The Frobenius number for sequences of triangular and tetrahedral numbers

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Abstract

We compute the Frobenius number for sequences of triangular and tetrahedral numbers. In addition, we study some properties of the numerical semigroups associated to those sequences.

Keywords: Frobenius number, triangular numbers, tetrahedral numbers, telescopic sequences, free numerical semigroups.

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1 Introduction

According to [4], Frobenius raised in his lectures the following question: given relatively prime positive integers $a_1, \dots, a_n$, compute the largest natural number that is not representable as a non-negative integer linear combination of $a_1, \dots, a_n$. Nowadays, it is known as the Frobenius (coin) problem. Moreover, the solution is called the Frobenius number of the set $\{a_1, \dots, a_n\}$ and it is denoted by $\mathrm{F}(a_1, \dots, a_n)$.

It is well known the solution for $n = 2$ (see [22, 23]). Namely, $\mathrm{F}(a_1, a_2) = a_1 a_2 - a_1 - a_2$. At present, the Frobenius problem is open for $n \geq 3$. More precisely, Curtis showed in [6] that it is impossible to find a polynomial formula (this is, a finite set of polynomials) that computes the Frobenius number if $n = 3$. In fact, Ramírez-Alfonsín proves in [15] that this problem is NP-hard for n variable.

Many papers are devoted to study this problem for particular cases (see [16] for more details). Specially, when $\{a_1, \dots, a_n\}$ is part of a "classic" integer sequences: arithmetic and almost arithmetic ([4, 17, 11, 21]), Fibonacci ([12]),

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geometric ([14]), Mersenne ([19]), squares and cubes ([10, 13]), Thabit ([18]), etcetera.

For example, in [4] Brauer proves that

$$\mathrm{F}(n, n+1, \dots, n+k-1) = \left(\left\lfloor \frac{n-2}{k-1} \right\rfloor + 1\right)n - 1, \tag{1.1}$$

where, if $x \in \mathbb{R}$, then $\lfloor x \rfloor \in \mathbb{Z}$ and $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$. On the other hand, denoting by $a(n) = \mathrm{F}\left(\frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2}, \frac{(n+2)(n+3)}{2}\right)$, for $n \in \mathbb{N} \setminus \{0\}$, C. Baker conjectured that (see `https://oeis.org/A069755/internal`)

$$a(n) = \frac{-14 + 6(-1)^n + (3 + 9(-1)^n)n + 3(5 + (-1)^n)n^2 + 6n^3}{8}; \tag{1.2}$$

$$a(n) = \frac{6n^3 + 18n^2 + 12n - 8}{8}, \quad \text{for } n \text{ even}; \tag{1.3}$$

$$a(n) = \frac{6n^3 + 12n^2 - 6n - 20}{8}, \quad \text{for } n \text{ odd}. \tag{1.4}$$

Let us observe that both of these examples are particular cases of combinatorial numbers (or binomial coefficients) sequences, that is,

- $\binom{n}{1}, \binom{n+1}{1}, \dots, \binom{n+k-1}{1}$ in the first case,
- $\binom{n+1}{2}, \binom{n+2}{2}, \binom{n+3}{2}$ in the second one.

Let us recall that $\binom{n+1}{2}$ is known as triangular (or triangle) number and that the tetrahedral numbers correspond to $\binom{n+2}{3}$. These classes of numbers are precisely the aims of this paper.

In order to achieve our purpose, we will use a well-known formula by Johnson ([8]): if a_1, a_2, a_3 are relatively prime numbers and $\gcd\{a_1, a_2\} = d$, then

$$\mathrm{F}(a_1, a_2, a_3) = d\mathrm{F}\left(\frac{a_1}{d}, \frac{a_2}{d}, a_3\right) + (d-1)a_3. \tag{1.5}$$

In fact, we will use the well-known generalization by Brauer y Shockley ([5]): if $a_1, \dots, a_n$ are relatively prime numbers and $d = \gcd\{a_1, \dots, a_{n-1}\}$, then

$$\mathrm{F}(a_1, \dots, a_n) = d\mathrm{F}\left(\frac{a_1}{d}, \dots, \frac{a_{n-1}}{d}, a_n\right) + (d-1)a_n. \tag{1.6}$$

An interesting situation, to apply these formulae, corresponds with the telescopic sequences ([9]) and leads to the free numerical semigroups, which were introduced by Bertin and Carbonne ([2, 3]) and previously used by Watanabe ([24]). Let us note that this idea does not coincide with the categorical concept of free object.

