Fixed-point-free involutions and Schur P-positivity

Zachary Hamaker
Department of Mathematics
University of Michigan
zachary.hamaker@gmail.com

Eric Marberg
Department of Mathematics
HKUST
eric.marberg@gmail.com

Brendan Pawlowski*
Department of Mathematics
University of Michigan
br.pawlowski@gmail.com

Abstract

We prove that the *fixed-point-free involution Stanley symmetric functions* $\hat{F}^{\mathtt{FPF}}_z$ are Schur P-positive, and classify the involutions z for which $\hat{F}^{\mathtt{FPF}}_z$ is a single Schur P-function. These symmetric functions arise as the stable limits of the analogues of Schubert polynomials for the orbits of the symplectic group in the flag variety. They are indexed by fixed-point-free involutions in the symmetric group. Our proof of Schur P-positivity is constructive, and provides an efficient algorithm to compute the expansion of $\hat{F}^{\mathtt{FPF}}_z$ into Schur P-summands. We prove that this expansion is unitriangular with respect to the dominance order on partitions. As a corollary, we prove that the fixed-point-free involution Stanley symmetric function of the reverse permutation is a Schur P-function indexed by a shifted staircase shape.

Contents

*This author was partially supported by NSF grant 1148634.

1 Introduction

This article is a self-contained continuation of [9]. The principal objects of interest will be certain symmetric functions associated to fixed-point-free involutions in symmetric groups. The definition of these symmetric functions can be motivated by enumerative problems involving analogues of reduced words for permutations, which we now describe.

Let n be an even positive integer and write S_n for the symmetric group on $[n] = \{1, 2, \dots, n\}$. Define $\mathcal{F}_n$ as the S_n-conjugacy class of the permutation $\Theta_n = (1, 2)(3, 4) \cdots (n-1, n) \in S_n$. This is the set of all fixed-point-free involutions of $[n]$. Let $s_i = (i, i+1) \in S_n$ be the permutation exchanging i and $i+1$, and recall that a *reduced word* for $w \in S_n$ is a sequence of simple transpositions $(s_{i_1}, s_{i_2}, \dots, s_{i_\ell})$ of minimal possible length ℓ such that $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$. Write $\mathcal{R}(w)$ for the set of reduced words for $w \in S_n$ and consider the following similar definition:

Definition 1.1. For $z \in \mathcal{F}_n$, let $\hat{\mathcal{R}}_{\mathtt{FPF}}(z)$ be the set of sequences $(s_{i_1}, s_{i_2}, \dots, s_{i_\ell})$ of minimal possible length ℓ such that $z = s_{i_\ell} \cdots s_{i_2} s_{i_1} \Theta_n s_{i_1} s_{i_2} \cdots s_{i_\ell}$.

The elements of $\hat{\mathcal{R}}_{\mathtt{FPF}}(z)$ are what one should consider as the "reduced words" of a fixed-point-free involution. They arise in geometry when one studies the action of the symplectic group on the flag variety [4, 23], and in representation theory when one considers the *quasi-parabolic* Iwarori-Hecke algebra modules defined by Rains and Vazirani in [18].

An old result of Stanley [21] states that the number r_n of reduced words for the reverse permutation $w_n = n \cdots 321 \in S_n$ is also the number of standard Young tableaux of the staircase shape $\delta_n = (n-1, \dots, 2, 1, 0)$, and therefore has the formula $r_n = \binom{n}{2}! \cdot 1^{1-n} \cdot 3^{2-n} \cdot 5^{3-n} \cdots (2n-3)^{-1}$. The reverse permutation w_n belongs to $\mathcal{F}_n$ when n is even. Let $\hat{r}_n^{\mathtt{FPF}}$ denote the size of $\hat{\mathcal{R}}_{\mathtt{FPF}}(w_n)$. It is natural to ask if these numbers have any nice formula, and in [6] we proved the following:

Theorem 1.2. If $n = 2k$ is even then $\hat{r}_n^{\mathtt{FPF}} = \binom{2N}{N} r_k^2$ where $N = \binom{k}{2}$.

The numbers $(\hat{r}_n^{\mathtt{FPF}})_{n=2,4,6,8,10,\dots}$ begin as $(1, 2, 80, 236544, 108973522944, \dots)$ and form a subsequence of [20, A066051]. The theorem shows that $\hat{r}_n^{\mathtt{FPF}}$ is the number of standard bitableaux of shape (δ_p, δ_q), which is also the dimension of the largest complex irreducible representation of the hyperoctahedral group of rank $P + Q$.

For an arbitrary sequence of simple transpositions $\mathbf{a} = (s_{a_1}, s_{a_2}, \dots, s_{a_\ell})$, let $f_{\mathbf{a}} \in \mathbb{Z}[[x_1, x_2, \dots]]$ denote the formal power series given by summing the monomials $x_{i_1} x_{i_2} \cdots x_{i_\ell}$ over all positive integers $i_1 \leq i_2 \leq \cdots \leq i_\ell$ satisfying $i_j < i_{j+1}$ whenever $a_j < a_{j+1}$. Stanley computed r_n in [21] by showing that for each permutation $w \in S_n$ the power series $F_w = \sum_{\mathbf{a} \in \mathcal{R}(w)} f_{\mathbf{a}}$ is a symmetric function, and that for the reverse permutation this symmetric function is the Schur function $F_{w_n} = s_{\delta_n}$. One gets the formula for r_n from this identity by comparing coefficients of $x_1 x_2 \cdots x_{n(n-1)/2}$. (The reader should note that our notation for F_w differs from Stanley's in [21] by an inversion of indices.) Nowadays, one calls F_w a *Stanley symmetric function*. To prove Theorem 1.2, we introduced in [6] this variant of F_w:

Definition 1.3. The *FPF-involution Stanley symmetric function* of $z \in \mathcal{F}_n$ is $\hat{F}_z^{\mathtt{FPF}} = \sum_{\mathbf{a} \in \hat{\mathcal{R}}_{\mathtt{FPF}}(z)} f_{\mathbf{a}}$.

Notation. Our definitions of $\hat{\mathcal{R}}_{\mathtt{FPF}}(z)$ and $\hat{F}_z^{\mathtt{FPF}}$ have analogues that apply to all involutions in S_n, which are denoted $\hat{\mathcal{R}}(z)$ and $\hat{F}_z$ in [6, 9]. This accounts for our somewhat clumsy notation.

The set $\hat{\mathcal{R}}_{\mathtt{FPF}}(z)$ is the union of the sets of reduced words $\mathcal{R}(w)$ for the permutations w of minimal length with $z = w^{-1}\Theta_n w$, and $\hat{F}^{\mathtt{FPF}}_z$ is the sum of the corresponding Stanley symmetric functions F_w; see the discussion in Section 2.3. Edelman and Greene [5] proved that each F_w is Schur positive, and therefore each $\hat{F}^{\mathtt{FPF}}_z$ is also *Schur positive*, that is, a nonnegative integer linear combination of Schur functions. Our results will show that the symmetric functions $\hat{F}^{\mathtt{FPF}}_z$ have a strong positivity property.

Our first main theorem goes as follows. For positive integers $0 < \phi_1 < \phi_2 < \cdots < \phi_r < m$, consider the involution $y = (\phi_1, m+1)(\phi_2, m+2)\cdots(\phi_r, m+r)$. Write n for whichever of $m+r$ or $m+r+1$ is even, and let $z = y(f_1, f_2)(f_3, f_4)\cdots(f_{n-2r-1}, f_{n-2r}) \in \mathcal{F}_n$ where $f_1 < f_2 < \cdots < f_{n-2r} \leq n$ are the fixed points of y. Define a permutation to be *FPF-Grassmannian* if it occurs as an involution z of this form, or is equal to Θ_n. We prove the following theorem in Section 3.19.

Theorem 1.4. For each $z \in \mathcal{F}_n$ it holds that $\hat{F}^{\mathtt{FPF}}_z \in \mathbb{N}\text{-span}\left\{\hat{F}^{\mathtt{FPF}}_v : v \text{ is FPF-Grassmannian}\right\}$.

Our proof of this result gives a finite algorithm for decomposing any $\hat{F}^{\mathtt{FPF}}_z$ into its FPF-Grassmannian summands. It turns out that we can easily identify the symmetric functions which give these summands.

