NEW ESTIMATES FOR THE n-TH PRIME NUMBER

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Abstract. In this paper we establish a new explicit upper and lower bound for the n-th prime number, which improve the currently best estimates given by Dusart in 2010. As the main tool we use some recently obtained explicit estimates for the prime counting function. A further main tool is the usage of estimates concerning the reciprocal of $\log p_n$. As an application we derive refined estimates for $\vartheta(p_n)$ in terms of n, where $\vartheta(x)$ is Chebyshev's ϑ-function.

1. Introduction

Let p_n denotes the n-th prime number and let $\pi(x)$ be the number of primes not exceeding x. In 1896, Hadamard [7] and de la Vallée-Poussin [15] proved, independently, the asymptotic formula $\pi(x) \sim x/\log x$ for $x \to \infty$, which is known as the *Prime Number Theorem*. Here, $\log x$ is the natural logarithm of x. As a consequence of the Prime Number Theorem, one gets an asymptotic expression for the n-th prime number, namely

$$p_n \sim n \log n \qquad (n \to \infty). \tag{1.1}$$

In 1902, Cipolla [3] proved a more precise result. He showed that for every positive integer m there exist unique monic polynomials $T_1, \ldots, T_m \in \mathbb{Q}[x]$ with rational coefficients and $\deg(T_k) = k$, such that

$$p_n = n\left(\log n + \log\log n - 1 + \sum_{k=1}^{m} \frac{(-1)^{k+1}T_k(\log\log n)}{k\log^k n}\right) + O\left(\frac{n(\log\log n)^{m+1}}{\log^{m+1} n}\right). \tag{1.2}$$

The polynomials T_k can be computed explicitly. In particular, $T_1(x) = x - 2$ and $T_2(x) = x^2 - 6x + 11$ (see Cipolla [3] or Salvy [13] for further terms). Since the computation of the n-th prime number p_n is difficult for large n, we are interested in upper and lower bounds for p_n. Cipolla's asymptotic formula (1.2) yields that the inequalities

$$p_n > n \log n, \tag{1.3}$$

$$p_n < n(\log n + \log\log n), \tag{1.4}$$

$$p_n > n(\log n + \log\log n - 1) \tag{1.5}$$

hold for all sufficiently large values of n, respectively. The first breakthrough concerning a lower bound for the n-th prime number is due to Rosser [10, Theorem 1]. In 1939, he showed that the inequality (1.3) holds for every positive integer n. In the literature, this result is often called *Rosser's theorem*. Further, he proved [10, Theorem 2] that the inequality

$$p_n < n(\log n + 2\log\log n) \tag{1.6}$$

holds for every positive integer $n \geq 4$. The next result concerning an upper bound which corresponds to the first three terms of (1.2) is due to Rosser and Schoenfeld [11, Theorem 3]. In 1962, they refined Rosser's theorem and the inequality (1.6) by showing that

$$p_n > n(\log n + \log\log n - 1.5)$$

holds for every positive integer $n \geq 2$, and that the inequality

$$p_n < n(\log n + \log\log n - 0.5) \tag{1.7}$$

holds for every positive integer $n \geq 20$, which implies that the inequality (1.4) is fulfilled for every positive integer $n \geq 6$. Based on their estimates for the Chebyshev functions $\psi(x)$ and $\vartheta(x)$, Rosser and Schoenfeld [12] announced to have new estimates for the n-th prime number p_n but they have never published the details. In the direction of (1.5), Robin [9, Lemme 3, Théorème 8] showed that

$$p_n \geq n(\log n + \log\log n - 1.0072629) \tag{1.8}$$

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for every positive integer $n \geq 2$, and that the inequality (1.5) holds for every positive integer n such that $2 \leq n \leq \pi(10^{11})$. In 1996, Massias and Robin [8, Théorème A] gave a series of improvements of (1.7) and (1.8). For example, they have found that

$$p_n \geq n(\log n + \log\log n - 1.002872)$$

for every positive integer $n \geq 2$. Under the assumption that the Riemann hypothesis is true, they [8, Théorème A(vi)] were able to show that the inequality (1.5) holds for every positive integer $n \geq 2$ and that the inequality

$$p_n \leq n\left(\log n + \log\log n - 1 + \frac{\log\log n - 1.8}{\log n}\right) \tag{1.9}$$

holds for every positive integer $n \geq 27\,076$. Two years later, Dusart [4, Théorème 1.7] showed in his thesis that the inequality (1.8) holds for every positive integer $n \geq 27\,076$ even without the assumption that the Riemann hypothesis is true. Further, he achieved a breakthrough concerning the inequality (1.5) by showing that this inequality holds for every positive integer $n \geq 2$. He even found a refinement of (1.5) by showing that the lower bound

$$p_n \geq n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2.25}{\log n}\right)$$

is valid for every positive integer $n \geq 2$. The current best estimates for the n-th prime are also given by Dusart [5, Proposition 6.6 and Proposition 6.7] (see also Dusart [6, Proposition 5.15 and Proposition 5.16]). In 2010, he used new estimates for Chebyshev's ϑ-function to show that the inequality

$$p_n \leq n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n}\right) \tag{1.10}$$

holds for every positive integer $n \geq 688\,383$, which corresponds to the four terms of (1.2), and that

$$p_n \geq n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2.1}{\log n}\right) \tag{1.11}$$

for every positive integer $n \geq 3$. The goal of this paper is to improve the inequalities (1.10) and (1.11) in the direction of (1.2). For this purpose, we first use some recently established estimates for the prime counting function $\pi(x)$ to construct $n_0, b_0(n)$, depending on some parameters, with $b_0(n) \to 10.7$ for $n \to \infty$ so that the following result holds.

Theorem 1.1 (See Theorem 4.3). *For every positive integer $n \geq n_0$, we have*

$$p_n < n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + b_0(n)}{2\log^2 n}\right).$$

By estimating $b_0(n)$, we obtain the following refinement of (1.10).

Corollary 1.2 (See Corollary 4.6). *For every positive integers $n \geq 46\,254\,381$, we have*

$$p_n < n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + 10.667}{2\log^2 n}\right).$$

Furthermore, we construct $n_1, b_1(n)$, depending on some parameters, with $b_1(n) \to 11.3$ for $n \to \infty$ so that the following upper bound is valid.

Theorem 1.3 (See Theorem 5.4). *For every positive integer $n \geq n_1$, we have*

$$p_n > n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + b_1(n)}{2\log^2 n}\right).$$

Finally, we use Theorem 1.3 to find the following improvement of (1.11).

Corollary 1.4 (See Corollary 5.5). *For every positive integer $n \geq 2$, we have*

$$p_n > n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + 11.508}{2\log^2 n}\right).$$

2. Throughout notations

Throughout this paper, n denotes a positive integer. In the majority of the proofs, we use, for a better readability, the notation

$$w = \log\log n, \quad y = \log n, \quad z = \log p_n.$$

Before we give the proofs of Theorem 1.1 and Theorem 1.3, we first establish some preliminary results concerning the reciprocal of $\log p_n$ in Section 3, in which we introduce the following polynomials $P_1, \ldots, P_{12} \in \mathbb{Q}[x]$:

- $P_1(x) = 3x^2 - 6x + 5$,
- $P_2(x) = 5x^3 - 24x^2 + 39x - 14$,
- $P_3(x) = 7x^4 - 48x^3 + 120x^2 - 124x + 51$,
- $P_4(x) = 9x^5 - 80x^4 + 280x^3 - 480x^2 + 405x - 124$,
- $P_5(x) = 11x^6 - 120x^5 + 540x^4 - 1280x^3 + 1680x^2 - 1146x + 325$,
- $P_6(x) = 13x^7 - 168x^6 + 924x^5 - 2800x^4 + 5040x^3 - 5376x^2 + 3143x - 762$,
- $P_7(x) = 4x^8 - 84x^7 + 630x^6 - 2492x^5 + 5915x^4 - 8764x^3 + 7966x^2 - 4064x + 896$,
- $P_8(x) = 3x^2 - 6x + 5.2$,
- $P_9(x) = x^3 - 6x^2 + 11.4x - 4.2$,
- $P_{10}(x) = 2x^3 - 7.2x^2 + 8.4x - 4.41$,
- $P_{11}(x) = x^3 - 4.2x^2 + 4.41x$,
- $P_{12}(x) = 9.3x^2 - 12.3x + 11.5$.

Furthermore, we define the polynomials $Q_1, \ldots, Q_9 \in \mathbb{Q}[x]$ by

- $Q_1(x) = 12x^7 - 138x^6 + 676x^5 - 1819x^4 + 2914x^3 - 2782x^2 + 1468x - 328$,
- $Q_2(x) = 90x^6 - 700x^5 + 2405x^4 - 4506x^3 + 4801x^2 - 2732x + 648$,
- $Q_3(x) = 50x^5 - 275x^4 + 662x^3 - 833x^2 + 538x - 140$,
- $Q_4(x) = 30x^4 - 114x^3 + 181x^2 - 136x + 40$,
- $Q_5(x) = 18x^3 - 43x^2 + 38x - 12$,
- $Q_6(x) = 7x^2 - 8x + 2$,
- $Q_7(x) = (x^2 - x + 1)P_{12}(x) + (x^2 - x + 1)P_8(x) - 3.15P_9(x) - P_{10}(x) + 12.85P_8(x)$,
- $Q_8(x) = 3.15P_{10}(x) + 12.85P_9(x)$,
- $Q_9(x) = 2(x^2 - x + 1)P_9(x) - P_8(x)P_{12}(x)$.

In addition, let A_0 be a real number such that $0.75 \leq A_0 < 1$ and let $F_0 : \mathbb{N}_{\geq 2} \to \mathbb{R}$ be defined by

$$F_0(n) = \log n - A_0 \log p_n.$$

From (1.1) follows that $F_0(n) \geq 0$ for all sufficiently large values of n, and we define

$$N_0 = N(A_0) = \min\{k \in \mathbb{N} \mid F_0(n) \geq 0 \text{ for every positive integer } n \geq k\}. \tag{2.1}$$

3. Some estimates for the quantity $1/\log p_n$

In 1902, Cipolla [3, p. 139] showed that an asypmtotic formula for $1/\log p_n$ is given by

$$\frac{1}{\log p_n} = \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + o\left(\frac{1}{\log^2 n}\right). \tag{3.1}$$

3.1. **New lower bounds.** Concerning (3.1), we show the following lower bound for $1/\log p_n$, where the polynomials $P_1, \ldots, P_4 \in \mathbb{Z}[x]$ are defined as in Section 2.

Proposition 3.1. *For every positive integer $n \geq 688\,383$, we have*

$$\frac{1}{\log p_n} \geq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n} + \frac{1}{\log p_n}\left(\frac{P_1(\log\log n)}{2\log^3 n} - \frac{P_2(\log\log n)}{6\log^4 n} + \frac{P_3(\log\log n)}{12\log^5 n} - \frac{P_4(\log\log n)}{20\log^6 n}\right).$$

Proof. Let n be a positive integer satisfying $n \geq 688\,383$. As mentioned in Section 2, we write, for convenience, $w = \log\log n$, $y = \log n$ and $z = \log p_n$. First, we note that $\log(1+x) \leq \sum_{k=1}^{7}(-1)^{k+1}x^k/k$ for every $x > -1$. Together with (1.10) and the fact that $(w-1)/y + (w-2)/y^2 > -1$ for every positive integer $n \geq 5$, we get

$$-y^2 + (y-w)z \leq -w^2 + (y-w)\sum_{k=1}^{7}\frac{(-1)^{k+1}}{k}\left(\frac{w-1}{y} + \frac{w-2}{y^2}\right)^k.$$

Extending the right-hand side of the last inequality, we obtain that the inequality

$$-y^2 + (y-w)z < -w^2 + w - 1 - \frac{P_1(w)}{2y} + \frac{P_2(w)}{6y^2} - \frac{P_3(w)}{12y^3} + \frac{P_4(w)}{20y^4} - \frac{P_5(w)}{30y^5} + \frac{P_6(w)}{42y^6} - \frac{P_7(w)}{28y^7} - \frac{(w-2)Q_1(w)}{12y^8} - \frac{(w-2)^2Q_2(w)}{30y^9} - \frac{(w-2)^3Q_3(w)}{10y^{10}} - \frac{(w-2)^4Q_4(w)}{6y^{11}} - \frac{(w-2)^5Q_5(w)}{6y^{12}} - \frac{(w-2)^6Q_6(w)}{7y^{13}} - \frac{w(w-2)^7}{7y^{14}} \tag{3.2}$$

holds, where the polynomials $P_5, P_6, P_7, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 \in \mathbb{Q}[x]$ are defined in Section 2. We have $Q_i(x) \geq 0$ for every positive integer i such that $1 \leq i \leq 6$ and every $x \geq 2$. Together with (3.2) and $x(x-2)^7 \geq 0$ for every $x \geq 2$, we get

$$-y^2 + (y-w)z < -w^2 + w - 1 - \frac{P_1(w)}{2y} + \frac{P_2(w)}{6y^2} - \frac{P_3(w)}{12y^3} + \frac{P_4(w)}{20y^4} - \frac{P_5(w)}{30y^5} + \frac{P_6(w)}{42y^6} - \frac{P_7(w)}{28y^7}.$$

Finally, it suffices to apply the inequality $P_5(w)/30 - P_6(w)/(42y) + P_7(w)/(28y^2) \geq 0$. □

We obtain the following weaker lower bound for $1/\log p_n$.

