Maximal subsemigroups of finite transformation and diagram monoids

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Abstract

We describe and count the maximal subsemigroups of many well-known monoids of transformations and monoids of partitions. More precisely, we find the maximal subsemigroups of the full spectrum of monoids of order- or orientation-preserving transformations and partial permutations considered by V. H. Fernandes and co-authors (12 monoids in total); the partition, Brauer, Jones, and Motzkin monoids; and certain further monoids.

Although descriptions of the maximal subsemigroups of some of the aforementioned classes of monoids appear in the literature, we present a unified framework for determining these maximal subsemigroups. This approach is based on a specialised version of an algorithm for determining the maximal subsemigroups of any finite semigroup, developed by the third and fourth authors. This allows us to concisely present the descriptions of the maximal subsemigroups, and to more clearly see their common features.

Contents

1 Introduction, definitions, and summary of results

A proper subsemigroup of a semigroup S is *maximal* if it is not contained in any other proper subsemigroup of S. Similarly, a proper subgroup of a group G is *maximal* if it is not contained in any other proper subgroup of G. If G is a finite group, then every non-empty subsemigroup of G is a subgroup, and so these notions are not really distinct in this case. The same is not true if G is an infinite group. For instance, the natural numbers form a subsemigroup, but not a subgroup, of the integers under addition.

Maximal subgroups of finite groups have been extensively studied, in part because of their relationship to primitive permutation representations, and, for example, the Frattini subgroup. The maximal subgroups of the symmetric group are described, in some sense, by the O'Nan-Scott Theorem [43] and the Classification of Finite Simple Groups. Maximal subgroups of infinite groups have also been extensively investigated; see [3–5, 7, 9, 10, 35, 36, 39, 42] and the references therein.

There are also many papers in the literature relating to maximal subsemigroups of semigroups that are not groups. We describe the finite case in more detail below; for the infinite case see [17] and the references therein. Maximal subgroups of infinite groups, and maximal subsemigroups of infinite semigroups, are very different from their finite counterparts. For example, there exist infinite groups with no maximal subgroups at all, infinite groups with as many maximal subgroups as subsets, and subgroups that are not contained in any maximal subgroup. Analogous statements hold for semigroups also.

In [26], Graham, Graham, and Rhodes showed that every maximal subsemigroup of a finite semigroup has certain features, and that every maximal subsemigroup must be one of a small number of types. As is often the case for semigroups, this classification depends on the description of maximal subgroups of certain finite groups. In [16], Donoven, Mitchell, and Wilson describe an algorithm for calculating the maximal subsemigroups of an arbitrary finite semigroup, starting from the results in [26]. In the current paper, we use the framework provided by this algorithm to describe and count the maximal subsemigroups of several families of finite monoids of partial transformations and monoids of partitions. The maximal subsemigroups of several of these transformation monoids were described or counted in [11–14, 24, 28]. However, these results have been somewhat disparate, their proofs rather ad hoc, and the methods devised in each instance did not lead to a general theory. To our knowledge, except for those of the dual symmetric inverse monoid [37], the maximal subsemigroups of the monoids of partitions considered here have not been previously determined. We also calculate the maximal subsemigroups of several transformation monoids related to those in the literature, which were not previously known. We approach this problem in a concise and consistent way, which could be applied to many further semigroups.

This paper is structured as follows. In Section 1.1, we describe the notation and definitions relating to semigroups in general that are used in the paper. In Sections 1.2 and 1.3, we define the monoids of transformations and partitions whose maximal subsemigroups we classify. These are monoids of order and orientation preserving and reversing partial transformations; the partition, Brauer, Jones, and Motzkin monoids; and some related monoids. In Section 1.4, we summarise the results in this paper. In Section 2, we present several results about the maximal subsemigroups of an arbitrary finite monoid. Many of the results in Section 2 follow from [16], and provide a foundation that is adapted to the specific monoids under consideration in the later sections. In Sections 3 and 4, we classify the maximal subsemigroups of the monoids defined in Sections 1.2 and 1.3, respectively.

1.1 Background and preliminaries for arbitrary semigroups

A *semigroup* is a set with an associative binary operation. A *subsemigroup* of a semigroup is a subset that is also a semigroup under the same operation. A subsemigroup of S is *proper* if it does not equal S and it is *maximal* if it is a proper subsemigroup of S that is not contained in any other proper subsemigroup of S. A *monoid* is a semigroup S with an *identity* element 1, which has the property that $1s = s1 = s$ for all $s \in S$, and a *submonoid* of a monoid S is a subsemigroup that contains 1. For a subset X of a semigroup S, the *subsemigroup of S generated by X*, denoted by $\langle X \rangle$, is the least subsemigroup of S, with respect to containment, containing X. More generally, for a collection of subsets $X_1, \ldots, X_m$ of S and a collection of elements $x_1, \ldots, x_n$ in S, we use the notation $\langle X_1, \ldots, X_m, x_1, \ldots, x_n \rangle$, or some reordering of this, to denote the subsemigroup of S generated by $X_1 \cup \cdots \cup X_m \cup \{x_1, \ldots, x_n\}$. A *generating set* for S is a subset X of S such that $S = \langle X \rangle$.

Let S be a semigroup. A *left ideal* of S is a subset I of S such that $SI = \{sx : s \in S,\ x \in I\} \subseteq I$. A *right ideal* is defined analogously, and an *ideal* of S is a subset of S that is both a left ideal and a right ideal. Let $x, y \in S$ be arbitrary. The *principal left ideal generated by* x is the set $Sx \cup \{x\}$, which is a left ideal of S, whereas the *principal ideal generated by* x is the set $SxS \cup Sx \cup xS \cup \{x\}$, and is an ideal. We say that x and y are $\mathscr{L}$-related if the principal left ideals generated by x and y in S are equal. Clearly $\mathscr{L}$ defines an equivalence relation on S — called *Green's $\mathscr{L}$-relation* on S. We write $x \mathscr{L} y$ to denote that (x, y) belongs to $\mathscr{L}$. Green's $\mathscr{R}$-relation is defined dually to Green's $\mathscr{L}$-relation; Green's $\mathscr{H}$-relation is the meet, in the lattice of equivalence relations on S, of $\mathscr{L}$ and $\mathscr{R}$. Green's $\mathscr{D}$-relation is the composition

$\mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$, and if $x, y \in S$, then $x \mathscr{J} y$ whenever the (two-sided) principal ideals generated by x and y are equal. In a finite semigroup $\mathscr{D} = \mathscr{J}$. We will refer to the equivalence classes of Green's $\mathscr{K}$-relation, where $\mathscr{K} \in \{\mathscr{H}, \mathscr{L}, \mathscr{R}, \mathscr{D}, \mathscr{J}\}$, as $\mathscr{K}$-classes where $\mathscr{K}$ is any of $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, or $\mathscr{J}$, and the $\mathscr{K}$-class of $x \in S$ will be denoted by K_x. We write K_x^S if it is necessary to explicitly refer to the semigroup S on which the relation is defined. For a $\mathscr{J}$-class J of S and a Green's relation $\mathscr{K} \in \{\mathscr{H}, \mathscr{L}, \mathscr{R}\}$, we denote by $J/\mathscr{K}$ the set of $\mathscr{K}$-classes of S contained in J. A partial order on the $\mathscr{J}$-classes of S is induced by containment of the corresponding principal ideals; more precisely, for arbitrary elements $x, y \in S$, $J_x \leq J_y$ if and only if the principal ideal generated by y contains the principal ideal generated by x. A semigroup S is *$\mathscr{H}$-trivial* if Green's $\mathscr{H}$-relation is the equality relation on S.

An *idempotent* is a semigroup element x such that $x^2 = x$, and the collection of all idempotents in a semigroup S is denoted by $E(S)$. An $\mathscr{H}$-class of S that contains an idempotent is a subgroup of S [31, Corollary 2.2.6]. An element $x \in S$ is *regular* if there exists $y \in S$ such that $xyx = x$, and a semigroup is called *regular* if each of its elements is regular. A $\mathscr{D}$-class is *regular* if it contains a regular element; in this case, each of its elements is regular, and each of its $\mathscr{L}$-classes and $\mathscr{R}$-classes contains an idempotent [31, Propositions 2.3.1 and 2.3.2].

A semigroup S is a *regular $*$-semigroup* [40] if it possesses a unary operation * that satisfies $(x^*)^* = x$, $(xy)^* = y^*x^*$, and $x = xx^*x$ for all $x, y \in S$. Clearly a regular $*$-semigroup is regular. Throughout this paper, for a subset X of a regular $*$-semigroup, we use the notation X^* to denote $\{x^* : x \in X\}$. For a regular $*$-semigroup S and elements $x, y \in S$, $x\mathscr{R}y$ if and only if $x^*\mathscr{L}y^*$. An idempotent x of a regular $*$-semigroup is called a *projection* if $x^* = x$. The idempotent xx^* is the unique projection in the $\mathscr{R}$-class of x, and the idempotent x^*x is the unique projection in the $\mathscr{L}$-class of x. When H is a subgroup of a regular $*$-semigroup, for any $x \in H$, the element x^* coincides with the group-theoretic inverse of x in H.

An *inverse* semigroup is a semigroup S in which for each element $x \in S$, there is a unique element $x^{-1} \in S$, called the *inverse* of x in S, that satisfies $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$. With the operation * on S defined by $x^* = x^{-1}$, an inverse semigroup is a regular $*$-semigroup in which every idempotent is a projection. A *semilattice* is a commutative semigroup in which every element is an idempotent; any semilattice is an inverse semigroup.

An element x in a monoid S with identity 1 is a *unit* if there exists $x' \in S$ such that $xx' = x'x = 1$. The collection of units in a monoid is the $\mathscr{H}$-class of the identity, and is called the *group of units* of the monoid. In a finite monoid, the $\mathscr{H}$-class of the identity is also a $\mathscr{J}$-class; in this case, it is the unique maximal $\mathscr{J}$-class in the partial order of $\mathscr{J}$-classes of S.

We also require the following graph theoretic notions. A *graph* $\Gamma = (V, E)$ is a pair of sets V and E, called the *vertices* and the *edges* of Γ, respectively. An *edge* $e \in E$ is a pair $\{u, v\}$ of distinct vertices $u, v \in V$. A vertex u is *adjacent* to a vertex v in Γ if $\{u, v\}$ is an edge of Γ. The *degree* of a vertex v in Γ is the number of edges in Γ that contain v. An *independent subset* of Γ is a subset K of V such that there are no edges in E of the form $\{k, l\}$, where $k, l \in K$. A *maximal independent subset* of Γ is an independent subset that is contained in no other independent subset of Γ. A *bipartite graph* is a graph whose vertices can be partitioned into two maximal independent subsets.

In this paper, we define $\mathbb{N} = \{1, 2, 3, \ldots\}$.

1.2 Partial transformation monoids — definitions

In this section, we introduce the cast of partial transformation monoids whose maximal subsemigroups we determine.

Let $n \in \mathbb{N}$. A *partial transformation of degree n* is a partial map from $\{1, \ldots, n\}$ to itself. We define $\mathcal{PT}_n$, the *partial transformation monoid of degree n*, to be the monoid consisting of all partial transformations of degree n, under composition as binary relations. The identity element of this monoid is id_n, the *identity transformation* of degree n.

Let $\alpha \in \mathcal{PT}_n$. We define

$$\mathrm{dom}(\alpha) = \big\{i \in \{1, \ldots, n\} : i\alpha \text{ is defined}\big\}, \quad \mathrm{im}(\alpha) = \big\{i\alpha : i \in \mathrm{dom}(\alpha)\big\}, \quad \text{and } \mathrm{rank}(\alpha) = |\,\mathrm{im}(\alpha)|,$$

which are called the *domain*, *image*, and *rank* of α, respectively. We also define the *kernel* of $\alpha \in \mathcal{PT}_n$ to be the equivalence

$$\ker(\alpha) = \big\{(i, j) \in \mathrm{dom}(\alpha) \times \mathrm{dom}(\alpha) : i\alpha = j\alpha\big\}$$

on $\mathrm{dom}(\alpha)$. If $\mathrm{dom}(\alpha) = \{1, \ldots, n\}$, then α is called a *transformation*. If $\ker(\alpha)$ is the equality relation on $\mathrm{dom}(\alpha)$, i.e. if α is injective, then α is called a *partial permutation*. A *permutation* is a partial permutation $\alpha \in \mathcal{PT}_n$ such that $\mathrm{dom}(\alpha) = \{1, \ldots, n\}$. We define the following:

- $\mathcal{T}_n = \{\alpha \in \mathcal{PT}_n : \alpha \text{ is a transformation}\}$, the *full transformation monoid of degree n*;
- $\mathcal{I}_n = \{\alpha \in \mathcal{PT}_n : \alpha \text{ is a partial permutation}\}$, the *symmetric inverse monoid of degree n*; and
- $\mathcal{S}_n = \{\alpha \in \mathcal{PT}_n : \alpha \text{ is a permutation}\}$, the *symmetric group of degree n*.

The full transformation monoids and the symmetric inverse monoids play a role analogous to that of the symmetric group, in that every semigroup is isomorphic to a subsemigroup of some full transformation monoid [31, Theorem 1.1.2], and every inverse semigroup is isomorphic to an inverse subsemigroup of some symmetric inverse monoid [31, Theorem 5.1.7].

Let α be a partial transformation of degree n. Then $\mathrm{dom}(\alpha) = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, for some $i_1 < \cdots < i_k$. Throughout this paper, where we refer to an ordering of natural numbers, we mean the usual ordering $1 < 2 < 3 < \ldots$. We say that α is *order-preserving* if $i_1\alpha \leq \cdots \leq i_k\alpha$, and *order-reversing* if $i_1\alpha \geq \cdots \geq i_k\alpha$. We say that α is *orientation-preserving* if there exists at most one value l, where $1 \leq l \leq k-1$, such that $i_l\alpha > i_{l+1}\alpha$, and similarly, we say that α is *orientation-reversing* if there exists at most one value $1 \leq l \leq k-1$ such that $i_l\alpha < i_{l+1}\alpha$. Note that an order-preserving partial transformation is orientation-preserving, and an order-reversing partial transformation is orientation-reversing. Having defined these notions, we can introduce the twelve monoids of partial transformations that we consider here. These monoids have been extensively studied, see [12,14] and the references therein, where the notation used in this paper originates.

We define the following submonoids of $\mathcal{PT}_n$:

- $\mathcal{PO}_n = \{\alpha \in \mathcal{PT}_n : \alpha \text{ is order-preserving}\}$,
- $\mathcal{POD}_n = \{\alpha \in \mathcal{PT}_n : \alpha \text{ is order-preserving or order-reversing}\}$,
- $\mathcal{POP}_n = \{\alpha \in \mathcal{PT}_n : \alpha \text{ is orientation-preserving}\}$, and
- $\mathcal{POR}_n = \{\alpha \in \mathcal{PT}_n : \alpha \text{ is orientation-preserving or orientation-reversing}\}$.

We also define the following submonoids of $\mathcal{T}_n$ as the intersections:

- $\mathcal{O}_n = \mathcal{PO}_n \cap \mathcal{T}_n, \quad \mathcal{OD}_n = \mathcal{POD}_n \cap \mathcal{T}_n, \quad \mathcal{OP}_n = \mathcal{POP}_n \cap \mathcal{T}_n, \quad \text{and} \quad \mathcal{OR}_n = \mathcal{POR}_n \cap \mathcal{T}_n;$

and we define the following inverse submonoids of $\mathcal{I}_n$ as the intersections:

- $\mathcal{POI}_n = \mathcal{PO}_n \cap \mathcal{I}_n, \quad \mathcal{PODI}_n = \mathcal{POD}_n \cap \mathcal{I}_n, \quad \mathcal{POPI}_n = \mathcal{POP}_n \cap \mathcal{I}_n, \quad \text{and} \quad \mathcal{PORI}_n = \mathcal{POR}_n \cap \mathcal{I}_n.$

We require the groups of units of these monoids. We define γ_n to be the permutation of degree n that reverses the usual order of $\{1, \ldots, n\}$, i.e. $i\gamma_n = n - i + 1$ for all $i \in \{1, \ldots, n\}$. Thus when $n \geq 2$, the group $\langle \gamma_n \rangle$ has order 2. We define $\mathcal{C}_n$ to be the cyclic group generated by the permutation $(1\ 2\ \ldots\ n)$, and $\mathcal{D}_n = \langle (1\ 2\ \ldots\ n),\ \gamma_n \rangle$. When $n \geq 3$, $\mathcal{D}_n$ is a dihedral group of order $2n$. Note that $\mathcal{C}_2 = \mathcal{D}_2 = \langle \gamma_2 \rangle$.

The groups of units of $\mathcal{PO}_n$, $\mathcal{O}_n$, and $\mathcal{POI}_n$ are trivial; the groups of units of $\mathcal{POD}_n$, $\mathcal{OD}_n$, and $\mathcal{PODI}_n$ are $\langle \gamma_n \rangle$; the groups of units of $\mathcal{POP}_n$, $\mathcal{OP}_n$, and $\mathcal{POPI}_n$ are $\mathcal{C}_n$; and the groups of units of $\mathcal{POR}_n$, $\mathcal{OR}_n$, and $\mathcal{PORI}_n$ are $\mathcal{D}_n$. Finally, the symmetric group $\mathcal{S}_n$ is the group of units of $\mathcal{PT}_n$, $\mathcal{T}_n$, and $\mathcal{I}_n$.

See Figure 1 for an illustration of how these submonoids of $\mathcal{PT}_n$ are interrelated by containment.

1.3 Diagram monoids — definitions

In this section, we define those monoids of partitions whose maximal subsemigroups we determine.

Let $n \in \mathbb{N}$ be arbitrary. A *partition of degree n* is an equivalence relation of the set $\{1, \ldots, n\} \cup \{1', \ldots, n'\}$. An equivalence class of a partition is called a *block*, and a block is *transverse* if it contains points from both $\{1, \ldots, n\}$ and $\{1', \ldots, n'\}$. A *block bijection* is a partition all of whose blocks are transverse, and a block bijection is *uniform* if each of its blocks contains an equal number of points of $\{1, \ldots, n\}$ and $\{1', \ldots, n'\}$. The *rank* of a partition is the number of transverse blocks that it contains. A partition may be represented visually; see [29], for instance.

Let α and β be partitions of degree n. To calculate the product $\alpha\beta$, we require three auxiliary partitions, each being a partition of a different set. From α we create α' by replacing every occurrence of every i' by i'' in α, so that α' is a partition of $\{1, \ldots, n\} \cup \{1'', \ldots, n''\}$. Similarly, by replacing i by i'', we obtain a partition β' of $\{1'', \ldots, n''\} \cup \{1', \ldots, n'\}$ from β. We define $(\alpha\beta)'$ to be the smallest equivalence on $\{1, \ldots, n\} \cup \{1', \ldots, n'\} \cup \{1'', \ldots, n''\}$ that contains the relation $\alpha' \cup \beta'$, which is the transitive closure of $\alpha' \cup \beta'$. The product $\alpha\beta$ is the intersection of $(\alpha\beta)'$ and $(\{1, \ldots, n\} \cup \{1', \ldots, n'\}) \times (\{1, \ldots, n\} \cup \{1', \ldots, n'\})$. This operation is associative, and so the collection $\mathcal{P}_n$ of all partitions of degree n forms a semigroup under this operation. The partition id_n, whose blocks are $\{i, i'\}$ for all $i \in \{1, \ldots, n\}$, is the identity element of this semigroup and so $\mathcal{P}_n$ is a monoid — called the *partition monoid of degree n*. A *diagram monoid* is simply a submonoid of $\mathcal{P}_n$ for some $n \in \mathbb{N}$.

Let α be a partition of degree n. We define α^* to be the partition of $\{1, \ldots, n\} \cup \{1', \ldots, n'\}$ created from α by replacing the point i by i' in the block in which it appears, and by replacing the point i' by i, for all $i \in \{1, \ldots, n\}$. For arbitrary partitions $\alpha, \beta \in \mathcal{P}_n$, $(\alpha^*)^* = \alpha$, $\alpha\alpha^*\alpha = \alpha$, and $(\alpha\beta)^* = \beta^*\alpha^*$. In particular, $\mathcal{P}_n$ is a regular $*$-monoid, as defined in Section 1.1.

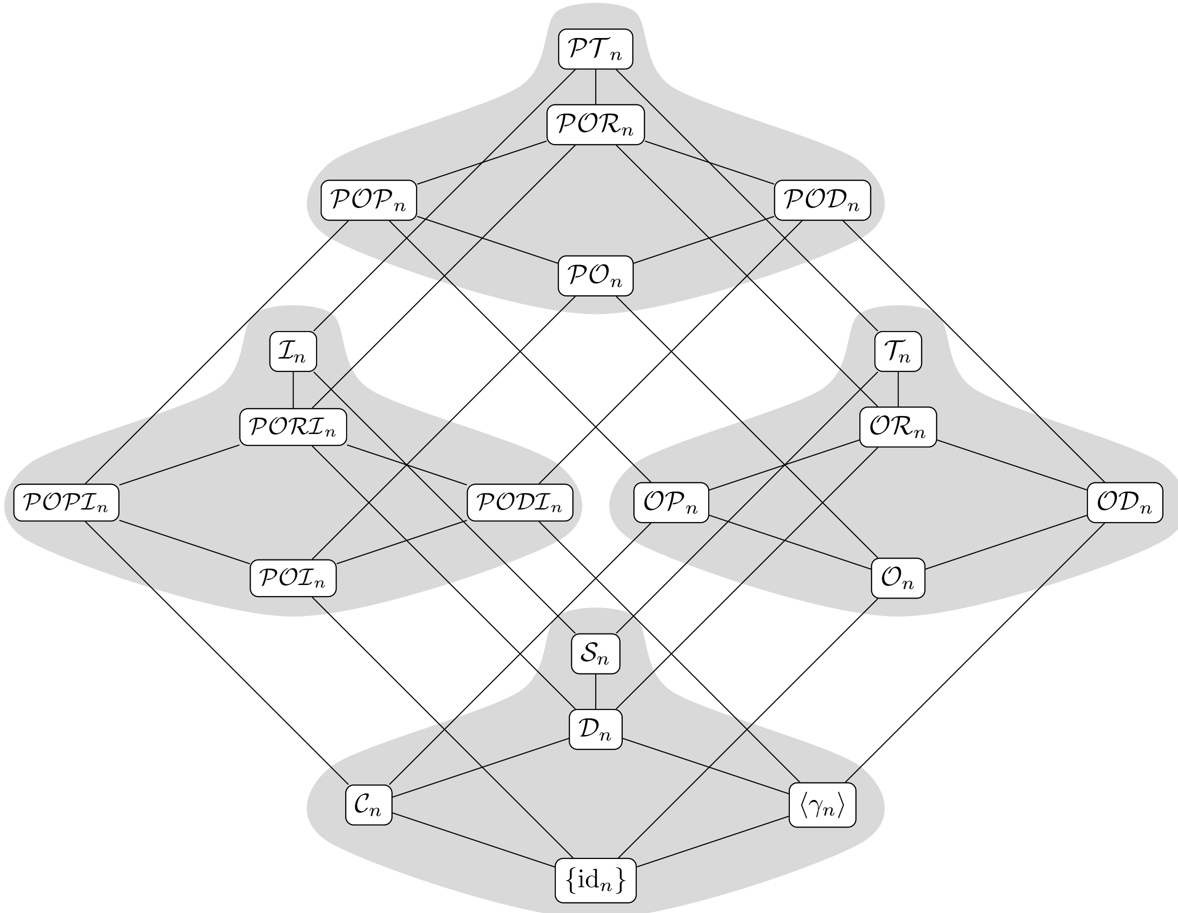

Figure 1: Part of the subsemigroup lattice of $\mathcal{PT}_n$. This diagram shows the monoids defined in Section 1.2, and their groups of units. For a monoid in the top/middle/left/right/bottom position of a shaded region, its group of units is shown in the corresponding place in the bottom shaded region.