Within the ring of symmetric functions is the subalgebra $\mathbb{Q}[p_1, p_3, p_5, \dots]$ generated by the odd power-sum functions. This algebra has a distinguished basis $\{P_\lambda\}$ indexed by strict integer partitions (that is, partitions with all distinct parts), whose elements P_λ are called *Schur P-functions*. See Section 2.4 for the precise definition. Define $z \in \mathcal{F}_n$ to be *FPF-vexillary* if $\hat{F}^{\mathtt{FPF}}_z$ is a Schur P-function. The next statement paraphrases Theorems 3.19 and 3.56.

Theorem 1.5. There is a pattern avoidance condition characterizing FPF-vexillary involutions. All FPF-Grassmannian involutions as well as the reverse permutations w_{2n} are FPF-vexillary.

See Corollary 3.57 for the minimal list of 16 patterns that must be avoided. Combining the last two theorems gives this corollary, which we conjectured in [6]:

Corollary 1.6. Each $\hat{F}^{\mathtt{FPF}}_z$ is *Schur P-positive*, i.e., $\hat{F}^{\mathtt{FPF}}_z \in \mathbb{N}\text{-span}\{P_\lambda : \lambda \text{ is a strict partition}\}$.

The preceding corollary has this refinement, which we prove in Section 3.3:

Theorem 1.7. Let $z \in \mathcal{F}_n$ and let c_i be the number of positive integers j with $j < i < y(j)$ and $j < y(i)$. If ν is the transpose of the partition given by sorting $(c_1, c_2, \dots, c_n)$, then ν is strict and $\hat{F}^{\mathtt{FPF}}_z \in P_\nu + \mathbb{N}\text{-span}\{P_\lambda : \lambda < \nu\}$ where $<$ is the dominance order on strict partitions.

As a corollary we get this result, which implies Theorem 1.2:

Corollary 1.8. If $n = 2k$ is even then $\hat{F}^{\mathtt{FPF}}_{w_n} = P_{(n-2,n-4,n-6,\dots)} = s^2_{\delta_k}$.

Proof. The first equality holds by Theorems 1.5 and 1.7. The second equality is a consequence of [6, Theorem 1.4] or [22, Theorem 9.3]. ☐

Theorems 1.4, 1.5, and 1.7 are "fixed-point-free" analogues of the main results in [9]. Our proofs depend on an interpretation of the symmetric functions $\hat{F}^{\mathtt{FPF}}_z$ as stable limits of polynomials introduced by Wyser and Yong [23] to represent the cohomology classes of certain orbit closures. Background material on these polynomials and other preliminaries appear in Section 2. We prove the theorems in this introduction in Section 3, along with a few other results.

The methods in this paper are primarily algebraic. In contrast to the situation in our previous work [9], it is an open problem to find a bijective proof for the fact that $\hat{F}^{\mathtt{FPF}}_z$ is Schur P-positive. The literature on Stanley symmetric functions suggests that there should exist a meaningful representation-theoretic interpretation of $\hat{F}^{\mathtt{FPF}}_z$, but this also remains to be found.

Acknowledgements

We thank Dan Bump, Michael Joyce, Vic Reiner, Alex Woo, Ben Wyser, and Alex Yong for many helpful conversations during the development of this paper.

2 Preliminaries

Let $\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z}$ denote the respective sets of positive, nonnegative, and all integers. For $n \in \mathbb{P}$, let $[n] = \{1, 2, \dots, n\}$. The *support* of a permutation $w : X \to X$ is the set $\mathrm{supp}(w) = \{i \in X : w(i) \neq i\}$. Define $S_\mathbb{Z}$ as the group of permutations of $\mathbb{Z}$ with finite support, and let $S_\infty \subset S_\mathbb{Z}$ be the subgroup of permutations with support contained in $\mathbb{P}$. We view S_n as the subgroup of permutations in S_∞ which fix all integers outside $[n]$.

Throughout, we let $s_i = (i, i+1) \in S_\mathbb{Z}$ for $i \in \mathbb{Z}$. Let $\mathcal{R}(w)$ be the set of reduced words for $w \in S_\mathbb{Z}$ and write $\ell(w)$ for the common length of these words. We let $\mathrm{Des}_L(w)$ and $\mathrm{Des}_R(w)$ denote the *left* and *right descent sets* of $w \in S_\mathbb{Z}$, consisting of the simple transpositions s_i such that $\ell(s_i w) < \ell(w)$ and $\ell(w s_i) < \ell(w)$, respectively.

2.1 Divided difference operators

We recall a few properties of *divided difference operators*. Our main references are [11, 14, 15, 16]. Let $\mathcal{L} = \mathbb{Z}\left[x_1, x_2, \dots, x_1^{-1}, x_2^{-1}, \dots\right]$ be the ring of Laurent polynomials over $\mathbb{Z}$ in a countable set of commuting indeterminates, and let $\mathcal{P} = \mathbb{Z}[x_1, x_2, \dots]$ be the subring of polynomials in $\mathcal{L}$. The group S_∞ acts on $\mathcal{L}$ by permuting variables, and one defines

$$\partial_i f = (f - s_i f)/(x_i - x_{i+1}) \qquad \text{for } i \in \mathbb{P} \text{ and } f \in \mathcal{L}.$$

The *divided difference operator* ∂_i defines a map $\mathcal{L} \to \mathcal{L}$ which restricts to a map $\mathcal{P} \to \mathcal{P}$. It is clear by definition that $\partial_i f = 0$ if and only if $s_i f = f$. If $f \in \mathcal{L}$ is homogeneous and $\partial_i f \neq 0$ then $\partial_i f$ is homogeneous of degree $\deg(f) - 1$. If $f, g \in \mathcal{L}$ then $\partial_i(fg) = (\partial_i f)g + (s_i f)\partial_i g$, and if $\partial_i f = 0$, then $\partial_i(fg) = f\partial_i g$.

For $i \in \mathbb{P}$ the *isobaric divided difference operator* $\pi_i : \mathcal{L} \to \mathcal{L}$ is defined by

$$\pi_i(f) = \partial_i(x_i f) = f + x_{i+1}\partial_i f \qquad \text{for } f \in \mathcal{L}.$$

Observe that $\pi_i f = f$ if and only if $s_i f = f$, in which case $\pi_i(fg) = f\pi_i(g)$ for $g \in \mathcal{L}$. If $f \in \mathcal{L}$ is homogeneous with $\pi_i f \neq 0$, then $\pi_i f$ is homogeneous of the same degree. The operators ∂_i and π_i both satisfy the braid relations for S_∞, so we may define $\partial_w = \partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}$ and $\pi_w = \pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ for any $(s_{i_1}, s_{i_2}, \dots, s_{i_k}) \in \mathcal{R}(w)$. Moreover, one has $\partial_i^2 = 0$ and $\pi_i^2 = \pi_i$ for all $i \in \mathbb{P}$.

Definition 2.1. Given $a, b \in \mathbb{P}$ with $a < b$, define $\partial_{b,a} = \partial_{b-1}\partial_{b-2}\cdots\partial_a$ and $\pi_{b,a} = \pi_{b-1}\pi_{b-2}\cdots\pi_a$. For numbers $a, b \in \mathbb{P}$ with $a \geq b$, set $\partial_{b,a} = \pi_{b,a} = \mathrm{id}$.

The following statement, which will be of use later, is [9, Lemma 2.2].

Lemma 2.2. If $a \leq b$ and $f \in \mathcal{L}$ are such that $\partial_i f = 0$ for $a < i < b$, then $\pi_{b,a} f = \partial_{b,a}\left(x_a^{b-a} f\right)$.