Definition 1.1. Let $(a_1, \dots, a_n)$ be a sequence of positive integers such that $\gcd\{a_1, \dots, a_n\} = 1$ (where $n \geq 2$). Let $d_i = \gcd\{a_1, \dots, a_i\}$ for $i = 1, \dots, n$. We will say that $(a_1, \dots, a_n)$ is a *telescopic sequence* if $\frac{a_i}{d_i}$ is representable as a non-negative integer linear combination of $\frac{a_1}{d_{i-1}}, \dots, \frac{a_{i-1}}{d_{i-1}}$ for $i = 2, \dots, n$.

Let us observe that, if $(a_1, \ldots, a_n)$ is a telescopic sequence, then the sequence $\left(\frac{a_1}{d_i}, \ldots, \frac{a_i}{d_i}\right)$ is also telescopic for $i = 2, \ldots, n-1$.

Let $(\mathbb{N}, +)$ be the additive monoid of non-negative integers. We will say that S is a *numerical semigroup* if it is an additive subsemigroup of $\mathbb{N}$ which satisfies that $0 \in S$ and that $\mathbb{N} \setminus S$ is a finite set.

Let $X = \{x_1, \ldots, x_n\}$ be a non-empty subset of $\mathbb{N} \setminus \{0\}$. We will denote by $\langle X \rangle = \langle x_1, \ldots, x_n \rangle$ the monoid generated by X, that is,

$$\langle X \rangle = \{\lambda_1 x_1 + \cdots + \lambda_n x_n \mid \lambda_1, \ldots \lambda_n \in \mathbb{N}\}.$$

It is well known (see [20]) that every submonoid S of $(\mathbb{N}, +)$ has a unique *minimal system of generators*, that is, there exists a unique X such that $S = \langle X \rangle$ and $S \neq \langle Y \rangle$ for any $Y \subsetneq X$. In addition, X is a system of generators of a numerical semigroup if and only if $\gcd(X) = 1$.

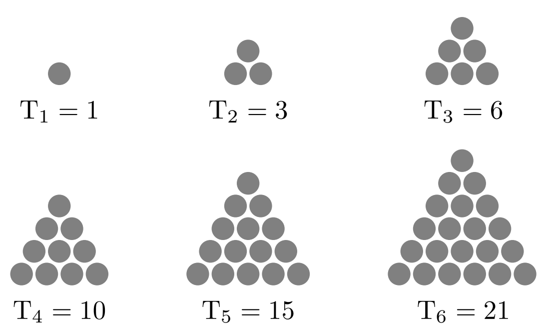
Let $X = \{x_1, \ldots, x_n\}$ be the minimal system of generators of a numerical semigroup S. Then n (that is, the cardinality of X) is called the *embedding dimension* of S and it is denoted by $\mathrm{e}(S)$.

Definition 1.2. We will say that S is a *free numerical semigroup* if there exists a telescopic sequence $(a_1, \ldots, a_n)$ such that $S = \langle a_1, \ldots, a_n \rangle$.

Before finish this introduction, we summarize the content of this paper. In Section 2 we compute the Frobenius number of three consecutive triangular numbers. In Section 3 we solve the analogue for four consecutive tetrahedral numbers. In the last section, we show some results on numerical semigroups generated by three consecutive triangular numbers or four consecutive tetrahedral numbers, taking advantage of they are free numerical semigroups.

2 Triangular numbers

Let us recall that a *triangular number* (or *triangle number*) is a positive integer which counts the number of dots composing an equilateral triangle. For example, in the next figure we show the first six triangular numbers.



It is well known that the nth triangular number is given by the combinatorial number $\mathrm{T}_n = \binom{n+1}{2}$.

In order to compute the Frobenius number of a sequence of three triangular numbers, we need to determine if we have a sequence of relatively prime integers. First, we give a technical lemma.

Lemma 2.1.

$$\gcd\{\mathrm{T}_n, \mathrm{T}_{n+1}\} = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd,} \\ n+1, & \text{if } n \text{ is even.} \end{cases}$$

Proof. If n is odd, then we have that

$$\gcd\{\mathrm{T}_n, \mathrm{T}_{n+1}\} = \gcd\left\{\frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2}\right\} = \frac{n+1}{2}\gcd\{n,2\} = \frac{n+1}{2}.$$

On the other hand, if n is even, then

$$\gcd\{\mathrm{T}_n, \mathrm{T}_{n+1}\} = \gcd\left\{\frac{n(n+1)}{2}, \frac{(n+1)(n+2)}{2}\right\} = (n+1)\gcd\left\{\frac{n}{2}, 1\right\} = n+1.$$

□

In the following lemma we show that three consecutive triangular numbers are always relatively prime.

Lemma 2.2. $\gcd\{\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2}\} = 1$.