There is a canonical embedding of the symmetric group of degree n in $\mathcal{P}_n$, where a permutation α is mapped to the partition with blocks $\{i, (i\alpha)'\}$ for all $i \in \{1, \ldots, n\}$. Since an element of $\mathcal{P}_n$ is a unit if and only if each of its blocks has the form $\{i, j'\}$ for some $i, j \in \{1, \ldots, n\}$, it follows that the image of this embedding is the group of units of $\mathcal{P}_n$. We reuse the notation $\mathcal{S}_n$ to refer to this group. We define ρ_n to be the partition whose blocks are $\{n, 1'\}$ and $\{i, (i+1)'\}$ for $i \in \{1, \ldots, n-1\}$; in the context of this embedding, ρ_n can be thought of as the n-cycle $(1\ 2\ \ldots\ n)$. Thus the subsemigroup generated by ρ_n is a cyclic group of order n. We define a canonical ordering

$$n' < (n-1)' < \cdots < 1' < 1 < 2 < \cdots < n$$

on $\{1, \ldots, n\} \cup \{1', \ldots, n'\}$. We say that $\alpha \in \mathcal{P}_n$ is *planar* if there do not exist distinct blocks A and X of α, and points $a, b \in A$ and $x, y \in X$, such that $a < x < b < y$. More generally, α is *annular* if $\alpha = \rho_n^k \beta \rho_n^l$ for some planar partition $\beta \in \mathcal{P}_n$ and for some $k, l \in \mathbb{Z}$ (note that $\rho_n^n = \mathrm{id}_n$). For a graphical description of these properties, see [2,29].

In Section 4, we determine the maximal subsemigroups of $\mathcal{P}_n$ and the following submonoids:

- $\mathcal{PB}_n = \{\alpha \in \mathcal{P}_n : \text{each block of } \alpha \text{ has size at most } 2\}$, the *partial Brauer monoid of degree n*, introduced in [38];
- $\mathcal{B}_n = \{\alpha \in \mathcal{P}_n : \text{each block of } \alpha \text{ has size exactly } 2\}$, the *Brauer monoid of degree n*, introduced in [38];
- $\mathcal{I}_n^* = \{\alpha \in \mathcal{P}_n : \alpha \text{ is a block bijection}\}$, the *dual symmetric inverse monoid of degree n*, introduced in [23];
- $\mathfrak{F}_n = \{\alpha \in \mathcal{P}_n : \alpha \text{ is a uniform block bijection}\}$, the *uniform block bijection monoid of degree n*, also known as the *factorisable dual symmetric inverse monoid of degree n*, see [22] for more details;
- $\mathcal{PP}_n = \{\alpha \in \mathcal{P}_n : \alpha \text{ is planar}\}$, the *planar partition monoid of degree n*, introduced in [29];

- $\mathcal{M}_n = \{\alpha \in \mathcal{PB}_n : \alpha \text{ is planar}\}$, the *Motzkin monoid of degree* n, see [6] for more details;
- $\mathcal{J}_n = \{\alpha \in \mathcal{B}_n : \alpha \text{ is planar}\}$, the *Jones monoid of degree* n, also known as the *Temperley-Lieb monoid*, introduced in [32]; and
- $\mathcal{AJ}_n = \{\alpha \in \mathcal{B}_n : \alpha \text{ is annular}\}$, the *annular Jones monoid of degree* n, introduced in [2].

By [29], the planar partition monoid of degree n is isomorphic to the Jones monoid of degree $2n$. Therefore, we will not determine the maximal subsemigroups of $\mathcal{PP}_n$ directly, since their description can be obtained from the results in Section 4.5.

The group of units of $\mathcal{PB}_n$, $\mathcal{B}_n$, $\mathcal{I}_n^*$, and $\mathfrak{F}_n$ is $\mathcal{S}_n$, the group of units of $\mathcal{M}_n$ and $\mathcal{J}_n$ is the trivial group $\{\mathrm{id}_n\}$, and the group of units of $\mathcal{AJ}_n$ is the cyclic group $\mathcal{C}_n = \langle \rho_n \rangle$.

1.4 Summary of results

A summary of the results of this paper is shown in Table 1; a description of the maximal subsemigroups that we count in Table 1 can be found in the referenced results.

Monoid	Group of units	Number of maximal subsemigroups	OEIS [44]	Result	
$\mathcal{POI}_n$	Trivial	$2^n - 1$	A000225	Theorem 3.9	cf. [24, Theorem 2]
$\mathcal{PO}_n$		$2^n + 2n - 2$	A131520	Theorem 3.2	cf. [14, Theorem 1]
$\mathcal{M}_n$		$2^n + 2n - 3$	A131898	Theorem 4.10	—
$\mathcal{O}_n$		$A_{2n-1} + 2n - 4$	A000931	Theorem 3.7	cf. [11, Theorem 2]
$\mathcal{J}_n$		$2F_{n-1} + 2n - 3$	A290140	Theorem 4.8	—
$\mathcal{PP}_n$		$2F_{2n-1} + 4n - 3$	A290140	Theorem 4.8	—
$\mathcal{PODI}_{2n}$	Order 2	$3 \cdot 2^{n-1} - 1$	A052955	Theorem 3.10	cf. [13, Theorem 4]
$\mathcal{PODI}_{2n-1}$		$2^n - 1$	A052955	Theorem 3.10	cf. [13, Theorem 4]
$\mathcal{POD}_n$		$2^{\lceil n/2 \rceil} + n - 1$	A016116	Theorem 3.3	—
$\mathcal{OD}_n$		$A_n + n - 3$	A000931	Theorem 3.8	cf. [28, Theorem 2]
$\mathcal{POPI}_n$	$\mathcal{C}_n$ (cyclic)	$\lvert\mathbb{P}_n\rvert + \lvert\mathbb{P}_{n-1}\rvert$	A059957	Theorem 3.15	—
$\mathcal{POP}_n$		$\lvert\mathbb{P}_n\rvert + 2$	A083399	Theorem 3.11	—
$\mathcal{AJ}_n$		$\lvert\mathbb{P}_n\rvert + 1$	A083399	Theorem 4.9	—
$\mathcal{OP}_n$		$\lvert\mathbb{P}_n\rvert + 1$	A083399	Theorem 3.13	cf. [12, Theorem 1.6]
$\mathcal{PORI}_n$	$\mathcal{D}_n$ (dihedral)	$1 + \lvert\mathbb{P}_{n-1}\rvert + \sum_{p \in \mathbb{P}_n} p$	A290289	Theorem 3.16	—
$\mathcal{POR}_n$		$3 + \sum_{p \in \mathbb{P}_n} p$	A008472	Theorem 3.12	—
$\mathcal{OR}_n$		$2 + \sum_{p \in \mathbb{P}_n} p$	A008472	Theorem 3.14	cf. [12, Theorem 2.6]
$\mathcal{T}_n$	$\mathcal{S}_n$ (symmetric)	$s_n + 1$	A290138	Theorem 3.1	—
$\mathcal{I}_n$		$s_n + 1$	A290138	Theorem 3.1	—
$\mathcal{I}_n^*$		$s_n + 1$	A290138	Theorem 4.6	cf. [37, Theorem 19]
$\mathfrak{F}_n^*$		$s_n + 1$	A290138	Theorem 4.5	—
$\mathcal{B}_n$		$s_n + 1$	A290138	Theorem 4.4	—
$\mathcal{PT}_n$		$s_n + 2$	A290138	Theorem 3.1	—
$\mathcal{PB}_n$		$s_n + 3$	A290138	Theorem 4.3	—
$\mathcal{P}_n$		$s_n + 4$	A290138	Theorem 4.2	—

Table 1: The maximal subsemigroups of the monoids from this paper, where n is sufficiently large (usually $n \geq 2$ or $n \geq 3$). For $k \in \mathbb{N}$, s_k is the number of maximal subgroups of the symmetric group of degree k [33]; $\mathbb{P}_k$ is the set of primes that divide k; A_k is the k^{th} term of the sequence defined by $A_1 = 1$, $A_2 = A_3 = 2$, and $A_k = A_{k-2} + A_{k-3}$ for $k \geq 4$, i.e. the $(k+6)^{\text{th}}$ term of the Padovan sequence [44, A000931]; and F_k is the k^{th} term of the Fibonacci sequence [44, A000045], defined by $F_1 = F_2 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for $k \geq 3$.

2 The maximal subsemigroups of an arbitrary finite monoid

In this section, we present some results about the maximal subsemigroups of an arbitrary finite monoid, which are related to those given in [16, 26] for an arbitrary finite semigroup. Since each of the semigroups to which we apply these results is a monoid, we state the following results in the context of finite monoids. While some of the results given in this section hold for an arbitrary finite semigroup, many of them they do not.

Let S be a finite monoid. By [26, Proposition 1], for each maximal subsemigroup M of S there exists a single $\mathscr{J}$-class J that contains $S \setminus M$, or equivalently $S \setminus J \subseteq M$. Throughout this paper, we call a maximal subsemigroup whose complement is contained in a $\mathscr{J}$-class J a *maximal subsemigroup arising from* J.

We consider the question of which $\mathscr{J}$-classes of S give rise to maximal subsemigroups. Let J be a $\mathscr{J}$-class of S. There exist maximal subsemigroups arising from J if and only if every generating set for S intersects J non-trivially. To see the truth of this statement, let M be a maximal subsemigroup of S arising from J, so that $S \setminus J \subseteq M$. For any subset A of S that is disjoint from J, it follows that

$$\langle A \rangle \leq \langle S \setminus J \rangle \leq M \neq S,$$

and A does not generate S. Conversely, if J intersects every generating set for S non-trivially, then certainly $S \setminus J$ does not generate S. Thus the subsemigroup $\langle S \setminus J \rangle$ of S is proper, and is contained in a maximal subsemigroup.

In order to calculate the maximal subsemigroups of S, we identify those $\mathscr{J}$-classes of S that intersect every generating set of S non-trivially. Then, for each of these $\mathscr{J}$-classes J, we find the maximal subsemigroups of S arising from J.

Let S be a finite monoid, let J be a regular $\mathscr{J}$-class of S, and let M be a maximal subsemigroup of S arising from J. By [16, Section 3], the intersection $M \cap J$ has one of the following mutually-exclusive forms:

(M1) $M \cap J = \varnothing$.

(M2) $M \cap J$ is a non-empty union of both $\mathscr{L}$- and $\mathscr{R}$-classes of J;

(M3) $M \cap J$ is a non-empty union of $\mathscr{L}$-classes of J;

(M4) $M \cap J$ is a non-empty union of $\mathscr{R}$-classes of J;

(M5) $M \cap J$ has non-empty intersection with every $\mathscr{H}$-class of J;

In general, the collection of maximal subsemigroups arising from a particular regular $\mathscr{J}$-class J can have any combination of types (M2), (M3), (M4), and (M5). However, if $S \setminus J$ is a maximal subsemigroup of S, then it is the only maximal subsemigroup to arise from J. In other words, there is at most one maximal subsemigroup of type (M1) arising from J, and its existence precludes the occurrence of maximal subsemigroups of types (M2)–(M5).

It can be most difficult to calculate the maximal subsemigroups of S that arise from J and have type (M5) — we consider a special case in Section 2.2.2, which covers the majority of instances in this paper. However, in many cases it can be easily shown that no maximal subsemigroups of type (M5) exist, such as when S is $\mathscr{H}$-trivial, or when S is idempotent generated. More generally, since a maximal subsemigroup of S of type (M5) contains $E(S)$, the following lemma holds.

Lemma 2.1. *Let S be a finite monoid with group of units G, and let J be a $\mathscr{J}$-class of S that is not equal to G. If*

(a) *each $\mathscr{H}$-class of J is trivial, or*

(b) $J \subseteq \langle G,\ E(S) \rangle$,

then there are no maximal subsemigroups of type (M5) arising from J.

When S is a finite regular $*$-semigroup and J is a $\mathscr{J}$-class of S, it is routine to verify that the * operation permutes the maximal subsemigroups of S that arise from J. In other words, for a subset M of S, M is a maximal subsemigroup of S arising from J if and only if M^* is a maximal subsemigroup of S arising from J. If M is a maximal subsemigroup of type (M1), (M2), or (M5), then M^* is a maximal subsemigroup of the same type. However, since the * operation of a regular $*$-semigroup transposes $\mathscr{L}$-classes and $\mathscr{R}$-classes, M is a maximal subsemigroup of type (M3) if and only if M^* is a maximal subsemigroup of type (M4). In particular, the maximal subsemigroups of type (M3) can be deduced from those of type (M4), and vice versa. These ideas are summarised in the following lemma.

Lemma 2.2. *Let S be a finite regular $*$-semigroup, let M be a subset of S, and let J be a $\mathscr{J}$-class of S. Then M is a maximal subsemigroup of S arising from J if and only if M^* is a maximal subsemigroup of S arising from J. In particular, M is a maximal subsemigroup of S arising from J of type (M3) if and only M^* is a maximal subsemigroup of S arising from J of type (M4).*

2.1 Maximal subsemigroups arising from the group of units

Let S be a finite monoid with group of units G. Since the subset of non-invertible elements of a monoid is an ideal, the $\mathscr{J}$-class G intersects every generating set for S non-trivially, and so there exist maximal subsemigroups of S arising from G. Note that G is the unique such $\mathscr{J}$-class only when S is a group. Another consequence of the fact that the non-invertible elements of S form an ideal is that the maximal subsemigroups arising from the group of units can be calculated from G in isolation. As shown in the following lemma, they are straightforward to describe in terms of the maximal subsemigroups of G.

Lemma 2.3. *Let S be a finite monoid with group of units G. Then the maximal subsemigroups of S arising from G are the sets $(S \setminus G) \cup U$, for each maximal subsemigroup U of G.*

A *subsemigroup* of a finite group is a *subgroup*, unless it is empty; the only group to possess the empty semigroup as a maximal subsemigroup is the trivial group. Thus, when the group of units of a particular monoid is known to be non-trivial, we may use the term "maximal subgroup" in place of "maximal subsemigroup", as appropriate. This observation permits the following corollary.

Corollary 2.4. *Let S be a finite monoid with group of units G. If G is trivial, then the unique maximal subsemigroup of S arising from G is $S \setminus G$, which has type (M1). If G is non-trivial, then the maximal subsemigroups of S arising from G are the sets $(S \setminus G) \cup U$, for each maximal subgroup U of G, which have type (M5).*

Only three families of non-trivial groups appear as the group of units of a monoid in this paper: the cyclic groups, the dihedral groups, and the symmetric groups. The conjugacy classes of maximal subgroups of the finite symmetric groups are described in [33] and counted in [34]; see [44, A066115]. However, no simple formula is known for the total number of maximal subgroups. Thus we use the notation s_k to denote the number of maximal subsemigroups of the symmetric group of degree k. For the maximal subgroups of the cyclic and dihedral groups, we present the following well-known results.

Lemma 2.5. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $G = \langle \alpha \mid \alpha^n \rangle$ be a cyclic group of order n. The maximal subgroups of G are the subgroups $\langle \alpha^p \rangle$, for each prime divisor p of n. In particular, the total number of maximal subgroups is the number of prime divisors of n.*

Lemma 2.6. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $G = \langle \sigma,\ \rho \mid \sigma^2,\ \rho^n,\ (\sigma\rho)^2 \rangle$ be a dihedral group of order $2n$. The maximal subgroups of G are $\langle \rho \rangle$ and the subgroups $\langle \rho^p,\ \rho^{-i}\sigma\rho^i \rangle$, for each prime divisor p of n and for each integer i with $0 \leq i \leq p-1$. In particular, the total number of maximal subgroups is one more than the sum of the prime divisors of n.*

2.2 Maximal subsemigroups arising from a regular $\mathscr{J}$-class covered by the group of units

Let S be a finite monoid with group of units G, and let J be a $\mathscr{J}$-class of S that is not equal to G. The maximal subsemigroups that arise from J are, in general, more complicated to describe than those maximal subsemigroups that arise from G. This is because the elements of S contained in $\mathscr{J}$-classes that are above J (in the $\mathscr{J}$-class partial order) may act on, or generate, elements within J. Therefore it is not possible to calculate the maximal subsemigroups that arise from J without considering these other $\mathscr{J}$-classes.

Certainly G is a $\mathscr{J}$-class of S that is strictly above J, since it is the unique maximal $\mathscr{J}$-class of S. When the group of units is the *only* $\mathscr{J}$-class strictly above J, the problem of finding the maximal subsemigroups that arise from J is simpler than the general case. We say that such a $\mathscr{J}$-class is *covered* by the group of units.

Suppose that J is a $\mathscr{J}$-class of S that is covered by G. Since the elements contained in $\mathscr{J}$-classes above J are units, their action on J is easier than understand than the action of arbitrary semigroup elements. Moreover, since the only $\mathscr{J}$-class above J is closed under multiplication, it follows that $S \setminus J$ is a subsemigroup of S. In particular, any generating set for S intersects J non-trivially, and there exist maximal subsemigroups that arise from J.

We summarise this discussion in the following proposition.

Proposition 2.7 (Maximal subsemigroups of type (M1))**.** *Let S be a finite monoid and let J be a $\mathscr{J}$-class of S that is covered by the group of units of S. Then $S \setminus J$ is a maximal subsemigroup of S if and only if no maximal subsemigroups of types (M2)–(M5) arise from J.*

For the majority of the monoids considered in this paper, the *only* $\mathscr{J}$-classes that give rise to maximal subsemigroups are the group of units, and regular $\mathscr{J}$-classes that are covered by the group of units.

By [26, Proposition 2], a maximal subsemigroup either:

- is a union of $\mathscr{H}$-classes of the semigroup, or

- intersects every $\mathscr{H}$-class of the semigroup non-trivially.

A maximal subsemigroup of type (M1), (M2), (M3), or (M4) is a union of $\mathscr{H}$-classes, whereas maximal subsemigroups of the second kind are those of type (M5). We consider these cases separately. We present versions of some results from [16] that are simplified to suit the current context, and that are used for calculating maximal subsemigroups of types (M2), (M3), and (M4), in Section 2.2.1. Only a few of the monoids considered in this paper exhibit maximal subsemigroups of type (M5), and we present results tailored to some of these monoids in Section 2.2.2.

2.2.1 Maximal subsemigroups that are unions of $\mathscr{H}$-classes: types (M2), (M3), and (M4)

Let S be a finite monoid and let J be a regular $\mathscr{J}$-class of S that is covered by the group of units G of S. To find the maximal subsemigroups of S arising from J that have types (M2)–(M4), we construct from J a bipartite graph $\Delta(S, J)$, and analyse its properties according to the forthcoming results. This bipartite graph was introduced by Donoven, Mitchell, and Wilson in [16, Section 3]. When the context unambiguously identifies the $\mathscr{J}$-class that is under consideration, i.e. when a monoid possesses only one $\mathscr{J}$-class covered by the group of units, we will often use the shorter notation $\Delta(S)$ in place of $\Delta(S, J)$.

By Green's Lemma [31, Lemmas 2.2.1 and 2.2.2], the group of units G of S acts on the $\mathscr{L}$-classes of J by right multiplication, and on the $\mathscr{R}$-classes of J by left multiplication. The vertices of $\Delta(S, J)$ are the orbits of $\mathscr{L}$-classes of J and the orbits of $\mathscr{R}$-classes of J, under these actions. In the special case that J consists of a single $\mathscr{H}$-class, we differentiate between the orbit of $\mathscr{L}$-classes $\{J\}$ and the orbit of $\mathscr{R}$-classes $\{J\}$, so that $\Delta(S, J)$ contains two vertices. There is an edge in $\Delta(S, J)$ between an orbit of $\mathscr{L}$-classes A and an orbit of $\mathscr{R}$-classes B if and only if there exists an $\mathscr{L}$-class $L \in A$ and an $\mathscr{R}$-class $R \in B$ such that the $\mathscr{H}$-class $L \cap R$ is a group. We define the two *bicomponents* of $\Delta(S, J)$ as follows: one bicomponent is the collection of all orbits of $\mathscr{L}$-classes of J, the other bicomponent is the collection of all orbits of $\mathscr{R}$-classes of J; the bicomponents of $\Delta(S, J)$ partition its vertices into two maximal independent subsets. Note that $\Delta(S, J)$ is isomorphic to a quotient of the Graham-Houghton graph of the principal factor of J, as defined in [19, 27, 30] — in the case that the orbits of $\mathscr{L}$- and $\mathscr{R}$-classes are trivial, these graphs are isomorphic.

The following results characterize the maximal subsemigroups of S of types (M2)–(M4) that arise from J in terms of the graph $\Delta(S, J)$. These lemmas follow from the results of [16, Section 3], having been simplified according to the assumption that J is covered by the group of units of S. More specifically, the results of [16, Section 3] are formulated in terms of two graphs Δ and Θ, and two coloured digraphs $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$, that are constructed from the relevant $\mathscr{J}$-class. When the semigroup in question is a monoid and the $\mathscr{J}$-class is covered by the group of units, the graph Θ and the digraphs $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$ have no edges, and each vertex of $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$ has colour 0. Thus the conditions on Θ, $\Gamma_{\mathscr{L}}$, and $\Gamma_{\mathscr{R}}$ are immediately satisfied. The graph Δ in [16, Section 3] is equivalent to $\Delta(S, J)$.

Proposition 2.8 (Maximal subsemigroups of type (M2); cf. [16, Corollary 3.13])**.** *Let T be a subset of S such that $S \setminus T \subseteq J$. Then T is a maximal subsemigroup of S of type (M2) if and only if there exist proper non-empty subsets $A \subsetneq J/\mathscr{L}$ and $B \subsetneq J/\mathscr{R}$ such that $T \cap J$ is the union of the $\mathscr{L}$-classes in A and the $\mathscr{R}$-classes in B, and A and B are unions of vertices that together form a maximal independent subset of $\Delta(S, J)$.*

By Proposition 2.8, the maximal subsemigroups of S of type (M2) arising from J are in bijective correspondence with the maximal independent subsets of $\Delta(S, J)$ — excluding the bicomponents of $\Delta(S, J)$. Thus we deduce the following corollary.