2.2 Schubert polynomials and Stanley symmetric functions

The *Schubert polynomial* (see [3, 11, 14, 16]) of $y \in S_n$ is the polynomial $\mathfrak{S}_y = \partial_{y^{-1}w_n} x^{\delta_n} \in \mathcal{P}$, where as above we let $w_n = n \cdots 321 \in S_n$ and $x^{\delta_n} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1$. This formula for $\mathfrak{S}_y$ is independent of the choice of n such that $y \in S_n$, and we consider the Schubert polynomials to be a family indexed by S_∞. Since $\partial_i^2 = 0$, it follows directly from the definition that

$$\mathfrak{S}_1 = 1 \qquad \text{and} \qquad \partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } s_i \in \mathrm{Des}_R(w) \\ 0 & \text{if } s_i \notin \mathrm{Des}_R(w) \end{cases} \qquad \text{for each } i \in \mathbb{P}. \tag{2.1}$$

Conversely, one can show that $\{\mathfrak{S}_w\}_{w \in S_\infty}$ is the unique family of homogeneous polynomials indexed by S_∞ satisfying (2.1); see [11, Theorem 2.3] or the introduction of [2]. Each $\mathfrak{S}_w$ has degree $\ell(w)$, and the polynomials $\mathfrak{S}_w$ for $w \in S_\infty$ form a $\mathbb{Z}$-basis for $\mathcal{P}$ [16, Proposition 2.5.4].

Write Λ for the usual subring of bounded degree *symmetric functions* in the ring of formula power series $\mathbb{Z}[[x_1, x_2, \dots]]$. A sequence of power series $f_1, f_2, \dots$ has a limit $\lim_{n\to\infty} f_n \in \mathbb{Z}[[x_1, x_2, \dots]]$ if the coefficient sequence of each fixed monomial is eventually constant.

Notation. For any map $w : \mathbb{Z} \to \mathbb{Z}$ and $N \in \mathbb{Z}$, let $w \gg N : \mathbb{Z} \to \mathbb{Z}$ be the map $i \mapsto w(i-N)+N$.

Theorem-Definition 2.3 (See [3, 14, 21])**.** If $w \in S_\mathbb{Z}$ then the limit $F_w = \lim_{N\to\infty} \mathfrak{S}_{w \gg N}$ is a well-defined, homogeneous symmetric function of degree $\ell(w)$. If $w \in S_\infty$ then $F_w = \lim_{n\to\infty} \pi_{w_n} \mathfrak{S}_w$. This power series coincides with the *Stanley symmetric function* F_w defined in the introduction.

This definition makes it clear that $F_w = F_{w \gg N}$ for any $N \in \mathbb{Z}$, though does not tell us how to efficiently compute these symmetric functions. It is well-known that each F_w is Schur positive; for a brief account of one way to compute the corresponding Schur expansion, see [9, §4.2].

2.3 FPF-involution Schubert polynomials

Let $\mathcal{F}_n$ for $n \in \mathbb{P}$ denote the set of permutations $z \in S_n$ with $z = z^{-1}$ and $z(i) \neq i$ for all $i \in [n]$. Define $\mathcal{F}_\infty$ and $\mathcal{F}_\mathbb{Z}$ as the S_∞- and $S_\mathbb{Z}$-conjugacy classes of the permutation $\Theta : \mathbb{Z} \to \mathbb{Z}$ given by

$$\Theta : i \mapsto i - (-1)^i.$$

We refer to elements of $\mathcal{F}_n$, $\mathcal{F}_\infty$, and $\mathcal{F}_\mathbb{Z}$ as *fixed-point-free (FPF) involutions*. Note that $\mathcal{F}_n$ is empty if n is odd, and that if $z \in \mathcal{F}_\mathbb{Z}$ and $N \in \mathbb{Z}$ then $z \gg N \in \mathcal{F}_\mathbb{Z}$ if and only if N is even, where $z \gg N$ denotes the map $i \mapsto z(i-N)+N$. While technically $\mathcal{F}_n \not\subset \mathcal{F}_\infty$, there is a natural inclusion

$$\iota : \mathcal{F}_n \hookrightarrow \mathcal{F}_\infty \tag{2.2}$$

given by the map which sends $z \in \mathcal{F}_n$ to the permutation of $\mathbb{Z}$ whose restrictions to $[n]$ and to $\mathbb{Z} \setminus [n]$ coincide respectively with those of z and Θ. In symbols, we have $\iota(z) = z \cdot \Theta \cdot s_1 \cdot s_3 \cdot s_5 \cdots s_{n-1}$. It is often convenient to identify elements of $\mathcal{F}_n$, $\mathcal{F}_\infty$, or $\mathcal{F}_\mathbb{Z}$ with the complete matchings on $[n]$, $\mathbb{P}$, or $\mathbb{Z}$ in which distinct vertices are connected by an edge whenever they form a nontrivial cycle. We draw such matchings so that the vertices are points on a horizontal axis, ordered from left to right, and the edges appear as convex curves in the upper half plane. For example,

$(1,6)(2,7)(3,4)(5,8) \in \mathcal{F}_8$ is represented as 

We may omit the numbers labeling the vertices in these matchings, if clear from context.

For each $z \in \mathcal{F}_\mathbb{Z}$, define $\mathrm{Inv}(z)$ (respectively, $\mathrm{Cyc}_\mathbb{Z}(z)$) as the set of pairs $(i,j) \in \mathbb{Z}\times\mathbb{Z}$ with $i<j$ and $z(i)>z(j)$ (respectively, $i<j=z(i)$). In turn, let $\mathrm{Des}_R(z) = \{s_i : (i,i+1) \in \mathrm{Inv}(z)\}$ and $\mathrm{Cyc}_\mathbb{P}(z) = \mathrm{Cyc}_\mathbb{Z}(z) \cap (\mathbb{P}\times\mathbb{P})$. The set $\mathrm{Inv}_{\mathsf{FPF}}(z) = \mathrm{Inv}(z) - \mathrm{Cyc}_\mathbb{Z}(z)$ is finite with an even number of elements, and is empty if and only if $z = \Theta$. We let

$$\hat\ell_{\mathsf{FPF}}(z) = \tfrac{1}{2}|\mathrm{Inv}_{\mathsf{FPF}}(z)| \qquad \text{and} \qquad \mathrm{Des}_R^{\mathsf{FPF}}(z) = \{s_i \in \mathrm{Des}_R(z) : (i,i+1) \notin \mathrm{Cyc}_\mathbb{Z}(z)\}.$$

These definitions are related by the following lemma, whose elementary proof is omitted.

Proposition 2.4. If $z \in \mathcal{F}_\mathbb{Z}$ then $\hat\ell_{\mathsf{FPF}}(szs) = \begin{cases} \hat\ell_{\mathsf{FPF}}(z) - 1 & \text{if } s \in \mathrm{Des}_R^{\mathsf{FPF}}(z) \\ \hat\ell_{\mathsf{FPF}}(z) & \text{if } s \in \mathrm{Des}_R(z) - \mathrm{Des}_R^{\mathsf{FPF}}(z) \\ \hat\ell_{\mathsf{FPF}}(z)+1 & \text{if } s \in \{s_i : i \in \mathbb{Z}\} - \mathrm{Des}_R(z). \end{cases}$

Define $\mathcal{A}_{\mathsf{FPF}}(z)$ for $z \in \mathcal{F}_\mathbb{Z}$ as the set of permutations $w \in S_\mathbb{Z}$ of minimal length with $z = w^{-1}\Theta w$. This set is nonempty and finite, and its elements all have length $\hat\ell_{\mathsf{FPF}}(z)$. Note that the set $\hat{\mathcal{R}}_{\mathsf{FPF}}(z)$ defined in the introduction is the union $\hat{\mathcal{R}}_{\mathsf{FPF}}(z) = \bigcup_{w\in\mathcal{A}_{\mathsf{FPF}}(z)} \mathcal{R}(w)$.

Definition 2.5. The *FPF-involution Schubert polynomial* of $z \in \mathcal{F}_\infty$ is $\hat{\mathfrak{S}}^{\mathsf{FPF}}_z = \sum_{w\in\mathcal{A}_{\mathsf{FPF}}(z)} \mathfrak{S}_w$.

For $z \in \mathcal{F}_n$, we set $\mathcal{A}_{\mathsf{FPF}}(z) = \mathcal{A}_{\mathsf{FPF}}(\iota(z))$ and $\hat{\mathfrak{S}}^{\mathsf{FPF}}_z = \hat{\mathfrak{S}}^{\mathsf{FPF}}_{\iota(z)}$.

