COMBINATORIAL IDENTITIES GENERATED BY DIFFERENCE ANALOGS OF HYPERBOLIC AND TRIGONOMETRIC FUNCTIONS OF ORDER n

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Abstract. We naturally obtain some combinatorial identities finding the difference analogs of hyperbolic and trigonometric functions of order n. In particular, we obtain the identities connected with the proved in the paper the addition formulas for these analogs.

1. Introduction

The original definitions of the hyperbolic and trigonometric functions of order n are the following (cf. [1], point 18.2).

Definition 1. *The n functions*

$$h_s(x,n)=\frac{1}{n}\sum_{t=1}^{n}\omega^{(1-s)t}\exp(\omega^t x),\quad s=1,...,n, \tag{1}$$

where $\omega=\exp(\frac{2\pi i}{n})$, are called hyperbolic functions of order n.

In particular,

$$h_1(x,1)=e^x, h_1(x,2)=\cosh x, h_2(x,2)=\sinh x. \tag{2}$$

Definition 2. *The n functions*

$$k_s(x,n)=\sum_{t=0}^{\infty}\frac{(-1)^t x^{nt+s-1}}{(nt+s-1)!},\quad s=1,...,n, \tag{3}$$

$n\geq 2$, *are called trigonometric functions of order n.*

In particular,

$$k_1(x,1)=e^{-x}, k_1(x,2)=\cos x, k_2(x,2)=\sin x. \tag{4}$$

We consider the following equivalent definitions which could be proved directly from Definitions 1, 2 and the uniqueness of the solution of the Cauchy problem.

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Proposition 1. *a) The functions* $\{h_s(x,n)\},\ \ s=1,...,n,$ *form the solution of the Cauchy problem for the following system of ordinary differential equations*

$$y_s' = y_{s-1},\ \ s=2,3,...,n,\ \ y_1' = y_n$$

with the initials $y_1(0)=1,\ \ y_s(0)=0,\ \ s=2,...,n.$
b) The functions $\{k_s(x,n)\},\ \ s=1,...,n,$ *form the solution of the Cauchy problem for the following system of ordinary differential equations*

$$y_s' = y_{s-1},\ \ s=2,3,...,n,\ \ y_1' = -y_n$$

with the initials $y_1(0)=1,\ \ y_s(0)=0,\ \ s=2,...,n.$

Note that also we have

$$h_s(x,n)=\sum_{t=0}^{\infty}\frac{x^{nt+s-1}}{(nt+s-1)!},\ \ s=1,...,n. \tag{5}$$

Proposition 1 allows to introduce the difference analogs of hyperbolic and trigonometric functions of order n. As usual, set $\Delta f(m)=f(m+1)-f(m)$.

Definition 3. *For a fixed* n *and nonnegative integer variation* m, *the functions* $\{H_s(m,n)\},\ \ s=1,...,n,$ *are called difference hyperbolic of order* n *if they form the solution of the following system of difference equations*

$$\Delta y_s(m)=y_{s-1}(m),\ \ s=2,3,...,n,\ \ \Delta y_1(m)=y_n(m) \tag{6}$$

with the initials $y_1(0)=1,\ \ y_s(0)=0,\ \ s=2,...,n.$

Definition 4. *For a fixed* n *and nonnegative integer variation* m, *the functions* $\{K_s(m,n)\},\ \ s=1,...,n,$ *are called difference trigonometric of order* n *if they form the solution of the following system of difference equations*

$$\Delta y_s(m)=y_{s-1}(m),\ \ s=2,3,...,n,\ \ \Delta y_1(m)=-y_n(m) \tag{7}$$

with the initials $y_1(0)=1,\ \ y_s(0)=0,\ \ s=2,...,n.$

Our goal is, using the properties of functions $H_s(m,n)$ and $K_s(m,n)$, to prove the following identities.

Theorem 1. *For* $m\geq 0$, *we have*

$$\sum_{t\geq 0}\binom{m}{nt+s-1}=\frac{1}{n}\sum_{j=1}^{n}(\omega^j+1)^m\omega^{j(1-s)},\ \ s=1,...,n; \tag{8}$$

$$\sum_{t\geq 0}(-1)^t\binom{m}{nt+s-1}=\frac{1}{n}\sum_{j=1}^{n}(\mu^{2j-1}+1)^m\mu^{(2j-1)(1-s)},\ \ s=1,...,n, \tag{9}$$

where $\mu=\exp(\frac{\pi i}{n})$.