Corollary 2.9. *The number of maximal subsemigroups of S of type (M2) arising from J is two less than the number of maximal independent subsets of $\Delta(S, J)$.*

The connection between the graph $\Delta(S, J)$ and the maximal subsemigroups of S of types (M3) and (M4) that arise from J is given in the following propositions.

Proposition 2.10 (Maximal subsemigroups of type (M3); cf. [16, Corollary 3.15])**.** *Let T be a subset of S such that $S \setminus T \subseteq J$. Then T is a maximal subsemigroup of S of type (M3) if and only if there exists a proper non-empty subset $A \subsetneq J/\mathscr{L}$ such that $T \cap J$ is the union of the $\mathscr{L}$-classes in A, and $(J/\mathscr{L}) \setminus A$ is a vertex in $\Delta(S, J)$ that is not adjacent to a vertex of degree 1.*

Proposition 2.11 (Maximal subsemigroups of type (M4))**.** *Let T be a subset of S such that $S \setminus T \subseteq J$. Then T is a maximal subsemigroup of S of type (M4) if and only if there exists a proper non-empty subset $B \subsetneq J/\mathscr{R}$ such that $T \cap J$ is the union of the $\mathscr{R}$-classes in B, and $(J/\mathscr{R}) \setminus B$ is a vertex in $\Delta(S, J)$ that is not adjacent to a vertex of degree 1.*

By Proposition 2.10, the number of maximal subsemigroups of S of type (M3) is the number of orbits of $\mathscr{L}$-classes that are adjacent in $\Delta(S, J)$ only to orbits of $\mathscr{R}$-classes with degree at least 2. In the case that every orbit of $\mathscr{R}$-classes has degree 2 or more in $\Delta(S,J)$, then the number of maximal subsemigroups of type (M3) is simply the number of orbits of $\mathscr{L}$-classes. By Proposition 2.11, the analogous statements hold for maximal subsemigroups of type (M4).

On the other hand, the number of maximal subsemigroups is restricted when the group of units acts transitively.

Lemma 2.12. *If G acts transitively on the $\mathscr{L}$-classes of J, then no maximal subsemigroups of types (M2) or (M3) arise from J. Similarly, if G acts transitively on the $\mathscr{R}$-classes of J, then no maximal subsemigroups of types (M2) or (M4) arise from J.*

Proof. Suppose that the group of units acts transitively on the $\mathscr{L}$-classes of J, so that the graph $\Delta(S,J)$ has a single vertex of $\mathscr{L}$-classes. Since J is regular, there are no isolated vertices in $\Delta(S,J)$. Therefore each vertex of $\mathscr{R}$-classes has degree 1, and is adjacent to the unique vertex of $\mathscr{L}$-classes. Thus $\Delta(S,J)$ has just two maximal independent subsets — its bicomponents. By Corollary 2.9, it follows that there are no maximal subsemigroups of type (M2) arising from J, and by Proposition 2.10, no maximal subsemigroups of type (M3) arise from J either. The proof of the second statement is dual. ∎

When S is a regular $*$-monoid, the graph $\Delta(S,J)$ is particularly easy to describe. Suppose there exist $\mathscr{L}$-classes L_x and L_y in J, and a unit $g \in G$, such that $L_x g = L_y$. Then

$$g^* R_{x^*} = g^* L_x^* = (L_x g)^* = L_y^* = R_{y^*}.$$

In this way, it is easy to see that the orbits of $\mathscr{L}$-classes of J are in bijective correspondence with the orbits of $\mathscr{R}$-classes of J. Specifically, the set $\{L_{x_1},\ldots,L_{x_n}\}$ is an orbit of $\mathscr{L}$-classes of J if and only if $\{L_{x_1}^*,\ldots,L_{x_n}^*\} = \{R_{x_1^*},\ldots,R_{x_n^*}\}$ is an orbit of $\mathscr{R}$-classes of J. These two orbits are adjacent vertices in $\Delta(S,J)$, since the $\mathscr{H}$-class $L_x \cap R_{x^*}$ contains the projection x^*x for each x, and is therefore a group. Furthermore, since an element $e \in S$ is an idempotent if and only if e^* is an idempotent, it follows that an $\mathscr{H}$-class H_e is a group if and only if

$$H_e^* = L_e^* \cap R_e^* = R_{e^*} \cap L_{e^*} = H_{e^*}$$

is a group. Thus the function that maps an orbit of $\mathscr{L}$-classes $\{L_{x_1},\ldots,L_{x_n}\}$ to the orbit of $\mathscr{R}$-classes $\{R_{x_1^*},\ldots,R_{x_n^*}\}$, and vice versa, is an automorphism of $\Delta(S,J)$ of order 2.

The situation is further simplified when every idempotent of J is a projection (such as when S is inverse). In this case, since the only group $\mathscr{H}$-class of L_x is $L_x \cap R_{x^*}$, it follows that an orbit of $\mathscr{L}$-classes $\{L_{x_1},\ldots,L_{x_n}\}$ is adjacent in $\Delta(S,J)$ only to the corresponding orbit of $\mathscr{R}$-classes $\{R_{x_1^*},\ldots,R_{x_n^*}\}$. Thus every vertex has degree one, and a maximal independent subset of $\Delta(S,J)$ is formed by choosing one vertex from each edge. Due to this observation, and using Propositions 2.10 and 2.11, we obtain the following corollary.

Corollary 2.13. *Let S be a finite regular $*$-monoid with group of units G, and let J be a $\mathscr{J}$-class of S that is covered by G and whose only idempotents are projections. Suppose that $\{O_1,\ldots,O_n\}$ are the orbits of the right action of G on the $\mathscr{L}$-classes of J. Then the maximal subsemigroups of S arising from J are of types (M1), (M2), or (M5). A maximal subsemigroup of type (M2) is the union of $S \setminus J$ and the union of the Green's classes*

$$\big\{L : L \in O_i,\ i \in A\big\} \cup \big\{L^* : L \in O_i,\ i \notin A\big\},$$

where A is any proper non-empty subset of $\{1,\ldots,n\}$. In particular, there are $2^n - 2$ maximal subsemigroups of type (M2), and no maximal subsemigroups of types (M3) or (M4).

2.2.2 Maximal subsemigroups that intersect every $\mathscr{H}$-class: type (M5)

To describe maximal subsemigroups of type (M5) — i.e. those that intersect each $\mathscr{H}$-class of S non-trivially — we must use a different approach from that in Section 2.2.1. Few of the monoids in this paper exhibit maximal subsemigroups of type (M5) that arise from a $\mathscr{J}$-class covered by the group of units. However, such maximal subsemigroups do occur in some instances, and in Proposition 2.17, we present a result that will be useful for these cases.

Let S be a finite regular $*$-monoid with group of units G. To prove Proposition 2.17, we require the following definition: for a subset $A \subseteq S$, define the *setwise stabilizer of A in G*, $\mathrm{Stab}_G(A)$, to be the subgroup $\{g \in G : Ag = A\}$ of G. Note that $\mathrm{Stab}_G(A)$ is defined to be the set of elements of G that stabilize A on the *right*. However, if we define $A^* = \{a^* : a \in A\}$, then the set of elements of G that stabilize A on the *left* is equal to $\mathrm{Stab}_G(A^*)$, since

$$\big\{g \in G : gA = A\big\} = \big\{g \in G : A^*g^* = A^*\big\} = \mathrm{Stab}_G(A^*)^* = \mathrm{Stab}_G(A^*)^{-1} = \mathrm{Stab}_G(A^*).$$

Thus, for a subset H of S that satisfies $H^* = H$, such as for a group $\mathscr{H}$-class,

$$\mathrm{Stab}_G(H) = \{g \in G : Hg = H = gH\}.$$

This observation is required in the proof of Proposition 2.17.

In Proposition 2.17, we require the set $e\,\mathrm{Stab}_G(H_e) = \{es : s \in \mathrm{Stab}_G(H_e)\}$, where e is a projection of the regular $*$-monoid S, and the $\mathscr{J}$-class J_e is covered by G. Any submonoid of S that contains both e and G also contains $e\,\mathrm{Stab}_G(H_e)$. In particular, every maximal subsemigroup of type (M5) arising from J_e contains G and all idempotents in J_e, and hence contains $e\,\mathrm{Stab}_G(H_e)$. A stronger result, necessary for the proof of Proposition 2.17, is given by the following lemma.

Lemma 2.14. *Let S be a finite monoid with group of units G, let $e \in E(S)$, and let T be a submonoid of S that contains both e and G. Then the set $e\,\mathrm{Stab}_G(H_e^S)$ is a subgroup of H_e^T.*

Proof. Since e is an idempotent, $H_e^T = T \cap H_e^S$. Clearly $e\,\mathrm{Stab}_G(H_e^S) \subseteq eG \subseteq T$. Let $g \in \mathrm{Stab}_G(H_e^S)$. Then $eg \in H_e^S$ by definition, and so $e\,\mathrm{Stab}_G(H_e^S) \subseteq H_e^S$. Thus $e\,\mathrm{Stab}_G(H_e^S) \subseteq T \cap H_e^S = H_e^T$, and the subset is non-empty since $e = e1 \in e\,\mathrm{Stab}_G(H_e^S)$, where 1 is the identity of S. Since S is finite, it remains to show that $e\,\mathrm{Stab}_G(H_e^S)$ is closed under multiplication. Let $g, g' \in \mathrm{Stab}_G(H_e^S)$. Since $eg \in H_e^S$ and e is the identity of H_e^S, it follows that $(eg)e = eg$. Thus

$$(eg)(eg') = (ege)g' = (eg)g' = e(gg') \in e\,\mathrm{Stab}_G(H_e^S). \qquad \square$$

The following two technical lemmas are also required for the proof of Proposition 2.17.

Lemma 2.15 ([41, Theorem A.2.4])**.** *Let S be a finite semigroup and let $x, y \in S$. Then $x \mathscr{J} xy$ if and only if $x \mathscr{R} xy$, and $x \mathscr{J} yx$ if and only if $x \mathscr{L} yx$.*

Lemma 2.16 (follows from [31, Proposition 2.3.7])**.** *Let R be an $\mathscr{R}$-class of an arbitrary semigroup, and let $x, y \in R$. Then $xy \in R$ if and only if H_x is a group.*

Proposition 2.17. *Let S be a finite regular $*$-monoid with group of units G, let J be a $\mathscr{J}$-class of S that is covered by G, and let H_e^S be the $\mathscr{H}$-class of a projection $e \in J$. Suppose that G acts transitively on the $\mathscr{R}$-classes or the $\mathscr{L}$-classes of J, and that J contains one idempotent per $\mathscr{L}$-class and one idempotent per $\mathscr{R}$-class (i.e. every idempotent of J is a projection). Then the maximal subsemigroups of S arising from J are either:*

(a) *$(S \setminus J) \cup GUG = \langle S \setminus J,\ U \rangle$, for each maximal subgroup U of H_e^S that contains $e\,\mathrm{Stab}_G(H_e^S)$ (type (M5)), or*

(b) *$S \setminus J$, if no maximal subsemigroups of type (M5) exist (type (M1)).*

Proof. Since S is a regular $*$-monoid, G acts transitively on the $\mathscr{L}$-classes of J if and only if G acts transitively on the $\mathscr{R}$-classes of J. Hence there are no maximal subsemigroups of types (M2), (M3), or (M4) arising from J, by Lemma 2.12. By Proposition 2.7, it remains to describe the maximal subsemigroups of type (M5).

Let U be a maximal subgroup of H_e^S that contains $e\,\mathrm{Stab}_G(H_e^S)$, and define $M_U = (S \setminus J) \cup GUG$. To prove that M_U is a maximal subsemigroup of S, we first show that M_U is a proper subset of S, then that it is a subsemigroup, and finally that it is maximal in S. Since G acts transitively on the $\mathscr{L}$- and $\mathscr{R}$-classes of J and M_U contains $S \setminus J$, it follows that the set M_U intersects every $\mathscr{H}$-class of S non-trivially. Given that M_U is a subsemigroup, and since $G \subseteq S \setminus J$, it is obvious that it is generated by $(S \setminus J) \cup U$.

To prove that M_U is a proper subset of S, it suffices to show that $GUG \cap H_e^S \subseteq U$. Let $x \in GUG \cap H_e^S$. Since $x \in GUG$, we may write $x = \alpha u \beta$ for some $\alpha, \beta \in G$ and $u \in U$. Since $u, \alpha u \beta \in H_e^S$, it is straightforward to show that $\alpha u,\ u\beta \in H_e^S$. Thus

$$\alpha H_e^S = \alpha(u H_e^S) = (\alpha u) H_e^S = H_e^S, \qquad \text{and} \qquad H_e^S \beta = (H_e^S u)\beta = H_e^S(u\beta) = H_e^S.$$

In other words, α and β stabilize H_e^S on the left and right, respectively. Thus $\alpha, \beta \in \mathrm{Stab}_G(H_e^S)$, and

$$x = ex = e\alpha u e \beta \in \big(e\,\mathrm{Stab}_G(H_e^S)\big) U \big(e\,\mathrm{Stab}_G(H_e^S)\big) \subseteq U^3 = U.$$

In order to show that S is a subsemigroup, it suffices to show that $xy \in M_U$ whenever $x, y \in G \cup GUG$, because $S \setminus (G \cup J)$ is an ideal of S. If $x \in G$ and $y \in G$, then certainly $xy \in G$. If $x \in G$ and $y \in GUG$, then $xy \in G^2UG = GUG$ and $yx \in GUG^2 = GUG$. For the final case, assume that $x, y \in GUG$ and that $xy \in J$. By definition, $x = \alpha u \beta$ and $y = \sigma v \tau$ for some $\alpha, \beta, \sigma, \tau \in G$ and $u, v \in U$. It suffices to show that $\beta\sigma \in \mathrm{Stab}_G(H_e^S)$, because then

$$xy = \alpha u \beta \sigma v \tau = \alpha(ue)\beta\sigma v\tau = \alpha u(e\beta\sigma)v\tau \in GU\left(e\,\mathrm{Stab}_G(H_e^S)\right)UG \subseteq GU^3G = GUG.$$

Since H_e^S is a group containing u and v, it follows that $u^*u = vv^* = e$. Thus

$$e\beta\sigma e = u^*u\beta\sigma v v^* = u^*\alpha^{-1}(\alpha u \beta \sigma v \tau)\tau^{-1}v^* = u^*\alpha^{-1}(xy)\tau^{-1}v^*.$$

Together with $xy = \alpha u(e\beta\sigma e)v\tau$, it follows that $e\beta\sigma e \in J$. By Lemma 2.15, $e\beta\sigma e \in R_e^S$. Since the elements $e\beta\sigma$ and e, and their product $e\beta\sigma e$, are all contained in R_e^S, Lemma 2.16 implies that $H_{e\beta\sigma}^S$ is a group. By assumption, R_e^S contains only one group $\mathscr{H}$-class, which is H_e^S. Thus $e\beta\sigma \in H_e^S$, and so $H_e^S\beta\sigma = (H_e^S e)\beta\sigma = H_e^S(e\beta\sigma) = H_e^S$, i.e. $\beta\sigma \in \operatorname{Stab}_G(H_e^S)$, as required.

Let M be a maximal subsemigroup of S that contains M_U. By [26, Proposition 4], $M \cap H_e^S$ is a maximal subgroup of H_e^S, and the intersection of M with any $\mathscr{H}$-class of J contains exactly $|M \cap H_e^S|$ elements. Since $M \cap H_e^S$ contains U, the maximality of U in H_e^S implies that $U = M \cap H_e^S$. Since the group G acts transitively on the $\mathscr{L}$- and $\mathscr{R}$-classes of J, the intersection of GUG with any $\mathscr{H}$-class of J contains at least $|U|$ elements. Thus $|M| \leq |M_U|$, and so $M = M_U$.

Conversely, suppose that M is a maximal subsemigroup of S of type (M5) arising from J. By [26, Proposition 4], the intersection $U = M \cap H_e^S = H_e^M$ is a maximal subgroup of H_e^S, and it contains $e\operatorname{Stab}_G(H_e^S)$ by Lemma 2.14. Since M contains G, U, and $S \setminus J$, it contains the maximal subsemigroup $M_U = (S \setminus J) \cup GUG$. But M is a proper subsemigroup, which implies that $M = M_U$. ∎

2.3 Maximal subsemigroups arising from other $\mathscr{J}$-classes

The following lemma can be used to find the maximal subsemigroups that arise from an arbitrary $\mathscr{J}$-class of a finite semigroup. In the later sections, for conciseness, we will sometimes use this lemma to find the maximal subsemigroups that arise from a $\mathscr{J}$-class of a monoid that is covered by the group of units. Additionally, a small number of the diagram monoids in Section 4 exhibit maximal subsemigroups arising from a $\mathscr{J}$-class that is neither equal to nor covered by the group of units. The following lemma will be particularly useful when we determine the maximal subsemigroups that arise in this case. Although the results of [16] are, in their full generality, applicable to such cases, the few examples in this paper do not warrant their use.

Lemma 2.18. *Let S be a finite semigroup, and let J be a $\mathscr{J}$-class of S. Suppose that there distinct subsets $X_1, \ldots, X_k \subseteq J$ such that for all $A \subseteq J$, $S = \langle S \setminus J,\ A\rangle$ if and only if $A \cap X_i \neq \varnothing$ for all $i \in \{1, \ldots, k\}$. Then the maximal subsemigroups of S arising from J are precisely the sets $S \setminus X_i$ for each $i \in \{1, \ldots, k\}$.*

Proof. Note that, by the definition of the sets X_i and the assumption that they are distinct, no set X_i is contained in a different set X_j. Let $i \in \{1, \ldots, k\}$. We show that $S \setminus X_i$ is a subsemigroup of S; its maximality is then obvious. Let $x, y \in S \setminus X_i$. Since $S \setminus X_i$ does not generate S, but it contains $S \setminus J$ and an element $x_j \in X_j$ for each $j \in \{1, \ldots, n\} \setminus \{i\}$, it follows that $xy \notin X_i$. Conversely, let M be a maximal subsemigroup of S arising from J. If $M \cap X_i \neq \varnothing$ for each i then, by assumption, $S = \langle M\rangle = M$, a contradiction. Thus $M \cap X_i = \varnothing$ for some i. In other words, $M \subseteq S \setminus X_i$. By the maximality of M in S, it follows that $M = S \setminus X_i$. ∎

Let S be a monoid with group of units G, and suppose there exists a non-empty subset $X \subseteq S \setminus G$ with the property that for any $A \subseteq S$, $S = \langle G,\ A\rangle$ if and only if $A \cap X \neq \varnothing$. This is equivalent to the property that for any $x \in S$, $S = \langle G,\ x\rangle$ if and only if $x \in X$. At several instances in Sections 3 and 4, we wish to determine the maximal subsemigroups of a monoid that has such a subset X. Let $x \in X$. By definition, the principal ideal generated by x consists of every element of S that can be written as a product involving x. Since $S = \langle G,\ x\rangle$ and G is closed under multiplication, this ideal is $S \setminus G$. Since x was arbitrary, every element of X generates the same principal ideal, and so X is contained in some $\mathscr{J}$-class J of S. Therefore the maximal subsemigroups of S arise from its $\mathscr{J}$-classes G and J; the maximal subsemigroup that arises from J, namely $S \setminus X$, can be found by applying Lemma 2.18 with $k = 1$ and $X_1 = X$. The preceding argument is summarised in the following corollary.

Corollary 2.19. *Let S be a finite monoid with group of units G, and suppose there exists a non-empty subset X of $S \setminus G$ with the property that $S = \langle G,\ x\rangle$ if and only if $x \in X$. Then the only maximal subsemigroup of S that does not arise from the group of units is $S \setminus X$.*

3 Partial transformation monoids

In this section, we find the maximal subsemigroups of the families of monoids of partial transformations defined in Section 1.2.

The maximal subsemigroups of several of the monoids considered in this section have been described in the literature. The maximal subsemigroups of $\mathcal{OP}_n$ and $\mathcal{OR}_n$ were described in [12], those of $\mathcal{POI}_n$ were found in [24], and those of $\mathcal{PODI}_n$ were found in [13]. The maximal subsemigroups of the singular ideal of $\mathcal{O}_n$ were found in [11], and those of the singular ideal of $\mathcal{PO}_n$ in [14]. Additionally, the maximal subsemigroups of the singular ideal of $\mathcal{OD}_n$ were described in [11] and [28]. However, since the group of units of $\mathcal{OD}_n$ is non-trivial, this is a fundamentally different problem than finding the maximal subsemigroups of $\mathcal{OD}_n$. The maximal subsemigroups of $\mathcal{PT}_n$, $\mathcal{T}_n$, and $\mathcal{I}_n$ are well-known folklore. To our knowledge, a description of the maximal subsemigroups of neither $\mathcal{POD}_n$, $\mathcal{OD}_n$, $\mathcal{POP}_n$, $\mathcal{POPI}_n$, $\mathcal{POR}_n$, nor $\mathcal{PORI}_n$ has appeared in the literature.

We reprove the known results in order to demonstrate that they may be obtained in a largely unified manner, using the tools described in Section 2. Furthermore, the descriptions of the maximal subsemigroups of some of these monoids — such as those of $\mathcal{PO}_n$ and $\mathcal{POD}_n$ — are very closely linked, and so it is instructive to present such results together, regardless of whether they were previously known.

Let $n \in \mathbb{N}$, $n \geq 2$. We require some facts and notation that are common to the submonoids of $\mathcal{PT}_n$ defined in Section 1.2; let S be such a monoid. Then S is regular, and any generating set for S contains elements of rank n and $n-1$, but needs not contain elements of smaller rank. The Green's relations on S can be characterised as follows:

- $\alpha \mathscr{L} \beta$ if and only if $\mathrm{im}(\alpha) = \mathrm{im}(\beta)$,
- $\alpha \mathscr{R} \beta$ if and only if $\ker(\alpha) = \ker(\beta)$, and
- $\alpha \mathscr{J} \beta$ if and only if $\mathrm{rank}(\alpha) = \mathrm{rank}(\beta)$,

for $\alpha, \beta \in S$. Note that $\ker(\alpha) = \ker(\beta)$ implies that $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$, by definition. It is well-known that this is the characterization of the Green's relations on $\mathcal{PT}_n$. Since S is regular submonoid of $\mathcal{PT}_n$, it follows that the characterization of Green's $\mathscr{L}$- and $\mathscr{R}$-relations on S is as described [31, Proposition 2.4.2]. It is straightforward to see that $\mathscr{J}$-equivalence in S is determined by rank.

By the previous paragraph, to describe the maximal subsemigroups of S, we must find those maximal subsemigroups that arise from the group of units, and those that arise from the $\mathscr{J}$-class containing elements of rank $n-1$. The results of Section 2.1 apply in the former case, and the results of Section 2.2 apply in the latter case.