Corollary 3.2. *For every positive integer $n \geq 456\,914$, we have*

$$\frac{1}{\log p_n} \geq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n} + \frac{P_1(\log\log n)}{2\log^3 n \log p_n} - \frac{P_2(\log\log n)}{6\log^4 n \log p_n}.$$

Proof. It is easy to see that $P_3(\log\log n)/(12\log^3 n) - P_4(\log\log n)/(20\log^4 n) \geq 0$ for every positive integer $n \geq \exp(\exp(2))$. Now we use Proposition 3.1 to get that the required inequality holds for every positive integer $n \geq 688\,383$. For the remaining cases we use a computer. □

Corollary 3.3. *For every positive integer $n \geq 71$, we have*

$$\frac{1}{\log p_n} \geq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n}.$$

Proof. Since the inequality

$$\frac{P_1(\log\log n)}{2\log n} - \frac{P_2(\log\log n)}{6\log^2 n} \geq 0 \tag{3.3}$$

holds for every positive integer $n \geq 3$, Corollary 3.2 implies the validity of the required inequality for every positive integer $n \geq 456\,914$. we conclude by checking the remaining cases with a computer. □

3.2. **New upper bounds.** Next, we establish the following upper bound for $1/\log p_n$. Here, we use a similar method as in the proof of Proposition 3.1.

Proposition 3.4. *For every positive integer $n \geq 2$, we have*

$$\frac{1}{\log p_n} \leq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n} + \frac{P_8(\log\log n)}{2\log^3 n \log p_n} - \frac{P_9(\log\log n)}{2\log^4 n \log p_n} - \frac{P_{10}(\log\log n)}{2\log^5 n \log p_n} - \frac{P_{11}(\log\log n)}{2\log^6 n \log p_n}.$$

Proof. First, we consider the case $n \geq 33$. For convenience, we write again $w = \log\log n$, $y = \log n$ and $z = \log p_n$. We note that the inequality $\log(1+t) \geq t - t^2/2$ holds for every $t \geq 0$. Together with (1.11) and $(w-1)/y + (w-2.1)/y^2 \geq 0$, we obtain that

$$-y^2 + (y-w)z \geq -w^2 + (y-w)\sum_{k=1}^{2} \frac{(-1)^{k+1}}{k}\left(\frac{w-1}{y} + \frac{w-2.1}{y^2}\right)^k,$$

which implies that the required inequality holds. Finally we use a computer for the remaining cases. □

Proposition 3.4 implies the following upper bounds for $1/\log p_n$.

Corollary 3.5. *For every positive integer $n \geq 2$, we have*

$$\frac{1}{\log p_n} \leq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n} + \frac{P_8(\log\log n)}{2\log^3 n \log p_n} - \frac{P_9(\log\log n)}{2\log^4 n \log p_n} - \frac{P_{10}(\log\log n)}{2\log^5 n \log p_n}.$$

Proof. If $n \geq 3$, we have $P_{11}(w) = w(w-2.1)^2 \geq 0$. So the claim follows from Proposition 3.4 for every positive integer $n \geq 3$. A computer check completes the proof. □

Corollary 3.6. *For every positive integer $n \geq 2$, we have*

$$\frac{1}{\log p_n} \leq \frac{1}{\log n} - \frac{\log\log n}{\log^2 n} + \frac{(\log\log n)^2 - \log\log n + 1}{\log^2 n \log p_n} + \frac{P_8(\log\log n)}{2\log^3 n \log p_n} - \frac{P_9(\log\log n)}{2\log^4 n \log p_n}.$$

Proof. Since $P_{10}(x) \geq 0$ if and only if $x \geq 2.1$, Corollary 3.5 implies the validity of the required inequality for every positive integer $n \geq 3\,520$. We check the remaining cases with a computer. □

4. A new upper bound for the n-th prime number

Before we formulate the main result of this section concerning an upper bound for the n-th prime number, which improves the upper bound (1.10), we introduce some preliminaries.

4.1. **Preliminaries.** Before we give the proof of Theorem 1.1, we first show two lemmata. We start with the following implicit upper bound for the n-th prime number.

Lemma 4.1. *For every positive integer $n \geq 841\,424\,868$, we have*

$$p_n < n\left(\log p_n - 1 - \frac{1}{\log p_n} - \frac{2.85}{\log^2 p_n} - \frac{13.15}{\log^3 p_n} - \frac{70.7}{\log^4 p_n} - \frac{458.7275}{\log^5 p_n} - \frac{3428.7225}{\log^6 p_n}\right).$$

Proof. If $n \geq 841\,424\,976$, the inequality follows from [2, Theorem 3.8]. So it remains to check with a computer that this inequality holds for every positive integer n with $841\,424\,868 \leq n \leq 841\,424\,975$. □

Lemma 4.2. *For every $x \geq 2.11$, we have*

$$\frac{(x^2 - 3.85x + 14.15)P_1(x)}{2} - 2 \cdot \frac{2.85P_2(x)}{6} + \frac{P_3(x)}{12} \geq \frac{(x^2 - 3.85x + 14.15)P_2(x)}{6e^x} + \frac{P_4(x)}{20e^x}.$$

Proof. We set $g_1(x) = 250x^4 - 2103x^3 + 8169x^2 - 11935x + 6351$ and $g_2(x) = -154x^5 + 1345x^4 - 5723x^3 + 12955x^2 - 14545x + 4706$. Further, we define $f(x) = g_1(x)(1+x) + g_2(x)$. It is easy to see that $f(x) \geq 0$ for every $x \geq 1.8$. Since $g_1(x) \geq 0$ for every $x \geq 2.11$, we use the inequality $e^t > 1 + t$ to get that the inequality $g_1(x)e^x + g_2(x) \geq 0$, which is equivalent to the desired inequality, holds for every $x \geq 2.11$. □

4.2. **Notations.** In the section, we use the following notation. Let A_1 be a real number such that $0 < A_1 \leq 458.7275$ and let $F_1 : \mathbb{N}_{\geq 2} \to \mathbb{R}$ be defined by

$$\begin{aligned} F_1(n) = {} & \frac{A_1}{\log^5 p_n} + \frac{((\log\log n)^2 - 3.85\log\log n + 14.15)((\log\log n)^2 - \log\log n + 1)}{\log^4 n \log p_n} \\ & + \left(\frac{13.15((\log\log n)^2 - \log\log n + 1)}{\log^2 n \log^2 p_n} - \frac{70.7\log\log n}{\log^2 n \log^2 p_n}\right)\left(\frac{1}{\log n} + \frac{1}{\log p_n}\right) \\ & + \frac{2.85P_1(\log\log n)}{2\log^3 n \log^2 p_n} + \frac{2.85P_1(\log\log n)}{2\log^4 n \log p_n} - \frac{P_2(\log\log n)}{6\log^4 n \log p_n}. \end{aligned}$$

Then $F_1(n) \geq 0$ for all sufficiently large values of n and we define

$$N_1 = N(A_1) = \min\{k \in \mathbb{N} \mid F_1(n) \geq 0 \text{ for every positive integer } n \geq k\}.$$

In the following let $a : \mathbb{N}_{\geq 2} \to \mathbb{R}$ be an arithmetic function so that

$$a(n) \geq -(\log\log n)^2 + 6\log\log n \tag{4.1}$$

and let $N_2, N_3, N_4 \geq 2$ be three constants depending on the arithmetic function a, so that

$$-1 < \frac{\log\log n - 1}{\log n} + \frac{\log\log n - 2}{\log^2 n} - \frac{(\log\log n)^2 - 6\log\log n + a(n)}{2\log^3 n} \leq 1 \tag{4.2}$$

for every positive integer $n \geq N_2$ and

$$\frac{\log\log n - 2}{\log^2 n} - \frac{(\log\log n)^2 - 6\log\log n + a(n)}{2\log^3 n} \geq 0 \tag{4.3}$$

for every positive integer $n \geq N_3$ as well as

$$p_n < n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + a(n)}{2\log^2 n}\right) \tag{4.4}$$

for every positive integer $n \geq N_4$.

4.3. **A new upper bound for the n-th prime number.** Now, we give the proof of Theorem 1.1. For this we set

$$G(x) = \frac{2x^3 - 21x^2 + 82.2x - 98.9}{6e^{3x}} - \frac{x^4 - 14x^3 + 53.4x^2 - 100.6x + 17}{4e^{4x}} + \frac{2x^5 - 10x^4 + 35x^3 - 110x^2 + 150x - 42}{10e^{5x}} - \frac{3x^4 - 44x^3 + 156x^2 - 96x + 64}{24e^{6x}}$$

and for $w = \log\log n$ let

$$b_0(n) = 10.7 + \frac{2A_2}{\log^3 n} + \frac{2A_3}{\log^4 n} + \frac{a(n)}{\log n}\left(1 - \frac{w-1}{\log n} - \frac{w-2}{\log^2 n} + \frac{2w^2 - 12w + a(n)}{4\log^3 n}\right) - 2\,G(w)\log^2 n + \frac{A_0((5.7A_1 + 8.7)w^2 - (32A_0 + 38)w + 147.1A_0 + 10.7)}{\log^2 n} + \frac{2\cdot 70.7A_0^3(w^2 - w + 1)}{\log^4 n} + \frac{2\cdot 70.7A_0^4(w^2 - w + 1)}{\log^4 n}, \tag{4.5}$$

where $A_2 = (458.7275 - A_1)A_0^5$ and $A_3 = 3428.7225A_0^6$.