Note that formula (8) is known ([3], [4]), but formula (9) probably is new (at least, it is neither in [3] nor in [4]).

Let us define the sets $\{H_s(m,n)\}, \{K_s(m,n)\}$ outside $s \in \{1,...,n\}$, putting for $s = 1,...,n$,

$$H_{-(s-1)}(m,n) = H_{n-s+1}(m,n), \ \ K_{-(s-1)}(m,n) = -K_{n-s+1}(m,n). \tag{10}$$

Below we show that the definition (10) is quite natural.

Theorem 2. *(The addition formulas) For integers $m, s \geq 0$ we have the identities:*

$$H_i(m+s,n) = \sum_{j=1}^{n} H_j(s,n)H_{i-j+1}(m,n), \ \ i = 1,...,n; \tag{11}$$

$$K_i(m+s,n) = \sum_{j=1}^{n} K_j(s,n)K_{i-j+1}(m,n), \ \ i = 1,...,n. \tag{12}$$

Finally, consider circulant matrices H_n and K_n with the first row $\{(-1)^{i-1}H_i(m,n)\}, \ \ i = 1,...,n$, and $\{(-1)^{i-1}K_i(m,n)\}, \ \ i = 1,...,n$, respectively.

Theorem 3. 1) *If n is even, then for every $m \geq 1$,* $\det \mathrm{H}_n = 0$;
2) *If n is odd, then for every $m \geq 1$,* $\det \mathrm{K}_n = 0$.

2. Proof of Theorem 1

Proof. Using $\Delta\binom{m}{k} = \binom{m}{k-1}$, it is easy to verify that $H_s(m,n)$ and $K_s(m,n)$ have the following form (such that the initials evidently hold):

$$H_s(m,n) = \sum_{t \geq 0} \binom{m}{nt+s-1}, \ \ s = 1,...,n; \tag{13}$$

$$K_s(m,n) = \sum_{t \geq 0} (-1)^t \binom{m}{nt+s-1}, \ \ s = 1,...,n. \tag{14}$$

Moreover, (13) and (14) agree with (10). For example, consider the equality from (10) for $s = 1$ $K_0(m,n) = -K_n(m,n)$. We have

$$K_n(m) = \binom{m}{n-1} - \binom{m}{2n-1} + \binom{m}{3n-1} - ...$$

and formally for "$s = 0$" we have

$$K_0(m,n) = \binom{m}{-1} - \binom{m}{n-1} + \binom{m}{2n-1} - \binom{m}{3n-1} - ....$$

Since $\binom{m}{-1} = 0$, the considered equality is evident.

Furthermore, note that, by Definition 3, 4, $H_s(m,n),\ \ s=0,...,n$ satisfies the difference equation $\Delta^n y - y = 0$, while $K_s(m,n),\ \ s=0,...,n$ satisfies the difference equation $\Delta^n y + y = 0$. The characteristic equations of these difference equations are (cf.[2])

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\lambda^k \mp 1 = (\lambda-1)^n \mp 1 = 0 \tag{15}$$

respectively. Thus we have

$$H_s(m,n) = \sum_{j=1}^{n} C_{s,j}^{(1)}(\omega^j+1)^m =$$

$$\sum_{j=1}^{n} C_{s,j}^{(2)}\omega^{(1-s)j}(\omega^j+1)^m,\ \ s=1,...,n. \tag{16}$$

Further, note that to obtain n distinct roots of $x^n=-1$ we could consider $\{\mu,\mu^3,...,\mu^{2n-1}\}$, where $\mu=\exp(\frac{\pi i}{n})$. So,

$$K_s(m,n) = \sum_{j=1}^{n} C_{s,j}^{(3)}(\mu^{2j-1}+1)^m =$$

$$\sum_{j=1}^{n} C_{s,j}^{(4)}\mu^{(1-s)(2j-1)}(\mu^{2j-1}+1)^m,\ \ s=1,...,n. \tag{17}$$

Let us show that

$$C_{s,j}^{(2)} = C_{s,j}^{(4)} = \frac{1}{n}$$

such that

$$H_s(m,n) = \frac{1}{n}\sum_{j=1}^{n}\omega^{(1-s)j}(\omega^j+1)^m,\ \ s=1,...,n; \tag{18}$$

$$K_s(m,n) = \frac{1}{n}\sum_{j=1}^{n}\mu^{(1-s)(2j-1)}(\mu^{2j-1}+1)^m,\ \ s=1,...,n. \tag{19}$$