Notation for the groups of units that appear in this section was defined in Section 1.2. In order to describe the remaining maximal subsemigroups, we require the following notation for the Green's classes that contain partial transformations of rank $n-1$. Define

$$J_{n-1} = \{\alpha \in \mathcal{PT}_n : \mathrm{rank}(\alpha) = n-1\}$$

to be the $\mathscr{J}$-class of $\mathcal{PT}_n$ consisting of partial transformations of rank $n-1$. A partial transformation of rank $n-1$ lacks exactly one element from its image, and is either a partial permutation that lacks one element from its domain, or is a transformation with a unique non-trivial kernel class, which contains two points. Thus for distinct $i, j \in \{1, \ldots, n\}$, we define the Green's classes

- $L_i = \{\alpha \in J_{n-1} : i \notin \mathrm{im}(\alpha)\}$, which is an $\mathscr{L}$-class;
- $R_i = \{\alpha \in J_{n-1} : i \notin \mathrm{dom}(\alpha)\}$, which is an $\mathscr{R}$-class consisting of partial permutations; and
- $R_{\{i,j\}} = \{\alpha \in J_{n-1} : (i, j) \in \ker(\alpha)\}$, which is an $\mathscr{R}$-class consisting of transformations.

An $\mathscr{H}$-class of the form $L_i \cap R_j$ is a group if and only if $i = j$, and an $\mathscr{H}$-class of the form $L_i \cap R_{\{j,k\}}$ is a group if and only if $i \in \{j, k\}$.

Let S be one of the submonoids of $\mathcal{PT}_n$ defined in Section 1.2. It follows that the set $J_{n-1} \cap S$ is a regular $\mathscr{J}$-class of S, that the $\mathscr{L}$-classes of $J_{n-1} \cap S$ are the sets of the form $L_i \cap S$, and that the $\mathscr{R}$-classes of $J_{n-1} \cap S$ are those non-empty sets of the form $R_i \cap S$ and $R_{\{i,j\}} \cap S$, for distinct $i, j \in \{1, \ldots, n\}$. Whenever we present a picture of the graph $\Delta(S, J_{n-1} \cap S)$, such as the picture of the graph $\Delta(\mathcal{PO}_n, J_{n-1} \cap \mathcal{PO}_n)$ given in Figure 3, we label an $\mathscr{L}$-class as L_i rather than as $L_i \cap S$, and so on, in order to avoid cluttering the image. This approach also has the advantage of emphasizing the similarities between the graphs of related monoids — indeed, some graphs may be obtained as induced subgraphs of others.

Note that the non-trivial kernel class of a order-preserving or -reversing transformation of rank $n-1$ has the form $\{i, i+1\}$ for some $i \in \{1, \ldots, n-1\}$, and that the non-trivial kernel class of an orientation-preserving or -reversing transformation of rank $n-1$ has the same form, or is equal to $\{1, n\}$. Any non-empty subset of $\{1, \ldots, n\}$ appears as the image of some partial transformation in each of the monoids defined in Section 1.2.

Often, the principal obstacle to describing the maximal subsemigroups of S is to determine the maximal independent subsets of $\Delta(S, J_{n-1} \cap S)$. Since the $\mathscr{J}$-class of S to be considered in each case is $J_{n-1} \cap S$, throughout the remainder of this section, we write $\Delta(S)$ in place of $\Delta(S, J_{n-1} \cap S)$. To determine the maximal independent subsets of $\Delta(S)$, we must calculate the actions of the group of units G of S on the $\mathscr{R}$-classes and $\mathscr{L}$-classes of $J_{n-1} \cap S$. The orbits of $\mathscr{L}$-classes correspond to the orbits of G on $\{1, \ldots, n\}$, in the following way: if $\Omega \subseteq \{1, \ldots, n\}$ is an orbit of G on $\{1, \ldots, n\}$, then $\{L_i \cap S : i \in \Omega\}$ is an orbit of G on $(J_{n-1} \cap S)/\mathscr{L}$, and vice versa. In the same way, the orbits of $\mathscr{R}$-classes that contain partial permutations of rank $n-1$ correspond to the orbits of G on the indices $\{i : R_i \cap S \neq \varnothing\}$. Finally, the orbits of $\mathscr{R}$-classes that contain transformations of rank $n-1$ correspond to orbits of G on the sets $\{\{i, j\} : i \neq j,\ R_{\{i,j\}} \cap S \neq \varnothing\}$. The actions of $\{\mathrm{id}_n\}$, $\langle \gamma_n \rangle$, $\mathcal{C}_n$, $\mathcal{D}_n$, and $\mathcal{S}_n$ on these sets are easy to understand.

3.1 $\mathcal{PT}_n$, $\mathcal{T}_n$, and $\mathcal{I}_n$

First we find the maximal subsemigroups of the monoids $\mathcal{PT}_n$, $\mathcal{T}_n$, and $\mathcal{I}_n$; their maximal subsemigroups are well-known folklore, but we include the following result for completeness, and as a gentle introduction to the application of the results of Section 2.

Theorem 3.1. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary.*

(a) *The maximal subsemigroups of $\mathcal{PT}_n$, the partial transformation monoid of degree n, are the sets:*

  (i) $(\mathcal{PT}_n \setminus \mathcal{S}_n) \cup U$, *where U is a maximal subgroup of $\mathcal{S}_n$ (type (M5));*

  (ii) $\mathcal{PT}_n \setminus \big\{\alpha \in \mathcal{T}_n : \operatorname{rank}(\alpha) = n-1\big\}$ *(type (M4)); and*

  (iii) $\mathcal{PT}_n \setminus \big\{\alpha \in \mathcal{I}_n : \operatorname{rank}(\alpha) = n-1\big\}$ *(type (M4)).*

(b) *The maximal subsemigroups of $\mathcal{T}_n$, the full transformation monoid of degree n, are the sets:*

  (i) $(\mathcal{T}_n \setminus \mathcal{S}_n) \cup U$, *where U is a maximal subgroup of $\mathcal{S}_n$ (type (M5)); and*

  (ii) $\mathcal{T}_n \setminus \{\alpha \in \mathcal{T}_n : \operatorname{rank}(\alpha) = n-1\}$ *(type (M1)).*

(c) *The maximal subsemigroups of $\mathcal{I}_n$, the symmetric inverse monoid of degree n, are the sets:*

  (i) $(\mathcal{I}_n \setminus \mathcal{S}_n) \cup U$, *where U is a maximal subgroup of $\mathcal{S}_n$ (type (M5)); and*

  (ii) $\mathcal{I}_n \setminus \{\alpha \in \mathcal{I}_n : \operatorname{rank}(\alpha) = n-1\}$ *(type (M1)).*

In particular, for $n \geq 2$, there are $s_n + 2$ maximal subsemigroups of $\mathcal{PT}_n$, and $s_n + 1$ maximal subsemigroups of both $\mathcal{T}_n$ and $\mathcal{I}_n$, where s_n is the number of maximal subgroups of the symmetric group of degree n. The monoid $\mathcal{PT}_1 = \mathcal{I}_1$ is a semilattice of order 2: its maximal subsemigroups are each of its singleton subsets.

Proof. Since the group of units of $\mathcal{PT}_n$, $\mathcal{T}_n$, and $\mathcal{I}_n$ is $\mathcal{S}_n$, it follows by Corollary 2.4 that the maximal subsemigroups arising from the group of units are, in each case, those described in the statement of the theorem. It is well-known that $\mathcal{T}_n$ and $\mathcal{I}_n$ can each be generated by the symmetric group $\mathcal{S}_n$, along with any one of their elements of rank $n-1$. Thus, by Corollary 2.19, for each of these two monoids, the unique maximal subsemigroup arising from its $\mathscr{J}$-class of rank $n-1$ has type (M1).

It remains to consider those maximal subsemigroups that arise from the $\mathscr{J}$-class J_{n-1} of $\mathcal{PT}_n$. Since $\mathcal{PT}_n = \langle \mathcal{T}_n, \mathcal{I}_n \rangle$, the partial transformation monoid of degree n can generated by $\mathcal{S}_n$, along with *any* partial permutation of rank $n-1$ and *any* transformation of rank $n-1$. Since $\mathcal{T}_n$ and $\mathcal{I}_n$ are both subsemigroups of $\mathcal{PT}_n$, any generating set for $\mathcal{PT}_n$ contains *both* a transformation and a partial permutation of rank $n-1$. Thus, using Lemma 2.18 with $k = 2$, $X_1 = J_{n-1} \cap \mathcal{T}_n$, and $X_2 = J_{n-1} \cap \mathcal{I}_n$, it follows that the maximal subsemigroups of $\mathcal{PT}_n$ arising from J_{n-1} are those given in the theorem. ☐

The description of the maximal subsemigroups of $\mathcal{PT}_n$ that arise from its $\mathscr{J}$-class J_{n-1} can also be obtained by using the graph $\Delta(\mathcal{PT}_n)$ and the results of Section 2.2.1. Since $\mathcal{PT}_n$ is generated by its units and its idempotents of rank $n-1$, Lemma 2.1 implies that no maximal subsemigroups of type (M5) arise from J_{n-1}. The action of $\mathcal{S}_n$ on the $\mathscr{L}$-classes of J_{n-1} is transitive, and so by Lemma 2.12 there are no maximal subsemigroups of types (M2) or (M3). However, there are two orbits under the action of $\mathcal{S}_n$ on the $\mathscr{R}$-classes of J_{n-1}: one orbit contains the $\mathscr{R}$-classes of transformations, and the other contains the $\mathscr{R}$-classes of partial permutations. These two orbits of $\mathscr{R}$-classes are adjacent in $\Delta(\mathcal{PT}_n)$ to the unique orbit of $\mathscr{L}$-classes; a picture of $\Delta(\mathcal{PT}_n)$ is shown in Figure 2. Thus, by Proposition 2.11, there are two maximal subsemigroups of type (M4) arising from J_{n-1}, each formed by removing the $\mathscr{R}$-classes from one of these orbits.

3.2 $\mathcal{PO}_n$ and $\mathcal{POD}_n$

The maximal subsemigroups of the singular ideal of $\mathcal{PO}_n$ were described in [14]. Since $\mathcal{PO}_n$ has a trivial group of units, this work essentially describes the maximal subsemigroups of $\mathcal{PO}_n$ itself. To our knowledge, the maximal subsemigroups of $\mathcal{POD}_n$ have not been described in the literature. Using our approach, we find that the maximal subsemigroups of $\mathcal{POD}_n$ are closely linked to those of $\mathcal{PO}_n$.

Let $n \in \mathbb{N}$, $n \geq 2$. To state the main results in this section we require the following notation. If $S \in \{\mathcal{PO}_n, \mathcal{POD}_n\}$, then $J_{n-1} \cap S$ is a regular $\mathscr{J}$-class of S, the $\mathscr{L}$-classes of $J_{n-1} \cap S$ are $\big\{L_i \cap S : i \in \{1, \ldots, n\}\big\}$, and the $\mathscr{R}$-classes are $\big\{R_i \cap S : i \in \{1, \ldots, n\}\big\}$ and $\big\{R_{\{i,i+1\}} \cap S : i \in \{1, \ldots, n-1\}\big\}$. Note that $\mathcal{POD}_n$ is generated by $\mathcal{PO}_n$ and the permutation γ_n that reverses the usual order on $\{1, \ldots, n\}$. More information about $\mathcal{PO}_n$ can be found in [25].

The main results of this section are the following theorems.

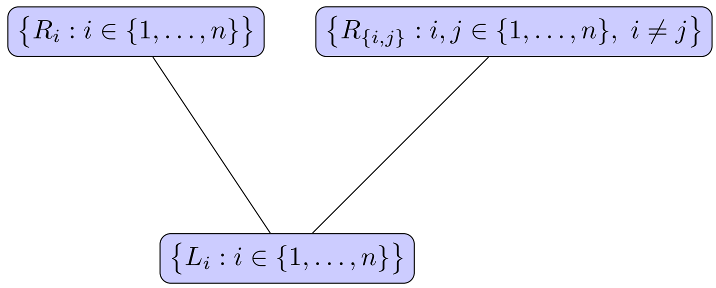

Figure 2: The graph $\Delta(\mathcal{PT}_n)$.

Theorem 3.2. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary and let $\mathcal{PO}_n$ be the monoid of order-preserving partial transformations on $\{1, \ldots, n\}$ with the usual order. Then the maximal subsemigroups of $\mathcal{PO}_n$ are:*

(a) *$\mathcal{PO}_n \setminus \{\mathrm{id}_n\}$ (type (M1));*

(b) *the union of $\mathcal{PO}_n \setminus J_{n-1}$ and the union of*

$$\{L_i \cap \mathcal{PO}_n : i \in A\} \cup \{R_i \cap \mathcal{PO}_n : i \notin A\} \cup \{R_{\{i,i+1\}} \cap \mathcal{PO}_n : i, i+1 \notin A\},$$

where A is any non-empty proper subset of $\{1, \ldots, n\}$ (type (M2)) and;

(c) *$\mathcal{PO}_n \setminus R$, where R is any $\mathscr{R}$-class in $J_{n-1} \cap \mathcal{PO}_n$ (type (M4)).*

The monoid $\mathcal{PO}_1 = \mathcal{PT}_1$ is a semilattice of order 2: its maximal subsemigroups are each of its singleton subsets. In particular, for $n \in \mathbb{N}$, there are $2^n + 2n - 2$ maximal subsemigroups of $\mathcal{PO}_n$.

Theorem 3.3. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary. Let $\mathcal{POD}_n$ be the monoid of order-preserving or -reversing partial transformations on $\{1, \ldots, n\}$ with the usual order, and let γ_n be the permutation of degree n that reverses this order. Then the maximal subsemigroups of $\mathcal{POD}_n$ are:*

(a) *$\mathcal{POD}_n \setminus \{\gamma_n\}$ (type (M5));*

(b) *the union of $\mathcal{POD}_n \setminus J_{n-1}$ and the union of*

$$\begin{aligned}\{(L_i \cup L_{n-i+1}) \cap \mathcal{POD}_n : i \in A\} \cup \{(R_i \cup R_{n-i+1}) \cap \mathcal{POD}_n : i \notin A\}& \\ \cup \{(R_{\{i,i+1\}} \cup R_{\{n-i,n-i+1\}}) \cap \mathcal{POD}_n : i, i+1 \notin A\}&,\end{aligned}$$

where A is any non-empty proper subset of $\{1, \ldots, \lceil n/2 \rceil\}$ (type (M2)) and;

(c) *$\mathcal{POD}_n \setminus (R \cup \gamma_n R)$, where R is any $\mathscr{R}$-class in $J_{n-1} \cap \mathcal{POD}_n$ (type (M4)); in particular:*

  (i) *$\mathcal{POD}_n \setminus (R_i \cup R_{n-i+1})$, for $i \in \{1, \ldots, \lceil n/2 \rceil\}$; and*

  (ii) *$\mathcal{POD}_n \setminus (R_{\{i,i+1\}} \cup R_{\{n-i,n-i+1\}})$, for $i \in \{1, \ldots, \lfloor n/2 \rfloor\}$.*

The monoid $\mathcal{POD}_1 = \mathcal{PT}_1$ is a semilattice of order 2: its maximal subsemigroups are each of its singleton subsets. In particular, for $n \in \mathbb{N}$, there are $2^{\lceil n/2 \rceil} + n - 1$ maximal subsemigroups of $\mathcal{POD}_n$.

The most substantial part of the proof of Theorems 3.2 and 3.3 is the description of the maximal independent subsets of $\Delta(\mathcal{PO}_n)$ and $\Delta(\mathcal{POD}_n)$, respectively. Since $\mathcal{PO}_n$ has a trivial group of units, the orbits of $\mathscr{L}$-classes and $\mathscr{R}$-classes of $J_{n-1} \cap \mathcal{PO}_n$ are singletons, and $\Delta(\mathcal{PO}_n)$ is isomorphic to the Graham-Houghton graph of the principal factor of $J_{n-1} \cap \mathcal{PO}_n$. A picture of $\Delta(\mathcal{PO}_n)$ is shown in Figure 3.

The group of units of $\mathcal{POD}_n$ is $\langle \gamma_n \rangle$, which equals $\{\mathrm{id}_n, \gamma_n\}$. Thus $\Delta(\mathcal{POD}_n)$ is isomorphic to a quotient of $\Delta(\mathcal{PO}_n)$. Since $\langle \gamma_n \rangle$ has $\lceil n/2 \rceil$ orbits on the set $\{1, \ldots, n\}$, there are $\lceil n/2 \rceil$ corresponding orbits of $\mathscr{L}$-classes, and $\lceil n/2 \rceil$ orbits of $\mathscr{R}$-classes of partial permutations. Furthermore, there are $\lfloor n/2 \rfloor$ orbits of $\langle \gamma_n \rangle$ on the set $\{\{i, i+1\} : i \in \{1, \ldots, n-1\}\}$, and these orbits correspond to $\lfloor n/2 \rfloor$ orbits of $\mathscr{R}$-classes of transformations. A picture of $\Delta(\mathcal{POD}_n)$ is shown in Figure 4 for odd n, and in Figure 5 for even n; see these pictures for a description of the edges of these graphs. In the case that n is odd, the graph $\Delta(\mathcal{POD}_n)$ is isomorphic to $\Delta(\mathcal{PO}_{\lceil n/2 \rceil})$.

Given these descriptions of $\Delta(\mathcal{PO}_n)$ and $\Delta(\mathcal{POD}_n)$, the following lemmas are established.

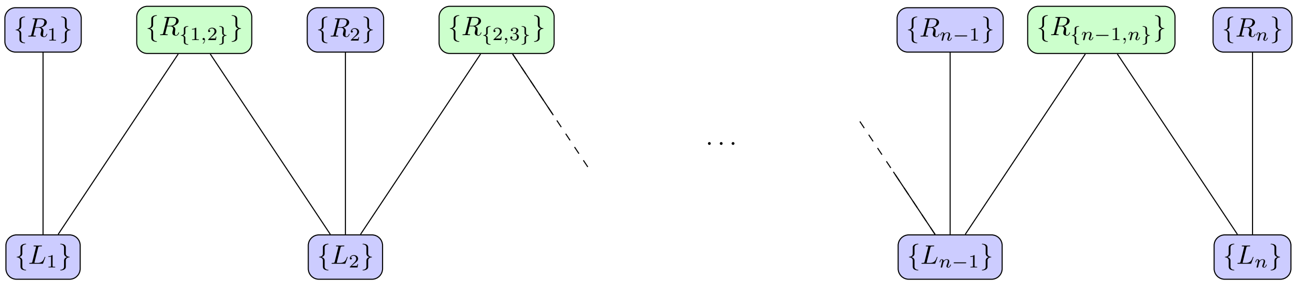

Figure 3: The graph $\Delta(\mathcal{PO}_n)$.

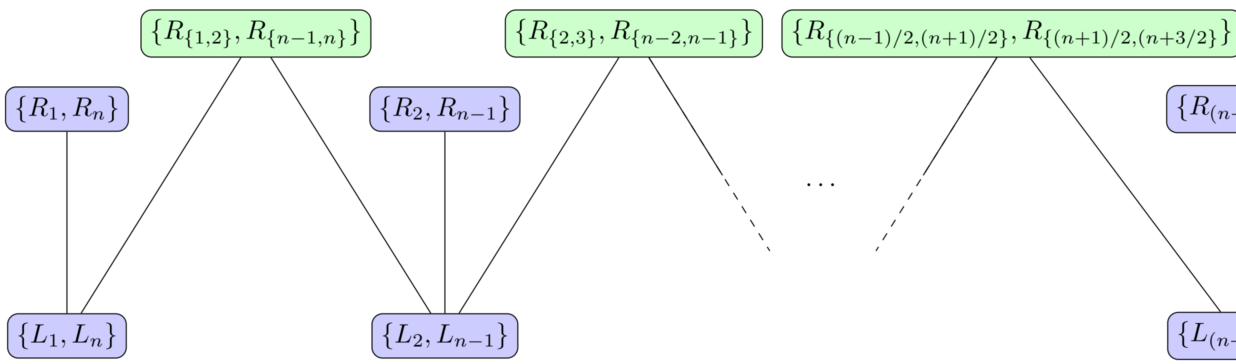

Figure 4: The graph $\Delta(\mathcal{POD}_n)$, when n is odd.

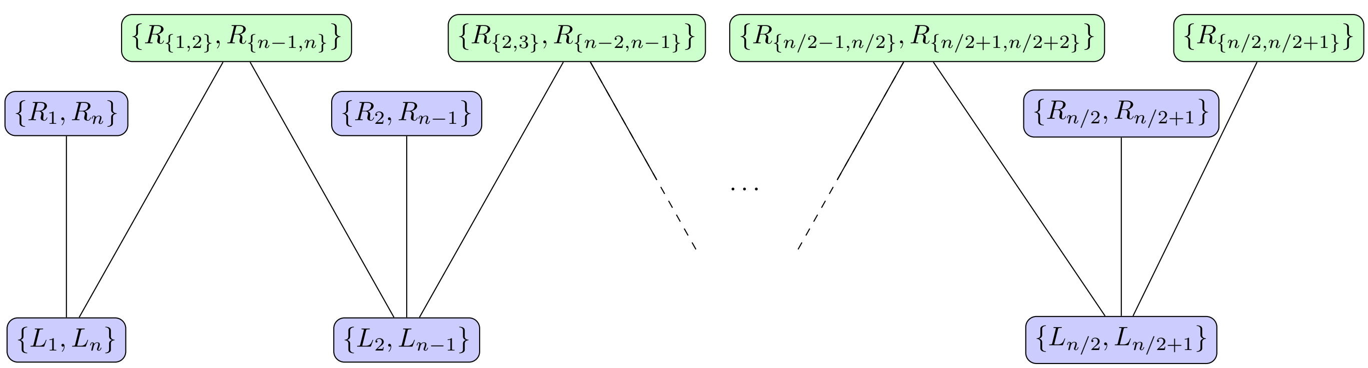

Figure 5: The graph $\Delta(\mathcal{POD}_n)$, when n is even.

Lemma 3.4. *Let K be any collection of vertices of the graph $\Delta(\mathcal{PO}_n)$. Then K is a maximal independent subset of $\Delta(\mathcal{PO}_n)$ if and only if*

$$K = \big\{\{L_i \cap \mathcal{PO}_n\} : i \in A\big\} \cup \big\{\{R_i \cap \mathcal{PO}_n\} : i \notin A\big\} \cup \big\{\{R_{\{i,i+1\}} \cap \mathcal{PO}_n\} : i, i+1 \notin A\big\}$$

for some subset A of $\{1, \ldots, n\}$.

Proof. ($\Rightarrow$) Suppose that K is a maximal independent subset of $\Delta(\mathcal{PO}_n)$. There exists a set $A \subseteq \{1, \ldots, n\}$ of indices such that $\big\{\{L_i \cap \mathcal{PO}_n\} : i \in A\big\}$ is the collection of $\mathscr{L}$-class vertices in K. Since a vertex of the form $\{R_i \cap \mathcal{PO}_n\}$ is adjacent in $\Delta(\mathcal{PO}_n)$ only to the vertex $\{L_i \cap \mathcal{PO}_n\}$, it follows by the maximality of K that $\{R_i \cap \mathcal{PO}_n\} \in K$ if and only if $i \notin A$. Similarly, since an orbit of the form $\{R_{\{i,i+1\}} \cap \mathcal{PO}_n\}$ is adjacent in $\Delta(\mathcal{PO}_n)$ only to the orbits $\{L_i \cap \mathcal{PO}_n\}$ and $\{L_{i+1} \cap \mathcal{PO}_n\}$, it follows that $\{R_{\{i,i+1\}} \cap \mathcal{PO}_n\} \in K$ if and only if $i \notin A$ and $i + 1 \notin A$. Since we have considered all vertices of $\Delta(\mathcal{PO}_n)$, it follows that K has the required form.