Theorem 4.3. *For every positive integer $n \geq \max\{N_0, N_1, N_2, N_3, N_4, 688\,383\}$, we have*

$$p_n < n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + b_0(n)}{2\log^2 n}\right).$$

Proof. For convenience, we write $w = \log\log n$, $y = \log n$ and $z = \log p_n$. We set

$$\Phi_1(n) = \frac{1}{y^2} - \frac{w}{y^3} + \frac{w^2 - w + 1}{y^3 z} + \frac{P_1(w)}{2y^4 z} - \frac{P_2(w)}{6y^5 z}$$

$$\Phi_2(n) = \frac{1}{y^2} - \frac{w}{y^3} - \frac{w}{y^2 z} + \frac{w^2 - w + 1}{y^3 z} + \frac{w^2 - w + 1}{y^2 z^2} + \left(\frac{P_1(w)}{2y^3 z} - \frac{P_2(w)}{6y^4 z}\right)\left(\frac{1}{y} + \frac{1}{z}\right)$$

$$\Phi_3(n) = \frac{1}{y^2} - \frac{w}{y^3} - \frac{w}{y^2 z} + \frac{w^2 - w + 1}{y^3 z} + \frac{w^2 - w + 1}{y^2 z^2}.$$

By Corollary 3.2, we get

$$\frac{1}{z^2} \geq \frac{1}{yz} - \frac{w}{y^2 z} + \frac{w^2 - w + 1}{y^2 z^2} + \frac{P_1(w)}{2y^3 z^2} - \frac{P_2(w)}{6y^4 z^2}. \tag{4.6}$$

Again by Corollary 3.2, we obtain

$$\frac{1}{yz} \geq \Phi_1(n). \tag{4.7}$$

We apply this inequality to (4.6) to obtain

$$\frac{1}{z^2} \geq \Phi_2(n) \tag{4.8}$$

and by using (3.3), we get

$$\frac{1}{z^2} \geq \Phi_3(n). \tag{4.9}$$

Since $F_0(n) \geq 0$ and $2.85x^2 - 16x + 73.55 \geq 0$ for every $x \geq 0$, we obtain

$$\frac{2.85w^2 - 16w + 73.55}{z^2} \geq \frac{A_0(5.7w^2 - 32w + 147.1)}{2yz}. \tag{4.10}$$

We define $f(x) = (5.7A_0 + 8.7)x^2 - (32A_0 + 38)x + 147.1A_0 + 10.7$. Since $0.75 \leq A_0 < 1$, it follows $f(x) \geq 12.975x^2 - 70x + 121.025 \geq 0$ for every $x \geq 0$. Using $F_0(n) \geq 0$ and (4.10), we get

$$\frac{2.85w^2 - 16w + 73.55}{z^2} + \frac{8.7w^2 - 38w + 10.7}{2yz} \geq \frac{A_0 f(w)}{2y^2}. \tag{4.11}$$

From the definition of A_2 and A_3 and from $F_1(n) \geq 0$ it follows that

$$\frac{A_2}{y^5} + \frac{A_3}{y^6} \leq \frac{458.7275 - A_1}{z^5} + \frac{3428.7225}{z^6} \tag{4.12}$$

and

$$\frac{70.7A_0^3}{y^6} + \frac{70.7A_0^4}{y^6} \leq \frac{70.7}{y^3 z^3} + \frac{70.7}{y^2 z^4}. \tag{4.13}$$

Now we use (4.11), (4.12) and (4.13) to obtain

$$\begin{aligned}\tag{4.14} &\frac{10.7 - b_0(n)}{2y^2} + \frac{2.85(w^2-w+1)}{y^2z^2} - \frac{13.15w}{y^2z^2} + \frac{70.7}{y^2z^2} + \frac{8.7w^2-38w+10.7}{2y^3z} + \frac{458.7275}{z^5} \\ &\quad - \frac{A_1}{z^5} + \frac{3428.7225}{z^6} + \frac{70.7(w^2-w+1)}{y^2z^3}\left(\frac{1}{y}+\frac{1}{z}\right) \\ &\quad \geq G(w) - \frac{a(n)}{2y^3}\left(1 - \frac{w-1}{y} - \frac{w-2}{y^2} + \frac{2w^2-12w+a(n)}{4y^3}\right).\end{aligned}$$

From (3.3) follows directly

$$\frac{13.15}{z}\left(\frac{P_1(w)}{2y^3z} - \frac{P_2(w)}{6y^4z}\right)\left(\frac{1}{y}+\frac{1}{z}\right) \geq 0.$$

We add the last inequality and the inequality obtained in Lemma 4.2 with $x = w$ to the left-hand side of (4.14) and get

$$\begin{aligned}&\frac{5.35}{y^2} - \frac{b_0(n)}{2y^2} + \frac{2.85(w^2-w+1)}{y^2z^2} - \frac{13.15w}{y^2z^2} + \frac{70.7}{y^2z^2} + \frac{8.7w^2-38w+10.7}{2y^3z} + \frac{458.7275}{z^5} + \frac{3428.7225}{z^6} \\ &\quad - \frac{A_1}{z^5} + \frac{70.7(w^2-w+1)}{y^2z^3}\left(\frac{1}{y}+\frac{1}{z}\right) + \frac{13.15}{z}\left(\frac{P_1(w)}{2y^3z} - \frac{P_2(w)}{6y^4z}\right)\left(\frac{1}{y}+\frac{1}{z}\right) - \frac{2.85P_2(w)}{6y^5z} \\ &\quad - \frac{2.85P_2(w)}{6y^4z^2} + \frac{(w^2-3.85w+14.15)P_1(w)}{2y^5z} - \frac{(w^2-3.85w+14.15)P_2(w)}{6y^6z} + \frac{P_3(w)}{12y^5z} - \frac{P_4(w)}{20y^6z} \\ &\geq G(w) - \frac{a(n)}{2y^3}\left(1 - \frac{w-1}{y} - \frac{w-2}{y^2} + \frac{2w^2-12w+a(n)}{4y^3}\right).\end{aligned}$$

Next we add $F_1(n) \geq 0$ into the left-hand side, use the identity $8.7w^2 - 38w + 10.7 = P_1(w) + 2\cdot 2.85(w^2 - w + 1) - 2\cdot 13.15w$ and collect all terms containing the number 70.7 and the term $w^2 - 3.85w + 14.15$, respectively, to obtain the inequality

$$\begin{aligned}&\frac{5.35}{y^2} - \frac{b_0(n)}{2y^2} + \frac{2.85(w^2-w+1)}{y^2z^2} - \frac{13.15w}{y^2z^2} + \frac{70.7}{z^2}\cdot\Phi_3(n) + \frac{458.7275}{z^5} + \frac{3428.7225}{z^6} + \frac{2.85(w^2-w+1)}{y^3z} \\ &\quad - \frac{13.15w}{y^3z} + \left(2.85 + \frac{13.15}{z}\right)\left(\frac{P_1(w)}{2y^3z} - \frac{P_2(w)}{6y^4z}\right)\left(\frac{1}{y}+\frac{1}{z}\right) + \frac{w^2-3.85w+14.15}{y}\cdot\Phi_1(n) \\ &\quad + \frac{P_1(w)}{2y^3z} - \frac{P_2(w)}{6y^4z} + \frac{P_3(w)}{12y^5z} - \frac{P_4(w)}{20y^6z} + \frac{13.15(w^2-w+1)}{y^2z^2}\left(\frac{1}{y}+\frac{1}{z}\right) - \frac{2.85w}{y^3} \\ &\geq H(w) - \frac{a(n)}{2y^3}\left(1 - \frac{w-1}{y} - \frac{w-2}{y^2} + \frac{2w^2-12w+a(n)}{4y^3}\right),\end{aligned}$$

where

$$H(x) = G(x) + \frac{x^2-3.85x+14.15}{e^{3x}} - \frac{x^3-3.85x^2+14.15x}{e^{4x}} - \frac{2.85x}{e^{3x}}.$$

Now we use (4.7) and (4.9) and collect all terms containing the numbers 2.85 and 13.15 to get

$$\begin{aligned}&\frac{2.5}{y^2} - \frac{b_0(n)}{2y^2} + \left(2.85 + \frac{13.15}{z}\right)\cdot\Phi_2(n) + \frac{70.7}{z^4} + \frac{458.7275}{z^5} + \frac{3428.7225}{z^6} + \frac{w^2-w+1}{y^2z} \\ &\quad + \frac{P_1(w)}{2y^3z} - \frac{P_2(w)}{6y^4z} + \frac{P_3(w)}{12y^5z} - \frac{P_4(w)}{20y^6z} \\ &\quad \geq H(w) - \frac{a(n)}{2y^3}\left(1 - \frac{w-1}{y} - \frac{w-2}{y^2} + \frac{2w^2-12w+a(n)}{4y^3}\right),\end{aligned}$$

Applying (4.8) and Proposition 3.1 to get

$$\begin{aligned}&\frac{2.5}{y^2} - \frac{b_0(n)}{2y^2} + \frac{2.85}{z^2} + \frac{13.15}{z^3} + \frac{70.7}{z^4} + \frac{458.7275}{z^5} + \frac{3428.7225}{z^6} - \frac{1}{y} + \frac{w}{y^2} + \frac{1}{z} \\ &\quad \geq H(w) - \frac{a(n)}{2y^3}\left(1 - \frac{w-1}{y} - \frac{w-2}{y^2} + \frac{2w^2-12w+a(n)}{4y^3}\right).\end{aligned}$$

A straightforward calculation shows that the last inequality is equivalent to

$$\begin{aligned}
&-\frac{1}{y} - \frac{w^2 - 4w - (4 - b_0(n))}{2y^2} + \frac{1}{z} + \frac{2.85}{z^2} + \frac{13.15}{z^3} + \frac{70.7}{z^4} + \frac{458.7275}{z^5} + \frac{3428.7225}{z^6} \\
&\quad \geq -\frac{w^2 - 6w + a(n)}{2y^3} - \frac{1}{2}\left(\frac{w-1}{y} + \frac{w-2}{y^2} - \frac{w^2 - 6w + a(n)}{2y^3}\right)^2 + \frac{1}{3}\left(\frac{w-1}{y} + \frac{w-2}{y^2}\right)^3 \\
&\qquad - \frac{1}{4}\left(\frac{w-1}{y}\right)^4 + \frac{1}{5}\left(\frac{w-1}{y}\right)^5.
\end{aligned}$$

We add both sides of this inequality by $(w-1)/y + (w-2)/y^2$. Since $g(x) = x^3/3$ is increasing and $h(x) = -x^4/4 + x^5/5$ is decreasing on the interval $[0,1]$, we use (4.1), (4.2) and (4.3) to get that

$$\begin{aligned}
&\frac{w-2}{y} - \frac{w^2 - 6w + b_0(n)}{2y^2} + \frac{1}{z} + \frac{2.85}{z^2} + \frac{13.15}{z^3} + \frac{70.7}{z^4} + \frac{458.7275}{z^5} + \frac{3428.7225}{z^6} \\
&\quad \geq \sum_{k=1}^{5} \frac{(-1)^{k+1}}{k}\left(\frac{w-1}{y} + \frac{w-2}{y^2} - \frac{w^2 - 6w + a(n)}{2y^3}\right)^k.
\end{aligned}$$

We have $\log(1+x) \leq \sum_{k=1}^{5}(-1)^{k+1}x/k$ for every $x > -1$. Using (4.2), we obtain

$$\begin{aligned}
&y + w - 1 + \frac{w-2}{y} - \frac{w^2 - 6w + b_0(n)}{2y^2} + \frac{1}{z} + \frac{2.85}{z^2} + \frac{13.15}{z^3} + \frac{70.7}{z^4} + \frac{458.7275}{z^5} + \frac{3428.7225}{z^6} \\
&\quad \geq y + w - 1 + \log\left(1 + \frac{w-1}{y} + \frac{w-2}{y^2} - \frac{w^2 - 6w + a(n)}{2y^3}\right).
\end{aligned}$$

Finally we use (4.4) and Lemma 4.1 to conclude the proof. □

Next, we use Theorem 4.3 to derive the exlicit upper bound for the n-th prime number stated in Corollary 1.2. For this purpose, we first prove the following both lemmata. In the first lemma we determine the value of N_0 for $A_0 = 0.87$.

Lemma 4.4. *For every positive integer $n \geq 1\,338\,564\,587$, we have*

$$\log n \geq 0.87 \log p_n.$$

Proof. We set

$$f(x) = e^x - 0.87\left(e^x + x + \log\left(1 + \frac{x-1}{e^x} + \frac{x-2}{e^{2x}}\right)\right).$$

Since $f'(x) \geq 0$ for every $x \geq 2.5$ and $f(3.046) \geq 0.00137$, it follows that $f(x) \geq 0$ for every $x \geq 3.046$. Substituting $x = \log\log n$ in $f(x)$, we use (1.10) to obtain that the desired inequality holds for every positive integer $n \geq \exp(\exp(3.046))$. We check the remaining cases with a computer. □

Next we use Lemma 4.4 to determine the value of N_1 for $A_1 = 155.32$.