Example 2.6. We have $\iota(4321) = s_1s_2\Theta s_2 s_1 = s_3s_2\Theta s_2s_3$ and $\mathcal{A}_{\mathsf{FPF}}(4321) = \{312, 1342\}$, so $\hat{\mathfrak{S}}^{\mathsf{FPF}}_{4321} = \mathfrak{S}_{312} + \mathfrak{S}_{1342} = x_1^2 + x_1x_2 + x_1x_3 + x_2x_3$.

The polynomials $\hat{\mathfrak{S}}^{\mathsf{FPF}}_z$ have the following characterization via divided differences.

Theorem 2.7 (See [6])**.** The FPF-involution Schubert polynomials $\{\hat{\mathfrak{S}}^{\mathsf{FPF}}_z\}_{z\in\mathcal{F}_\infty}$ are the unique family of homogeneous polynomials indexed by $\mathcal{F}_\infty$ such that if $i \in \mathbb{P}$ and $s = s_i$ then

$$\hat{\mathfrak{S}}^{\mathsf{FPF}}_\Theta = 1 \qquad \text{and} \qquad \partial_i \hat{\mathfrak{S}}^{\mathsf{FPF}}_z = \begin{cases} \hat{\mathfrak{S}}^{\mathsf{FPF}}_{szs} & \text{if } s \in \mathrm{Des}_R(z) \text{ and } (i,i+1) \notin \mathrm{Cyc}_\mathbb{Z}(z) \\ 0 & \text{otherwise.}\end{cases} \tag{2.3}$$

Wyser and Yong first considered these polynomials in [23], where they were denoted $\Upsilon_{z;(\mathrm{GL}_n,\mathrm{Sp}_n)}$. They showed, when n is even, that the FPF-involution Schubert polynomials indexed by $\mathcal{F}_n$ are cohomology representatives for the $\mathrm{Sp}_n(\mathbb{C})$-orbit closures in the flag variety $\mathrm{Fl}(n) = \mathrm{GL}_n(\mathbb{C})/B$, with $B \subset \mathrm{GL}_n(\mathbb{C})$ denoting the Borel subgroup of lower triangular matrices. The symmetric functions $\hat F^{\mathsf{FPF}}_z$ are related to the polynomials $\hat{\mathfrak{S}}^{\mathsf{FPF}}_z$ by the following identity.

Theorem-Definition 2.8. The *FPF-involution Stanley symmetric function* of $z \in \mathcal{F}_\mathbb{Z}$ is the power series $\hat F^{\mathsf{FPF}}_z = \sum_{w\in\mathcal{A}_{\mathsf{FPF}}(z)} F_w = \lim_{N\to\infty} \hat{\mathfrak{S}}^{\mathsf{FPF}}_{z\gg 2N} \in \Lambda$. This coincides with the symmetric function $\hat F^{\mathsf{FPF}}_z$ defined in the introduction. If $z \in \mathcal{F}_\infty$ then $\hat F^{\mathsf{FPF}}_z = \lim_{n\to\infty} \pi_{w_n} \hat{\mathfrak{S}}^{\mathsf{FPF}}_z$.

Proof. This follows from Theorem-Definition 2.3. □

2.4 Schur P-functions

Our main results will relate $\hat{F}_z^{\mathtt{FPF}}$ to the *Schur P-functions* in Λ, which were introduced in work of Schur [19] but have since arisen in a variety of other contexts (see, e.g., [2, 10, 17]). Good references for these symmetric functions include [22, §6] and [15, §III.8]. For integers $0 \le r \le n$, let

$$G_{r,n} = \prod_{i\in[r]} \prod_{j\in[n-i]} \left(1 + x_i^{-1}x_{i+j}\right) \in \mathcal{L}.$$

For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, let $\ell(\lambda)$ denote the largest index $i \in \mathbb{P}$ with $\lambda_i \neq 0$. The partition λ is *strict* if $\lambda_i \neq \lambda_{i+1}$ for all $i < \ell(\lambda)$. Define $x^\lambda = x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_r^{\lambda_r}$ where $r = \ell(\lambda)$.

Theorem-Definition 2.9. Let λ be a strict partition with $r = \ell(\lambda)$ parts. The power series $P_\lambda = \lim_{n\to\infty} \pi_{w_n}\left(x^\lambda G_{r,n}\right) \in \Lambda$ is then a well-defined, homogeneous symmetric function of degree $\sum_i \lambda_i$, which one calls the *Schur P-function* of λ.

We present this slightly unusual definition of P_λ for its compatibility with Theorem-Definition 2.3. The symmetric functions P_λ may be described more concretely as generating functions for certain shifted tableaux; see [9, §5.1] or [15, 22] for details.

Whereas the Schur functions form a $\mathbb{Z}$-basis for Λ, the Schur P-functions form a $\mathbb{Z}$-basis for the subring $\mathbb{Q}[p_1, p_3, p_5, \dots] \cap \Lambda$ generated by the odd-indexed power sum symmetric functions [22, Corollary 6.2(b)]. Each Schur P-function P_λ is itself Schur positive [15, Eq. (8.17), §III.8].

2.5 Transition formulas

The *Bruhat order* $<$ on $S_\mathbb{Z}$ is the weakest partial order with $w < wt$ when $w \in S_\mathbb{Z}$ and $t \in S_\mathbb{Z}$ is a transposition such that $\ell(w) < \ell(wt)$. We define the *Bruhat order* $<$ on $\mathcal{F}_\mathbb{Z}$ as the weakest partial order with $z < tzt$ when $z \in \mathcal{F}_\mathbb{Z}$ and $t \in S_\mathbb{Z}$ is a transposition such that $\hat{\ell}_{\mathtt{FPF}}(z) < \hat{\ell}_{\mathtt{FPF}}(tzt)$. Rains and Vazirani's results in [18] imply the following theorem, which is also [8, Theorem 4.6].

Theorem 2.10 (See [18])**.** Let $n \in 2\mathbb{P}$. The following properties hold:

(a) $(\mathcal{F}_\mathbb{Z}, <)$ is a graded poset with rank function $\hat{\ell}_{\mathtt{FPF}}$.

(b) If $y, z \in \mathcal{F}_n$ then $y \le z$ holds in $(S_\mathbb{Z}, <)$ if and only if $\iota(y) \le \iota(z)$ holds in $(\mathcal{F}_\mathbb{Z}, <)$.

(c) Fix $y, z \in \mathcal{F}_\mathbb{Z}$ and $w \in \mathcal{A}_{\mathtt{FPF}}(z)$. Then $y \le z$ if and only if there exists $v \in \mathcal{A}_{\mathtt{FPF}}(y)$ with $v \le w$.

Note that both $\iota(\mathcal{F}_n)$ and $\mathcal{F}_\infty$ are lower ideals in $(\mathcal{F}_\mathbb{Z}, <)$. We write $y \lessdot_\mathcal{F} z$ for $y, z \in \mathcal{F}_\mathbb{Z}$ if $\{w \in \mathcal{F}_\mathbb{Z} : y \le w < z\} = \{y\}$. If $y, z \in \mathcal{F}_n$ for some $n \in 2\mathbb{P}$, then we write $y \lessdot_\mathcal{F} z$ if $\iota(y) \lessdot_\mathcal{F} \iota(z)$.

Example 2.11. The set $\mathcal{F}_4 = \{(1,2)(3,4) < (1,3)(2,4) < (1,4)(2,3)\}$ is totally ordered by $<$.

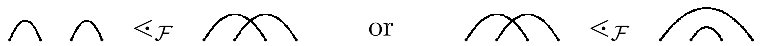
Let $z \in \mathcal{F}_\mathbb{Z}$. Cycles $(a,b), (i,j) \in \mathrm{Cyc}_\mathbb{Z}(z)$ with $a < i$ are *crossing* if $a < i < b < j$ and *nesting* if $a < i < j < b$. One can check that $\hat{\ell}_{\mathtt{FPF}}(z) = 2n + c$ where n and c are the respective numbers of unordered pairs of nesting and crossing cycles of z. If $E \subset \mathbb{Z}$ has size $n \in \mathbb{P}$ then we write ϕ_E and ψ_E for the unique order-preserving bijections $[n] \to E$ and $E \to [n]$, and define $[z]_E = \psi_{z(E)} \circ z \circ \phi_E \in S_n$. The following is equivalent to [1, Corollary 2.3].