($\Leftarrow$) From the definition of $\Delta(\mathcal{PO}_n)$, it is easy to verify that K is a maximal independent subset of $\Delta(\mathcal{PO}_n)$. □

The following lemma can be proved using an argument similar to that used in the proof of Lemma 3.4.

Lemma 3.5. *Let K be any collection of vertices of the graph $\Delta(\mathcal{POD}_n)$. Then K is a maximal independent subset of $\Delta(\mathcal{POD}_n)$ if and only if K is equal to*

$$\big\{\{L_i \cap \mathcal{POD}_n, L_{n-i+1} \cap \mathcal{POD}_n\} : i \in A\big\} \cup \big\{\{R_i \cap \mathcal{POD}_n, R_{n-i+1} \cap \mathcal{POD}_n\} : i \notin A\big\}$$
$$\cup \big\{\{R_{\{i,i+1\}} \cap \mathcal{POD}_n, R_{\{n-i,n-i+1\}} \cap \mathcal{POD}_n\} : i, i+1 \notin A\big\},$$

for some subset A of $\{1, \ldots, \lceil n/2 \rceil\}$.

Proof of Theorems 3.2 and 3.3. The group of units of $\mathcal{PO}_n$ is the trivial group $\{\mathrm{id}_n\}$, and the group of units of $\mathcal{POD}_n$ is $\langle \gamma_n \rangle$, which has order 2. Thus by Corollary 2.4, the maximal subsemigroups that arise from the group of units in each instance are as described.

Let $S \in \{\mathcal{PO}_n, \mathcal{POD}_n\}$. Since $\mathcal{PO}_n$ is idempotent generated [25, Theorem 3.13], and since $\mathcal{POD}_n = \langle \mathcal{PO}_n, \gamma_n \rangle$, it follows by Lemma 2.1 that there are no maximal subsemigroups of type (M5) arising from $J_{n-1} \cap S$. It follows directly from Proposition 2.8, and Lemmas 3.4 and 3.5, that the maximal subsemigroups of type (M2) are those described in the theorems. There exist vertices of degree 1 in $\Delta(S)$ — the orbits of $\mathscr{R}$-classes of partial permutations. Each orbit of $\mathscr{L}$-classes is adjacent to such a vertex. Thus by Proposition 2.10, there are no maximal subsemigroups of type (M3) arising from S, but by Proposition 2.11, each orbit of $\mathscr{R}$-classes can be removed to provide a maximal subsemigroup of type (M4). If $S = \mathcal{PO}_n$, then there are $2n - 1$ maximal subsemigroups of this type, and if $S = \mathcal{POD}_n$, then there are n. By Proposition 2.7, there is no maximal subsemigroup of type (M1). □

3.3 $\mathcal{O}_n$ and $\mathcal{OD}_n$

The maximal subsemigroups of the singular ideal of $\mathcal{O}_n$ were incorrectly described and counted in [46]: the given formula for the number of maximal subsemigroups of the singular ideal of $\mathcal{O}_n$ is correct for $2 \le n \le 5$, but gives only a lower bound when $n \ge 6$. A correct description, although no number, was subsequently given in [11]. The maximal subsemigroups of the singular ideal of $\mathcal{OD}_n$ were described in [28]. The group of units of $\mathcal{O}_n$ is trivial, and so the maximal subsemigroups of its singular ideal correspond in an obvious way to the maximal subsemigroups of $\mathcal{O}_n$. However, the group of units of $\mathcal{OD}_n$ is non-trivial, and the units act on the monoid in such a way as to break the correspondence between the maximal subsemigroups of the singular part, and the maximal subsemigroups of $\mathcal{OD}_n$ itself. Thus [11, 28] solve an essentially different problem than the description of the maximal subsemigroups of $\mathcal{OD}_n$.

Let $S \in \{\mathcal{O}_n, \mathcal{OD}_n\}$. Then S is a regular monoid and, by definition, $\mathcal{O}_n = \mathcal{PO}_n \cap \mathcal{T}_n$, and $\mathcal{OD}_n = \mathcal{POD}_n \cap \mathcal{T}_n$. From the description of the Green's classes of $\mathcal{PO}_n$ and $\mathcal{POD}_n$ given in Section 3.2, $J_{n-1} \cap S$ is a regular $\mathscr{J}$-class of S, the set of $\mathscr{L}$-classes of $J_{n-1} \cap S$ is $\big\{L_i \cap S : i \in \{1, \ldots, n\}\big\}$, and the set of $\mathscr{R}$-classes of $J_{n-1} \cap S$ is $\big\{R_{\{i,i+1\}} \cap S : i \in \{1, \ldots, n-1\}\big\}$.

We may identify the Green's classes of $J_{n-1} \cap \mathcal{O}_n$ with those of $J_{n-1} \cap \mathcal{PO}_n$ that contain transformations, so that $L_i \cap \mathcal{O}_n$ corresponds with $L_i \cap \mathcal{PO}_n$ and $R_{\{i,i+1\}} \cap \mathcal{O}_n$ corresponds with $R_{\{i,i+1\}} \cap \mathcal{PO}_n$. In this way, we obtain the graph $\Delta(\mathcal{O}_n)$ as the subgraph of $\Delta(\mathcal{PO}_n)$ induced on those orbits of Green's classes that contain transformations. A similar statement relates $\Delta(\mathcal{OD}_n)$ to the induced subgraph of $\Delta(\mathcal{POD}_n)$ on its orbits of Green's classes that contain transformations. Thus the definitions of $\Delta(\mathcal{O}_n)$ and $\Delta(\mathcal{OD}_n)$ are contained in those of $\Delta(\mathcal{PO}_n)$ and $\Delta(\mathcal{POD}_n)$.

For $k \in \mathbb{N}$, we define the path graph of order k to be the graph with vertices $\{1, \ldots, k\}$ and edges

$$\big\{\{i, i+1\} : i \in \{1, \ldots, k-1\}\big\}.$$

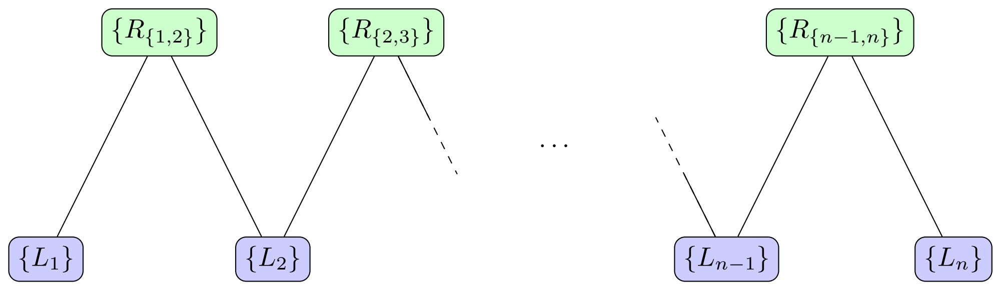

Figure 6: The graph $\Delta(\mathcal{O}_n)$.

The vertices of degree 1 in the path graph of order k are the end-points, 1 and k. It is easy to see that $\Delta(\mathcal{O}_n)$ is isomorphic to the path graph of order $2n-1$, via the isomorphism that maps the orbit $\{L_i \cap \mathcal{O}_n\}$ to the vertex $2i-1$, and maps the orbit $\{R_{\{i,i+1\}} \cap \mathcal{O}_n\}$ to the vertex $2i$. Similarly, $\Delta(\mathcal{OD}_n)$ is isomorphic to the path graph of order n. A picture of $\Delta(\mathcal{O}_n)$ is shown in Figure 6.

We can count the number of maximal independent subsets of $\Delta(\mathcal{O}_n)$ and $\Delta(\mathcal{OD}_n)$ using a recurrence relation.

Lemma 3.6. *The number A_n of maximal independent subsets of the path graph of order n satisfies the recurrence relation $A_1 = 1$, $A_2 = A_3 = 2$, and $A_n = A_{n-2} + A_{n-3}$ for $n \geq 4$.*

Proof. For $n \in \mathbb{N}$, define Γ_n to be the path graph of order n. It is straightforward to verify that $A_1 = 1$, and $A_2 = A_3 = 2$.

Suppose that $n \geq 4$, and that K is a maximal independent subset of Γ_{n-3}. Then $K \cup \{n-1\}$ is a maximal independent subset of Γ_n. Similarly, if K is a maximal independent subset of Γ_{n-2}, then $K \cup \{n\}$ is a maximal independent subset of Γ_n. In this way, distinct maximal independent subsets of Γ_{n-3} and Γ_{n-2} give rise to distinct maximal independent subsets of Γ_n. Thus $A_n \geq A_{n-2} + A_{n-3}$.

Conversely, suppose that $n \geq 4$ and that K is a maximal independent subset of Γ_n. Since the vertex n has degree 1 in Γ_n, it follows by the maximality of K that precisely one of $n-1$ and n is a member of K. If $n-1 \in K$, then $n-2 \notin K$, which implies that $K \setminus \{n-1\}$ is a maximal independent subset of Γ_{n-3}. Otherwise, $K \setminus \{n\}$ is a maximal independent subset of Γ_{n-2}. Thus $A_n \leq A_{n-2} + A_{n-3}$. $\square$

Note that A_n is equal to the $(n+6)^{\text{th}}$ term of the Padovan sequence [44, A000931].

Given the proof of Lemma 3.6, it is clear that a subset $K \subseteq \{1, \ldots, n\}$ is a maximal independent subset of Γ_n if and only if K contains exactly one of 1 and 2, and for any $i \in K$ with $i \leq n-2$, K contains exactly one of $i+2$ and $i+3$. There are two special maximal independent subsets of Γ: the subset of all even vertices, and the subset of all odd vertices. These maximal independent subsets correspond to the bicomponents of $\Delta(\mathcal{O}_n)$ and $\Delta(\mathcal{OD}_n)$, and so they are the unique maximal independent subsets that do not give rise to maximal subsemigroups of $\mathcal{O}_n$ and $\mathcal{OD}_n$ of type (M2) — see Proposition 2.8 and Corollary 2.9.

The following theorems are the main results of this section.

Theorem 3.7. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary and let $\mathcal{O}_n$ be the monoid of order-preserving transformations on $\{1, \ldots, n\}$ with the usual order. Then the maximal subsemigroups of $\mathcal{O}_n$ are:*

(a) *$\mathcal{O}_n \setminus \{\mathrm{id}_n\}$ (type (M1));*

(b) *The union of $\mathcal{O}_n \setminus J_{n-1}$ and the union of the Green's classes in*

$$\big\{L_{(i+1)/2} \cap \mathcal{O}_n : i \in A,\ i \text{ is odd}\big\} \cup \big\{R_{\{i/2,(i/2)+1\}} \cap \mathcal{O}_n : i \in A,\ i \text{ is even}\big\},$$

where A is a maximal independent subset of the path graph of order $2n-1$ that contains both odd and even numbers (type (M2));

(c) *$\mathcal{O}_n \setminus L$, where L is any $\mathscr{L}$-class in $J_{n-1} \cap \mathcal{O}_n$ (type (M3)); and*

(d) *$\mathcal{O}_n \setminus R_{\{i,i+1\}}$, where $i \in \{2, \ldots, n-2\}$ (type (M4)).*

In particular, there are 3 maximal subsemigroups of $\mathcal{O}_2$, and for $n \geq 3$, there are $A_{2n-1} + 2n - 4$ maximal subsemigroups of $\mathcal{O}_n$, where A_{2n-1} is the $(2n-1)^{th}$ term of the sequence defined by $A_1 = 1$, $A_2 = A_3 = 2$, and $A_k = A_{k-2} + A_{k-3}$ for $k \geq 4$.

Theorem 3.8. *Let $n \in \mathbb{N}$, $n \geq 3$, be arbitrary. Let $\mathcal{OD}_n$ be the monoid of order-preserving and -reversing transformations on $\{1, \ldots, n\}$ with the usual order, and let γ_n be the permutation of degree n that reverses this order. Then the maximal subsemigroups of $\mathcal{OD}_n$ are:*

(a) *$\mathcal{OD}_n \setminus \{\gamma_n\}$ (type (M5));*

(b) *The union of $\mathcal{OD}_n \setminus J_{n-1}$ and the union of the Green's classes in*

$$\big\{(L_{(i+1)/2} \cup L_{n+1-(i+1)/2}) \cap \mathcal{OD}_n : i \in A,\ i \text{ is odd}\big\} \cup \big\{(R_{\{i/2,(i/2)+1\}} \cup R_{\{n-(i/2),n+1-(i/2)\}}) \cap \mathcal{OD}_n : i \in A,\ i \text{ is even}\big\},$$

where A is a maximal independent subset of the path graph of order n that contains both odd and even numbers (type (M2));

(c) *$\mathcal{OD}_n \setminus (L_i \cup L_{n-i+1})$, where $\begin{cases} i \in \{1, \ldots, (n+1)/2\} & \text{if } n \text{ is odd,} \\ i \in \{1, \ldots, n/2 - 1\} & \text{if } n \text{ is even} \end{cases}$ (type (M3)); and*

(d) *$\mathcal{OD}_n \setminus (R_{\{i,i+1\}} \cup R_{\{n-i,n-i+1\}})$, where $\begin{cases} i \in \{2, \ldots, (n-3)/2\} & \text{if } n \text{ is odd,} \\ i \in \{2, \ldots, n/2\} & \text{if } n \text{ is even} \end{cases}$ (type (M4)).*

In particular, there are 3 maximal subsemigroups of $\mathcal{OD}_3$, and for $n \geq 4$, there are $A_n + n - 3$ maximal subsemigroups of $\mathcal{OD}_n$, where $A_1 = 1$, $A_2 = A_3 = 2$, and $A_n = A_{n-2} + A_{n-3}$ for $n \geq 4$. The monoid $\mathcal{OD}_2 = \mathcal{T}_2$ has 2 maximal subsemigroups.

Proof of Theorems 3.7 and 3.8. Let $S \in \{\mathcal{O}_n, \mathcal{OD}_n\}$. The group of units of $\mathcal{O}_n$ is trivial and the group of units of $\mathcal{OD}_n$ is $\langle \gamma_n \rangle$, so by Corollary 2.4, the maximal subsemigroups that arise from the group group of units of S are as stated.

Since $\mathcal{O}_n$ is generated by its idempotents of rank $n-1$ [1] and $\mathcal{OD}_n = \langle \mathcal{O}_n, \gamma_n \rangle$ [21], it follows by Lemma 2.1 there are no maximal subsemigroups arising from $J_{n-1} \cap S$ of type (M5) arising from $J_{n-1} \cap S$.

We have already noted that $\Delta(\mathcal{O}_n)$ and $\Delta(\mathcal{OD}_n)$ are paths of length $2n-1$ and n, respectively. It follows from Proposition 2.8 that the maximal subsemigroups of type (M2) are those described in the theorems. By Corollary 2.9 and Lemma 3.6, the number of maximal subsemigroups of type (M2) is $A_{2n-1} - 2$ for $\mathcal{O}_n$, and $A_n - 2$ for $\mathcal{OD}_n$.

To describe the maximal subsemigroups of types (M3) and (M4), it suffices to identify the two vertices of $\Delta(S)$ that are adjacent to the end-points of $\Delta(S)$. From this, the description of the maximal subsemigroups of type (M3) follows from Proposition 2.10, and the description of the maximal subsemigroups of type (M4) follows from Proposition 2.11. In particular, the total number of both types of maximal subsemigroups is two less than the number of vertices of $\Delta(S)$.

By Proposition 2.7, and since $n \geq 3$, there is no maximal subsemigroup of S of type (M1). $\square$

3.4 $\mathcal{POI}_n$ and $\mathcal{PODI}_n$

The maximal subsemigroups of $\mathcal{POI}_n$ are described and counted in [24, Theorem 2], and the maximal subsemigroups of $\mathcal{PODI}_n$ are described and counted in [13, Theorem 4]. Additional information about $\mathcal{POI}_n$ may be found in [20, 24].

Let $S \in \{\mathcal{POI}_n, \mathcal{PODI}_n\}$. Then S is an inverse monoid, and $J_{n-1} \cap S$ is a $\mathscr{J}$-class of S. By definition, $\mathcal{POI}_n = \mathcal{PO}_n \cap \mathcal{I}_n$, and $\mathcal{PODI}_n = \mathcal{POD}_n \cap \mathcal{I}_n$, and so given the description of the Green's classes of $\mathcal{PO}_n$ and $\mathcal{POD}_n$ from Section 3.2, the set of $\mathscr{L}$-classes of $J_{n-1} \cap S$ is $\big\{L_i \cap S : i \in \{1, \ldots, n\}\big\}$, and the set of $\mathscr{R}$-classes of $J_{n-1} \cap S$ is $\big\{R_i \cap S : i \in \{1, \ldots, n\}\big\}$.

In Theorem 3.9 we describe the maximal subsemigroups of $\mathcal{POI}_n$, and in Theorem 3.10 we describe those of $\mathcal{PODI}_n$.

Theorem 3.9. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary and let $\mathcal{POI}_n$ be the inverse monoid of order-preserving partial permutations on $\{1, \ldots, n\}$ with the usual order. Then the maximal subsemigroups of $\mathcal{POI}_n$ are:*

(a) *$\mathcal{POI}_n \setminus \{\mathrm{id}_n\}$ (type (M1)); and*

(b) *the union of $\mathcal{POI}_n \setminus J_{n-1}$ and the union of*

$$\big\{L_i \cap \mathcal{POI}_n : i \in A\big\} \cup \big\{R_i \cap \mathcal{POI}_n : i \notin A\big\},$$

where A is any non-empty proper subset of $\{1, \ldots, n\}$ (type (M2)).

In particular, for $n \geq 2$, there are $2^n - 1$ maximal subsemigroups of $\mathcal{POI}_n$. The monoid $\mathcal{POI}_1 = \mathcal{PT}_1$ is a semilattice of order 2: its maximal subsemigroups are each of its singleton subsets.

Proof. Since $\mathcal{POI}_n$ is $\mathscr{H}$-trivial, there are no maximal subsemigroups of type (M5) by Lemma 2.1, and by Corollary 2.4, the maximal subsemigroup arising from the group of units is formed by removing the identity. The fact that the group of units is trivial implies that there are n orbits of $\mathscr{L}$-classes of $J_{n-1} \cap \mathcal{POI}_n$, each one being a singleton. Since the $\mathscr{R}$-class $R_i \cap \mathcal{POI}_n$ is equal to $(L_i \cap \mathcal{POI}_n)^{-1}$, it follows by Corollary 2.13 that the maximal subsemigroups of $\mathcal{POI}_n$ arising from $J_{n-1} \cap \mathcal{POI}_n$ are those described in the theorem. ◻

Theorem 3.10. *Let $n \in \mathbb{N}$, $n \geq 3$, be arbitrary. Let $\mathcal{PODI}_n$ be the inverse monoid of order-preserving and order-reversing partial permutations on $\{1, \ldots, n\}$ with the usual order, and let γ_n be the permutation of degree n that reverses this order. For $i, j \in \{1, \ldots, n\}$ define $\alpha_{i,j}$ to be the order-preserving partial permutation with domain $\{1, \ldots, n\} \setminus \{i\}$ and image $\{1, \ldots, n\} \setminus \{j\}$, and define $\beta_{i,j}$ to be the order-reversing partial permutation with this domain and image. Then the maximal subsemigroups of $\mathcal{PODI}_n$ are:*

(a) *$\mathcal{PODI}_n \setminus \{\gamma_n\}$ (type (M5));*

(b) *$(\mathcal{PODI}_n \setminus J_{n-1}) \cup I_A$, where n is even,*

$$I_A = \big\{\alpha_{i,j},\ \beta_{i,n-j+1},\ \beta_{n-i+1,j},\ \alpha_{n-i+1,n-j+1} : i, j \in A \text{ or } i, j \notin A\big\} \\ \cup \big\{\beta_{i,j},\ \alpha_{i,n-j+1},\ \alpha_{n-i+1,j},\ \beta_{n-i+1,n-j+1} : i \in A, j \notin A \text{ or } i \notin A, j \in A\big\},$$

and A is any subset of $\{2, \ldots, n/2\}$ (type (M5)); and

(c) *the union of $\mathcal{PODI}_n \setminus J_{n-1}$ and the union of*

$$\big\{(L_i \cup L_{n-i+1}) \cap \mathcal{PODI}_n : i \in A\big\} \cup \big\{(R_i \cup R_{n-i+1}) \cap \mathcal{PODI}_n : i \notin A\big\},$$

where A is any non-empty proper subset of $\{1, \ldots, \lceil n/2 \rceil\}$ (type (M2)).

There are 2 maximal subsemigroups of $\mathcal{PODI}_2 = \mathcal{I}_2$. In particular, for $n \geq 2$ there are $3 \cdot 2^{(n/2)-1} - 1$ maximal subsemigroups of $\mathcal{PODI}_n$ when n is even, and $2^{(n+1)/2} - 1$ when n is odd. The monoid $\mathcal{PODI}_1 = \mathcal{PT}_1$ is a semilattice of order 2: its maximal subsemigroups are each of its singleton subsets.

Proof. Since the group of units of $\mathcal{PODI}_n$ is $\langle \gamma_n \rangle$, which contains two elements, it follows by Corollary 2.4 that the maximal subsemigroup arising from the group of units of $\mathcal{PODI}_n$ is $\mathcal{PODI}_n \setminus \{\gamma_n\}$.

The graph $\Delta(\mathcal{PODI}_n)$ may be obtained as the induced subgraph of $\Delta(\mathcal{POD}_n)$ on those orbits of Green's classes that contain partial permutations. In particular, the orbits of $\langle \gamma_n \rangle$ on the $\mathscr{L}$-classes of $J_{n-1} \cap \mathcal{PODI}_n$ are the sets $\big\{L_i \cap \mathcal{PODI}_n,\ L_{n-i+1} \cap \mathcal{PODI}_n\big\}$, for $i \in \{1, \ldots, \lceil n/2 \rceil\}$. By Corollary 2.13, the maximal subsemigroups of $\mathcal{PODI}_n$ that arise from $J_{n-1} \cap \mathcal{PODI}_n$ are those of type (M2) described in the theorem — of which there are $2^{\lceil n/2 \rceil} - 2$ — along with any maximal subsemigroups of type (M5).