Indeed, it is easy to verify that $\Delta H_s(m,n) = H_{s-1}$ (in particular, $\Delta H_1(m,n) = H_0(m,n) = H_n(m,n)$); $\Delta K_s(m,n) = K_{s-1}$ (in particular, $\Delta K_1(m,n) = K_0(m,n) = -K_n(m,n)$). Initials also hold, in view of identities for $s>1:$

$$\sum_{j=1}^{n}\omega^{j(1-s)} = 0,\ \ \sum_{j=1}^{n}\mu^{(2j-1)(1-s)} = \mu^{s-1}\sum_{j=1}^{n}\omega^{j(1-s)} = 0.$$

Comparing (13) with (18) and (14) with (19) we obtain (8) and (9) respectively. □

Using (18), (19) and simple transformations, we obtain, for example, formulas:

$$K_2(m,2) = (\sqrt{2})^m \sin\frac{\pi m}{4},$$
$$H_1(m,3) = \frac{1}{3}(2^m + 2\cos\frac{\pi m}{3}).$$

These are A009545, A024493 [5] respectively. In particular, $(\sqrt{2})^m \sin\frac{\pi m}{4}$ is the difference analog of $k_2(x,2) = \sin x$.

Further examples: for $m \geq 1$, using (9), we have

$$K_1(m,5) = (2/5)(\varphi+2)^{m/2}(cos(\pi m/10) + (\varphi-1)^m cos(3\pi m/10)),$$
$$K_2(m,5) = (2/5)(\varphi+2)^{m/2}(cos(\pi(m-2)/10) + (\varphi-1)^m cos(3\pi(m-2)/10)),$$
$$K_3(m,5) = (2/5)(\varphi+2)^{m/2}(cos(\pi(m-4)/10) + (\varphi-1)^m cos(3\pi(m-4)/10)),$$
$$K_4(m,5) = (2/5)(\varphi+2)^{m/2}(cos(\pi(m-6)/10) + (\varphi-1)^m cos(3\pi(m-6)/10)),$$
$$K_5(m,5) = (2/5)(\varphi+2)^{m/2}(cos(\pi(m-8)/10) + (\varphi-1)^m cos(3\pi(m-8)/10)),$$

where φ is the golden ratio. These sequences are A289306, A289321, A289387, A289388, A289389 [5] respectively. Note that in case $n = 5$ in (9) the third summand is 0, but if $m = 0$, 0^0 is accepted as 1. It is the reason why the formulas hold only for $m \geq 1$. Note also that, using these formulas, it is easy to find all zeros of the functions $K_i(m,5)$. So, we find that
$K_1(m,5) = 0$ if and only if $m \equiv 5 \pmod{10}$;
$K_2(m,5) = 0$ if and only if $m = 0$ or $m \equiv 7 \pmod{10}$;
$K_3(m,5) = 0$ if and only if $m = 0, m = 1$ or $m \equiv 9 \pmod{10}$;
$K_4(m,5) = 0$ if and only if $m = 0, m = 2$ or $m \equiv 1 \pmod{10}$;
$K_5(m,5) = 0$ if and only if $m = 0, m = 1, m = 2$ or $m \equiv 3 \pmod{10}$.

3. Proof of Theorem 2

Proof. Since the proofs for the formulas of Theorem 2 are identical, we prove the latter one. Using Definition 4 and (10), let us find the values $K_i(1) = y_i(1)$ (Here we write $K_i(m,n) = K_i(m)$ for a fixed n). Since $\Delta y_i(m) = y_{i-1}(m)$, then

$$y_i(m+1) = y_i(m) + y_{i-1}(m). \tag{20}$$

Hence, for m=0, we have $y_i(1) = 0$, except for $i = 1$ and $i = 2 : y_1(1) = 1$ and $y_2(1) = 1$. Consequently, the sum $\sum_{j=1}^{n} K_j(1)K_{i-j+1}(m)$ contains only two positive summands for $j = 1, 2$. So, by (20), we have

$$K_i(m+1) = K_i(m) + K_{i-1}(m) = \sum_{j=1}^{n} K_j(1)K_{i-j+1}(m). \tag{21}$$

It is formula (12) for $s = 1$. Further we use induction. Suppose, for every $m \geq 0$, we have

$$K_i(m+s) = \sum_{j=1}^{n} K_j(s)K_{i-j+1}(m), \;\; i = 1, ..., n. \tag{22}$$

Then we show that

$$K_i(m+(s+1)) = \sum_{j=1}^{n} K_j(s+1)K_{i-j+1}(m), \;\; i = 1, ..., n. \tag{23}$$