Lemma 4.5. *Let $A_1 = 155.32$. Then $N_1 = 100\,720\,878$.*

Proof. First, let $n \geq \exp(\exp(3.05))$. Since $f(x) = 6x^4 - 34.1x^3 + 163.65x^2 - 198.3x + 141.65 \geq 0$ for every $x \geq 0$, we get $f(w)/(6y^4z) \geq f(w)/(6y^3z^2)$ and it suffices to show that

$$\frac{155.32}{z^5} + \frac{6w^4 - 34.1w^3 + 268.2w^2 - 752.7w + 263.3}{6y^3z^2} \geq -\frac{13.15w^2 - 83.85w + 13.15}{y^2z^3}. \tag{4.15}$$

In order to do this, we set

$$\begin{aligned}
g(x) &= (6x^4 - 34.1x^3 + 268.2x^2 - 752.7x + 263.3)(e^x + x) \\
&\quad + 6e^x(13.15x^2 - 83.85x + 13.15 + 155.32 \cdot 0.87^2).
\end{aligned}$$

It is easy to see that $h_1(x) = 6x^4 - 10.1x^3 + 244.8x^2 - 561.6x - 208.229752 \geq 0$ for every $x \geq 2.6$ and that $h_2(x) = 30x^4 - 136.4x^3 + 804.6x^2 - 1505.4x + 263.3 \geq 0$ for every $x \geq 2.2$. Hence $g'(x) = h_1(x)e^x + h_2(x) \geq 0$ for every $x \geq 2.6$. Since $g(3.05) \geq 0.9$, we obtain that $g(x) \geq 0$ for every $x \geq 3.05$. Since $6x^4 - 34.1x^3 + 268.2x^2 - 752.7x + 263.3 \geq 0$ for every $x \geq 3.05$, we use (1.3) to get

$$\frac{155.32 \cdot 0.87^2}{y^2z^3} + \frac{6w^4 - 34.1w^3 + 268.2w^2 - 752.7w + 263.3}{6y^3z^2} \geq -\frac{13.15w^2 - 83.85w + 13.15}{y^2z^3}.$$

Now we apply Lemma 4.4 to obtain (4.15). So, the desired inequality holds for every positive integer $n \geq \exp(\exp(3.05))$. For every positive integer n satisfying $100\,720\,878 \leq n \leq \exp(\exp(3.05))$ we check the asserted inequality with a computer. □

In view of Cipolla's asymptotic expansion (1.2), we find the following upper bound for the n-th prime numnber, which refines the upper bound (1.10) from Dusart [5, Proposition 6.6].

Corollary 4.6. *For every positive integers $n \geq 46\,254\,381$, we have*

$$p_n < n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + 10.667}{2\log^2 n}\right).$$

Proof. For convenience, we write $w = \log\log n$ and $y = \log n$. We set $A_0 = 0.87$ and $A_1 = 155.32$. By Lemma 4.4 and Lemma 4.5, we have $N_0 = 1\,338\,564\,587$ and $N_1 = 100\,720\,878$. The proof is divided into two steps.

First step: We set $a(n) = -w^2 + 6w$. Then we can chose $N_2 = 5$ and $N_3 = 1\,619$. By (1.10), we obtain that $N_4 = 688\,383$ is a suitable choice for N_4. Using (4.5) we obtain

$$b_0(n) \geq 10.7 + g(n), \tag{4.16}$$

where

$$g(n) = -\frac{2w^3 - 18w^2 + 64.2w - 98.9}{3y} + \frac{w^4 - 12w^3 + 63.16w^2 - 203.17w + 258.29}{2y^2} - \frac{2w^5 - 10w^4 + 30w^3 - 70w^2 + 90w - 1554.24}{5y^3} - \frac{8w^3 - 2137.44w^2 + 2185.45w - 37836.25}{12y^4}.$$

Now we show that $g(n) \geq -0.059$ for every positive integer $n \geq 3$. In order to do this, we define

$$\begin{aligned} g_1(x,t) = {} & 3.54e^{4x} + 20(18x^2 + 98.9)e^{3x} - 20(2t^3 + 64.2t)e^{3t} + 30(x^4 + 63.16x^2 + 258.29)e^{2x} \\ & - 30(12t^3 + 203.17t)e^{2t} + 12(10x^4 + 70x^2 + 1554.24)e^x - 12(2t^5 + 30t^3 + 90t)e^t \\ & + 5(2137.44x^2 + 37836.25) - 5(8t^3 + 2185.45t). \end{aligned}$$

If $t_0 \leq x \leq t_1$, then $g_1(x,x) \geq g_1(t_0,t_1)$. We check with a computer that $g_1(i \cdot 10^{-5}, (i+1)\cdot 10^{-5}) \geq 0$ for every nonnegative integer i with $0 \leq i \leq 699\,999$. Therefore,

$$g(n) + 0.059 = \frac{g_1(w,w)}{60y^4} \geq 0 \qquad (0 \leq w \leq 7). \tag{4.17}$$

Next, we prove that $g_1(x,x) \geq 0$ for every $x \geq 7$. In order to do this, let $W_1(x) = 3.54e^x - 20(2x^3 - 18x^2 + 64.2x - 98.9)$. It is easy to show that $W_1(x) \geq 792$ for every $x \geq 7$. Hence, we get

$$\begin{aligned} g_1(x,x) \geq {} & (792e^x + 30(x^4 - 12x^3 + 63.16x^2 - 203.17x + 258.29))e^{2x} \\ & - 12(2x^5 - 10x^4 + 30x^3 - 70x^2 + 90x - 1554.24)e^x \\ & - 5(8x^3 - 2137.44x^2 + 2185.45x - 37836.25). \end{aligned}$$

Since $792e^t + 30(t^4 - 12t^3 + 63.16t^2 - 203.17t + 258.29) \geq 875\,011$ for every $t \geq 7$, we obtain that $g(n) + 0.059 = g_1(w,w)/(60y^4) \geq 0$ for every positive integer n satisfying $w \geq 7$. Together with (4.17) and (4.16), we get that $b(n) \geq 10.641$ for every positive integer $n \geq 3$. Applying this to Theorem 4.3, we obtain that the inequality

$$p_n < n\left(y + w - 1 + \frac{w-2}{y} - \frac{w^2 - 6w + 10.641}{2y^2}\right)$$

holds for every positive integer $n \geq 1\,338\,564\,587$. For every positive integer n such that $39\,529\,802 \leq n \leq 1\,338\,564\,586$ we check the last inequality with a computer.

Second step: We set $a(n) = 10.641$. Then, we can chose $N_2 = 8$ and $N_3 = 4914$. Further, it follows from the first step that $N_4 = 39\,529\,802$. By (4.5), we have

$$b_0(n) \geq 10.7 + h(n), \tag{4.18}$$

where $h(n)$ is given by

$$\begin{aligned} h(n) = {} & -\frac{2w^3 - 21w^2 + 82.2w - 130.823}{3y} + \frac{w^4 - 14w^3 + 77.16w^2 - 236.45w + 279.57}{2y^2} \\ & - \frac{2w^5 - 10w^4 + 35w^3 - 110w^2 + 203.205w - 1660.65}{5y^3} \\ & + \frac{3w^4 - 44w^3 + 2309.28w^2 - 2568.52w + 38175.947}{12y^4}. \end{aligned}$$

Now we show that $h(n) \geq -0.033$ for every positive integer $n \geq 3$. We set

$$\begin{aligned} h_1(x,t) = 1.98e^{4x} &+ 20(21x^2 + 130.823)e^{3x} - 20(2t^3 + 82.2t)e^{3t} + 30(x^4 + 77.16x^2 + 279.57)e^{2x} \\ &- 30(14t^3 + 236.45t)e^{2t} + 12(10x^4 + 110x^2 + 1660.65)e^x - 12(2t^5 + 35t^3 + 203.205t)e^t \\ &+ 5(3x^4 + 2309.28x^2 + 38175.947) - 5(44t^3 + 2568.52t). \end{aligned}$$

Clearly, $h_1(x,x) \geq h_1(t_0,t_1)$ for every x such that $t_0 \leq x \leq t_1$. We use a computer to verify that $h_1(i \cdot 10^{-6}, (i+1) \cdot 10^{-6}) \geq 0$ for every nonnegative integer i with $0 \leq i \leq 7\,999\,999$. Therefore,

$$h(n) + 0.033 = \frac{h_1(w,w)}{60y^4} \geq 0 \qquad (0 \leq w \leq 8). \tag{4.19}$$

Next, we show that $h_1(x,x) \geq 0$ for every $x \geq 8$. Since $1.98e^t - 20(2t^3 - 21t^2 + 82.2t - 130.823) \geq 1766$ for every $t \geq 8$, we have

$$\begin{aligned} h_1(x,x) \geq 1766e^{3x} &+ 30(x^4 - 14x^3 + 77.16x^2 - 236.45x + 279.57)e^{2x} \\ &- 12(2x^5 - 10x^4 + 35x^3 - 110x^2 + 203.205x - 1660.65)e^x \\ &+ 5(3x^4 - 44x^3 + 2309.28x^2 - 2568.52x + 38175.947). \end{aligned}$$

Note that $1766e^t + 30(t^4 - 14t^3 + 77.16t^2 - 236.45t + 279.57) \geq 5\,271\,998$ for every $t \geq 8$. Hence, we get that

$$h(n) + 0.033 = \frac{h_1(w,w)}{60y^4} \geq 0 \qquad (w \geq 8).$$

Combined with (4.19) and (4.18), we get that $b(n) \geq 10.667$ for every positive integer $n \geq 3$. Finally, we apply this to Theorem 4.3 and obtain that the required inequality holds for every positive integer $n \geq 1\,338\,564\,587$. We verify the remaining cases with a computer. □

In the following example we compare the error term of the approximation from Corollary 4.6 with Dusart's approximation from (1.10) for the 10^n-th prime number.

Example. Denoting the right-hand side of (1.10) by $D_{up}(n)$ and the right-hand side from Corollary 4.6 by $A_{up}(n)$, we use [14] to obtain the following table:

n	p_n	$\lceil D_{up}(n) - p_n \rceil$	$\lceil A_{up}(n) - p_n \rceil$
10^8	2 038 074 743	299 689	52 949
10^9	22 801 763 489	2 522 619	580 644
10^{10}	252 097 800 623	20 510 784	4 613 984
10^{11}	2 760 727 302 517	172 884 400	38 768 198
10^{12}	29 996 224 275 833	1 469 932 710	311 593 524
10^{13}	323 780 508 946 331	12 732 767 836	2 542 231 421
10^{14}	3 475 385 758 524 527	112 026 014 682	21 049 069 521
10^{15}	37 124 508 045 065 437	998 861 791 991	176 995 293 694
10^{16}	394 906 913 903 735 329	9 004 342 407 404	1 507 803 850 451
10^{17}	4 185 296 581 467 695 669	81 924 060 077 026	12 998 658 322 559
10^{18}	44 211 790 234 832 169 331	751 154 982 343 786	113 204 602 033 556
10^{19}	465 675 465 116 607 065 549	6 932 757 377 044 651	994 838 584 902 026
10^{20}	4 892 055 594 575 155 744 537	64 346 895 915 006 554	8 812 315 669 274 243
10^{21}	51 271 091 498 016 403 471 853	600 148 288 357 489 952	78 609 427 974 695 423
10^{22}	536 193 870 744 162 118 627 429	5 621 157 733 905 567 326	705 633 790 460 554 787
10^{23}	5 596 564 467 986 980 643 073 683	52 844 890 559 120 248 010	6 369 721 461 578 220 680
10^{24}	58 310 039 994 836 584 070 534 263	498 427 891 603 997 785 646	57 790 587 904 575 515 367

5. A new lower bound for the n-th prime number

The goal of this section is to prove Theorem 1.3 and Corollary 1.4. In order to do this, we first note some useful inequalities.

5.1. **Preliminaries.** First, we use a recently obtained upper bound for $\pi(x)$ to derive the following implicit lower bound for the n-th prime number.