Suppose that M is a maximal subsemigroup of $\mathcal{PODI}_n$ of type (M5) arising from $J_{n-1} \cap \mathcal{PODI}_n$. By [26, Proposition 4, Case 1], the intersection of M with each $\mathscr{H}$-class of $J_{n-1} \cap \mathcal{PODI}_n$ is non-empty, and each of these intersections has some common size, q. Since an $\mathscr{H}$-class in $J_{n-1} \cap \mathcal{PODI}_n$ contains two elements, and M is a proper subsemigroup of $\mathcal{PODI}_n$ that lacks only elements from $J_{n-1} \cap \mathcal{PODI}_n$, it follows that $q = 1$. In other words, the intersection of M with each $\mathscr{H}$-class of $J_{n-1} \cap \mathcal{PODI}_n$ contains a single element. For $i, j \in \{1, \ldots, n\}$, let $\delta_{i,j}$ denote the unique element of M that is contained in the $\mathscr{H}$-class $\mathcal{PODI}_n \cap (L_i \cap R_j) = \{\alpha_{i,j}, \beta_{i,j}\}$ of $\mathcal{PODI}_n$. In other words, $M \cap (L_i \cap R_j) = \{\delta_{i,j}\}$.

Since M contains γ_n, it follows that $\delta_{i,j} \in M$ if and only if $\delta_{i,j}\gamma_n,\ \gamma_n\delta_{i,j},\ \gamma_n\delta_{i,j}\gamma_n \in M$. In particular, $\alpha_{i,j} \in M$ if and only if $\alpha_{i,j},\ \beta_{i,n-j+1},\ \beta_{n-i+1,j},\ \alpha_{n-i+1,n-j+1} \in M$, and $\beta_{i,j} \in M$ if and only if $\beta_{i,j},\ \alpha_{i,n-j+1},\ \alpha_{n-i+1,j},\ \beta_{n-i+1,n-j+1} \in M$. For odd n, this leads to the contradictory statement that $\alpha_{(n+1)/2,(n+1)/2} \in M$ if and only if $\beta_{(n+1)/2,(n+1)/2} \in M$, and so n is even.

Given these observations, in order to describe M, it suffices to specify $\delta_{i,j}$ for each $i, j \in \{1, \ldots, n/2\}$. Indeed, our description can be even more concise. We observe that $\delta_{i,i} = \alpha_{i,i}$, since M contains every idempotent of $\mathcal{PODI}_n$. This implies that $\delta_{i,j}\delta_{j,i} = \delta_{i,i} = \alpha_{i,i}$, and so $\delta_{i,j} = \alpha_{i,j}$ if and only if $\delta_{j,i} = \alpha_{j,i}$. Furthermore, $\delta_{i,j} = \delta_{i,1}\delta_{1,j}$. Thus, to specify $\delta_{i,j}$ for each $i, j \in \{1, \ldots, n\}$, it suffices to specify $\delta_{1,i}$ for each $i \in \{2, \ldots, n/2\}$. Let $A = \big\{i \in \{2, \ldots, n/2\} : \delta_{1,i} = \beta_{1,j}\big\}$. A routine calculation shows that $M = (\mathcal{PODI}_n \setminus J_{n-1}) \cup I_A$, where I_A is the set defined in the statement of the theorem.

Conversely, for an even number $n \geq 4$ and a subset $A \subseteq \{2, \ldots, n/2\}$, it is tedious, but routine, to verify that $(\mathcal{PODI}_n \setminus J_{n-1}) \cup I_A$ is a subsemigroup of $\mathcal{PODI}_n$; by construction, it intersects every $\mathscr{H}$-class of $\mathcal{PODI}_n$ non-trivially. Any maximal subsemigroup of $\mathcal{PODI}_n$ that contains $(\mathcal{PODI}_n \setminus J_{n-1}) \cup I_A$ has type (M5), and so by the preceding arguments, we see that it is equal to $(\mathcal{PODI}_n \setminus J_{n-1}) \cup I_A$. Thus $(\mathcal{PODI}_n \setminus J_{n-1}) \cup I_A$ is a maximal subsemigroup of $\mathcal{PODI}_n$ of type (M5).

For two subsets $A, A' \subseteq \{2, \ldots, n/2\}$, it is clear from the definitions that $I_A = I_{A'}$ if and only if $A = A'$. Thus there are $2^{(n/2)-1}$ maximal subsemigroups of type (M5) when n is even, and none when n is odd. ◻

3.5 $\mathcal{POP}_n$ and $\mathcal{POR}_n$

As far as we are aware, the maximal subsemigroups of $\mathcal{POP}_n$ and $\mathcal{POR}_n$ have not been previously considered in the literature. To state the results of this section, we require the following notation. Let $S \in \{\mathcal{POP}_n, \mathcal{POR}_n\}$. Then $J_{n-1} \cap S$ is a regular $\mathscr{J}$-class of S. The $\mathscr{L}$-classes of $J_{n-1} \cap S$ are the sets $L_i \cap S$ for each $i \in \{1, \ldots, n\}$, and the $\mathscr{R}$-classes of $J_{n-1} \cap S$ are the sets $R_i \cap S$ for each $i \in \{1, \ldots, n\}$ and $R_{\{i,i+1\}} \cap S$ for each $i \in \{1, \ldots, n-1\}$, along with the set $R_{\{1,n\}} \cap S$. The group of units of $\mathcal{POP}_n$ is $\mathcal{C}_n$, and the group of units of $\mathcal{POR}_n$ is $\mathcal{D}_n$ — see Section 1.2 for the definitions of these groups.

The following theorems are the main results of this section.

Theorem 3.11. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary and let $\mathcal{POP}_n$ be the monoid of orientation-preserving partial transformations on $\{1, \ldots, n\}$ with the usual order. Then the maximal subsemigroups of $\mathcal{POP}_n$ are:*

(a) *$(\mathcal{POP}_n \setminus \mathcal{C}_n) \cup L$, where L is a maximal subgroup of the cyclic group $\mathcal{C}_n$ (type (M5));*

(b) *$\mathcal{POP}_n \setminus \{\alpha \in \mathcal{OP}_n : \operatorname{rank}(\alpha) = n-1\}$ (type (M4)); and*

(c) *$\mathcal{POP}_n \setminus \{\alpha \in \mathcal{POPI}_n : \operatorname{rank}(\alpha) = n-1\}$ (type (M4)).*

The monoid $\mathcal{POP}_1 = \mathcal{PT}_1$ is a semilattice of order 2: its maximal subsemigroups are each of its singleton subsets. In particular, for $n \in \mathbb{N}$, there are $|\mathbb{P}_n| + 2$ maximal subsemigroups of $\mathcal{POP}_n$, where $\mathbb{P}_n$ is the set of primes that divide n.

Theorem 3.12. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary and let $\mathcal{POR}_n$ be the monoid of orientation-preserving and -reversing partial transformations on $\{1, \ldots, n\}$ with the usual order. Then the maximal subsemigroups of $\mathcal{POR}_n$ are:*

(a) *$(\mathcal{POR}_n \setminus \mathcal{D}_n) \cup L$, where L is a maximal subgroup of the group $\mathcal{D}_n$ (type (M5));*

(b) *$\mathcal{POR}_n \setminus \{\alpha \in \mathcal{OR}_n : \operatorname{rank}(\alpha) = n-1\}$ (type (M4)); and*

(c) *$\mathcal{POR}_n \setminus \{\alpha \in \mathcal{PORI}_n : \operatorname{rank}(\alpha) = n-1\}$ (type (M4)).*

In particular, there are 3 maximal subsemigroups of $\mathcal{POR}_2$, and for $n \geq 3$, there are $\sum_{p \in \mathbb{P}_n} p + 3$ maximal subsemigroups of $\mathcal{POR}_n$, where $\mathbb{P}_n$ is the set of the primes that divide n. The monoid $\mathcal{POR}_1 = \mathcal{PT}_1$ is a semilattice of order 2: its maximal subsemigroups are each of its singleton subsets.

Proof of Theorems 3.11 and 3.12. Let $S \in \{\mathcal{POP}_n, \mathcal{POR}_n\}$, and let G be the group of units of S. The maximal subsemigroups arising from the group of units follow by Lemma 2.5, Lemma 2.6, and Corollary 2.4. Since $\mathcal{PO}_n$ is idempotent generated [25, Theorem 3.13], and $S = \langle \mathcal{PO}_n, G \rangle$, it follows by Lemma 2.1 that there are no maximal subsemigroups of type (M5) arising from $J_{n-1} \cap S$.

The remainder of the proof is similar to the proof of Theorem 3.1 concerning the maximal subsemigroups of $\mathcal{PT}_n$. The group of units G of S acts transitively on the $\mathscr{L}$-classes of $J_{n-1} \cap S$, and so there are no maximal subsemigroups of types (M2) and (M3) by Lemma 2.12. On the other hand, G has two orbits on the set of $\mathscr{R}$-classes of $J_{n-1} \cap S$: it transitively permutes the $\mathscr{R}$-classes of transformations, and it transitively permutes the $\mathscr{R}$-classes of partial permutations. By Proposition 2.11, the two maximal subsemigroups of S of type (M4) are found by removing either the partial permutations, or the transformations, of rank $n-1$. By Proposition 2.7, there is no maximal subsemigroup of type (M1). ☐

3.6 $\mathcal{OP}_n$ and $\mathcal{OR}_n$

The maximal subsemigroups of $\mathcal{OP}_n$ and $\mathcal{OR}_n$ were originally described in [12]. We restate these results in the following theorems.

Theorem 3.13. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary and let $\mathcal{OP}_n$ be the monoid of orientation-preserving transformations on $\{1, \ldots, n\}$ with the usual order. Then the maximal subsemigroups of $\mathcal{OP}_n$ are:*

(a) *$(\mathcal{OP}_n \setminus \mathcal{C}_n) \cup L$, where L is a maximal subgroup of the cyclic group $\mathcal{C}_n$ (type (M5)); and*

(b) *$\mathcal{OP}_n \setminus \{\alpha \in \mathcal{OP}_n : \operatorname{rank}(\alpha) = n-1\}$ (type (M1)).*

In particular, for $n \geq 2$, there are $|\mathbb{P}_n| + 1$ maximal subsemigroups of $\mathcal{OP}_n$, where $\mathbb{P}_n$ is the set of primes that divide n

Theorem 3.14. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary and let $\mathcal{OR}_n$ be the monoid of orientation-preserving and orientation-reversing transformations on $\{1, \ldots, n\}$ with the usual order. Then the maximal subsemigroups of $\mathcal{OR}_n$ are:*

(a) $(\mathcal{OR}_n \setminus \mathcal{D}_n) \cup L$, *where* L *is a maximal subgroup of the group* $\mathcal{D}_n$ *(type (M5)); and*

(b) $\mathcal{OR}_n \setminus \{\alpha \in \mathcal{OR}_n : \operatorname{rank}(\alpha) = n-1\}$ *(type (M1)).*

In particular, there are 4 maximal subsemigroups of $\mathcal{OR}_2$*, and for* $n \geq 3$*, there are* $\sum_{p \in \mathbb{P}_n} p + 2$ *maximal subsemigroups of* $\mathcal{OR}_n$*, where* $\mathbb{P}_n$ *is the set of the primes that divide* n*.*

Proof of Theorems 3.13 and 3.14. Let $S \in \{\mathcal{OP}_n, \mathcal{OR}_n\}$, and let G be the group of units of S. The group of units of $\mathcal{OP}_n$ is $\mathcal{C}_n$ and the group of units of $\mathcal{OR}_n$ is $\mathcal{D}_n$. Therefore the description of the maximal subsemigroups arising from G follows by Corollary 2.4 and Lemma 2.5, or Lemma 2.6, as appropriate. Let $\alpha \in J_{n-1} \cap S$, and let ε be an idempotent of $\mathcal{O}_n$ of rank $n-1$. Clearly there exist permutations $\sigma, \tau \in G$ such that $\ker(\sigma\alpha\tau) = \ker(\varepsilon)$ and $\operatorname{im}(\sigma\alpha\tau) = \operatorname{im}(\varepsilon)$, i.e. $\sigma\alpha\tau \in H_\varepsilon^S$. Therefore $\varepsilon = (\sigma\alpha\tau)^k$ for some $k \in \mathbb{N}$. Since $\mathcal{O}_n$ is generated by its idempotents of rank $n-1$ [1] and $S = \langle G,\ \mathcal{O}_n \rangle$, it follows that $S = \langle G,\ \alpha \rangle$ if and only if $\alpha \in J_{n-1} \cap S$. The result follows by Corollary 2.19. ∎

3.7 $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$

The maximal subsemigroups of the inverse monoids $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$ have not been previously determined in the literature, as far as we are aware. These monoids exhibit maximal subsemigroups of type (M5) arising from a $\mathscr{J}$-class covered by the group of units, and to which we can apply the results of Section 2.2.2.

Let $S \in \{\mathcal{POPI}_n, \mathcal{PODI}_n\}$. Then $J_{n-1} \cap S$ is a regular $\mathscr{J}$-class of S consisting of partial permutations. By definition, $\mathcal{POPI}_n = \mathcal{POP}_n \cap \mathcal{I}_n$ and $\mathcal{PORI}_n = \mathcal{POR}_n \cap \mathcal{I}_n$. Therefore the group of units of $\mathcal{POPI}_n$ is $\mathcal{C}_n$ and the group of units of $\mathcal{PODI}_n$ is $\mathcal{D}_n$, and given the description of the Green's classes of $\mathcal{POP}_n$ and $\mathcal{POR}_n$ in Section 3.5, it follows that the $\mathscr{L}$-classes and $\mathscr{R}$-classes of $J_{n-1} \cap S$ are $\{L_i \cap S : i \in \{1, \ldots, n\}\}$ and $\{R_i \cap S : i \in \{1, \ldots, n\}\}$, respectively. See Section 1.2 for more information about this notation.

In the following theorems, which are the main results of this section, we use Proposition 2.17 to describe the maximal subsemigroups of $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$.

Theorem 3.15. *Let* $n \in \mathbb{N}$*,* $n \geq 3$*, be arbitrary and let* $\mathcal{POPI}_n$ *be the inverse monoid of orientation-preserving partial permutations of* $\{1, \ldots, n\}$ *with the usual order. For* $k \in \mathbb{N}$*, let* $\mathbb{P}_k$ *denote the set of all primes that divide* k*. Then the maximal subsemigroups of* $\mathcal{POPI}_n$ *are:*

(a) $(\mathcal{POPI}_n \setminus \mathcal{C}_n) \cup U$*, where* U *is a maximal subgroup of* $\mathcal{C}_n$ *(type (M5)), and*

(b) $\langle \mathcal{POPI}_n \setminus J_{n-1},\ \zeta^p \rangle$*, where* $p \in \mathbb{P}_{n-1}$ *and the partial permutation* ζ *is defined by*

$$\zeta = \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 & n \\ 2 & 3 & \cdots & n-1 & 1 & - \end{pmatrix}$$

(type (M5)).

In particular, there are $|\mathbb{P}_n| + |\mathbb{P}_{n-1}|$ *maximal subsemigroups of* $\mathcal{POPI}_n$ *for* $n \geq 3$*. For* $i \in \{1, 2\}$*,* $\mathcal{POPI}_n = \mathcal{I}_n$*; see Theorem 3.1.*

Proof. By Corollary 2.4 and Lemma 2.5, the maximal subsemigroups arising from the group of units are those described in the statement of the theorem, and there are $|\mathbb{P}_n|$ of them. It remains to describe the maximal subsemigroups that arise from $J_{n-1} \cap \mathcal{POPI}_n$.

Define

$$H = H_{\mathrm{id}_{n-1}}^{\mathcal{POPI}_n} = \{\alpha \in \mathcal{POPI}_n : \operatorname{dom}(\alpha) = \operatorname{im}(\alpha) = \{1, \ldots, n-1\}\}.$$

Then H is a group $\mathscr{H}$-class of $\mathcal{POPI}_n$ contained in the $\mathscr{J}$-class $J_{n-1} \cap \mathcal{POPI}_n$. Note that H is isomorphic to the cyclic group of order $n-1$, and is generated by ζ. The setwise stabilizer $\operatorname{Stab}_{\mathcal{C}_n}(H)$ is equal to the pointwise stabilizer of n in $\mathcal{C}_n$. Since this stabilizer is trivial, Proposition 2.17 implies that any maximal subgroup U of H gives rise to a maximal subsemigroup $\langle \mathcal{POPI}_n \setminus J_{n-1},\ U \rangle$ of $\mathcal{POPI}_n$. By Lemma 2.5, the maximal subgroups of H are $\langle \zeta^p \rangle$ for each $p \in \mathbb{P}_{n-1}$. It follows from Proposition 2.17 that these are the only maximal subsemigroups to arise from $J_{n-1} \cap \mathcal{POPI}_n$. ∎

Theorem 3.16. *Let* $n \in \mathbb{N}$*,* $n \geq 4$*, be arbitrary and let* $\mathcal{PORI}_n$ *be the inverse monoid of orientation-preserving or -reversing partial permutations of* $\{1, \ldots, n\}$ *with the usual order. For* $k \in \mathbb{N}$*, let* $\mathbb{P}_k$ *denote the set of all primes that divide* k*. Then the maximal subsemigroups of* $\mathcal{PORI}_n$ *are:*

(a) $(\mathcal{PORI}_n \setminus \mathcal{D}_n) \cup U$*, where* U *is a maximal subgroup of* $\mathcal{D}_n$ *(type (M5)), and*

(b) $\langle \mathcal{PORI}_n \setminus J_{n-1},\ \zeta^p,\ \tau\rangle$, *where* $p \in \mathbb{P}_{n-1}$, *and the partial permutations* ζ *and* τ *are defined by*

$$\zeta = \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 & n \\ 2 & 3 & \cdots & n-1 & 1 & - \end{pmatrix}, \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n-1 & n-2 & \cdots & 1 & - \end{pmatrix}$$

(type (M5)).

For $i \in \{1,2,3\}$, $\mathcal{PORI}_n = \mathcal{I}_n$. *In particular, there are* $1 + |\mathbb{P}_{n-1}| + \sum_{p\in\mathbb{P}_n} p$ *maximal subsemigroups of* $\mathcal{PORI}_n$ *for* $n \geq 3$.

Proof. The group of units of $\mathcal{PORI}_n$ is $\mathcal{D}_n$, and so by Corollary 2.4 and Lemma 2.6, the maximal subsemigroups arising from the group of units are as stated in the theorem, and there are $1 + \sum_{p\in\mathbb{P}_n} p$ of them.

Define

$$H = H^{\mathcal{PORI}_n}_{\mathrm{id}_{n-1}} = \big\{\alpha \in \mathcal{PORI}_n : \mathrm{dom}(\alpha) = \mathrm{im}(\alpha) = \{1, \ldots, n-1\}\big\}.$$

Then H is a group $\mathscr{H}$-class of $\mathcal{PORI}_n$ contained in the $\mathscr{J}$-class $J_{n-1} \cap \mathcal{POPI}_n$. Note that since $n \geq 4$, H is a dihedral group of order $2(n-1)$, and it is generated by the partial permutations ζ and τ. An element of $\mathcal{D}_n$ belongs to the setwise stabilizer $\mathrm{Stab}_{\mathcal{D}_n}(H)$ if and only if it stabilizes the set $\{1,\ldots,n-1\}$. Equivalently, $\mathrm{Stab}_{\mathcal{D}_n}(H)$ consists of those permutations in $\mathcal{D}_n$ that fix the point n. Thus $\mathrm{Stab}_{\mathcal{D}_n}(H)$ contains only id_n and the permutation in $\mathcal{D}_n$ that fixes n and reverses the order of $\{1,\ldots,n-1\}$. In particular,

$$\mathrm{id}_{n-1} \mathrm{Stab}_{\mathcal{D}_n}(H) = \big\{\mathrm{id}_{n-1} \cdot h : h \in \mathrm{Stab}_{\mathcal{D}_n}(H)\big\} = \{\mathrm{id}_{n-1}, \tau\}.$$

Since any subgroup of H contains id_{n-1}, it follows from Proposition 2.17 that the maximal subsemigroups arising from $\mathcal{PORI}_n \cap J_{n-1}$ are $\langle \mathcal{PORI}_n \setminus J_{n-1},\ U\rangle$, for each maximal subgroup U of H that contains τ. By Lemma 2.6, the maximal subgroups of H are $\langle\zeta\rangle$ and the subgroups $\langle \zeta^p,\ \zeta^{-i}\tau\zeta^i\rangle$, where $p \in \mathbb{P}_{n-1}$ and $0 \leq i \leq p-1$. Thus the maximal subgroups of H that contain τ are those in the latter form where $i = 0$. It follows that the maximal subsemigroups arising from $\mathcal{PORI}_n \cap J_{n-1}$ are those stated in the theorem, and that there are $|\mathbb{P}_{n-1}|$ such maximal subsemigroups. ∎

4 Diagram monoids

In this section, we determine the maximal subsemigroups of those monoids of partitions defined in Section 1.3. It is clear that each of these monoids is closed under the * operation, and is therefore a regular $*$-monoid. Furthermore, the monoid $\mathcal{I}_n^*$ and its submonoid $\mathfrak{F}_n$ are inverse; the remaining monoids are not inverse, in general.

There exists a natural injective function from the partial transformation monoid of degree n to the partition monoid of degree n. Thus we may think of partitions as generalisations of transformations. In this way, the notion of a *planar* partition is a generalisation of the notion of an *order-preserving* partial transformation, and the notion of an *annular* partition is a generalisation of the notion of an *orientation-preserving* partial transformation.

Let $\alpha \in \mathcal{P}_n$. We define $\ker(\alpha)$, the *kernel* of α, to be the restriction of the equivalence α to $\{1,\ldots,n\}$. We also define $\mathrm{dom}(\alpha)$, the *domain* of α, to be the subset of $\{1,\ldots,n\}$ comprising those points that are contained in a transverse block of α. Given these definitions, we define $\mathrm{coker}(\alpha) = \ker(\alpha^*)$ and $\mathrm{codom}(\alpha) = \mathrm{dom}(\alpha^*)$, the *cokernel* and *codomain* of α, respectively. For the majority of the monoids defined in Section 1.3, the Green's relations are completely determined by domain, kernel, and rank.

Lemma 4.1 ([38], [45, Theorem 17], [37, Theorem 5], and [15, Theorem 2.4])**.** *Let* $S \in \{\mathcal{P}_n, \mathcal{PB}_n, \mathcal{B}_n, \mathcal{I}_n^*, \mathcal{M}_n, \mathcal{J}_n\}$ *and let* $\alpha, \beta \in S$. *Then:*

(a) $\alpha \mathscr{R} \beta$ *if and only if* $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$ *and* $\ker(\alpha) = \ker(\beta)$*;*

(b) $\alpha \mathscr{L} \beta$ *if and only if* $\alpha^* \mathscr{R} \beta^*$, *i.e.* $\alpha \mathscr{L} \beta$ *if and only if* $\mathrm{codom}(\alpha) = \mathrm{codom}(\beta)$ *and* $\mathrm{coker}(\alpha) = \mathrm{coker}(\beta)$*; and*

(c) $\alpha \mathscr{J} \beta$ *if and only if* $\mathrm{rank}(\alpha) = \mathrm{rank}(\beta)$.