Proof. By Lemma 2.1, if n is odd, then

$$\gcd\{\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2}\} = \gcd\left\{\gcd\{\mathrm{T}_n, \mathrm{T}_{n+1}\}, \gcd\{\mathrm{T}_{n+1}, \mathrm{T}_{n+2}\}\right\} =$$

$$\gcd\left\{\frac{n+1}{2}, n+2\right\} = \gcd\left\{\frac{n+1}{2}, \frac{n+1}{2}+1\right\} = 1.$$

The proof is similar if n is even. Therefore, we omit it. □

In the next result, we show the key to obtain the answer to our question.

Proposition 2.3. *The sequences* $(\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2})$ *and* $(\mathrm{T}_{n+2}, \mathrm{T}_{n+1}, \mathrm{T}_n)$ *are telescopic.*

Proof. Let n be an odd integer. From Lemmas 2.1 and 2.2, $\gcd\{\mathrm{T}_n, \mathrm{T}_{n+1}\} = \frac{n+1}{2}$ and $\gcd\{\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2}\} = 1$. Now, it is obvious that

$$\frac{\mathrm{T}_{n+2}}{1} = \frac{n+3}{2}(n+2) \in \left\langle \frac{\mathrm{T}_n}{\frac{n+1}{2}}, \frac{\mathrm{T}_{n+1}}{\frac{n+1}{2}} \right\rangle = \langle n, n+2\rangle.$$

Therefore, $(\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2})$ is telescopic if n is odd.

In a similar way, we can show that $(\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2})$ is telescopic if n is even. And the same comment can be applied to the sequence $(\mathrm{T}_{n+2}, \mathrm{T}_{n+1}, \mathrm{T}_n)$. □

Now we are ready to give the main result of this section.

Proposition 2.4. *Let $n \in \mathbb{N} \setminus \{0\}$. Then*

$$\mathrm{F}(\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2}) = \begin{cases} \frac{3n^3+6n^2-3n-10}{4}, & \text{if } n \text{ is odd,} \\ \frac{3n^3+9n^2+6n-4}{4}, & \text{if } n \text{ is even.} \end{cases}$$

Equivalently,

$$\mathrm{F}(\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2}) = \left\lfloor \frac{n}{2} \right\rfloor (\mathrm{T}_n + \mathrm{T}_{n+1} + \mathrm{T}_{n+2} - 1) - 1. \tag{2.1}$$

Proof. Let n be an odd positive integer. From (1.5) (or (1.6)) and the proof of Proposition 2.3, we have that

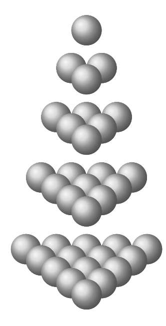
$$\mathrm{F}(\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2}) = \frac{n+1}{2}\mathrm{F}\left(\frac{\mathrm{T}_n}{\frac{n+1}{2}}, \frac{\mathrm{T}_{n+1}}{\frac{n+1}{2}}, \mathrm{T}_{n+2}\right) + \frac{n-1}{2}\mathrm{T}_{n+2} =$$

$$\frac{n+1}{2}\mathrm{F}\left(n, n+2\right) + \frac{n-1}{2}\frac{(n+2)(n+3)}{2}$$

and, having in mind that $\mathrm{F}\left(n, n+2\right) = n^2 - 2$, then the conclusion is obvious. On the other hand, the reasoning for even n is similar. Finally, an straightforward computation leads to (2.1). □

3 Tetrahedral numbers

Let us recall that a *tetrahedral number* (or *triangular pyramidal number*) is a positive integer which counts the number of balls composing a regular tetrahedron. The nth tetrahedral number is given by the combinatorial number $\mathrm{TH}_n = \binom{n+2}{3}$. Thus, in the following figure, we see the pyramid (by layers) associated to the 5th tetrahedral number ($\mathrm{TH}_5 = 35$).



In this section, our purpose is compute the Frobenius number for a sequence of four consecutive tetrahedral numbers.

We need a previous lemma with an easy proof.

Lemma 3.1. *Let $(a_1, a_2, \dots, a_n)$ be a sequence of positive integers such that $d_1 = \gcd\{a_1, a_2, \dots, a_n\}$. If $d_2 = \gcd\{a_2 - a_1, \dots, a_n - a_{n-1}\}$, then $d_1 | d_2$. In particular, if $d_2 = 1$, then $d_1 = 1$.*

Now, let us see that four consecutive tetrahedral numbers are always relatively prime.

Lemma 3.2. $\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}, \mathrm{TH}_{n+3}\} = 1$.

Proof. It is clear that

$$(\mathrm{TH}_{n+1} - \mathrm{TH}_n, \mathrm{TH}_{n+2} - \mathrm{TH}_{n+1}, \mathrm{TH}_{n+3} - \mathrm{TH}_{n+2}) = (\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2}).$$

Therefore, by applying Lemmas 2.2 and 3.1, we have the conclusion. □

The following lemma has an easy proof too. So, we omit it.