Lemma 5.1. *For every positive integer n, we have*

$$p_n > n\left(\log p_n - 1 - \frac{1}{\log p_n} - \frac{3.15}{\log^2 p_n} - \frac{12.85}{\log^3 p_n} - \frac{71.3}{\log^4 p_n} - \frac{463.2275}{\log^5 p_n} - \frac{4585}{\log^6 p_n}\right).$$

Proof. If $n \geq 49$, the claim follows from [2, Theorem 3.2]. For smaller values of n, we use a computer. □

Lemma 5.2. *For every positive integer $n \geq 6$, we have*

$$\frac{12.85P_9(\log\log n)}{2\log^6 n \log p_n} + \frac{3.15P_{10}(\log\log n)}{2\log^6 n \log p_n} + \frac{P_{11}(\log\log n)}{2\log^6 n\log p_n} \geq 0.$$

Proof. It is easy to see that $f(x) = 12.85P_9(x) + 3.15P_{10}(x) + P_{11}(x) \geq 0$ for every $x \geq 2$. Hence, $f(w) \geq 0$ for every positive integer $n \geq \exp(\exp(2))$. We check the remaining cases with a computer. □

Lemma 5.3. *Let $w = \log\log n$. For every positive integer $n \geq 17$, we have*

$$\frac{P_9(w)P_{12}(w)}{4\log^7 n\log p_n} + \frac{12.85P_{10}(w)}{2\log^7 n\log p_n} + \frac{3.15P_{11}(w)}{2\log^7 n\log p_n} + \frac{3.15P_{11}(w)}{2\log^6 n\log^2 p_n} \geq \frac{(w-2)^4}{4\log^8 n}.$$

Proof. Using Lemma 4.4 and a computer, we get that the inequality

$$\log m \geq 0.75\log p_m \tag{5.1}$$

holds for every positive integers $m \geq 255$. Let $n \geq 255$. We set $f(x) = P_9(x)P_{12}(x) + 2\cdot 12.85P_{10}(x) + 2\cdot 3.15P_{11}(x) + 0.75\cdot 2\cdot 3.15P_{11}(x)$ and $g(x) = 0.75f(x) - (x-2)^4$. Note that $g(x) \geq 0$ and $f(x) \geq 0$ for every $x \geq 1.5$. Together with (5.1), we get $f(w)/(4\log^7 n\log p_n) - (w-2)^4/(4\log^8 n) \geq g(w)/(4\log^8 n) \geq 0$. Further, we have $P_{11}(x) \geq 0$ for every $x \geq 0$. Hence, by (5.1), we get that the required inequality holds for every positive integer $n \geq 255$. We conclude by direct computation. □

5.2. **Notations.** Beside the notation from Section 2, we use in this section the following further notations. Let $B_1, \ldots, B_{10}$ be real positive constants satisfying

$$B_6 + B_7 + B_8 + B_9 + B_{10} \leq 3.15. \tag{5.2}$$

Writing $w = \log\log n$, $y = \log n$ and $z = \log p_n$, we define the arithmetic functions $H_i : \mathbb{N}_{\geq 2} \to \mathbb{R}$, $1 \leq i \leq 10$, by

- $H_1(n) = \dfrac{B_1 w}{y^3 z} - \dfrac{Q_7(w)}{2y^5 z} + \dfrac{Q_8(w)}{2y^5z^2} + \dfrac{Q_9(w)}{4y^6 z} + \dfrac{12.85P_9(w)}{2y^4z^3}$,
- $H_2(n) = \dfrac{B_2 w}{y^3 z} + \dfrac{12.85w}{y^2z^2} - \dfrac{71.3}{z^4}$,
- $H_3(n) = \dfrac{B_3 w}{y^3 z} - \dfrac{3.15P_8(w)}{2y^3z^2} - \dfrac{12.85(w^2-w+1)}{y^3z^2}$,
- $H_4(n) = \dfrac{B_4 w}{y^3 z} + \dfrac{3.15P_9(w) - 12.85P_8(w)}{2y^4z^2}$,
- $H_5(n) = \dfrac{B_5 w}{y^3 z} + \dfrac{P_9(w) - 3.15P_8(w)}{2y^4 z} - \dfrac{12.85(w^2-w+1)}{y^4 z} - \dfrac{(w^2-w+1)^2}{y^4 z}$,
- $H_6(n) = \dfrac{B_6 w}{y^2 z} + \dfrac{(12.85 - B_1 - B_2 - B_3 - B_4 - B_5)w}{y^3 z} - \dfrac{3.15(w^2-w+1)}{y^2z^2}$,
- $H_7(n) = \dfrac{B_7 w}{y^2 z} - \dfrac{12.85P_8(w)}{2y^3z^3}$,
- $H_8(n) = \dfrac{B_8 w}{y^2 z} - \dfrac{12.85(w^2-w+1)}{y^2z^3}$,
- $H_9(n) = \dfrac{B_9 w}{y^2 z} - \dfrac{463.2275}{z^5}$,
- $H_{10}(n) = \dfrac{B_{10} w}{y^2 z} - \dfrac{4585}{z^6}$.

Since $H_i(n) \geq 0$ for every $1 \leq i \leq 10$ and all sufficiently large values of n, respectively, we define

$$M_i(B_i) = \min\{k \in \mathbb{N} \mid H_i(n) \geq 0 \text{ for every } n \geq k\}$$

and set

$$K_1 = \max_{1\leq i\leq 10} M_i(B_i).$$

Let $a : \mathbb{N}_{\geq 2} \to \mathbb{R}$ be an arithmetic function and let K_2, K_3, K_4 be positive integers, which depend on a, so that

$$p_n > n \left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + a(n)}{2\log^2 n} \right) \tag{5.3}$$

for every positive integer $n \geq K_2$ and

$$a(n) > -(\log\log n)^2 + 6\log\log n \tag{5.4}$$

for every positive integer $n \geq K_3$, as well as

$$0 \leq \frac{\log\log n - 1}{\log n} + \frac{\log\log n - 2}{\log^2 n} - \frac{(\log\log n)^2 - 6\log\log n + a(n)}{2\log^3 n} \leq 1 \tag{5.5}$$

for every positive integer $n \geq K_4$. Furthermore, we define the function $G : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} G(x) = {} & \frac{2x^3 - 15x^2 + 42x - 14}{6e^{3x}} + \frac{3.15x}{e^{3x}} - \frac{12.85}{e^{3x}} - \frac{x^2 - x + 1}{e^{3x}} + \frac{(x^2 - x + 1)x}{e^{4x}} - \frac{P_{12}(x)}{2e^{4x}} + \frac{12.85x}{e^{4x}} \\ & + \frac{P_{12}(x)x}{2e^{5x}} + \frac{(x-1)^2}{2e^{2x}} - \frac{x^3 - 6x^2 + 12x - 7}{3e^{3x}} - \sum_{k=2}^{4} \frac{(-1)^k}{k} \left(\frac{x-1}{e^x} + \frac{x-2}{e^{2x}} \right)^k + \frac{(x-2)^4}{4e^{8x}}. \end{aligned}$$

5.3. **A new lower bound for the n-th prime number.** In order to find the explicit lower bound for the n-th prime number stated in Corollary 1.4, we set

$$b_1(n) = 11.3 - 2\,G(\log\log n)\log^2 n + \frac{a(n)}{\log n} - \frac{2A_0(3.15 - (B_6 + B_7 + B_8 + B_9 + B_{10}))\log\log n}{\log n} \tag{5.6}$$

and first show the following theorem.

Theorem 5.4. *For every positive integer $n \geq \max\{N_0, K_1, K_2, K_3, K_4, 3\,520\}$, we have*

$$p_n > n \left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + b_1(n)}{2\log^2 n} \right).$$

Proof. Let $n \geq \max\{N_0, K_1, K_2, K_3, K_4, 3520\}$. For convenience, we write $w = \log\log n$, $y = \log n$ and $z = \log p_n$. We set

$$\begin{aligned} \Psi_1(n) &= -\frac{1}{y} + \frac{w}{y^2} - \frac{w^2 - w + 1}{y^2 z} - \frac{P_8(w)}{2y^3 z} + \frac{P_9(w)}{2y^4 z} \\ \Psi_2(n) &= -\frac{1}{y^2} + \frac{w}{y^3} + \frac{w}{y^2 z} - \left(\frac{1}{y} + \frac{1}{z} \right) \left(\frac{w^2 - w + 1}{y^2 z} + \frac{P_8(w)}{2y^3 z} - \frac{P_9(w)}{2y^4 z} - \frac{P_{10}(w)}{2y^5 z} - \frac{P_{11}(w)}{2y^6 z} \right) \\ \Psi_3(n) &= -\frac{1}{y^3} + \frac{w}{y^4} + \frac{w}{y^3 z} + \frac{w}{y^2 z^2} - \frac{w^2 - w + 1}{y^4 z} - \frac{w^2 - w + 1}{y^3 z^2} - \frac{w^2 - w + 1}{y^2 z^3} \\ &\quad - \frac{P_8(w)}{2y^5 z} - \frac{P_8(w)}{2y^4 z^2} - \frac{P_8(w)}{2y^3 z^3} + \frac{P_9(w)}{2y^6 z} + \frac{P_9(w)}{2y^5 z^2} + \frac{P_9(w)}{2y^4 z^3} + \frac{P_{10}(w)}{2y^7 z}. \end{aligned}$$

By Corollary 3.6, we have

$$-\frac{1}{z} \geq \Psi_1(n). \tag{5.7}$$

Similar to the proof of (4.8), we use Proposition 3.4 to get that

$$-\frac{1}{z^2} \geq \Psi_2(n). \tag{5.8}$$

Together with $P_{11}(x) = x(x - 2.1)^2 \geq 0$ for every $x \geq 0$, $P_{10}(x) = 2(x - 2.1)(x^2 - 1.5x + 1.05) \geq 0$ for every $x \geq 2.1$ and Corollary 3.5, we get

$$-\frac{1}{z^3} \geq \Psi_3(n). \tag{5.9}$$

We have $F_0(n) \geq 0$ and, by (5.2), $3.15 - (B_6 + B_7 + B_8 + B_9 + B_{10}) \geq 0$. Together with the definition of $b_1(n)$, we obtain that the inequality

$$\frac{11.3 - b_1(n)}{2y^2} \leq G(w) - \frac{a(n)}{2y^3} + \frac{(3.15 - (B_6 + B_7 + B_8 + B_9 + B_{10}))w}{y^2 z}$$

holds. Now we add the right-hand side of the last inequality with $\sum_{i=1}^{10} H_i(n) \geq 0$ and use Lemma 5.2 and Lemma 5.3 to get

$$\begin{aligned}\frac{d(n)}{2y^2} \leq G(w) &- \frac{a(n)}{2y^3} + \frac{3.15w}{y^2z} + \frac{12.85P_9(w)}{2y^6z} + \frac{3.15P_{10}(w)}{2y^6z} + \frac{P_{11}(w)}{2y^6z} - \frac{Q_7(w)}{2y^5z} + \frac{Q_8(w)}{2y^5z^2} + \frac{Q_9(w)}{4y^6z} \\ &+ \frac{12.85P_9(w)}{2y^4z^3} + \frac{12.85w}{y^2z^2} - \frac{71.3}{z^4} - \frac{3.15P_8(w)}{2y^3z^2} - \frac{12.85(w^2-w+1)}{y^3z^2} + \frac{3.15P_9(w)}{2y^4z^2} \\ &- \frac{12.85P_8(w)}{2y^4z^2} + \frac{P_9(w)}{2y^4z} - \frac{3.15P_8(w)}{2y^4z} - \frac{12.85(w^2-w+1)}{y^4z} - \frac{(w^2-w+1)^2}{y^4z} + \frac{12.85w}{y^3z} \\ &- \frac{3.15(w^2-w+1)}{y^2z^2} - \frac{12.85P_8(w)}{2y^3z^3} - \frac{12.85(w^2-w+1)}{y^2z^3} - \frac{463.2275}{z^5} - \frac{4585}{z^6} \\ &+ \frac{P_9(w)P_{12}(w)}{4y^7z} + \frac{12.85P_{10}(w)}{2y^7z} + \frac{3.15P_{11}(w)}{2y^7z} + \frac{3.15P_{11}(w)}{2y^6z^2} - \frac{(w-2)^4}{4y^8},\end{aligned}$$

where $d(n) = 11.3 - b_1(n)$. Substituting the definitions of $Q_7(x), Q_8(x), Q_9(x)$ and $G(x)$, it follows that

$$\begin{aligned}\frac{d(n)}{2y^2} \leq &-\frac{a(n)}{2y^3} + \frac{2w^3-15w^2+42w-14}{6y^3} + \frac{3.15w}{y^3} + 12.85\cdot\Psi_3(n) + \left(\frac{w^2-w+1}{y^2} + \frac{P_{12}(w)}{2y^3}\right)\cdot\Psi_1(n) \\ &+ \frac{(w-1)^2}{2y^2} - \frac{w^3-6w^2+12w-7}{3y^3} - \sum_{k=2}^{4}\frac{(-1)^k}{k}\left(\frac{w-1}{y} + \frac{w-2}{y^2}\right)^k + \frac{3.15w}{y^2z} + \frac{3.15P_{10}(w)}{2y^6z} \\ &+ \frac{P_{11}(w)}{2y^6z} + \frac{3.15P_9(w)}{2y^5z} + \frac{P_{10}(w)}{2y^5z} + \frac{3.15P_{10}(w)}{2y^5z^2} - \frac{71.3}{z^4} - \frac{3.15P_8(w)}{2y^3z^2} + \frac{3.15P_9(w)}{2y^4z^2} + \frac{P_9(w)}{2y^4z} \\ &- \frac{3.15P_8(w)}{2y^4z} - \frac{3.15(w^2-w+1)}{y^2z^2} - \frac{463.2275}{z^5} - \frac{4585}{z^6} + \frac{3.15P_{11}(w)}{2y^7z} + \frac{3.15P_{11}(w)}{2y^6z^2}.\end{aligned}$$