Proposition 2.12 (See [1])**.** Let $y \in \mathcal{F}_\mathbb{Z}$. Fix integers $i < j$ and define $A = \{i, j, y(i), y(j)\}$ and $z = (i,j)y(i,j)$. Then $\hat{\ell}_{\mathtt{FPF}}(z) = \hat{\ell}_{\mathtt{FPF}}(y) + 1$ if and only if the following conditions hold:

(a) One has $y(i) < y(j)$ but no $e \in \mathbb{Z}$ exists with $i < e < j$ and $y(i) < y(e) < y(j)$.

(b) Either $[y]_A = (1,2)(3,4) \lessdot_{\mathcal{F}} [z]_A = (1,3)(2,4)$ or $[y]_A = (1,3)(2,4) \lessdot_{\mathcal{F}} [z]_A = (1,4)(2,3)$.

Remark 2.13. If condition (a) holds then $(i,j) \notin \mathrm{Cyc}_{\mathbb{Z}}(y)$ so necessarily $|A| = 4$, and condition (b) asserts that $[y]_A \lessdot_{\mathcal{F}} [z]_A$, which occurs if and only if $[y]_A$ and $[z]_A$ coincide with



In the first case $[(i,j)]_A \in \{(1,4),(2,3)\}$, and in the second $[(i,j)]_A \in \{(1,2),(3,4)\}$.

Given $y \in \mathcal{F}_{\mathbb{Z}}$ and $r \in \mathbb{Z}$, let

$$\begin{aligned}
\hat\Psi^+(y,r) &= \left\{ z \in \mathcal{F}_{\mathbb{Z}} : \hat\ell_{\mathtt{FPF}}(z) = \hat\ell_{\mathtt{FPF}}(y) + 1 \text{ and } z = (r,j)y(r,j) \text{ for an integer } j > r \right\} \\
\hat\Psi^-(y,r) &= \left\{ z \in \mathcal{F}_{\mathbb{Z}} : \hat\ell_{\mathtt{FPF}}(z) = \hat\ell_{\mathtt{FPF}}(y) + 1 \text{ and } z = (i,r)y(i,r) \text{ for an integer } i < r \right\}.
\end{aligned} \tag{2.4}$$

These sets are both nonempty, and if z belongs to either of them then $y \lessdot_{\mathcal{F}} z$. The following transition formula for FPF-involution Schubert polynomials is [8, Theorem 4.17].

Theorem 2.14 (See [8])**.** If $y \in \mathcal{F}_\infty$ and $(p,q) \in \mathrm{Cyc}_{\mathbb{P}}(y)$ then $(x_p + x_q)\hat{\mathfrak{S}}^{\mathtt{FPF}}_y = \sum_{z\in\hat\Psi^+(y,q)} \hat{\mathfrak{S}}^{\mathtt{FPF}}_z - \sum_{z\in\hat\Psi^-(y,p)} \hat{\mathfrak{S}}^{\mathtt{FPF}}_z$ where we set $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z = 0$ for all $z \in \mathcal{F}_{\mathbb{Z}} - \mathcal{F}_\infty$.

Example 2.15. Set $\hat\Psi^\pm(y,r) = \hat\Psi^\pm(\iota(y),r)$ for $y \in \mathcal{F}_n$. If $y = (1,2)(3,7)(4,5)(6,8) \in \mathcal{F}_8$ then

$$\begin{aligned}
\hat\Psi^+(y,7) &= \{(7,8)y(7,8)\} = \{(1,2)(3,8)(4,5)(6,7)\} \\
\hat\Psi^-(y,3) &= \{(2,3)y(2,3)\} = \{(1,3)(2,7)(4,5)(6,8)\}
\end{aligned}$$

so $(x_3 + x_7)\hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,2)(3,7)(4,5)(6,8)} = \hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,2)(3,8)(4,5)(6,7)} - \hat{\mathfrak{S}}^{\mathtt{FPF}}_{(1,3)(2,7)(4,5)(6,8)}$.

Taking limits and invoking Theorem-Definition 2.8 gives the following identity in Λ.

Theorem 2.16. If $y \in \mathcal{F}_{\mathbb{Z}}$ and $(p,q) \in \mathrm{Cyc}_{\mathbb{Z}}(y)$ then $\sum_{z\in\hat\Psi^-(y,p)} \hat F^{\mathtt{FPF}}_z = \sum_{z\in\hat\Psi^+(y,q)} \hat F^{\mathtt{FPF}}_z$.

Proof. We have $\hat\Psi^\pm(y \gg 2N, r+2N) = \{w \gg 2N : w \in \hat\Psi^\pm(y,r)\}$ for $y \in \mathcal{F}_{\mathbb{Z}}$ and $r, N \in \mathbb{Z}$, so it follows that $\sum_{z\in\hat\Psi^+(y,q)} \hat F^{\mathtt{FPF}}_z - \sum_{z\in\hat\Psi^-(y,p)} \hat F^{\mathtt{FPF}}_z = \lim_{N\to\infty}(x_{p+2N} + x_{q+2N})\hat{\mathfrak{S}}^{\mathtt{FPF}}_{y\gg 2N} = 0$. ☐

3 Results

3.1 FPF-Grassmannian involutions

In this section we identify a class of "Grassmannian" elements of $\mathcal{F}_{\mathbb{Z}}$ for which $\hat F^{\mathtt{FPF}}_z$ is a Schur P-function. Recall that the *(Rothe) diagram* of a permutation $w \in S_\infty$ is the set $D(w) = \{(i, w(j)) : (i,j) \in \mathrm{Inv}(w)\}$ where $\mathrm{Inv}(w)$ is the set of pairs $(i,j) \in \mathbb{Z}\times\mathbb{Z}$ with $i < j$ and $w(i) > w(j)$. The following similar definitions were introduced in [6, Section 3.2]:

Definition 3.1. The *(FPF-involution) diagram* of $z \in \mathcal{F}_\infty$ is the set $\hat D_{\mathtt{FPF}}(z)$ of pairs $(i,j) \in \mathbb{P}\times\mathbb{P}$ with $j < i < z(j)$ and $j < z(i)$.

Note that $\hat{D}_{\mathtt{FPF}}(z) = \{(i, z(j)) : (i,j) \in \mathrm{Inv}_{\mathtt{FPF}}(z),\ z(j) < i\}$. The *code* of $w \in S_\infty$ is the sequence $c(w) = (c_1, c_2, c_3, \dots)$ where c_i is the number of positions in the ith row of $D(w)$.

Definition 3.2. The *(FPF-involution) code* of $z \in \mathcal{F}_\infty$ is the sequence $\hat{c}_{\mathtt{FPF}}(z) = (c_1, c_2, \dots)$ in which c_i is the number of positions in the ith row of $\hat{D}_{\mathtt{FPF}}(z)$.

Define $\hat{D}_{\mathtt{FPF}}(z) = \hat{D}_{\mathtt{FPF}}(\iota(z))$ and $\hat{c}_{\mathtt{FPF}}(z) = \hat{c}_{\mathtt{FPF}}(\iota(z))$ for $z \in \mathcal{F}_n$ when $n \in 2\mathbb{P}$; then $\hat{D}_{\mathtt{FPF}}(z)$ is the subset of positions in $D(z)$ strictly below the diagonal.

Example 3.3. If $z = (1,4)(2,5)(3,6)$ then $\hat{D}_{\mathtt{FPF}}(z) = \{(2,1), (3,1), (3,2)\}$ and $\hat{c}_{\mathtt{FPF}}(z) = (0,1,2)$.

The shifted shape of a strict partition μ is the set $\{(i, i+j-1) \in \mathbb{P} \times \mathbb{P} : 1 \leq j \leq \mu_i\}$. An involution z in $\mathcal{F}_n$ or $\mathcal{F}_\infty$ is *FPF-dominant* if $\{(i-1, j) : (i,j) \in \hat{D}_{\mathtt{FPF}}(z)\}$ is the transpose of the shifted shape of a strict partition. (Note that $\hat{D}_{\mathtt{FPF}}(z)$ contains no positions in its first row.)

Example 3.4. Both $(1,7)(2,4)(3,5)(6,8)$ and $(1,7)(2,5)(3,4)(6,8)$ are FPF-dominant. The only elements of $\mathcal{F}_{2k}$ for $k \in \mathbb{P}$ which are 132-avoiding (i.e., *dominant*) are those of the form $(1, k+1)(2, k+2)\cdots(k, 2k)$. These dominant involutions are also FPF-dominant.