Lemma 3.3. *Let* $n \in \mathbb{N} \setminus \{0\}$.

1. *If* $n = 6k$, *then* $\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}\} = (6k+1)(3k+1)$.
2. *If* $n = 6k+1$, *then* $\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}\} = (3k+1)(2k+1)$.
3. *If* $n = 6k+2$, *then* $\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}\} = (2k+1)(3k+2)$.
4. *If* $n = 6k+3$, *then* $\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}\} = (3k+2)(6k+5)$.
5. *If* $n = 6k+4$, *then* $\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}\} = (6k+5)(k+1)$.
6. *If* $n = 6k+5$, *then* $\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}\} = (k+1)(6k+7)$.

In the next two results, we give the tool for getting the answer to our problem.

Proposition 3.4. *The sequence* $(\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}, \mathrm{TH}_{n+3})$ *is telescopic if and only if* $n \not\equiv 4 \bmod 6$ *or* $n \not\equiv 5 \bmod 6$.

Proof. We are going to study the six possible cases $n = 6k + r$ with $k \in \mathbb{N}$ and $r \in \{0, 1, \ldots, 5\}$.

1. Let $n = 6k$. Since $\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}\}$ is equal to

   $$\gcd\big\{ \gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}\}, \gcd\{\mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}\}\big\},$$

   from items 1 and 2 of Lemma 3.3, we have $\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}\} = \gcd\{(6k+1)(3k+1), (3k+1)(2k+1)\} = 3k+1$. Now, it is easy to check that $\mathrm{TH}_{n+3} = 0\frac{\mathrm{TH}_n}{3k+1} + (3k+2)\frac{\mathrm{TH}_{n+1}}{3k+1} + 2\frac{\mathrm{TH}_{n+2}}{3k+1}$.
   On the other hand, since

   $$\frac{(\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2})}{3k+1} = \big(2k(6k+1), (6k+1)(2k+1), 2(2k+1)(3k+2)\big),$$

   $\gcd\{2k(6k+1), (6k+1)(2k+1)\} = 6k+1$, and

   $$2(2k+1)(3k+2) = 0\frac{2k(6k+1)}{6k+1} + 2(3k+2)\frac{(6k+1)(2k+1)}{6k+1},$$

   we conclude that $(\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}, \mathrm{TH}_{n+3})$ is telescopic.

2. Having in mind items 2 and 3 of Lemma 3.3, if $n = 6k+1$, then we get that $\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}\} = 2k+1$. In addition,

$$\mathrm{TH}_{n+3} = 0\frac{\mathrm{TH}_n}{2k+1} + 0\frac{\mathrm{TH}_{n+1}}{2k+1} + 2(k+1)\frac{\mathrm{TH}_{n+2}}{2k+1}.$$

Since $\gcd\left\{\frac{\mathrm{TH}_n}{2k+1}, \frac{\mathrm{TH}_{n+1}}{2k+1}\right\} = 3k+1$ and

$$\frac{\mathrm{TH}_{n+2}}{2k+1} = (3k+2)\frac{\mathrm{TH}_n}{(2k+1)(3k+1)} + 2\frac{\mathrm{TH}_{n+1}}{(2k+1)(3k+1)},$$

we have the result.

3. For $n = 6k+2$, we have that $\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}\} = 3k+2$,

$$\mathrm{TH}_{n+3} = 0\frac{\mathrm{TH}_n}{3k+2} + 3(k+1)\frac{\mathrm{TH}_{n+1}}{3k+2} + 2\frac{\mathrm{TH}_{n+2}}{3k+2},$$

$\gcd\left\{\frac{\mathrm{TH}_n}{3k+2}, \frac{\mathrm{TH}_{n+1}}{3k+2}\right\} = 2k+1$, and

$$\frac{\mathrm{TH}_{n+2}}{3k+2} = 0\frac{\mathrm{TH}_n}{(3k+2)(2k+1)} + 2(k+1)\frac{\mathrm{TH}_{n+1}}{(3k+2)(2k+1)}.$$

4. For $n = 6k+3$, we have that $\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}\} = 6k+5$,

$$\mathrm{TH}_{n+3} = 0\frac{\mathrm{TH}_n}{6k+5} + 0\frac{\mathrm{TH}_{n+1}}{6k+5} + 2(3k+4)\frac{\mathrm{TH}_{n+2}}{6k+5},$$

$\gcd\left\{\frac{\mathrm{TH}_n}{6k+5}, \frac{\mathrm{TH}_{n+1}}{6k+5}\right\} = 3k+2$, and

$$\frac{\mathrm{TH}_{n+2}}{6k+5} = 3(k+1)\frac{\mathrm{TH}_n}{(6k+5)(3k+2)} + 2\frac{\mathrm{TH}_{n+1}}{(6k+5)(3k+2)}.$$