Note that $w^2 - w + 1 \geq 0$ und $P_{12}(w) \geq 0$. Then, by using (5.7) and (5.9), we get

$$\begin{aligned}\frac{d(n)}{2y^2} \leq &\frac{2w^3-15w^2+42w-14}{6y^3} + \frac{3.15w}{y^3} - \frac{12.85}{z^3} - \frac{w^2-w+1}{y^2z} - \frac{P_{12}(w)}{2y^3z} + \frac{(w-1)^2}{2y^2} \\ &- \frac{w^3-6w^2+12w-7}{3y^3} - \sum_{k=2}^{4}\frac{(-1)^k}{k}\left(\frac{w-1}{y} + \frac{w-2}{y^2}\right)^k - \frac{a(n)}{2y^3} + \frac{3.15w}{y^2z} + \frac{3.15P_{10}(w)}{2y^6z} \\ &+ \frac{P_{11}(w)}{2y^6z} + \frac{3.15P_9(w)}{2y^5z} + \frac{P_{10}(w)}{2y^5z} + \frac{3.15P_{10}(w)}{2y^5z^2} - \frac{71.3}{z^4} - \frac{3.15P_8(w)}{2y^3z^2} + \frac{3.15P_9(w)}{2y^4z^2} + \frac{P_9(w)}{2y^4z} \\ &- \frac{3.15P_8(w)}{2y^4z} - \frac{3.15(w^2-w+1)}{y^2z^2} - \frac{463.2275}{z^5} - \frac{4585}{z^6} + \frac{3.15P_{11}(w)}{2y^7z} + \frac{3.15P_{11}(w)}{2y^6z^2}.\end{aligned}$$

Since $P_{12}(x) = P_8(x) + 2\cdot 3.15(x^2 - x + 1)$ and $d(n) = 11.3 - b_1(n)$, we obtain

$$\begin{aligned}\frac{5-b_1(n)}{2y^2} \leq 3.15\cdot\Psi_2(n) &+ \frac{2w^3-15w^2+42w-14}{6y^3} - \frac{12.85}{z^3} - \frac{w^2-w+1}{y^2z} - \frac{P_8(w)}{2y^3z} + \frac{(w-1)^2}{2y^2} \\ &- \frac{w^3-6w^2+12w-7}{3y^3} - \sum_{k=2}^{4}\frac{(-1)^k}{k}\left(\frac{w-1}{y} + \frac{w-2}{y^2}\right)^k - \frac{a(n)}{2y^3} + \frac{P_9(w)}{2y^4z} + \frac{P_{10}(w)}{2y^5z} \\ &+ \frac{P_{11}(w)}{2y^6z} - \frac{71.3}{z^4} - \frac{463.2275}{z^5} - \frac{4585}{z^6}.\end{aligned}$$

Together with (5.8) and Proposition 3.4, we get

$$\begin{aligned}\frac{5-b_1(n)}{2y^2} \leq &-\frac{1}{z} - \frac{3.15}{z^2} - \frac{12.85}{z^3} - \frac{71.3}{z^4} - \frac{463.2275}{z^5} - \frac{4585}{z^6} + \frac{2w^3-15w^2+42w-14}{6y^3} + \frac{1}{y} - \frac{w}{y^2} \\ &+ \frac{(w-1)^2}{2y^2} - \frac{w^3-6w^2+12w-7}{3y^3} - \sum_{k=2}^{4}\frac{(-1)^k}{k}\left(\frac{w-1}{y} + \frac{w-2}{y^2}\right)^k - \frac{a(n)}{2y^3}.\end{aligned}$$

This inequality is equivalent to

$$\begin{aligned}\frac{w-2}{y} \leq &\frac{w-1}{y} + \frac{w-2}{y^2} - \frac{w^2-6w+a(n)}{2y^3} - \sum_{k=2}^{4}\frac{(-1)^k}{k}\left(\frac{w-1}{y} + \frac{w-2}{y^2}\right)^k \\ &+ \frac{w^2-6w+b_1(n)}{2y^2} - \frac{1}{z} - \frac{3.15}{z^2} - \frac{12.85}{z^3} - \frac{71.3}{z^4} - \frac{463.2275}{z^5} - \frac{4585}{z^6}.\end{aligned} \tag{5.10}$$

The functions $g_1(x) = -x^2/2 + x^3/3$ and $g_2(x) = -x^4/4$ are monotonic decreasing on the interval $[0,1]$. Together with (5.4), (5.5) and the inequality $\log(1+t) \geq \sum_{k=1}^4 (-1)^{k+1} t^k/k$, we get that the inequality (5.10) implies

$$\frac{w-2}{y} - \frac{w^2-6w+b_1(n)}{2y^2} \leq \log\left(1 + \frac{w-1}{y} + \frac{w-2}{y^2} - \frac{w^2-6w+a(n)}{2y^3}\right) - \frac{1}{z} - \frac{3.15}{z^2} - \frac{12.85}{z^3} - \frac{71.3}{z^4} - \frac{463.2275}{z^5} - \frac{4585}{z^6}.$$

Now we add $y+w-1$ to both sides of the last inequality und use (5.3) to get

$$y + w - 1 + \frac{w-2}{\log n} - \frac{w^2-6w+b_1(n)}{y} \leq z - 1 - \frac{1}{z} - \frac{3.15}{z^2} - \frac{12.85}{z^3} - \frac{71.3}{z^4} - \frac{463.2275}{z^5} - \frac{4585}{z^6}.$$

Finally we multiply the last inequality by n and apply Lemma 5.1 to conclude the proof. □

Now, we use Theorem 5.4 to establish the following explicit lower bound for the n-th prime number, which refines Dusart's lower bound given in (1.11).

Corollary 5.5. *For every positive integer $n \geq 2$, we have*

$$p_n > n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + 11.508}{2\log^2 n}\right).$$

Proof. Let $A_0 = 0.87$. Then, by Lemma 4.4, $N_0 = 1\,338\,564\,587$. In the following table we note the explicit values of $M_i(B_i)$ for given B_i:

i	1	2	3	4	5
B_i	0.27	4.23	1.575	0.058	2.24
$M_i(B_i)$	1 359 056 314	1 471 247 583	1 468 111 666	1 383 728 153	1 462 324 835
i	6	7	8	9	10
B_i	0.105	0.0026	0.052	0.1955	0.08
$M_i(B_i)$	5	1 075 859 481	1 445 815 789	1 479 240 488	1 447 605 594

The respective proof that $H_i(n) \geq 0$, where $1 \leq i \leq 10$, holds for every positive integer $n \geq M_i(B_i)$ can be found in the appendix. The above table indicates

$$3.15 - (B_6 + B_7 + B_8 + B_9 + B_{10}) = 2.7149 \tag{5.11}$$

and $K_1 = \max_{1 \leq i \leq 10} M_i(B_i) = 1\,479\,240\,488$. The proof of the required lower bound for the n-th prime number p_n consists of three steps.

First step: We set $a(n) = 0.2y - w^2 + 6w$. By (1.11), we can chose $K_2 = 3$. Further it is easy to see that $K_3 = 2$ and $K_4 = 33$ are suitable choices for K_3 and K_4, respectiviely. Using (5.6) and (5.11), we obtain the identity

$$\begin{aligned} b_1(n) = 11.5 &- \frac{2w^3 - 18w^2 + 63.071778w - 97.1}{3y} + \frac{w^4 - 12w^3 + 46.6w^2 - 112w + 40}{2y^2} \\ &+ \frac{2w^4 - 21.3w^3 + 40.3w^2 - 41.5w + 12}{y^3} + \frac{9w^4 - 56w^3 + 129w^2 - 132w + 52}{3y^4} \\ &+ \frac{2w^4 - 14w^3 + 36w^2 - 40w + 16}{y^5}. \end{aligned}$$

In this first step, we show that $b_1(n) \leq 11.589$ for every positive integer $n \geq \exp(\exp(3.05))$. For this purpose, we set

$$\begin{aligned} \alpha(x,t) = 0.534e^{5x} &+ 2(2x^3 + 63.071778x)e^{4x} - 2(18t^2 + 97.1)e^{4t} + 3(12x^3 + 112x)e^{3x} \\ &- 3(t^4 + 46.6t^2 + 40)e^{3t} + 6(21.3x^3 + 41.5x)e^{2x} - 6(2t^4 + 40.3t^2 + 12)e^{2t} \\ &+ 2(56x^3 + 132x)e^x - 2(9t^4 + 129t^2 + 52)e^t + 6(14x^3 + 40x) - 6(2t^4 + 36t^2 + 16). \end{aligned}$$

and notice the identity

$$\alpha(w,w) = 6(11.589 - b_1(n))y^5. \tag{5.12}$$

If $t_0 \leq x \leq t_1$, then $\alpha(x,x) \geq \alpha(t_0,t_1)$. We check with a computer that $\alpha(3.05 + i \cdot 10^{-5}, 3.05 + (i+1) \cdot 10^{-5}) \geq 0$ for every nonnegative integer i with $0 \leq i \leq 394\,999$. Hence, by (5.12),

$$b_1(n) \leq 11.589 \qquad (3.05 \leq w \leq 7). \tag{5.13}$$

Next, we show that $\alpha(x,x) \geq 0$ for every $x \geq 7$. Since $0.534e^x + 2(2x^3 - 18x^2 + 63.071778x - 97.1) \geq 882$ for every $x \geq 7$, we have

$$\alpha(x,x) \geq 882e^{4x} - 3(x^4 - 12x^3 + 46.6x^2 - 112x + 40)e^{3x} - 6(2x^4 - 21.3x^3 + 40.3x^2 - 41.5x + 12)e^{2x} - 2(9x^4 - 56x^3 + 129x^2 - 132x + 52)e^x - 6(2x^4 - 14x^3 + 36x^2 - 40x + 16).$$

Note that $882e^x - 3(x^4 - 12x^3 + 46.6x^2 - 112x + 40) \geq 967\,757$ for every $x \geq 7$. So, $\alpha(x,x) \geq 0$ for every $x \geq 7$. Combined with (5.12) and (5.13), we get that $b_1(n) \leq 11.589$ for every positive integer $n \geq \exp(\exp(3.05))$. Applying this to Theorem 5.4, we get that the inequality

$$p_n > n\left(y + w - 1 + \frac{w-2}{y} - \frac{w^2 - 6w + 11.589}{2y^2}\right)$$

is fulfilled for every positive integer $n \geq \exp(\exp(3.05))$. We check with a computer that the last inequality holds for every positive integer n such that $2 \leq n \leq \exp(\exp(3.05))$.

