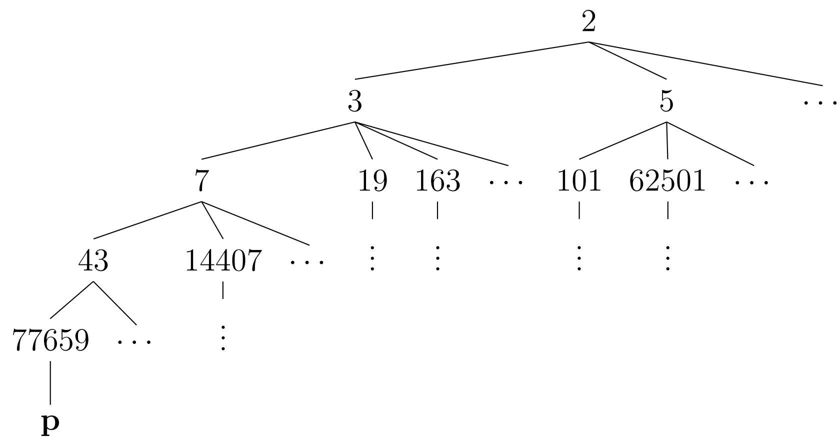

Figure 2: A portion of the tree of extended Fermat primes.

extended Fermat primes. Extended Fermat primes are sequence A286499. It is thought that there are only finitely many Fermat primes. However, we believe there are infinitely many extended Fermat primes and offer the following conjectures.

Conjecture 7. There exists an extended Fermat prime p such that $(p-1)p^k+1$ is an extended Fermat prime for infinitely many values of k.

Conjecture 8. Given any positive integer ℓ there exists a level-ℓ extended Fermat prime.

Towards an answer to Conjecture 7, the prime 3 may give a example. Computation suggests there are many primes of the form $2\cdot 3^k+1$. The values of k for which $2\cdot 3^k+1$ is prime is sequence A003306. In the direction of Conjecture 8, we have found a level-5 extended Fermat prime

$$\mathbf{p} = 2\cdot 3\cdot 7\cdot 43^2\cdot 77659^{197}+1.$$

We can form a rooted tree of extended Fermat primes with root 2 as follows. Let p and q be two extended Fermat primes, then q is a descendent of p if and only if $p \mid (q-1)$. A portion of this tree, including a path to the level-5 extended Fermat prime $\mathbf{p}$, is shown in Figure 2.

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2010 *Mathematics Subject Classification*: Primary 11D68; Secondary 11A41, 11A51.
Keywords: Egyptian fractions, pseudoperfect numbers, primary pseudoperfect numbers, Giuga numbers.

(Concerned with sequences A000668, A003306, A005835, A007850, A019434, A054377, A073932, A073935, A283423, A286497, and A286499.)