Parts (a) and (b) of Lemma 4.1 hold for all of the submonoids of $\mathcal{P}_n$ defined in Section 1.3, since each is a regular submonoid of $\mathcal{P}_n$ [31, Proposition 2.4.2]. The condition $\alpha \mathscr{J} \beta$ if and only if $\mathrm{rank}(\alpha) = \mathrm{rank}(\beta)$ can be shown to hold for the remaining monoids defined in Section 1.3, with the exception of $\mathfrak{F}_n$ — although here the condition does hold for uniform block bijections of ranks n or $n-1$. For $n \in \mathbb{N}$ and $k \in \{0,1,\ldots,n\}$, we define $J_k = \{\alpha \in \mathcal{P}_n : \mathrm{rank}(\alpha) = k\}$ to be the $\mathscr{J}$-class of $\mathcal{P}_n$ that comprises the partitions of rank k.

In general, if S is any of the diagram monoids defined in Section 1.3, then S has a unique $\mathscr{J}$-class J that is covered by the group of units of S. In several cases, to determine the maximal subsemigroups of S that arise from J, we require the graph $\Delta(S,J)$, as defined in Section 2.2.1. Given a description of the $\mathscr{L}$-classes and $\mathscr{R}$-classes of J, to describe $\Delta(S,J)$ it remains to describe the action of the group of units on the $\mathscr{R}$-classes of J. Since S is a regular $*$-monoid, a description of the action of the group of units on the $\mathscr{L}$-classes of J is obtained as a consequence; see Section 2.2.1. We observe that, for $\alpha \in \mathcal{P}_n$ and $\sigma \in \mathcal{S}_n$,

$$\operatorname{dom}(\sigma\alpha) = \{i\sigma^{-1} : i \in \operatorname{dom}(\alpha)\} \quad \text{and} \quad \ker(\sigma\alpha) = \{(i\sigma^{-1}, j\sigma^{-1}) : (i,j) \in \ker(\alpha)\}. \tag{1}$$

Given this description and Lemma 4.1, the action of a subgroup of $\mathcal{S}_n$ on the $\mathscr{R}$-classes of a particular $\mathscr{J}$-class is straightforward to determine.

4.1 The partition monoid $\mathcal{P}_n$

Let $n \in \mathbb{N}$, $n \geq 2$. We require the following information about the Green's classes of $\mathcal{P}_n$ in the $\mathscr{J}$-class J_{n-1}. Let $\alpha \in J_{n-1}$. By definition, α contains $n-1$ transverse blocks. Since each transverse block contains at least two points, and there are only $2n$ points in $\{1,\ldots,n\} \cup \{1',\ldots,n'\}$, there are few possible combinations of kernel and domain for α. In particular, either $\ker(\alpha)$ is trivial and $\operatorname{dom}(\alpha) = \{1,\ldots,n\} \setminus \{i\}$ for some $i \in \{1,\ldots,n\}$, or $\operatorname{dom}(\alpha) = \{1,\ldots,n\}$ and $\{i,j\}$ is the unique non-trivial kernel class of α, for some distinct $i,j \in \{1,\ldots,n\}$. By Lemma 4.1, these properties describe the $\mathscr{R}$-classes of J_{n-1}. Since the $\mathscr{L}$-classes and $\mathscr{R}$-classes of a regular $*$-semigroup correspond via the * operation, analogous statements hold for the $\mathscr{L}$-classes of J_{n-1}. Thus, for distinct $i,j \in \{1,\ldots,n\}$, we make the following definitions:

- $R_i = \{\alpha \in J_{n-1} : \operatorname{dom}(\alpha) = \{1,\ldots,n\} \setminus \{i\}\}$, an $\mathscr{R}$-class;
- $R_{\{i,j\}} = \{\alpha \in J_{n-1} : (i,j) \in \ker(\alpha)\}$, an $\mathscr{R}$-class;
- $L_i = R_i^* = \{\alpha \in J_{n-1} : \operatorname{codom}(\alpha) = \{1,\ldots,n\} \setminus \{i\}\}$, an $\mathscr{L}$-class;
- $L_{\{i,j\}} = R_{\{i,j\}}^* = \{\alpha \in J_{n-1} : (i,j) \in \operatorname{coker}(\alpha))\}$, an $\mathscr{L}$-class.

An $\mathscr{H}$-class of the form $L_i \cap R_j$ is a group if and only if $i = j$, an $\mathscr{H}$-class of the form $L_i \cap R_{\{j,k\}}$ or $R_i \cap L_{\{j,k\}}$ is a group if and only if $i \in \{j,k\}$, and an $\mathscr{H}$-class of the form $L_{\{i,j\}} \cap R_{\{k,l\}}$ is a group if and only if $\{i,j\} = \{k,l\}$.

The main result of this section is the following theorem.

Theorem 4.2. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\mathcal{P}_n$ be the partition monoid of degree n. Then the maximal subsemigroups of $\mathcal{P}_n$ are:*

(a) $(\mathcal{P}_n \setminus \mathcal{S}_n) \cup U$, *where U is a maximal subgroup of $\mathcal{S}_n$ (type (M5));*

(b) $\mathcal{P}_n \setminus \{\alpha \in \mathcal{P}_n : \operatorname{rank}(\alpha) = n-1$ *and* $\ker(\alpha)$ *is trivial*$\}$ *(type (M4));*

(c) $\mathcal{P}_n \setminus \{\alpha \in \mathcal{P}_n : \operatorname{rank}(\alpha) = n-1$ *and* $\operatorname{dom}(\alpha) = \{1,\ldots,n\}\}$ *(type (M4));*

(d) $\mathcal{P}_n \setminus \{\alpha \in \mathcal{P}_n : \operatorname{rank}(\alpha) = n-1$ *and* $\operatorname{coker}(\alpha)$ *is trivial*$\}$ *(type (M3)); and*

(e) $\mathcal{P}_n \setminus \{\alpha \in \mathcal{P}_n : \operatorname{rank}(\alpha) = n-1$ *and* $\operatorname{codom}(\alpha) = \{1,\ldots,n\}\}$ *(type (M3)).*

In particular, for $n \geq 2$, there are $s_n + 4$ maximal subsemigroups of $\mathcal{P}_n$, where s_n denotes the number of maximal subgroups of the symmetric group of degree n. The partition monoid of degree 1 is a semilattice of order 2: its maximal subsemigroups are each of its singleton subsets.

Proof. By [18, Section 6], the ideal $\mathcal{P}_n \setminus \mathcal{S}_n$ is generated by its idempotents of rank $n-1$. Thus, since $\mathscr{J}$-equivalence in $\mathcal{P}_n$ is determined by rank, the maximal subsemigroups of $\mathcal{P}_n$ arise from its group of units, $\mathcal{S}_n$, and the $\mathscr{J}$-class J_{n-1}. It follows by Corollary 2.4 that the maximal subsemigroups that arise from $\mathcal{S}_n$ are those stated in the theorem.

By Lemma 2.1, there are no maximal subsemigroups arising from J_{n-1} of type (M5). It is clear from (1) that $\mathcal{S}_n$ transitively permutes the $\mathscr{R}$-classes of J_{n-1} with trivial kernel, and it transitively permutes the $\mathscr{R}$-classes of J_{n-1} with domain $\{1,\ldots,n\}$. Thus there are two orbits of $\mathscr{R}$-classes of J_{n-1} under the action of $\mathcal{S}_n$; therefore there are two corresponding orbits of $\mathscr{L}$-classes. A picture of $\Delta(\mathcal{P}_n, J_{n-1})$ is shown in Figure 7.

Since the two bicomponents are the only maximal independent subsets of $\Delta(\mathcal{P}_n, J_{n-1})$, Corollary 2.9 implies that there are no maximal subsemigroups of type (M2). Each vertex of $\Delta(\mathcal{P}_n, J_{n-1})$ has degree 2, and so by Proposition 2.11, there are two maximal subsemigroups of type (M4), formed by removing each orbit of $\mathscr{R}$-classes in turn, and by Lemma 2.2, the corresponding maximal subsemigroups of type (M3) are those stated in the theorem. By Proposition 2.7, there are no maximal subsemigroups of type (M1). ∎

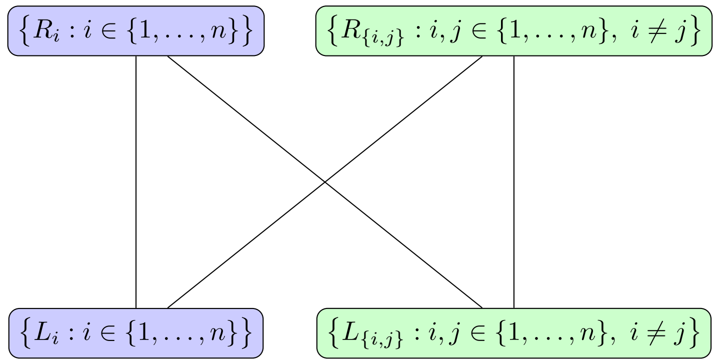

Figure 7: The graph $\Delta(\mathcal{P}_n, J_{n-1})$.

4.2 The partial Brauer monoid $\mathcal{PB}_n$

The symmetric inverse monoid degree n embeds in the partition monoid $\mathcal{P}_n$, via the injective homomorphism ϕ that maps a partial permutation α to the partition whose non-trivial blocks are $\{i,(i\alpha)'\}$ for each $i \in \mathrm{dom}(\alpha)$. Thus we define $\mathcal{I}_n$ to be copy of the symmetric inverse monoid of degree n embedded in $\mathcal{P}_n$ by ϕ. Clearly $\mathcal{I}_n$ is a submonoid of $\mathcal{PB}_n$.

To describe the maximal subsemigroups of $\mathcal{PB}_n$, we require a description of the elements of $\mathcal{PB}_n$ whose rank is at least $n-2$. Partitions of degree n that have rank n are units, and the group of units of $\mathcal{PB}_n$ is $\mathcal{S}_n$.

Let $\alpha \in J_{n-1} \cap \mathcal{PB}_n$. By definition, α contains precisely $n-1$ transverse blocks of size two, and two singleton blocks $\{i\}$ and $\{j'\}$, for some $i,j \in \{1,\dots,n\}$. Therefore α is the image of some partial permutation under the embedding ϕ. Since α was arbitrary, and $\mathcal{I}_n \subseteq \mathcal{PB}_n$, it follows that $J_{n-1} \cap \mathcal{PB}_n = J_{n-1} \cap \mathcal{I}_n$.

Let $\alpha \in \mathcal{PB}_n$, and suppose that $\mathrm{rank}(\alpha) = n-2$. Then α contains $n-2$ transverse blocks, which leaves a pair of points of $\{1,\dots,n\}$ and a pair of points of $\{1',\dots,n'\}$ that are not contained in transverse blocks. Each of these pairs forms either a block of size 2, or two singleton blocks. In particular, $\mathrm{dom}(\alpha)$ lacks some two points i and j, and either $\ker(\alpha)$ is trivial, or $\{i,j\}$ is the unique non-trivial kernel class of α. A similar statement holds for the codomain and cokernel of α.

Theorem 4.3. *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary and let $\mathcal{PB}_n$ be the partial Brauer monoid of degree n. Then the maximal subsemigroups of $\mathcal{PB}_n$ are:*

(a) *$(\mathcal{PB}_n \setminus \mathcal{S}_n) \cup U$, where U is a maximal subgroup of $\mathcal{S}_n$ (type (M5));*

(b) *$\mathcal{PB}_n \setminus \{\alpha \in \mathcal{PB}_n : \mathrm{rank}(\alpha) = n-1\}$ (type (M1));*

(c) *$\mathcal{PB}_n \setminus \{\alpha \in \mathcal{PB}_n : \mathrm{rank}(\alpha) = n-2$ and $\ker(\alpha)$ is non-trivial$\}$ (type (M4)); and*

(d) *$\mathcal{PB}_n \setminus \{\alpha \in \mathcal{PB}_n : \mathrm{rank}(\alpha) = n-2$ and $\mathrm{coker}(\alpha)$ is non-trivial$\}$ (type (M3)).*

In particular, for $n \geq 2$, there are $s_n + 3$ maximal subsemigroups of $\mathcal{PB}_n$, where s_n denotes the number of maximal subgroups of the symmetric group of degree n. The partial Brauer monoid $\mathcal{PB}_1 = \mathcal{P}_1$ is a semilattice of order 2: its maximal subsemigroups are each of its singleton subsets.

Proof. By [15], $\mathcal{PB}_n$ is generated by its elements with rank at least $n-2$, and any generating set contains elements of ranks n, $n-1$, and $n-2$. By Lemma 4.1, Green's $\mathscr{J}$-relation in $\mathcal{PB}_n$ is determined by rank. Thus, the $\mathscr{J}$-classes of $\mathcal{PB}_n$ from which there arise maximal subsemigroups are its group of units, $J_{n-1} \cap \mathcal{PB}_n$, and $J_{n-2} \cap \mathcal{PB}_n$. The group of units of $\mathcal{PB}_n$ is $\mathcal{S}_n$, and so by Corollary 2.4, the maximal subsemigroups that arise from the group of units are those stated.

Since the $\mathscr{J}$-class $J_{n-1} \cap \mathcal{PB}_n$ is covered by the group of units, $\mathcal{PB}_n \setminus J_{n-1}$ is a subsemigroup of $\mathcal{PB}_n$. Let $\alpha \in J_{n-1} \cap \mathcal{PB}_n = J_{n-1} \cap \mathcal{I}_n$ be arbitrary. As stated in the proof of Theorem 3.1, $\mathcal{I}_n$ is generated by its group of units $\mathcal{S}_n$ along with any element of rank $n-1$. Thus $\langle \mathcal{PB}_n \setminus J_{n-1},\ \alpha \rangle \supseteq \langle \mathcal{PB}_n \setminus J_{n-1},\ \mathcal{I}_n \rangle = \mathcal{PB}_n$. By using Lemma 2.18 with $k=1$ and $X_1 = J_{n-1} \cap \mathcal{PB}_n$, we find that the unique maximal subsemigroup of $\mathcal{PB}_n$ arising from this $\mathscr{J}$-class has type (M1).

In order to determine the maximal subsemigroups of $\mathcal{PB}_n$ that arise from its $\mathscr{J}$-class of rank $n-2$, we define the subsets

$$X = \{\alpha \in \mathcal{PB}_n : \mathrm{rank}(\alpha) = n-2 \text{ and } \ker(\alpha) \text{ is non-trivial}\}, \text{ and}$$
$$X^* = \{\alpha^* : \alpha \in X\} = \{\alpha \in \mathcal{PB}_n : \mathrm{rank}(\alpha) = n-2 \text{ and } \mathrm{coker}(\alpha) \text{ is non-trivial}\}.$$

Note that X is a union of $\mathscr{R}$-classes of $\mathcal{PB}_n$, and X^* is a union of $\mathscr{L}$-classes. Let A be a subset of $J_{n-2} \cap \mathcal{PB}_n$ such that $(\mathcal{PB}_n \setminus J_{n-2}) \cup A$ generates $\mathcal{PB}_n$. Let $\alpha \in X$ be arbitrary. Then α can the written as a product $\alpha = \beta_1 \cdots \beta_k$ of some of these generators. Clearly the generators $\beta_1, \ldots, \beta_k$ have rank at least $n-2$. Every element in $\mathcal{PB}_n$ of rank n and $n-1$ has a trivial kernel, and the subset of partitions with trivial kernel in $\mathcal{P}_n$ forms a subsemigroup. Thus there exists some $r \in \{1, \ldots, k\}$ such that $\operatorname{rank}(\beta_r) = n-2$ and $\ker(\beta_r)$ is non-trivial — in other words, $\beta_r \in X$. A dual argument shows that $A \cap X^* \neq \varnothing$. Conversely, for any subset A of $J_{n-2} \cap \mathcal{PB}_n$ that intersects X and X^* non-trivially, we have $\mathcal{PB}_n = \langle \mathcal{PB}_n \setminus J_{n-2},\ A \rangle$. By Lemma 2.18, the maximal subsemigroups of $\mathcal{PB}_n$ arising from $J_{n-2} \cap \mathcal{PB}_n$ are $\mathcal{PB}_n \setminus X$ and $\mathcal{PB}_n \setminus X^*$; these maximal subsemigroups have types (M4) and (M3), respectively. □

4.3 The Brauer monoid $\mathcal{B}_n$ and the uniform block bijection monoid $\mathfrak{F}_n$

The main results of this section are the following theorems, which describe the maximal subsemigroup of $\mathcal{B}_n$ and $\mathfrak{F}_n$.

Theorem 4.4. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\mathcal{B}_n$ be the Brauer monoid of degree n. Then the maximal subsemigroups of $\mathcal{B}_n$ are:*

(a) *$(\mathcal{B}_n \setminus \mathcal{S}_n) \cup U$, where U is a maximal subgroup of $\mathcal{S}_n$ (type (M5)); and*

(b) *$\mathcal{B}_n \setminus \{\alpha \in \mathcal{B}_n : \operatorname{rank}(\alpha) = n-2\}$ (type (M1)).*

In particular, for $n \geq 2$, there are $s_n + 1$ maximal subsemigroups of $\mathcal{B}_n$, where s_n denotes the number of maximal subgroups of the symmetric group of degree n.

Theorem 4.5. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\mathfrak{F}_n$ be the uniform block bijection monoid of degree n. Then the maximal subsemigroups of $\mathfrak{F}_n$ are:*

(a) *$(\mathfrak{F}_n \setminus \mathcal{S}_n) \cup U$, where U is a maximal subgroup of $\mathcal{S}_n$ (type (M5)); and*

(b) *$\mathfrak{F}_n \setminus \{\alpha \in \mathfrak{F}_n : \operatorname{rank}(\alpha) = n-1\}$ (type (M1)).*

In particular, for $n \geq 2$, there are $s_n + 1$ maximal subsemigroups of $\mathfrak{F}_n$, where s_n is the number of maximal subgroups of the symmetric group of degree n.

Let $n \in \mathbb{N}$, $n \geq 2$. By [2], $\mathcal{B}_n$ is generated by $\mathcal{S}_n$ and any projection of rank $n-2$, and by [37, Section 5], $\mathfrak{F}_n$ is generated by $\mathcal{S}_n$ and any projection of rank $n-1$. These facts are used in the following proof.

Proof of Theorems 4.4 and 4.5. The group of units of $\mathcal{B}_n$ and $\mathfrak{F}_n$ is $\mathcal{S}_n$. By Corollary 2.4, the maximal subsemigroups that arise from the group of units in each case are those described in the theorems.

Let $\alpha \in J_{n-2} \cap \mathcal{B}_n$. The non-transverse blocks of α are $\{i, j\}$ and $\{k', l'\}$ for some $i, j, k, l \in \{1, \ldots, n\}$ with $i \neq j$ and $k \neq l$. Let $\tau \in \mathcal{S}_n$ be a permutation that contains the blocks $\{k, i'\}$ and $\{l, j'\}$. Therefore the non-transverse blocks of $\alpha\tau$ are $\{i, j\}$ and $\{i', j'\}$, and so $(\alpha\tau)^m$ is a projection of rank $n-2$ for some $m \in \mathbb{N}$. Thus $\langle \mathcal{S}_n,\ \alpha \rangle \supseteq \langle \mathcal{S}_n,\ (\alpha\tau)^m \rangle = \mathcal{B}_n$, and so $\mathcal{B}_n = \langle \mathcal{S}_n,\ \alpha \rangle$. By a similar argument, $\mathfrak{F}_n = \langle \mathcal{S}_n,\ \beta \rangle$ for any uniform block bijection of rank $n-1$. By Corollary 2.19, the remaining maximal subsemigroups are those stated in the theorems. □

4.4 The dual symmetric inverse monoid $\mathcal{I}_n^*$

The maximal subsemigroups of the dual symmetric inverse monoid were first described in [37, Theorem 19]. We reprove this result in the following theorem.

Theorem 4.6. *Let $n \in \mathbb{N}$, $n \geq 3$, and let $\mathcal{I}_n^*$ be the dual symmetric inverse monoid of degree n. Then the maximal subsemigroups of $\mathcal{I}_n^*$ are:*

(a) *$(\mathcal{I}_n^* \setminus \mathcal{S}_n) \cup U$, where U is a maximal subgroup of $\mathcal{S}_n$ (type (M5)); and*

(b) *$\mathcal{I}_n^* \setminus \{\alpha \in \mathcal{I}_n^* : \operatorname{rank}(\alpha) = n-1$ and α is not uniform$\}$ (type (M5)).*

In particular, for $n \geq 3$ there are $s_n + 1$ maximal subsemigroups of $\mathcal{I}_n^*$, where s_n is the number of maximal subgroups of the symmetric group of degree n. For $n \in \{1, 2\}$, $\mathcal{I}_n^* = \mathfrak{F}_n$; see Theorem 4.5.

Proof. The group of units of $\mathcal{I}_n^*$ is $\mathcal{S}_n$. By Corollary 2.4, the maximal subsemigroups arising from the group of units are those described. By [37, Proposition 16], $\mathcal{I}_n^* = \langle \mathcal{S}_n,\ \alpha \rangle$ if and only if $\alpha \in J_{n-1} \cap (\mathcal{I}_n^* \setminus \mathfrak{F}_n)$. Using Corollary 2.19 with $X = J_{n-1} \cap (\mathcal{I}_n^* \setminus \mathfrak{F}_n)$, the result follows. □

The maximal subsemigroup of $\mathcal{I}_n^*$ that arises from its $\mathscr{J}$-class $J_{n-1} \cap \mathcal{I}_n^*$ can also be found by using Proposition 2.17, since $\mathcal{S}_n$ acts transitively on the $\mathscr{R}$-classes of $J_{n-1} \cap \mathcal{I}_n^*$, and each of the idempotents in this $\mathscr{J}$-class is a projection.

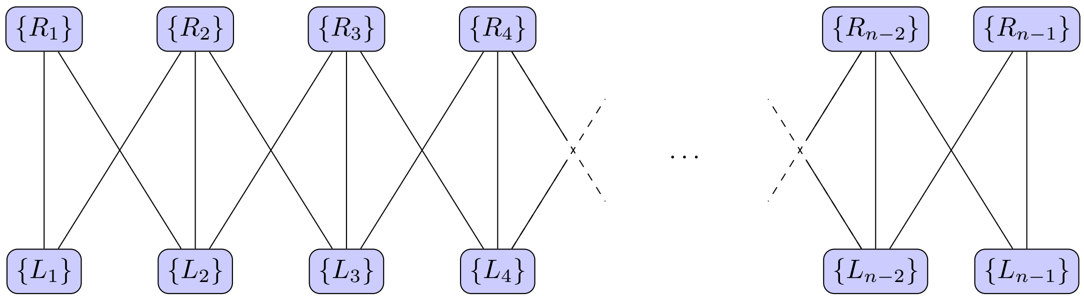

Figure 8: The graph $\Delta(\mathcal{J}_n, \mathcal{J}_n \cap J_{n-2})$.