Theorem 3.5 (See [6])**.** If $z \in \mathcal{F}_\infty$ is FPF-dominant then $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z = \prod_{(i,j)\in \hat{D}_{\mathtt{FPF}}(z)} (x_i + x_j)$.

Proof sketch. This is a slightly stronger statement than the result we proved as [6, Theorem 3.26], which gave the same formula but only for the dominant (i.e., 132-avoiding) elements of $\mathcal{F}_n$. However, the more general formula follows by the same argument with minor changes. ☐

The *lexicographic order* on S_∞ is the total order induced by identifying $w \in S_\infty$ with its one-line representation $w(1)w(2)w(3)\cdots$. For z in $\mathcal{F}_n$ or $\mathcal{F}_\infty$, we let $\beta_{\min}(z)$ denote the lexicographically minimal element of $\mathcal{A}_{\mathtt{FPF}}(z)$. The next lemma is a consequence of [7, Theorem 6.22].

Lemma 3.6 (See [7])**.** Suppose $z \in \mathcal{F}_\infty$ and $\mathrm{Cyc}_{\mathbb{P}}(z) = \{(a_i, b_i) : i \in \mathbb{P}\}$ where $a_1 < a_2 < \cdots$. The lexicographically minimal element $\beta_{\min}(z) \in \mathcal{A}_{\mathtt{FPF}}(z)$ is the inverse of the permutation whose one-line representation is $a_1 b_1 a_2 b_2 a_3 b_3 \cdots$.

The same statement with "$a_1 b_1 a_2 b_2 \cdots$" replaced by "$a_1 b_1 a_2 b_2 \cdots a_k b_k$" holds if $z \in \mathcal{F}_{2k}$.

Example 3.7. If $z = (1,4)(2,3) \in \mathcal{F}_4$ then $a_1 b_1 a_2 b_2 = 1423$ and $\beta_{\min}(z) = 1423^{-1} = 1342$.

Typically $\hat{D}_{\mathtt{FPF}}(z) \neq D(\beta_{\min}(z))$, but the following holds by [6, Lemma 3.8].

Lemma 3.8 (See [6])**.** If $z \in \mathcal{F}_\infty$ then $\hat{c}_{\mathtt{FPF}}(z) = c(\beta_{\min}(z))$.

A pair $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ is an *FPF-visible inversion* of $z \in \mathcal{F}_{\mathbb{Z}}$ if $i < j$ and $z(j) < \min\{i, z(i)\}$.

Lemma 3.9. The set of FPF-visible inversions of $z \in \mathcal{F}_\infty$ is $\mathrm{Inv}(\beta_{\min}(z))$.

Proof. Suppose $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ is an FPF-visible inversion of $z \in F_\infty$. Either $z(j) < i < z(i)$ or $z(j) < z(i) < i$, and in both cases j appears before i in the one-line representation of $\beta_{\min}(z)^{-1}$ so $(i,j) \in \mathrm{Inv}(\beta_{\min}(z))$. Conversely let $(i,j) \in \mathrm{Inv}(\beta_{\min}(z))$. Then $i < j$ and $j \in \{a, b\}$ and $i \in \{a', b'\}$ where $a < b = z(a)$ and $a' < b' = z(a')$ and $a < a'$. The only way this can happen is if $j = b$ in which case (i,j) is evidently an FPF-visible inversion of z. ☐

We say that $i \in \mathbb{Z}$ is an *FPF-visible descent* of $z \in \mathcal{F}_\mathbb{Z}$ if $(i, i+1)$ is an FPF-visible inversion, and define $\mathrm{Des}_V^{\mathsf{FPF}}(z) = \{s_i : i \in \mathbb{Z} \text{ is an FPF-visible descent of } z\} \subset \mathrm{Des}_R^{\mathsf{FPF}}(z)$. Since $s_i \in \mathrm{Des}_R(w)$ for $w \in S_\mathbb{Z}$ if and only if $(i, i+1) \in \mathrm{Inv}(w)$, the following is immediate from the previous lemma.

Lemma 3.10. If $z \in \mathcal{F}_\infty$ then $\mathrm{Des}_V^{\mathsf{FPF}}(z) = \mathrm{Des}_R(\beta_{\min}(z))$.

The *essential set* of a subset $D \subset \mathbb{P} \times \mathbb{P}$ is the set $\mathrm{Ess}(D)$ of positions $(i,j) \in D$ such that $(i+1, j) \notin D$ and $(i, j+1) \notin D$. A permutation $w \in S_\infty$ is n*-Grassmannian* if $\mathrm{Des}_R(w) = \{s_n\}$.

Proposition 3.11. For $z \in \mathcal{F}_\infty$ and $n \in \mathbb{P}$, the following are equivalent:

(a) $\mathrm{Des}_V^{\mathsf{FPF}}(z) = \{s_n\}$.

(b) $\hat{c}_{\mathsf{FPF}}(z)$ has the form $(0, c_2, \dots, c_n, 0, 0, \dots)$ where $c_2 \leq \dots \leq c_n \neq 0$.

(c) $\mathrm{Ess}(\hat{D}_{\mathsf{FPF}}(z))$ is nonempty and contained in $\{(n, j) : j \in \mathbb{P}\}$.

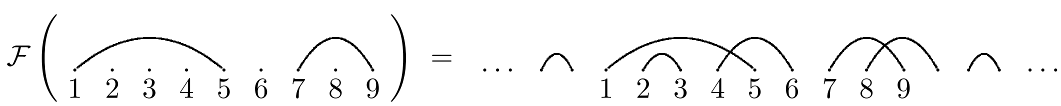
(d) The lexicographically minimal atom $\beta_{\min}(z) \in \mathcal{A}_{\mathsf{FPF}}(z)$ is n-Grassmannian.

Proof. Lemma 3.10 implies that (a) $\Leftrightarrow$ (d). It follows as an elementary exercise that the ith row of $\mathrm{Ess}(\hat{D}_{\mathsf{FPF}}(z))$ is nonempty if and only if $s_i \in \mathrm{Des}_V^{\mathsf{FPF}}(z)$; we omit the details, which are similar to the proof of [9, Lemma 4.14]. This shows that (a) $\Leftrightarrow$ (c). Lemma 3.8 implies that (b) $\Leftrightarrow$ (d), since $w \in S_\infty$ is n-Grassmannian if and only if the first n terms of $c(w)$ are weakly increasing and the last nonzero entry of $c(w)$ occurs in position n (see [9, Proposition-Definition 4.3]). $\square$

Algebraically, the preceding conditions suggest a natural concept of a "Grassmannian" fixed-point-free involution, but it turns out that this definition is slightly too restrictive. Define $\mathcal{I}_\mathbb{Z} = \{w \in S_\mathbb{Z} : w = w^{-1}\}$. Now consider the following maps $\mathcal{F} : \mathcal{I}_\mathbb{Z} \to \mathcal{F}_\mathbb{Z}$ and $\mathcal{I} : \mathcal{F}_\mathbb{Z} \to \mathcal{I}_\mathbb{Z}$:

Definition 3.12. For $y \in \mathcal{I}_\mathbb{Z}$, let m be any even integer with $m < i$ for all $i \in \mathrm{supp}(y)$, write ϕ for the order-preserving bijection $\mathbb{Z} \to \mathbb{Z} \setminus \mathrm{supp}(y)$ with $\phi(0) = m$, and define $\mathcal{F}(y)$ as the unique element of $\mathcal{F}_\mathbb{Z}$ with $\mathcal{F}(y)(i) = y(i)$ for $i \in \mathrm{supp}(y)$ and $\mathcal{F}(y) \circ \phi = \phi \circ \Theta$.