5. If $n = 6k+4$, then $\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}\} = k+1$. Let us suppose that there exist $\alpha, \beta, \gamma \in \mathbb{N}$ such that $\mathrm{TH}_{n+3} = \alpha\frac{\mathrm{TH}_n}{k+1} + \beta\frac{\mathrm{TH}_{n+1}}{k+1} + \gamma\frac{\mathrm{TH}_{n+2}}{k+1}$. Then

$$(6k+7)\big((3k+4)(2k+3) - (6k+5)\beta - 2(3k+4)\gamma\big) = 2(3k+2)(6k+5)\alpha.$$

Since $\gcd\{6k+7, 2\} = \gcd\{6k+7, 3k+2\} = \gcd\{6k+7, 6k+5\} = 1$, we have that $\alpha = (6k+7)\tilde{\alpha}$. Consequently,

$$(3k+4)(2k+3) - (6k+5)\beta - 2(3k+4)\gamma = 2(3k+2)(6k+5)\tilde{\alpha} \Rightarrow$$

$$(3k+4)(2k+3-2\gamma) = (6k+5)\big(\beta + 2(3k+2)\tilde{\alpha}\big).$$

Now, since $\gcd\{3k+4, 6k+5\} = 1$, we conclude that $(6k+5)|(2k+3-2\gamma)$ and, thereby, $2k+3-2\gamma = 0$. That is, we have a contradiction.

6. If $n = 6k+5$, then $\gcd\{\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}\} = 6k+7$,

$$\mathrm{TH}_{n+3} = 0\frac{\mathrm{TH}_n}{6k+7} + 0\frac{\mathrm{TH}_{n+1}}{6k+7} + 2(3k+5)\frac{\mathrm{TH}_{n+2}}{6k+7},$$

and $\gcd\left\{\frac{\mathrm{TH}_n}{3k+5}, \frac{\mathrm{TH}_{n+1}}{3k+5}\right\} = k+1$. Let us suppose that there exist $\alpha, \beta \in \mathbb{N}$ such that $\frac{\mathrm{TH}_{n+2}}{6k+7} = \alpha\frac{\mathrm{TH}_n}{(6k+7)(k+1)} + \beta\frac{\mathrm{TH}_{n+1}}{(6k+7)(k+1)}$. In such a case,

$$(3k+4)(2k+3) = (6k+5)\alpha + 2(3k+4)\beta.$$

Thus, since $\gcd\{3k+4, 6k+5\} = 1$, we have that $\alpha = (3k+4)\tilde{\alpha}$ and, therefore, $2k+3-2\beta = (6k+5)\tilde{\alpha}$. That is, once again, we get a contradiction. □

Using the same techniques of the previous proof, we have the next result.

Proposition 3.5. *The sequence* $(\mathrm{TH}_{n+3}, \mathrm{TH}_{n+2}, \mathrm{TH}_{n+1}, \mathrm{TH}_n)$ *is telescopic if and only if* $n \equiv 4 \bmod 6$ *or* $n \equiv 5 \bmod 6$.

By combining Propositions 3.4 and 3.5 with (1.6), it is clear that we can obtain the Frobenius number for every sequence $(\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}, \mathrm{TH}_{n+3})$. Thus we get the following result.

Proposition 3.6. *Let* $n \in \mathbb{N} \setminus \{0\}$. *Then* $\mathrm{F}(\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}, \mathrm{TH}_{n+3}) =$

1. $\frac{n-3}{3}\mathrm{TH}_{n+1} + n\mathrm{TH}_{n+2} + \frac{n}{2}\mathrm{TH}_{n+3} - \mathrm{TH}_n$, *if* $n = 6k$,
2. $(n-1)\mathrm{TH}_{n+1} + \frac{n-1}{2}\mathrm{TH}_{n+2} + \frac{n-1}{3}\mathrm{TH}_{n+3} - \mathrm{TH}_n$, *if* $n = 6k+1$,
3. $(n-1)\mathrm{TH}_{n+1} + \frac{n-2}{3}\mathrm{TH}_{n+2} + \frac{n}{2}\mathrm{TH}_{n+3} - \mathrm{TH}_n$, *if* $n = 6k+2$,
4. $\frac{n-3}{3}\mathrm{TH}_{n+1} + \frac{n-1}{2}\mathrm{TH}_{n+2} + (n+1)\mathrm{TH}_{n+3} - \mathrm{TH}_n$, *if* $n = 6k+3$,
5. $\frac{n+2}{3}\mathrm{TH}_{n+2} + \frac{n+2}{2}\mathrm{TH}_{n+1} + (n+2)\mathrm{TH}_n - \mathrm{TH}_{n+3}$, *if* $n = 6k+4$,
6. $(n+4)\mathrm{TH}_{n+2} + \frac{n+1}{3}\mathrm{TH}_{n+1} + \frac{n+1}{2}\mathrm{TH}_n - \mathrm{TH}_{n+3}$, *if* $n = 6k+5$.