Second step: We set $a(n) = 11.589$. Then, we can chose $K_2 = 2$ and $K_3 = 2$. Further, it is easy to see that $K_4 = 48$ is a suitable choice for K_4. Together with (5.6) and (5.11), we get that

$$b_1(n) = 11.3 - \frac{2w^3 - 21w^2 + 81.071778w - 131.867}{3y} + \frac{w^4 - 12w^3 + 46.6w^2 - 112w + 40}{2y^2} + \frac{2w^4 - 21.3w^3 + 40.3w^2 - 41.5w + 12}{y^3} + \frac{9w^4 - 56w^3 + 129w^2 - 132w + 52}{3y^4} + \frac{2w^4 - 14w^3 + 36w^2 - 40w + 16}{y^5}.$$

Our goal in this step is to show that the inequality $b_1(n) \leq 11.512$ holds for every positive integer $n \geq \exp(\exp(3.05))$. For this, we set

$$\beta(x,t) = 1.272e^{5x} + 2(2x^3 + 81.071778x)e^{4x} - 2(21t^2 + 131.867)e^{4t} + 3(12x^3 + 112x)e^{3x} - 3(t^4 + 46.6t^2 + 40)e^{3t} + 6(21.3x^3 + 41.5x)e^{2x} - 6(2t^4 + 40.3t^2 + 12)e^{2t} + 2(56x^3 + 132x)e^x - 2(9t^4 + 129t^2 + 52)e^t + 6(14x^3 + 40x) - 6(2t^4 + 36t^2 + 16).$$

Then, we get

$$\beta(w,w) = 6(11.512 - b_1(n))y^5. \tag{5.14}$$

Similar to the first step, we get that

$$b_1(n) \leq 11.512 \qquad (3.05 \leq w \leq 7). \tag{5.15}$$

So, it suffices to verify that $\beta(x,x) \geq 0$ for every $x \geq 7$. We notice that $1.272e^x + 2(2x^3 - 21x^2 + 81.071778x - 131.867) \geq 1580$ for every $x \geq 7$. Hence,

$$\beta(x,x) \geq 1580e^{4x} - 3(x^4 - 12x^3 + 46.6x^2 - 112x + 40)e^{3x} - 6(2x^4 - 21.3x^3 + 40.3x^2 - 41.5x + 12)e^{2x} - 2(9x^4 - 56x^3 + 129x^2 - 132x + 52)e^x - 6(2x^4 - 14x^3 + 36x^2 - 40x + 16).$$

Since $1580e^x - 3(x^4 - 12x^3 + 46.6x^2 - 112x + 40) \geq 1\,733\,207$ for every $x \geq 7$, we conclude that $\beta(x,x) \geq 0$ for every $x \geq 7$. Together with (5.14) and (5.15), we establish that $b_1(n) \leq 11.512$ for every positive integer $n \geq \exp(\exp(3.05))$. Applying this to Theorem 5.4, we get that

$$p_n > n\left(y + w - 1 + \frac{w-2}{y} - \frac{w^2 - 6w + 11.512}{2y^2}\right)$$

for every positive integer $n \geq \exp(\exp(3.05))$. Finally, we use a computer to verify that the last inequality also holds for every positive integer n such that $2 \leq n \leq \exp(\exp(3.05))$.

Third step: In this last step, we set $a(n) = 11.512$. Then, we can chose $K_2 = 2$ and $K_3 = 2$. Further, $K_4 = 47$ is a suitable value for K_4. Now, we use (5.6) and (5.11) to get

$$b_1(n) = 11.3 - \frac{2w^3 - 21w^2 + 81.071778w - 131.636}{3y} + \frac{w^4 - 12w^3 + 46.6w^2 - 112w + 40}{2y^2} + \frac{2w^4 - 21.3w^3 + 40.3w^2 - 41.5w + 12}{y^3} + \frac{9w^4 - 56w^3 + 129w^2 - 132w + 52}{3y^4} + \frac{2w^4 - 14w^3 + 36w^2 - 40w + 16}{y^5}.$$

To show that $b_1(n) \leq 11.508$ for every positive integer $n \geq \exp(\exp(3.05))$, we set

$$\begin{aligned}\gamma(x,t) = 1.248e^{5x} &+ 2(2x^3 + 81.071778x)e^{4x} - 2(21t^2 + 131.636)e^{4t} + 3(12x^3 + 112x)e^{3x} \\ &- 3(t^4 + 46.6t^2 + 40)e^{3t} + 6(21.3x^3 + 41.5x)e^{2x} - 6(2t^4 + 40.3t^2 + 12)e^{2t} \\ &+ 2(56x^3 + 132x)e^{x} - 2(9t^4 + 129t^2 + 52)e^{t} + 6(14x^3 + 40x) - 6(2t^4 + 36t^2 + 16).\end{aligned}$$

Notice that

$$\gamma(w,w) = 6(11.508 - b_1(n))y^5. \tag{5.16}$$

Analogously to the first step, we obtain that

$$b_1(n) \leq 11.508 \qquad (3.05 \leq w \leq 7). \tag{5.17}$$

Next, we find that $\gamma(x,x) \geq 0$ for every $x \geq 7$. Note that $1.248e^x + 2(2x^3 - 21x^2 + 81.071778x - 131.636) \geq 1554$ for every $x \geq 7$. Therefore

$$\begin{aligned}\gamma(x,x) \geq 1554e^{4x} &- 3(x^4 - 12x^3 + 46.6x^2 - 112x + 40)e^{3x} - 6(2x^4 - 21.3x^3 + 40.3x^2 - 41.5x + 12)e^{2x} \\ &- 2(9x^4 - 56x^3 + 129x^2 - 132x + 52)e^{x} - 6(2x^4 - 14x^3 + 36x^2 - 40x + 16).\end{aligned}$$

Since $1554e^x - 3(x^4 - 12x^3 + 46.6x^2 - 112x + 40) \geq 1\,704\,694$ for every $x \geq 7$, we get that $\gamma(x,x) \geq 0$ for every $x \geq 7$. Combined with (5.16) and (5.17), we conclude that $b_1(n) \leq 11.508$ for every positive integer $n \geq \exp(\exp(3.05))$. Then, Theorem 5.4 implies that the required inequality holds for every positive integer $n \geq \exp(\exp(3.05))$. A direct computation for smaller values of n completes the proof. □

In the following example we compare the error term of the approximation from Corollary 5.5 with the approximation from (1.11) for the 10^n-th prime number.

Example. Denoting the right-hand side of (1.11) by $D_{low}(n)$ and the right-hand side of Corollary 5.5 by $A_{low}(n)$, we use [14] to obtain the following table:

n	p_n	$\lceil p_n - D_{low}(n) \rceil$	$\lceil p_n - A_{low}(n) \rceil$
10^6	15 485 863	6 503	6 202
10^7	179 424 673	22 441	9 600
10^8	2 038 074 743	243 180	70 976
10^9	22 801 763 489	2 302 876	398 508
10^{10}	252 097 800 623	22 918 665	3 317 139
10^{11}	2 760 727 302 517	221 928 766	26 778 265
10^{12}	29 996 224 275 833	2 149 187 973	239 178 828
10^{13}	323 780 508 946 331	20 674 500 003	2 150 740 095
10^{14}	3 475 385 758 524 527	198 184 329 536	19 415 837 925
10^{15}	37 124 508 045 065 437	1 896 434 754 032	175 499 011 167
10^{16}	394 906 913 903 735 329	18 139 062 711 550	1 590 290 625 854
10^{17}	4 185 296 581 467 695 669	173 543 282 219 005	14 444 669 910 447
10^{18}	44 211 790 234 832 169 331	1 661 592 139 340 947	131 583 109 674 419
10^{19}	465 675 465 116 607 065 549	15 924 846 933 652 812	1 202 148 079 734 641
10^{20}	4 892 055 594 575 155 744 537	152 800 345 036 619 338	11 015 488 979 071 672
10^{21}	51 271 091 498 016 403 471 853	1 467 920 673 086 566 371	101 234 605 116 877 497
10^{22}	536 193 870 744 162 118 627 429	14 119 500 534 424 061 205	933 027 750 725 057 967
10^{23}	5 596 564 467 986 980 643 073 683	135 978 797 224 902 285 752	8 622 947 459 666 327 003
10^{24}	58 310 039 994 836 584 070 534 263	1 311 132 449 659 551 496 235	79 902 499 931 160 000 670

Remark. The asymptotic expansion (1.2) for the n-th prime number implies that the inequality

$$p_n > n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + 11}{2\log^2 n}\right) \tag{5.18}$$

holds for all sufficiently large values of n. Let r_3 denotes the smallest positive integer such that the inequality (5.18) holds for every positive integer $n \geq r_3$. Under the assumption that the Riemann hypothesis is true, Arias de Reyna and Toulisse [1, Theorem 6.4] proved that

$$39 \cdot 10^{29} < r_3 \leq 39.58 \cdot 10^{29}.$$

6. Estimates for $\vartheta(p_n)$ in terms of n

Chebyshev's ϑ-function is defined by

$$\vartheta(x) = \sum_{p \leq x} \log p,$$

where p runs over primes not exceeding x. Notice that the Prime Number Theorem is equivalent to

$$\vartheta(x) \sim x \qquad (x \to \infty). \tag{6.1}$$

By proving the existence of a zero-free region for the Riemann zeta-function $\zeta(s)$ to the left of the line $\mathrm{Re}(s) = 1$, de la Vallée-Poussin [16] was able to bound the error term in (6.1) by proving

$$\vartheta(x) = x + O(x \exp(-c\sqrt{\log x})),$$

where c is a positive absolute constant. Together with Cipolla's asymptotic expansion (1.2) we get

$$\vartheta(p_n) = n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + 11}{2\log^2 n} + O\left(\frac{(\log\log n)^3}{\log^3 n}\right)\right).$$

In this direction, many estimates for $\vartheta(p_n)$ were obtained (see for example Massias and Robin [8, Théorème B], and Dusart [5, Proposition 6.2 and Proposition 6.3]). Using Corollary 1.2 and Corollary 1.4, we are able to establish the following estimates for $\vartheta(p_n)$ in terms of n, which improve the current best estimates found by Dusart [6, Proposition 5.11 and Proposition 5.12], namely

$$\vartheta(p_n) \geq n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2.04}{\log n}\right)$$

for every positive integer $n \geq \pi(10^{15}) + 1 = 29\,844\,570\,422\,670$, and

$$\vartheta(p_n) \leq n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{0.782}{\log^2 n}\right)$$

for every positive integer $n \geq 781$.

Proposition 6.1. *For every positive integer $n \geq 2$, we have*

$$\vartheta(p_n) > n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + 11.808}{2\log^2 n}\right),$$

and for every positive integer $n \geq 2\,581$, we have

$$\vartheta(p_n) < n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + 10.367}{2\log^2 n}\right).$$

Proof. From [2, Theorem 1.1], it follows that the inequality

$$\vartheta(p_n) > p_n - \frac{0.15 p_n}{\log^3 p_n} \tag{6.2}$$

holds for every positive integer $n \geq 841\,508\,302$, and that

$$\vartheta(p_n) < p_n + \frac{0.15 p_n}{\log^3 p_n} \tag{6.3}$$

for every positive integer n. By Rosser and Schoenfeld [11, Corollary 1], we have $n > p_n/\log p_n$ for every positive integer $n \geq 7$. Applying this inequality to (6.2), we get $\vartheta(p_n) > p_n - 0.15n/\log^2 n$ for every positive integer $n \geq 841\,508\,302$. Together with Corollary 5.5, we obtain that the desired lower bound for $\vartheta(p_n)$ holds for every positive integer $n \geq 841\,508\,302$. We check the remaining cases for n with a computer.

Similar to the first part of the proof, we apply the inequality $n > p_n/\log p_n$ to (6.3) and obtain that the inequality $\vartheta(p_n) < p_n + 0.15n/\log^2 n$ holds for every positive integer $n \geq 7$. Now, we use Corollary 4.6 to get that the required upper bound for $\vartheta(p_n)$ holds for every positive integer $n \geq 46\,254\,381$. For smaller values of n, we verify the required upper bound with a computer. □

7. Appendix

In the proof of Corollary 5.5, we note the following table, which indicates the explicit values of $M_i(B_i)$ for given B_i:

i	1	2	3	4	5
B_i	0.27	4.23	1.575	0.058	2.24
$M_i(B_i)$	1 359 056 314	1 471 247 583	1 468 111 666	1 383 728 153	1 462 324 835
i	6	7	8	9	10
B_i	0.105	0.0026	0.052	0.1955	0.08
$M_i(B_i)$	5	1 075 859 481	1 445 815 789	1 479 240 488	1 447 605 594

In this appendix, we show that this table is indeed correct; i.e. $H_i(n) \geq 0$ for every positive integer $n \geq M_i(B_i)$ for the given values of B_i. We start with the claim concerning $H_1(n)$.

Proposition 7.1. *We have* $M_1(0.27) = 1\,359\,056\,314$.