By (21),

$$K_i(m+s+1) = K_i((m+s)+1) = K_i(m+s) + K_{i-1}(m+s). \tag{24}$$

Further, again by (21), the right hand side of (23) equals

$$\sum_{j=1}^{n} K_j(s+1)K_{i-j+1}(m) = \sum_{j=1}^{n}(K_j(s) + K_{j-1}(s))K_{i-j+1}(m) =$$

$$\sum_{j=1}^{n} K_j(s)K_{i-j+1}(m) + \sum_{j=1}^{n} K_{j-1}(s)K_{i-j+1}(m) = \Sigma_1 + \Sigma_2. \tag{25}$$

According to the induction supposition (22), we have $\Sigma_1 = K_i(m+s)$ and, by (24), it is left to prove that $\Sigma_2 = K_{i-1}(m+s)$. Again by the induction supposition (22), we have

$$K_{i-1}(m+s) = \sum_{j=1}^{n} K_j(s)K_{i-j}(m). \tag{26}$$

But for Σ_2 we have

$$\Sigma_2 = \sum_{j=1}^{n} K_{j-1}(s)K_{i-j+1}(m) \;\; (j-1 := j) = \sum_{j=0}^{n-1} K_j(s)K_{i-j}(m). \tag{27}$$

So, by (26), (27) and (10) we find

$$K_{i-1}(m+s) - \Sigma_2 = K_n(s)K_{i-n}(m) - K_0(s)K_i(m) =$$

$$(-K_0(s))(-K_i(m)) - K_0(m)K_i(m) = 0,$$

which completes the proof. □

For example, using (10), for $n = 3, \;\; i = 1$, we have

$$H_1(m+s) = H_1(s)H_1(m) + H_2(s)H_3(m) + H_3(s)H_2(m),$$

$$K_1(m+s) = K_1(s)K_1(m) - K_2(s)K_3(m) - K_3(s)K_2(m).$$

So, in particular, using (13) and (14), we obtain the corresponding identities for the binomial coefficients of the form $\binom{r}{3t+i-1}, \;\; r = s, m, s+m, \;\; t \geq 0, \;\; i = 1, 2, 3.$

4. Proof of Theorem 3

Proof. By the well known classic result, the determinant of a circulant matrix H equals

$$\prod_{t=1}^{n}\sum_{i=1}^{n}(-1)^{i-1}H_i(m,n)\omega_t^{i-1},$$

where $\{\omega_t\},\ \ t=1,...,n$, are all distinct roots of order n from 1. The factor corresponding $\omega_t=1$ equals $H_1-H_2+...-H_n$ since n is even. By (13), we have

$$H_1-H_2+...-H_n=$$
$$\binom{m}{0}+\binom{m}{n}+\binom{m}{2n}+\binom{m}{3n}+...$$
$$-\binom{m}{1}-\binom{m}{n+1}-\binom{m}{2n+1}-\binom{m}{3n+1}-...+$$
$$\binom{m}{2}+\binom{m}{n+2}+\binom{m}{2n+2}+\binom{m}{3n+2}+...-$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$-\binom{m}{n-1}-\binom{m}{2n-1}-\binom{m}{3n-1}-\binom{m}{4n-1}-...$$

Reading over columns, we see that all consecutive binomial coefficients occur with alternative signs. It is clear that $\binom{m}{m}$ occurs in the r-th row, if $m\equiv r \pmod n$, $\ 0\leq r\leq n-1$, and other summands are zeros. So, for $m\geq 1$, $H_1-H_2+...-H_n=\sum_{l=0}^{m}(-1)^l\binom{m}{l}=0$ and also $\det \mathrm{H}=0$. Analogously, using (14), for odd n we find that $K_1-K_2+K_3-...+K_n=0$ and so also $\det \mathrm{K}=0$.

□

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