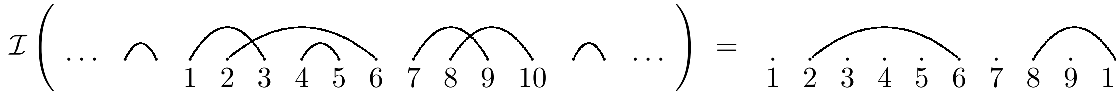
Note that $\mathcal{F}(y) = \iota(y)$ for $y \in \mathcal{F}_n$. The involution $\mathcal{F}(z)$ is formed from z by turning every pair of adjacent fixed points into a cycle; there are two ways of doing this, and we choose the way which makes $(2i-1, 2i)$ into a cycle for all sufficiently large $i \in \mathbb{Z}$. For example, we have



The map $\mathcal{F}$ has a right inverse:

Definition 3.13. For $z \in \mathcal{F}_\mathbb{Z}$, define $\mathcal{I}(z) \in \mathcal{I}_\mathbb{Z}$ as the involution whose nontrivial cycles are precisely the pairs $(p, q) \in \mathrm{Cyc}_\mathbb{Z}(z)$ for which there exists $(a, b) \in \mathrm{Cyc}_\mathbb{Z}(z)$ with $p < b < q$.

Equivalently, $\mathcal{I}(z)$ is the involution which restricts to the same map as z on its support, and whose fixed points are the integers $i \in \mathbb{Z}$ such that $\max\{i, z(i)\} < z(j)$ for all $j \in \mathbb{Z}$ with $\min\{i, z(i)\} < j < \max\{i, z(i)\}$. For example, we have



We see in these examples that $\mathcal{I}$ and $\mathcal{F}$ restrict to maps $\mathcal{F}_\infty \to \mathcal{I}_\infty$ and $\mathcal{I}_\infty \to \mathcal{F}_\infty$, respectively.

Proposition 3.14. Let $z \in \mathcal{F}_\mathbb{Z}$. Then $\mathcal{I}(z) = 1$ if and only if $z = \Theta$.

Proof. If $z \neq \Theta$ and i is the largest integer such that $i < z(i) \neq i+1$, then necessarily $z(i+1) < z(i)$, so $(i, z(i))$ is a nontrivial cycle of $\mathcal{I}(z)$, which is therefore not the identity. □

Proposition 3.15. The composition $\mathcal{F} \circ \mathcal{I}$ is the identity map $\mathcal{F}_\mathbb{Z} \to \mathcal{F}_\mathbb{Z}$.

Proof. Fix $z \in \mathcal{I}_\infty$ and let $\mathcal{C}$ be the set of cycles $(p, q) \in \mathrm{Cyc}_\mathbb{Z}(z)$ such that p and q are fixed points in $\mathcal{I}(z)$. By definition, if (p, q) and (p', q') are distinct elements of $\mathcal{C}$ then $p < q < p' < q'$ or $p' < q' < p < q$. The claim that $\mathcal{F} \circ \mathcal{I}(z) = z$ is a straightforward consequence of this fact. □

An involution $y \in \mathcal{I}_\mathbb{Z}$ is *I-Grassmannian* if $y = 1$ or $y = (\phi_1, n+1)(\phi_2, n+2)\cdots(\phi_r, n+r)$ for some integers $r \in \mathbb{P}$ and $\phi_1 < \phi_2 < \cdots < \phi_r \leq n$. See [9, Proposition-Definition 4.16] for several equivalent characterizations of such involutions.

Definition 3.16. An involution $z \in \mathcal{F}_\mathbb{Z}$ is *FPF-Grassmannian* if $\mathcal{I}(z) \in \mathcal{I}_\mathbb{Z}$ is I-Grassmannian.

Define an element of $\mathcal{F}_n$ to be FPF-Grassmannian if its image under $\iota : \mathcal{F}_n \to \mathcal{F}_\infty \subset \mathcal{F}_\mathbb{Z}$ is FPF-Grassmannian. This definition is equivalent to the one in the introduction.

Remark 3.17. The sequence $(g^{\mathtt{FPF}}_{2n})_{n\geq 1} = (1, 3, 12, 41, 124, 350, 952, 2540, \dots)$ with $g^{\mathtt{FPF}}_n$ the number of FPF-Grassmannian elements of $\iota(\mathcal{F}_n) \subset \mathcal{F}_\mathbb{Z}$ seems unrelated to any existing sequence in [20].

Suppose $z \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$ is FPF-Grassmannian, so that

$$\mathcal{I}(z) = (\phi_1, n+1)(\phi_2, n+2)\cdots(\phi_r, n+r) \in \mathcal{I}_\infty$$

for integers $r \in \mathbb{P}$ and $\phi_1 < \phi_2 < \cdots < \phi_r \leq n$. We define the *shape* of z to be the strict partition

$$\nu(z) = (n - \phi_1, n - \phi_2, \dots, n - \phi_r).$$

We also set $\nu(\Theta) = \emptyset = (0, 0, \dots)$. One can show that $\nu(z)$ is a partition of $\hat{\ell}_{\mathtt{FPF}}(z)$. Recall the definition of the operator $\pi_{b,a}$ from Section 2.1.

Lemma 3.18. Maintain the notation just given, but assume z is an FPF-Grassmannian element of $\mathcal{F}_\infty - \{\Theta\}$ so that $1 \leq \phi_1 < \phi_2 < \cdots < \phi_r \leq n$. Then $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z = \pi_{\phi_1,1}\pi_{\phi_2,2}\cdots\pi_{\phi_r,r}\left(x^{\nu(z)}G_{r,n}\right)$.

Proof. The proof is similar to that of [9, Lemma 4.19]. If $f_1 < f_2 < \cdots < f_k$ are the fixed points of $\mathcal{I}(z)$ in $[n]$, then k is even and $(f_1, f_2), (f_3, f_4), \dots, (f_{k-1}, f_k) \in \mathrm{Cyc}_\mathbb{Z}(z)$. It follows that if $\phi_i = i$ for all $i \in [r]$ then z is FPF-dominant and $\hat{D}_{\mathtt{FPF}}(z) = \{(i+j, i) : i \in [r] \text{ and } j \in [n-i]\}$. In this case the lemma reduces to $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z = x_1^{n-1}x_2^{n-2}\cdots x_r^{n-r}G_{r,n}$ which is evident from Theorem 3.5.

Alternatively, suppose there exists $i \in [r]$ such that $i < \phi_i$. Assume i is minimal with this property. Then $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z = \partial_{\phi_i,i}\hat{\mathfrak{S}}^{\mathtt{FPF}}_v$ for the FPF-Grassmannian involution $v \in \mathcal{F}_\infty$ with

$$\mathcal{I}(v) = (1, n+1)(2, n+2)\cdots(i, n+i)(\phi_{i+1}, n+i+1)(\phi_{i+2}, n+i+2)\cdots(\phi_r, n+r).$$

By induction $\hat{\mathfrak{S}}^{\mathtt{FPF}}_v = \pi_{\phi_{i+1},i+1}\pi_{\phi_{i+2},i+2}\cdots\pi_{\phi_r,r}\left(x^{\nu(v)}G_{r,n}\right)$. Since $x^{\nu(v)} = x_i^{\phi_i - i}x^{\nu(z)}$ and since multiplication by x_i commutes with π_j when $i < j$, it follows by Lemma 2.2 that $\hat{\mathfrak{S}}^{\mathtt{FPF}}_z = \partial_{\phi_i,i}\hat{\mathfrak{S}}^{\mathtt{FPF}}_v = \partial_{\phi_i,i}(x_i^{\phi_i-i}\pi_{\phi_{i+1},i+1}\pi_{\phi_{i+2},i+2}\cdots\pi_{\phi_r,r}(x^{\nu(z)}G_{r,n})) = \pi_{\phi_1,1}\pi_{\phi_2,2}\cdots\pi_{\phi_r,r}(x^{\nu(z)}G_{r,n})$. □

Theorem 3.19. If $z \in \mathcal{F}_\mathbb{Z}$ is FPF-Grassmannian, then $\hat{F}^{\mathtt{FPF}}_z = P_{\nu(z)}$.