Proof. We have $Q_8(x) \geq 0$ for every $x \geq 0.6$ and $P_9(x) \geq 0$ for every $x \geq 0.6$. Using Lemma 4.4, we get that the inequality

$$H_1(n) \geq \frac{f_1(w(n))}{4\log^6 n \log p_n} \tag{7.1}$$

for every positive integer $n \geq 1\,338\,564\,587$, where $f_1(x) = 4 \cdot 0.27xe^{3x} - 2Q_7(x)e^x + 2 \cdot 0.87Q_8(x) + Q_9(x) + 2 \cdot 12.85 \cdot 0.87^2 P_9(x)$. We show that $f(x) \geq 0$ for every $x \geq 3.05$. For this purpose, we set $g(x) = (116.64 + 87.48x)e^x + (-24.6x^4 - 322.1x^3 - 1137.1x^2 - 1265.98x - 512.24)$. It is easy to show that $g(x) \geq 212$ for every $x \geq 1.7$. So, $f_1^{(4)}(x) = g(x)e^x + 240x - 1005.6 \geq 212e^x + 240x - 1034.688 \geq 0$ for every $x \geq 1.7$. Now, it is easy to see that $f(x) \geq 0$ for every $x \geq 3.05$. Applying this to (7.1), we get that $H_1(n) \geq 0$ for every positive integer $n \geq \exp(\exp(3.05))$. Finally, it suffices to verify the remaining cases with a computer. □

Before we verify that $M_2(4.23) = 1\,471\,247\,583$, we introduce the following function.

Definition. For $x \geq 1$, let

$$\Phi(x) = e^x + x + \log\left(1 + \frac{x-1}{e^x} + \frac{x-2.1}{e^{2x}}\right).$$

We notice the following three properties of the function $\Phi(x)$.

Lemma 7.2. *For every* $x \geq 1$, *we have* $\Phi'(x) \geq e^x + 3/4$.

Proof. We have $\Phi'(x) \geq e^x + 3/4$ if and only if $g(x) = e^{2x} - 3xe^x + 7e^x - 7x + 18.7 \geq 0$. Since $g''(x) = 4e^{2x} - (3x-1)e^x \geq 0$ for every $x \geq 0$ and $g'(1) \geq 10.49$, we obtain that $g'(x) \geq 0$ for every $x \geq 1$. Together with $g(1) \geq 29.96$, we get that $g(x) \geq 0$ for every $x \geq 1$. □

Lemma 7.3. *For every* $x \geq 1.25$, *we have* $\Phi(x) \geq e^x + x$.

Proof. The desired inequality holds if and only if $(x-1)e^x + x - 2.1 \geq 0$. Since the last inequality holds for every $x \geq 1.25$, we conclude the proof. □

Lemma 7.4. *For every positive integer* $n \geq 3$, *we have* $\Phi(\log\log n) \leq \log p_n$.

Proof. The claim follows directly from [6, Proposition 5.16]. □

Next, we use these properties to determine the value $M_2(4.23)$.

Proposition 7.5. *We have* $M_2(4.23) = 1\,471\,247\,583$.

Proof. We set $f_2(x) = 4.23x\Phi^3(x) + 12.85xe^x\Phi^2(x) - 71.3e^{3x}$ and use Lemma 7.2 and Lemma 7.3 to obtain that the inequality

$$f_2'(x) \geq 4.23(e^x + x)^3 + 25.54xe^x(e^x + x)^2 + 12.85e^x(e^x + x)^2 + 25.7xe^{2x}(e^x + x) - 213.9e^{3x} \tag{7.2}$$

holds for every $x \geq 1.25$. We denote the right-hand side of the last inequality by $g_2(x)$. A straightforward calculation gives $g_2^{(3)}(x) \geq (1383.48x - 3930.66)e^{3x} \geq 0$ for every $x \geq 2.85$. Now, it is easy to see that $g_2(x) \geq 0$ for every $x \geq 3.02$. Applying this to (7.2), we conclude that $f_2'(x) \geq 0$ for every $x \geq 3.02$. Since $f_2(3.05) \geq 16.797$, we obtain that $f_2(\log\log n) \geq 0$ for every positive integer $n \geq \exp(\exp(3.05))$. Finally, we apply Lemma 7.4. For smaller values of n, we use a computer. □

Proposition 7.6. *We have* $M_3(1.575) = 1\,468\,111\,666$.

Proof. Let $f_3(x) = 3.15x\Phi(x) - 35.15x^2 + 44.6x - 42.08$. Using Lemma 7.2 and Lemma 7.3, we get that the inequality $f_3'(x) \geq (3.15e^x + 3.15 - 67.15)x \geq 0$ holds for every $x \geq 3.02$. Combined with $f_3(3.05) \geq 0.044$ and Lemma 7.4, we get that $H_3(n) \geq 0$ for every positive integer $n \geq \exp(\exp(3.05))$. We conclude by a computer check. □

Proposition 7.7. *We have* $M_4(0.058) = 1\,383\,728\,153$.

Proof. We set $f_4(x) = 0.116xe^x\Phi(x) + 3.15x^3 - 57.45x^2 + 113.01x - 80.05$. We have $f_4(3.05) \geq 0.812$ and, by Lemma 7.2 and Lemma 7.3, $f_4'(x) \geq (0.116(e^x(e^x + x) + e^{2x}) + 9.45x - 114.9)x \geq 0$ for every $x \geq 2.92$. Hence, $f_4(\log\log n) \geq 0$ for every positive integer $n \geq \exp(\exp(3.05))$. Finally it suffices to apply Lemma 7.4. For smaller values of n, we check the required inequality with a computer. □

Proposition 7.8. *We have* $M_5(2.24) = 1\,462\,324\,835$.

Proof. To proof the claim, we define $f_5(x) = 4.48xe^x - 2x^4 + 5x^3 - 37.7x^2 + 41.1x - 31.9$. Since $f_5^{(3)}(x) \geq 0$ for every $x \geq 2.1$ and $f_5''(2.1) \geq 31.756$, we obtain that $f_5''(x) \geq 0$ for every $x \geq 2.1$. Together with $f_5'(2.4) \geq 3.853$, we get that $f_5'(x) \geq 0$ for every $x \geq 2.4$. Combined with $f_5(3.05) \geq 0.06$, we conclude that $f_5(\log\log n) \geq 0$, and thus $H_5(n) \geq 0$, for every positive integer $n \geq \exp(\exp(3.05))$. For smaller values of n, we verify the inequality $H_5(n) \geq 0$ with a computer. □

By adding the constants $B_1, \ldots, B_5$ given in Proposition 7.1, Proposition 7.5, Proposition 7.6, Proposition 7.7 und Proposition 7.8, we get $12.85 - B_1 - B_2 - B_3 - B_4 - B_5 = 4.477$. Now we set $B_6 = 0.12$ to obtain the following explicit value for $M_6(B_6)$.

Proposition 7.9. *We have* $M_6(0.105) = 5$.

Proof. Let $r(x,t) = (0.105e^x + 4.477)x\Phi(x) + 3.15xe^x - 3.15(t^2+1)e^t$, let $f_6(x) = r(x,x)$. If $t_0 \leq x \leq t_1$, then $f_6(x) \geq r(t_0, t_1)$. We check with a computer that $r(0.7 + i \cdot 10^{-3}, 0.7 + (i+1)\cdot 10^{-3}) \geq 0$ for every nonnegative integer i such that $0 \leq i \leq 2\,799$. Hence $f_6(x) \geq 0$ for every x such that $0.7 \leq x \leq 3.5$. To show, that $f_6(x) \geq 0$ for every $x \geq 3.5$, we set

$$g(x) = (0.105xe^x + 0.105e^x + 4.477)(e^x + x) + (0.105e^x + 4.477)xe^x - 3.15xe^x(1 + x).$$

Then $g'(x) = h(x)e^x + 4.477$, where $h(x) = 0.42(1+x)e^x - 3.045x^2 - 4.658x + 5.909$. Since $h(x) \geq 0$ for every $x \geq 3.09$, we get that $g'(x) \geq 0$ for every $x \geq 3.09$. Together with $g(3.47) \geq 0$, we conclude that $g(x) \geq 0$ for every $x \geq 3.47$. Using Lemma 7.2 and Lemma 7.3, we obtain that $f_6'(x) \geq g(x) \geq 0$ for every $x \geq 3.47$. Combined with $f_6(3.5) \geq 4.35411$, we have $f_6(x) \geq 0$ for every $x \geq 3.5$. Hence, $f_6(x) \geq 0$ for every $x \geq 0.7$. Now, we apply Lemma 7.4 to get that $H_6(n) \geq 0$ for every positive integer $n \geq \exp(\exp(0.7))$. We conclude by direct computation. □

Proposition 7.10. *We have* $M_7(0.0026) = 1\,075\,859\,481$.

Proof. Substituting the definition of $P_8(x)$, we get

$$H_7(n) = \frac{0.0026w}{y^2z} - \frac{38.55w^2 - 77.1w + 66.82}{2y^3z^3}.$$

To show that $H_7(n) \geq 0$ for every positive integer $n \geq 1\,075\,859\,481$, we first consider the function $f_7(x) = 0.0052xe^x\Phi^2(x) - 38.55x^2 + 77.1x - 66.82$. We have $f_7(3.05) \geq 6.821$. Additionally, we use Lemma 7.2 and Lemma 7.3 to get that $f_7'(x) \geq (0.0052(e^x+x)^2(1+e^x) + 0.0104e^{2x}(e^x+x) - 77.1)x \geq 0$ for every $x \geq 2.76$. Hence, $f_7(\log\log n) \geq 0$ for every positive integer $n \geq \exp(\exp(3.05))$. Finally, we apply Lemma 7.4. For the remaining cases, we use a computer. □

Proposition 7.11. *We have* $M_8(0.052) = 1\,445\,815\,789$.

Proof. We set $f_8(x) = 0.052x\,\Phi^2(x) - 12.85(x^2 - x + 1)$. Then $f_8(3.05) \geq 0.148$ and, by Lemma 7.2 and Lemma 7.3, we obtain $f_8'(x) \geq (0.052(e^x + x) + 0.104(e^x + x)e^x - 25.7)x \geq 0$ for every $x \geq 2.66$. Hence $f_8(\log\log n) \geq 0$ for every positive integer $n \geq \exp(\exp(3.05))$. Now we use Lemma 7.4 to obtain that $H_8(n) \geq 0$ for every positive integer $n \geq \exp(\exp(3.05))$. For smaller values of n, we use a computer. □

Proposition 7.12. *We have* $M_9(0.1955) = 1\,479\,240\,488$.

Proof. We define $f_9(x) = 0.1955x\,\Phi^4(x) - 463.2275e^{2x}$. Then, by using Lemma 7.2 and Lemma 7.3, we have $f_9'(x) \geq (0.1955(e^x + x)^2 + 0.782x(e^x + x)^2 - 926.455)e^{2x} \geq 0$ for every $x \geq 2.83$. Combined with $f_9(3.05) \geq 7.11$, we conclude that $f_9(x) \geq 0$ for every $x \geq 3.05$. Substituting $x = \log\log n$ in $f_9(x)$, we get, by Lemma 7.4, that $H_9(n) \geq 0$ for every positive integer $n \geq \exp(\exp(3.05))$. For every positive integer n such that $1\,479\,240\,488 \leq n \leq \exp(\exp(3.05))$ we check the desired inequality with a computer. □

Finally, we determine the value of $M_{10}(0.08)$.

Proposition 7.13. *We have* $M_{10}(0.08) = 1\,447\,605\,594$.

Proof. Let $f_{10}(x) = 0.08x\,\Phi^5(x) - 4585e^{2x}$. Applying Lemma 7.2 and Lemma 7.3, we get $f_{10}'(x) \geq (0.4x(e^x + x)^3 - 9170)e^{2x} \geq 0$ for every $x \geq 2.9$. Together with $f_{10}(3.05) \geq 6142.27$, we obtain that $f_{10}(\log\log n) \geq 0$ for every positive integer $n \geq \exp(\exp(3.05))$. Now, we use Lemma 7.4 to conclude that $H_{10}(n) \geq 0$ for every positive integer $n \geq \exp(\exp(3.05))$. Finally, it suffices to verify the remaining cases with a computer. □

Acknowledgement

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