Remark 3.7. From the contents of this section and the previous one, it looks like that the problem is going to became more and more longueur as soon as we consider combinatorial numbers $\binom{n}{m}$ with bigger m. Anyway, it is not difficult to see that the sequence $\left(\binom{n+3}{4}, \binom{n+4}{4}, \binom{n+5}{4}, \binom{n+6}{4}, \binom{n+7}{4}\right)$ is telescopic if and only if $n \equiv x \bmod 6$ for $x \in \{0,1,2\}$. On the other hand, for $n \geq 9$, the sequence $\left(\binom{n+7}{4}, \binom{n+6}{4}, \binom{n+5}{4}, \binom{n+4}{4}, \binom{n+3}{4}\right)$ is telescopic if and only if $n \equiv x \bmod 6$ for $x \in \{3,4,5\}$. In addition, if $n \in \{3,4,5\}$, both of above sequences are telescopic. (In order to study all these cases, it is better to consider $n \equiv x \bmod 12$ for $x \in \{0,1,\ldots,11\}$.)

Remark 3.8. Let n, m be positive integers. At this moment, we could conjecture that the sequence $\left(\binom{n+m-1}{m}, \dots, \binom{n+2m-1}{m}\right)$ is telescopic if and only if the sequence $\left(\binom{n+2m-1}{m}, \dots, \binom{n+m-1}{m}\right)$ is not telescopic and, consequently, we would have an easy algorithmic process to compute $\mathrm{F}\left(\binom{n+m-1}{m}, \dots, \binom{n+2m-1}{m}\right)$. Unfortunately, neither $\left(\binom{12}{5}, \dots, \binom{17}{5}\right) = (792, 1287, 2002, 3003, 4368, 6188)$ nor $(6188, 4368, 3003, 2002, 1287, 792)$ are telescopic. In fact, all possible permutations of $(792, 1287, 2002, 3003, 4368, 6188)$ are not telescopic.

4 Consequences on numerical semigroups

As we comment in the introduction, if $(a_1, \dots, a_n)$ is a sequence of relatively prime positive integers, then the monoid $\langle a_1, \dots, a_n \rangle$ is a numerical semigroup. In this section we are interested in those numerical semigroups which are generated by three consecutive triangular numbers or by four consecutive tetrahedral numbers and have embedding dimension equal to three or four, respectively.

First of all, having in mind that

- $(\mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2})$ is always telescopic,
- either $(\mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}, \mathrm{TH}_{n+3})$ or $(\mathrm{TH}_{n+3}, \mathrm{TH}_{n+2}, \mathrm{TH}_{n+1}, \mathrm{TH}_n)$ is telescopic,

we can use the ideas of [20, Chapter 8] to obtain several results for the numerical semigroups $\mathcal{T}_n = \langle \mathrm{T}_n, \mathrm{T}_{n+1}, \mathrm{T}_{n+2} \rangle$ and $\mathcal{TH}_n = \langle \mathrm{TH}_n, \mathrm{TH}_{n+1}, \mathrm{TH}_{n+2}, \mathrm{TH}_{n+3} \rangle$.

Since we want that $\mathrm{e}(\mathcal{T}_n) = 3$, then we are going to take $n \geq 3$ along this section in this case. In the same way, we are going to consider $n \geq 4$ in order to have $\mathrm{e}(\mathcal{TH}_n) = 4$. Thus, we have the next result.

Proposition 4.1.

1. *If $n \geq 3$, then $\mathcal{T}_n$ is a free numerical semigroup with embedding dimension equal to three.*
2. *If $n \geq 4$, then $\mathcal{TH}_n$ is a free numerical semigroup with embedding dimension equal to four.*

We begin with a minimal presentation of $\mathcal{T}_n$.

Proposition 4.2. *A minimal presentation of $\mathcal{T}_n$ is*

1. *if n is odd: $\frac{n+1}{2}\mathrm{T}_{n+2} = \frac{n+3}{2}\mathrm{T}_{n+1}$ and $n\mathrm{T}_{n+1} = (n+2)\mathrm{T}_n$;*
2. *if n is even: $(n+1)\mathrm{T}_{n+2} = (n+3)\mathrm{T}_{n+1}$ and $\frac{n}{2}\mathrm{T}_{n+1} = \frac{n+2}{2}\mathrm{T}_n$.*

Proof. It is immediate from [20, Corollary 9.18]. □

From the above proposition, we deduce the following result on the Apéry set of T_n in $\mathcal{T}_n$ (see [1] for more information about Apéry sets).