4.5 The Jones monoid $\mathcal{J}_n$ and the annular Jones monoid $\mathcal{AJ}_n$

Let $n \in \mathbb{N}$. In this section, we determine the maximal subsemigroups of the Jones monoid $\mathcal{J}_n$ (also known as the Temperley-Lieb monoid), and the annular Jones monoid $\mathcal{AJ}_n$. Since the planar partition monoid of degree n is isomorphic to the Jones monoid of degree $2n$ [29], by determining the maximal subsemigroups of $\mathcal{J}_n$ we obtain those of $\mathcal{PP}_n$.

Suppose that $n \geq 2$. By [8], $\mathcal{J}_n$ is generated by the identity partition id_n and its projections of rank $n-2$. By Lemma 4.1, the set $J_{n-2} \cap \mathcal{J}_n$ is a $\mathscr{J}$-class of $\mathcal{J}_n$, and since there are no elements of $\mathcal{J}_n$ with rank $n-1$, it follows that this $\mathscr{J}$-class is covered by the group of units $\{\mathrm{id}_n\}$. Note that $\mathcal{J}_n$ is $\mathscr{H}$-trivial, since it consists of planar partitions.

To describe the maximal subsemigroups of $\mathcal{J}_n$ that arise from its $\mathscr{J}$-class of rank $n-2$ partitions, we require the graph $\Delta(\mathcal{J}_n, J_{n-2} \cap \mathcal{J}_n)$, which we hereafter refer to as $\Delta(\mathcal{J}_n)$. Thus we require a description of the Green's classes of $J_{n-2} \cap \mathcal{J}_n$. Let $\alpha \in J_{n-2} \cap \mathcal{J}_n$. Then α has $n-2$ transverse blocks, and these contain two points. By planarity, the remaining blocks are of the form $\{i, i+1\}$ and $\{j', (j+1)'\}$ for some $i, j \in \{1, \ldots, n-1\}$. By Lemma 4.1, the $\mathscr{J}$-class $J_{n-2} \cap \mathcal{J}_n$ contains $n-1$ $\mathscr{R}$-classes and $n-1$ $\mathscr{L}$-classes. For $i \in \{1, \ldots, n-1\}$, we define

- $R_i = \big\{\alpha \in \mathcal{J}_n : \mathrm{rank}(\alpha) = n-2 \text{ and } \{i, i+1\} \text{ is a block of } \alpha\big\}$, an $\mathscr{R}$-class; and
- $L_i = \big\{\alpha \in \mathcal{J}_n : \mathrm{rank}(\alpha) = n-2 \text{ and } \{i', (i+1)'\} \text{ is a block of } \alpha\big\}$, an $\mathscr{L}$-class.

The intersection of the $\mathscr{L}$-class L_i and the $\mathscr{R}$-class R_j is a group if and only if $|i-j| \leq 1$. Since the group of units of $\mathcal{J}_n$ is trivial, its action on the $\mathscr{L}$-classes and $\mathscr{R}$-classes of $J_{n-2} \cap \mathcal{J}_n$ is trivial. A picture of $\Delta(\mathcal{J}_n)$ is shown in Figure 8.

The maximal independent subsets of $\Delta(\mathcal{J}_n)$ are counted in the following lemma.

Lemma 4.7. *Let $n \geq 2$. The number of maximal independent subsets of $\Delta(\mathcal{J}_n)$ is $2F_{n-1}$, where F_{n-1} is the $(n-1)^{th}$ term of the Fibonacci sequence, defined by $F_1 = F_2 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for $k \geq 3$.*

Proof. The result may be verified directly for $n \in \{2, 3\}$. Suppose that $n \geq 4$, and let K be a maximal independent subset of $\Delta(\mathcal{J}_n)$. We first show that precisely one of $\{L_{n-1}\}$ and $\{R_{n-1}\}$ is contained in K. Since $\{L_{n-1}\}$ and $\{R_{n-1}\}$ are adjacent in $\Delta(\mathcal{J}_n)$, they are not both contained in K. Similarly, at least one of $\{L_{n-2}\}$ and $\{R_{n-2}\}$ is not contained in K. If $\{L_{n-2}\} \notin K$, then either $\{L_{n-1}\} \in K$, or the maximality of K implies that $\{R_{n-1}\} \in K$. If instead $\{R_{n-2}\} \notin K$, then it follows similarly that either $\{L_{n-1}\} \in K$ or $\{R_{n-1}\} \in K$. Therefore we shall count $a(n)$, the number of maximal independent subsets of $\Delta(\mathcal{J}_n)$ that contain $\{L_{n-1}\}$. By symmetry, the total number of maximal independent subsets is $2a(n)$.

For $i \in \{1, 2\}$, define Λ_{n-i} be the induced subgraph of $\Delta(\mathcal{J}_n)$ on the vertices $\big\{\{L_1\}, \ldots, \{L_{n-1-i}\}, \{R_1\}, \ldots, \{R_{n-1-i}\}\big\}$. Clearly Λ_{n-i} is isomorphic to $\Delta(\mathcal{J}_{n-i})$, and so the number of maximal independent subsets of Λ_{n-1} that contain $\{L_{n-2}\}$ is $a(n-1)$, and the number of maximal independent subsets of Λ_{n-2} that contain $\{R_{n-3}\}$ is $a(n-2)$.

Let K be a maximal independent subset of $\Delta(\mathcal{J}_n)$ containing $\{L_{n-1}\}$. Certainly K contains neither $\{R_{n-1}\}$ nor $\{R_{n-2}\}$, since these vertices are adjacent to $\{L_{n-1}\}$ in $\Delta(\mathcal{J}_n)$. However, K contains precisely one of $\{L_{n-2}\}$ or $\{R_{n-3}\}$: it does not contain both, since $\{L_{n-2}\}$ and $\{R_{n-3}\}$ are adjacent, but if $\{R_{n-3}\} \notin K$, then $K \cup \big\{\{L_{n-2}\}\big\}$ is a maximal independent subset containing K, and so $\{L_{n-2}\} \in K$.

If K contains $\{L_{n-2}\}$, then $K \setminus \big\{\{L_{n-1}\}\big\}$ is a maximal independent subset of Λ_{n-1} that contains $\{L_{n-2}\}$, while if K contains $\{R_{n-3}\}$, then $K \setminus \big\{\{L_{n-1}\}\big\}$ is a maximal independent subset of Λ_{n-2} that contains $\{R_{n-3}\}$. Conversely, maximal independent subsets of Λ_{n-1} containing $\{L_{n-2}\}$, and maximal independent subsets of Λ_{n-2} containing $\{R_{n-3}\}$, give rise to distinct maximal independent subsets of $\Delta(\mathcal{J}_n)$ that contain $\{L_{n-1}\}$, via the addition of $\{L_{n-1}\}$. It follows that $a(n) = a(n-1) + a(n-2)$. By this recurrence, and since $a(2) = F_1$ and $a(3) = F_2$, it follows that $a(n) = F_{n-1}$. $\square$

Whilst the maximal independent subsets of $\Delta(\mathcal{J}_n)$ may be readily counted, it is more difficult to describe them. Let $n \in \mathbb{N}$, $n \geq 3$ be arbitrary, and let K be a maximal independent subset of $\Delta(\mathcal{J}_n)$. As described in the proof of Lemma 4.7, K contains precisely one of $\{L_{n-1}\}$ and $\{R_{n-1}\}$, and a similar argument shows that K contains precisely one of $\{L_1\}$ and $\{R_1\}$. Let $i \in \{1, \ldots, n-2\}$. If $\{L_i\} \in K$, then either $\{L_{i+1}\} \in K$ or $\{R_{i+2}\} \in K$, while if $\{R_i\} \in K$, then either $\{R_{i+1}\} \in K$, or $\{L_{i+2}\} \in K$. In other words, K contains either $\{L_1\}$ or $\{R_1\}$, and each subsequent vertex contains either the same type of Green's class of index one higher, or it contains the other type of Green's class of index two higher; the final vertex is either $\{L_{n-1}\}$ or $\{R_{n-1}\}$. Conversely, any subset of vertices of $\Delta(\mathcal{J}_n)$ satisfying these requirements is a maximal independent subset of $\Delta(\mathcal{J}_n)$.

We may now describe the maximal subsemigroups of $\mathcal{J}_n$.

Theorem 4.8. *Let $n \in \mathbb{N}$, $n \geq 3$, and let $\mathcal{J}_n$ be the Jones monoid of degree n. Then the maximal subsemigroups of $\mathcal{J}_n$ are:*

(a) $\mathcal{J}_n \setminus \{\mathrm{id}_n\}$ *(type (M1));*

(b) *The union of $\mathcal{J}_n \setminus J_{n-2}$ and the union of the Green's classes contained in a maximal independent subset of $\Delta(\mathcal{J}_n)$ that is not a bicomponent of $\Delta(\mathcal{J}_n)$ (type (M2));*

(c) *$\mathcal{J}_n \setminus L$, where L is any $\mathscr{L}$-class in $\mathcal{J}_n$ of rank $n-2$ (type (M3)); and*

(d) *$\mathcal{J}_n \setminus R$, where R is any $\mathscr{R}$-class in $\mathcal{J}_n$ of rank $n-2$ (type (M4)).*

In particular, for $n \geq 3$, there are $2F_{n-1} + 2n - 3$ maximal subsemigroups of $\mathcal{J}_n$, where F_{n-1} is the $(n-1)^{th}$ term of the Fibonacci sequence, defined by $F_1 = F_2 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for $k \geq 3$. The Jones monoid $\mathcal{J}_2$ is a semilattice of order 2: its maximal subsemigroups are each of its singleton subsets.

Proof. Since the Jones monoid $\mathcal{J}_n$ is $\mathscr{H}$-trivial, by Corollary 2.4 the unique maximal subsemigroup arising from the group of units is formed by removing the identity of $\mathcal{J}_n$, and by Lemma 2.1, there are no maximal subsemigroups of type (M5). By Lemma 4.7 and Corollary 2.9, there are $2F_{n-1} - 2$ maximal subsemigroups of type (M2) arising from the $\mathscr{J}$-class of rank $n-2$ in $\mathcal{J}_n$; their description is a restatement of Proposition 2.8. Since each vertex of $\Delta(\mathcal{J}_n)$ has degree at least 2, it follows by Proposition 2.10 that any $\mathscr{L}$-class of rank $n-2$ can be removed to form a maximal subsemigroup of type (M3), and similarly any $\mathscr{R}$-class of rank $n-2$ can be removed to form a maximal subsemigroup of type (M4). Thus there are $n-1$ maximal subsemigroups of each of these types. By Proposition 2.7, there is no maximal subsemigroup of type (M1). ∎

It remains to describe the maximal subsemigroups of $\mathcal{A}\mathcal{J}_n$, the annular Jones monoid of degree n, which was defined in Section 1.3. Recall that $\rho_n \in \mathcal{A}\mathcal{J}_n$ is the partition of degree n with blocks $\{n, 1'\}$ and $\{i, (i+1)'\}$ for $i \in \{1, \ldots, n-1\}$, and that a partition is annular if it equals $\rho_n^i \beta \rho_n^j$ for some planar partition β and indices $i, j \in \mathbb{Z}$; see Section 1.3. To determine the maximal subsemigroups of $\mathcal{A}\mathcal{J}_n$, we require a small generating set for $\mathcal{A}\mathcal{J}_n$.

Let $\alpha = \rho_n^i \beta \rho_n^j \in \mathcal{A}\mathcal{J}_n$ be arbitrary, where β is planar and $i, j \in \mathbb{Z}$. It follows that $\beta = \rho_n^{-i} \alpha \rho_n^{-j} \in \mathcal{A}\mathcal{J}_n$, and so β is a planar partition whose blocks have size two. Therefore $\beta \in \mathcal{J}_n$, and

$$\mathcal{A}\mathcal{J}_n = \left\{\rho_n^i \beta \rho_n^j : i, j \in \{1, \ldots, n\},\ \beta \in \mathcal{J}_n\right\} = \langle \mathcal{J}_n,\ \rho_n \rangle.$$

Given projections $\xi, \zeta \in J_{n-2} \cap \mathcal{A}\mathcal{J}_n$, there exists some $i \in \{1, \ldots, n\}$ such that $\xi = \rho_n^{-i} \zeta \rho_n^i$. Since $\mathcal{J}_n$ is generated by the identity partition id_n and its projections of rank $n-2$ [8], it follows that $\mathcal{A}\mathcal{J}_n = \langle \rho_n,\ \xi \rangle$, where ξ is an arbitrary projection in $\mathcal{A}\mathcal{J}_n$ of rank $n-2$. We proceed by using the same technique as was used in the proof of Theorems 4.4 and 4.5. Let $\alpha \in J_{n-2} \cap \mathcal{A}\mathcal{J}_n$. There exists some index $i \in \{1, \ldots, n\}$ such that the non-transverse blocks of $\alpha \rho_n^i$ are either $\{k, k+1\}$ and $\{k', (k+1)'\}$ for some $k \in \{1, \ldots, n-1\}$, or are $\{1, n\}$ and $\{1', n'\}$. Thus $(\alpha \rho_n^i)^m$ is a projection for some $m \in \mathbb{N}$. It follows that $\langle \rho_n,\ \alpha \rangle \supseteq \langle \rho_n,\ (\alpha \rho_n^i)^m \rangle = \mathcal{A}\mathcal{J}_n$.

We may now state and prove the following theorem.

Theorem 4.9. *Let $n \in \mathbb{N}$, $n \geq 2$, let $\mathcal{A}\mathcal{J}_n$ be the annular Jones monoid of degree n, let ρ_n be the partition of degree n with blocks $\{n, 1'\}$ and $\{i, (i+1)'\}$ for $i \in \{1, \ldots, n-1\}$, and let $\mathbb{P}_n$ be the set of primes that divide n. Then the maximal subsemigroups of $\mathcal{A}\mathcal{J}_n$ are:*

(a) $(\mathcal{A}\mathcal{J}_n \setminus \langle \rho_n \rangle) \cup \langle \rho_n^d \rangle$, *where $d \in \mathbb{P}_n$ (type (M5)); and*

(b) $\mathcal{A}\mathcal{J}_n \setminus \{\alpha \in \mathcal{A}\mathcal{J}_n : \mathrm{rank}(\alpha) = n-2\}$ *(type (M1)).*

In particular, for $n \geq 2$ there are $|\mathbb{P}_n| + 1$ maximal subsemigroups of $\mathcal{A}\mathcal{J}_n$.

Proof. Since the group of units of $\mathcal{A}\mathcal{J}_n$ is $\langle \rho_n \rangle$, a cyclic group of order n, it follows by Lemma 2.5 and Corollary 2.4 that the maximal subsemigroups arising from the group of units are those given in the theorem. As described above, $\mathcal{A}\mathcal{J}_n = \langle \rho_n,\ \alpha \rangle$ if and only if $\alpha \in J_{n-2} \cap \mathcal{A}\mathcal{J}_n$. By Corollary 2.19, the sole remaining maximal subsemigroup is $\mathcal{A}\mathcal{J}_n \setminus J_{n-2}$, as required. ∎

4.6 The Motzkin monoid $\mathcal{M}_n$

Finally, in this section, we describe and count the maximal subsemigroups of the Motzkin monoid $\mathcal{M}_n$. Let $n \in \mathbb{N}$, $n \geq 2$. By [15, Proposition 4.2], $\mathcal{M}_n$ is generated by its elements of rank at least $n-2$, and any generating set for $\mathcal{M}_n$ contains elements of ranks n, $n-1$, and $n-2$. By Lemma 4.1, Green's $\mathscr{J}$-relation on $\mathcal{M}_n$ is determined by rank, and so the maximal subsemigroups of $\mathcal{M}_n$ arise from the $\mathscr{J}$-classes that correspond to these ranks. To describe the maximal subsemigroups of $\mathcal{M}_n$, we therefore require a description of its elements that have rank at least $n-2$.

Clearly the unique element of $\mathcal{M}_n$ of rank n is id_n.

An arbitrary element of rank $n-1$ in $\mathcal{M}_n$ has trivial kernel and cokernel, and is uniquely determined by the point i that it lacks from its domain and the point j that it lacks from its codomain. By Lemma 4.1, this information determines the $\mathscr{L}$- and $\mathscr{R}$-classes of $J_{n-1} \cap \mathcal{M}_n$. In other words, for any element in $J_{n-1} \cap \mathcal{M}_n$, there exist points $i, j \in \{1, \ldots, n\}$ such that $\{i\}$ and $\{j'\}$ are its unique singleton blocks. An element of $J_{n-1} \cap \mathcal{M}_n$ is an idempotent if its domain and codomain are equal, and so every idempotent in $J_{n-1} \cap \mathcal{M}_n$ is a projection.

Let α be an arbitrary element of rank $n-2$ in $\mathcal{M}_n$. Then α lacks two points $i, j \in \{1, \ldots, n\}$ from its domain, and either $\ker(\alpha)$ is trivial, or it contains the unique non-trivial kernel class $\{i, j\}$, in which case $|i-j| = 1$. Similarly, α lacks two points from its codomain, and either $\mathrm{coker}(\alpha)$ is trivial, or it contains a unique non-trivial class with two consecutive points. By Lemma 4.1, we obtain a description of the Green's classes of $\mathcal{M}_n$ of rank $n-2$.

Let $\alpha, \beta \in \mathcal{M}_n$ be elements of rank $n-1$. There exist numbers $i, j, k, l \in \{1, \ldots, n\}$ such that $\{i\}$ and $\{j'\}$ are the singleton blocks of α and $\{k\}$ and $\{l'\}$ are the singleton blocks of β. If $j = k$, then $\alpha\beta$ has rank $n-1$, and its singleton blocks are $\{i\}$ and $\{l'\}$. Otherwise, $\alpha\beta \in \mathcal{M}_n$ has rank $n-2$, and has trivial kernel and cokernel. Conversely, any element of $\mathcal{M}_n$ of rank $n-2$ with trivial kernel and cokernel can be written as the product of two elements of $\mathcal{M}_n$ of rank $n-1$. By [15, Lemma 4.11], it follows that $\mathcal{M}_n$ is generated by its elements of ranks n and $n-1$, along with the collection of projections of rank $n-2$ that have non-trivial kernel and cokernel.

The main result of this section is the following theorem.

Theorem 4.10. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\mathcal{M}_n$ be the Motzkin monoid of degree n. Then the maximal subsemigroups of $\mathcal{M}_n$ are:*

(a) $\mathcal{M}_n \setminus \{\mathrm{id}_n\}$ *(type (M1));*

(b) *The union of $\mathcal{M}_n \setminus J_{n-1}$ and*

$$\bigcup_{i \in A} \{\alpha \in \mathcal{M}_n : \mathrm{rank}(\alpha) = n-1 \text{ and } \{i\} \text{ is a block of } \alpha\} \cup \bigcup_{i \notin A} \{\alpha \in \mathcal{M}_n : \mathrm{rank}(\alpha) = n-1 \text{ and } \{i'\} \text{ is a block of } \alpha\},$$

where A is any non-empty proper subset of $\{1, \ldots, n\}$ (type (M2));

(c) $\mathcal{M}_n \setminus \big\{\alpha \in J_{n-2} : \{i, i+1\} \text{ is a block of } \alpha\big\}$ *for $i \in \{1, \ldots, n-1\}$ (type (M4)); and*

(d) $\mathcal{M}_n \setminus \big\{\alpha \in J_{n-2} : \{i', (i+1)'\} \text{ is a block of } \alpha\big\}$ *for $i \in \{1, \ldots, n-1\}$ (type (M3)).*

In particular, for $n \geq 2$, there are $2^n + 2n - 3$ maximal subsemigroups of the Motzkin monoid of degree n. The Motzkin monoid $\mathcal{M}_1 = \mathcal{P}_1$ is a semilattice of order 2: its maximal subsemigroups are each of its singleton subsets.

Proof. The group of units of $\mathcal{M}_n$ is the trivial group $\{\mathrm{id}_n\}$, and so by Corollary 2.4 the sole maximal subsemigroup of $\mathcal{M}_n$ that arises from its group of units is formed by removing it. Given the above description of the $\mathscr{J}$-class $J_{n-1} \cap \mathcal{M}_n$, it follows from Corollary 2.13 that the maximal subsemigroups that arise from this $\mathscr{J}$-class are those described in the theorem of type (M2), and that there are $2^n - 2$ of them; there are no maximal subsemigroups of type (M5) since $\mathcal{M}_n$ is $\mathscr{H}$-trivial. It remains to describe the maximal subsemigroups that arise from the $\mathscr{J}$-class of rank $n-2$.

For $i \in \{1, \ldots, n-1\}$, we define the subsets

$$X_i = \{\alpha \in \mathcal{M}_n : \mathrm{rank}(\alpha) = n-2 \text{ and } \{i, i+1\} \text{ is a block of } \alpha\}, \text{ and}$$
$$X_i^* = \{\alpha^* : \alpha \in X_i\} = \{\alpha \in \mathcal{M}_n : \mathrm{rank}(\alpha) = n-2 \text{ and } \{i', (i+1)'\} \text{ is a block of } \alpha\}$$

of the $\mathscr{J}$-class $J_{n-2} \cap \mathcal{M}_n$. Note that X_i is an $\mathscr{R}$-class of $\mathcal{M}_n$ and X_i^* is an $\mathscr{L}$-class of $\mathcal{M}_n$.

Let A be a subset of $J_{n-2} \cap \mathcal{M}_n$ such that $(\mathcal{M}_n \setminus J_{n-2}) \cup A$ generates $\mathcal{M}_n$. Let $i \in \{1, \ldots, n-1\}$ and $\alpha \in X_i$ be arbitrary. Then α can be written as a product $\alpha = \beta_1 \cdots \beta_k$ of the generators that have rank $n-1$ or $n-2$. If $\mathrm{rank}(\beta_1) = n-1$, then β_1, and each of its right-multiples, contains a singleton block of the form $\{j\}$ for some $j \in \{1, \ldots, n\}$. However, α is a right-multiple of β_1 and α contains no such block; thus $\mathrm{rank}(\beta_1) = n-2 = \mathrm{rank}(\alpha)$. Lemmas 2.15 and 4.1 imply that $\ker(\alpha) = \ker(\beta_1)$, i.e. $A \cap X_i \neq \varnothing$. A dual argument shows that $A \cap X_i^* \neq \varnothing$.

Conversely, for any subset A of $J_{n-2} \cap \mathcal{M}_n$ that intersects X_i and X_i^* non-trivially for all $i \in \{1, \ldots, n-1\}$, it is straightforward to see that $\langle \mathcal{M}_n \setminus J_{n-2},\ \cup A \rangle$ contains every projection in $J_{n-2} \cap \mathcal{M}_n$, and hence is equal to $\mathcal{M}_n$. By Lemma 2.18, the maximal subsemigroups of $\mathcal{M}_n$ arising from its $\mathscr{J}$-class of rank $n-2$ are the sets $\mathcal{M}_n \setminus X_i$ and $\mathcal{M}_n \setminus X_i^*$ for $i \in \{1, \ldots, n-1\}$; these maximal subsemigroups have types (M4) and (M3), respectively. $\square$

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