Corollary 4.3. *We have that*

1. *if n is odd, then*
$$\mathrm{Ap}(\mathcal{T}_n, \mathrm{T}_n) = \left\{ a\mathrm{T}_{n+1} + b\mathrm{T}_{n+2} \;\middle|\; a \in \{0, \ldots, n-1\},\ b \in \left\{0, \ldots, \frac{n-1}{2}\right\} \right\};$$

2. *if n is even, then*
$$\mathrm{Ap}(\mathcal{T}_n, \mathrm{T}_n) = \left\{ a\mathrm{T}_{n+1} + b\mathrm{T}_{n+2} \;\middle|\; a \in \left\{0, \ldots, \frac{n-2}{2}\right\},\ b \in \{0, \ldots, n\} \right\}.$$

Proof. It is enough to use the ideas developed in the proofs of Lemmas 9.14 and 9.15 in [20]. □

We finish the serie of results for $\mathcal{T}_n$ computing the Betti elements of these numerical semigroups (see [7] for more details about these elements of a numerical semigroup).

Corollary 4.4. *The Betti elements of $\mathcal{T}_n$ are*

1. $\frac{3}{2}\binom{n+3}{3}$ *and* $3\binom{n+2}{3}$*, if n is odd;*
2. $3\binom{n+3}{3}$ *and* $\frac{3}{2}\binom{n+2}{3}$*, if n is even.*

Remark 4.5. Let us observe that the Betti elements of $\mathcal{T}_n$ are given in terms of tetrahedral numbers. An analogue property can be observed in the case of $\mathcal{TH}_n$ (see Corollary 4.8): the Betti elements of $\mathcal{TH}_n$ can be expressed in terms of the combinatorial numbers $\binom{m}{4}$.

We finish this section with the serie of results relative to the numerical semigroups $\mathcal{TH}_n$ with $n \geq 4$.

Proposition 4.6. *A minimal presentation of $\mathcal{TH}_n$ is*

1. *if $n = 6k$:* $\frac{n+2}{2}\mathrm{TH}_{n+3} = 2\mathrm{TH}_{n+2} + \frac{n+4}{2}\mathrm{TH}_{n+1},\ (n+1)\mathrm{TH}_{n+2} = (n+4)\mathrm{TH}_{n+1},$ *and* $\frac{n}{3}\mathrm{TH}_{n+1} = \frac{n+3}{3}\mathrm{TH}_n$*;*
2. *if $n = 6k+1$:* $\frac{n+2}{3}\mathrm{TH}_{n+3} = \frac{n+5}{3}\mathrm{TH}_{n+2},\ \frac{n+1}{2}\mathrm{TH}_{n+2} = 2\mathrm{TH}_{n+1} + \frac{n+3}{2}\mathrm{TH}_n,$ *and* $n\mathrm{TH}_{n+1} = (n+3)\mathrm{TH}_n$*;*
3. *if $n = 6k+2$:* $\frac{n+2}{2}\mathrm{TH}_{n+3} = 2\mathrm{TH}_{n+2} + \frac{n+4}{2}\mathrm{TH}_{n+1},\ \frac{n+1}{3}\mathrm{TH}_{n+2} = \frac{n+4}{3}\mathrm{TH}_{n+1},$ *and* $n\mathrm{TH}_{n+1} = (n+3)\mathrm{TH}_n$*;*
4. *if $n = 6k+3$:* $(n+2)\mathrm{TH}_{n+3} = (n+5)\mathrm{TH}_{n+2},\ \frac{n+1}{2}\mathrm{TH}_{n+2} = 2\mathrm{TH}_{n+1} + \frac{n+3}{2}\mathrm{TH}_n,$ *and* $\frac{n}{3}\mathrm{TH}_{n+1} = \frac{n+3}{3}\mathrm{TH}_n$*;*
5. *if $n = 6k+4$:* $(n+3)\mathrm{TH}_n = n\mathrm{TH}_{n+1},\ \frac{n+4}{2}\mathrm{TH}_{n+1} = \frac{n-1}{3}\mathrm{TH}_{n+2} + \frac{n+2}{6}\mathrm{TH}_{n+3},$ *and* $\frac{n+5}{3}\mathrm{TH}_{n+2} = \frac{n+2}{3}\mathrm{TH}_{n+3}$*;*

6. *if* $n = 6k+5$*:* $\frac{n+3}{2}\mathrm{TH}_n = \frac{n-2}{3}\mathrm{TH}_{n+1} + \frac{n+1}{6}\mathrm{TH}_{n+2}$, $\frac{n+4}{3}\mathrm{TH}_{n+1} = \frac{n+1}{3}\mathrm{TH}_{n+2}$, *and* $(n+5)\mathrm{TH}_{n+2} = (n+2)\mathrm{TH}_{n+3}$.

Corollary 4.7.

1. *If* $n = 6k$*, then*

$$\mathrm{Ap}(\mathcal{TH}_n, \mathrm{TH}_n) = \left\{ a\mathrm{TH}_{n+1} + b\mathrm{TH}_{n+2} + c\mathrm{TH}_{n+3} \,\middle|\, a \in \left\{0, \ldots, \frac{n-3}{3}\right\}, \right.$$
$$\left. b \in \{0, \ldots, n\},\ c \in \left\{0, \ldots, \frac{n}{2}\right\} \right\}.$$

2. *If* $n = 6k+1$*, then*

$$\mathrm{Ap}(\mathcal{TH}_n, \mathrm{TH}_n) = \left\{ a\mathrm{TH}_{n+1} + b\mathrm{TH}_{n+2} + c\mathrm{TH}_{n+3} \,\middle|\, a \in \{0, \ldots, n-1\}, \right.$$
$$\left. b \in \left\{0, \ldots, \frac{n-1}{2}\right\},\ c \in \left\{0, \ldots, \frac{n-1}{3}\right\} \right\}.$$

3. *If* $n = 6k+2$*, then*

$$\mathrm{Ap}(\mathcal{TH}_n, \mathrm{TH}_n) = \left\{ a\mathrm{TH}_{n+1} + b\mathrm{TH}_{n+2} + c\mathrm{TH}_{n+3} \,\middle|\, a \in \{0, \ldots, n-1\}, \right.$$
$$\left. b \in \left\{0, \ldots, \frac{n-2}{3}\right\},\ c \in \left\{0, \ldots, \frac{n}{2}\right\} \right\}.$$

4. *If* $n = 6k+3$*, then*

$$\mathrm{Ap}(\mathcal{TH}_n, \mathrm{TH}_n) = \left\{ a\mathrm{TH}_{n+1} + b\mathrm{TH}_{n+2} + c\mathrm{TH}_{n+3} \,\middle|\, a \in \left\{0, \ldots, \frac{n-3}{3}\right\}, \right.$$
$$\left. b \in \left\{0, \ldots, \frac{n-1}{2}\right\},\ c \in \{0, \ldots, n+1\} \right\}.$$

5. *If* $n = 6k+4$*, then*

$$\mathrm{Ap}(\mathcal{TH}_n, \mathrm{TH}_{n+3}) = \left\{ a\mathrm{TH}_n + b\mathrm{TH}_{n+1} + c\mathrm{TH}_{n+2} \,\middle|\, a \in \{0, \ldots, n+2\}, \right.$$
$$\left. b \in \left\{0, \ldots, \frac{n+2}{2}\right\},\ c \in \left\{0, \ldots, \frac{n+2}{3}\right\} \right\}.$$

6. *If $n = 6k+5$, then*

$$\mathrm{Ap}(\mathcal{TH}_n, \mathrm{TH}_{n+3}) = \left\{ a\mathrm{TH}_n + b\mathrm{TH}_{n+1} + c\mathrm{TH}_{n+2} \;\middle|\; a \in \left\{0, \ldots, \frac{n+1}{2}\right\}, \right.$$

$$\left. b \in \left\{0, \ldots, \frac{n+1}{3}\right\},\ c \in \{0, \ldots, n+4\} \right\}.$$

Corollary 4.8. *The Betti elements of $\mathcal{TH}_n$ are*

1. $2\binom{n+5}{4}$, $4\binom{n+4}{4}$ *and* $\frac{4}{3}\binom{n+3}{4}$, *if* $n = 6k$;
2. $\frac{4}{3}\binom{n+5}{4}$, $2\binom{n+4}{4}$ *and* $4\binom{n+3}{4}$, *if* $n = 6k+1$;
3. $2\binom{n+5}{4}$, $\frac{4}{3}\binom{n+4}{4}$ *and* $4\binom{n+3}{4}$, *if* $n = 6k+2$;
4. $4\binom{n+5}{4}$, $2\binom{n+4}{4}$ *and* $\frac{4}{3}\binom{n+3}{4}$, *if* $n = 6k+3$;
5. $\frac{4}{3}\binom{n+5}{4}$, $2\binom{n+4}{4}$ *and* $4\binom{n+3}{4}$, *if* $n = 6k+4$;
6. $4\binom{n+5}{4}$, $\frac{4}{3}\binom{n+4}{4}$ *and* $2\binom{n+3}{4}$, *if* $n = 6k+5$.

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