Proof. Since $\hat{F}^{\mathsf{FPF}}_z = \hat{F}^{\mathsf{FPF}}_{z\gg N}$ for all $N \in 2\mathbb{Z}$, we may assume that $z \in \mathcal{F}_\infty$ and that $\mathcal{I}(z)$ is I-Grassmannian. Since $\pi_{w_n}\pi_i = \pi_{w_n}$ for all $i \in [n-1]$, Lemma 3.18 implies that if $\nu(z)$ has r parts and $n \geq r$ then $\pi_{w_n}\hat{\mathfrak{S}}^{\mathsf{FPF}}_z = \pi_{w_n}\left(x^{\nu(z)}G_{r,n}\right)$. The theorem follows by taking the limit as $n \to \infty$. □

Let us clarify the difference between FPF-Grassmannian involutions and elements of $\mathcal{F}_\mathbb{Z}$ with at most one FPF-visible descent. Define $\mathcal{I}_\infty = S_\infty \cap \mathcal{I}_\mathbb{Z}$ and for each $y \in \mathcal{I}_\infty$ let $\mathrm{Des}_V(y)$ be the set of integers $i \in \mathbb{Z}$ with $z(i+1) \leq \min\{i, z(i)\}$. Elements of $\mathrm{Des}_V(y)$ are *visible descents* of y.

Lemma 3.20. Fix $z \in \mathcal{F}_\infty$. Let $E = \{i \in \mathbb{P} : |z(i) - i| \neq 1\}$ and define $y \in \mathcal{I}_\infty$ as the involution with $y(i) = z(i)$ if $i \in E$ and $y(i) = i$ otherwise. Then $z = \mathcal{F}(y)$ and $\mathrm{Des}^{\mathsf{FPF}}_V(z) = \mathrm{Des}_V(y)$.

Proof. It is evident that $z = \mathcal{F}(y)$. Suppose $s_i \in \mathrm{Des}_V(y)$. Since $y(i+1) \neq i$ for all $i \in \mathbb{P}$ by definition, we must have $y(i+1) < \min\{i, y(i)\}$, so $i+1 \in E$, and therefore either $i \in E$ or $z(i) = i-1$. It follows in either case that $z(i+1) < \min\{i, z(i)\}$ so $s_i \in \mathrm{Des}^{\mathsf{FPF}}_V(z)$. Conversely, suppose $s_i \in \mathrm{Des}^{\mathsf{FPF}}_V(z)$ so that $i+1 \in E$. If $i \in E$ then $s_i \in \mathrm{Des}_V(y)$ holds immediately, and if $i \notin E$ then $z(i+1) < z(i) = i-1$, in which case $y(i+1) = z(i+1) < i = y(i)$ so $s_i \in \mathrm{Des}_V(y)$. □

In our previous work, we showed that $y \in \mathcal{I}_\mathbb{Z}$ is I-Grassmannian if and only if $|\mathrm{Des}_V(y)| \leq 1$ [9, Proposition-Definition 4.16]. Using this fact, we deduce the following:

Proposition 3.21. An involution $z \in \mathcal{F}_\mathbb{Z}$ has $|\mathrm{Des}^{\mathsf{FPF}}_V(z)| \leq 1$ if and only if z is FPF-Grassmannian and $\nu(z)$ is a strict partition whose consecutive parts each differ by odd numbers.

Proof. We may assume that $z \in \mathcal{F}_\infty - \{\Theta\}$. If z is FPF-Grassmannian and the consecutive parts of $\nu(z)$ differ by odd numbers then one can check that $|\mathrm{Des}^{\mathsf{FPF}}_V(z)| \leq 1$. Conversely, define $y \in \mathcal{I}_\infty$ as in Lemma 3.20 so that $z = \mathcal{F}(y)$. We have $\mathrm{Des}^{\mathsf{FPF}}_V(z) = \mathrm{Des}_V(y) = \{s_n\}$ if and only if $y = (\phi_1, n+1)(\phi_2, n+2)\cdots(\phi_r, n+r)$ for integers $r \in \mathbb{P}$ and $0 = \phi_0 < \phi_1 < \phi_2 < \cdots < \phi_r \leq n$. If y has this form then each $\phi_i - \phi_{i-1}$ is necessarily odd, and $\mathcal{I}(z) = y$ or $\mathcal{I}(z) = (\phi_2, n+2)(\phi_3, n+3)\cdots(\phi_r, n+r)$, so z is FPF-Grassmannian and the consecutive parts of $\nu(z)$ differ by odd numbers. □

Remark 3.22. Using the previous result, one can show that the number k_n of elements of $\mathcal{F}_n$ with at most one FPF-visible descent satisfies the recurrence $k_{2n} = 2k_{2n-2} + 2n - 3$ for $n \geq 2$. The corresponding sequence $(k_{2n})_{n\geq 1} = (1, 3, 9, 23, 53, 115, 241, 495, \dots)$ is [20, A183155].

3.2 Schur P-positivity

In this section we describe a recurrence for expanding $\hat{F}^{\mathsf{FPF}}_z$ into FPF-Grassmannian summands, and use this to deduce that each $\hat{F}^{\mathsf{FPF}}_z$ is Schur P-positive. Our strategy is similar to the one used in [9, §4.2], though with some added technical complications.

Order the set $\mathbb{Z} \times \mathbb{Z}$ lexicographically. Recall that $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ is an FPF-visible inversion of $z \in \mathcal{F}_\mathbb{Z}$ if $i < j$ and $z(j) < \min\{i, z(i)\}$, and that $i \in \mathbb{Z}$ is an FPF-visible descent of z if $(i, i+1)$ is an FPF-visible inversion. By Lemma 3.10, every $z \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$ has an FPF-visible descent.

Lemma 3.23. Let $z \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$ and suppose $j \in \mathbb{Z}$ is the smallest integer such that $z(j) < j - 1$. Then $j - 1$ is the minimal FPF-visible descent of z.

Proof. By hypothesis, either $z(j) < j - 2 = z(j-1)$ or $z(j) < j - 1 < z(j-1)$, so $j - 1$ is an FPF-visible descent of z. If $k-1$ is another FPF-visible descent of z, then $z(k) < k-1$ so $j \leq k$. □

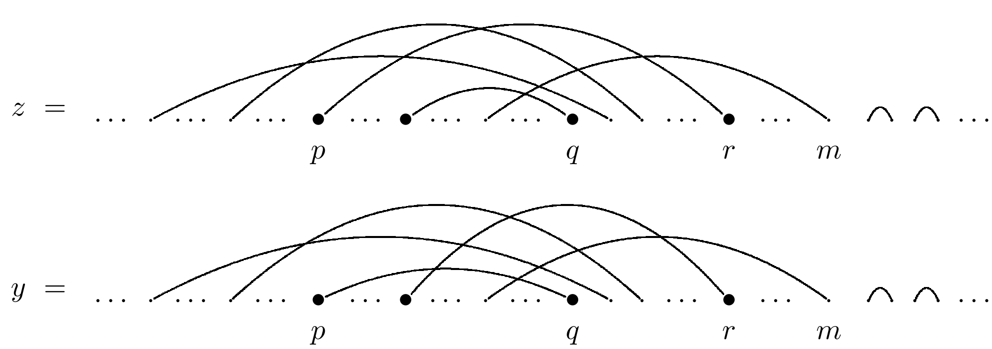
Lemma 3.24. Suppose $(q, r) \in \mathbb{Z} \times \mathbb{Z}$ is the maximal FPF-visible inversion of $z \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$. Let m be the largest even integer such that $z(m) \neq m - 1$. Then q is the maximal FPF-visible descent of z while r is the maximal integer with $z(r) < \min\{q, z(q)\}$, and $z(q+1) < z(q+2) < \cdots < z(m) \leq q$. In addition, we have either (a) $z(q) < q < r \leq m$ or (b) $q < z(q) = r + 1 = m$.

Proof. The first assertion is easy to check directly; we omit the details, which are similar to the proof of [9, Lemma 4.25]. Assume $q < z(q)$. It remains to show that $z(q) = r + 1 = m$. It cannot hold that $r < z(q) - 1$, since then either $(q, r+1)$ or $(r+1, z(q))$ would be an FPF-visible inversion of z, contradicting the maximality of (q, r). It also cannot hold that $z(q) < r$, as then $(z(q), r)$ would be an FPF-visible inversion of z. Hence $r = z(q) - 1$. If $j > z(q)$, then since $z(i) < q$ for all $q < i < z(q)$ and since $(z(q), j)$ cannot be an FPF-visible inversion of z, we must have $z(j) > z(q)$. From this observation and the fact that z has no FPF-visible descents greater than q, we deduce that $z(j) = \Theta(j)$ for all $j > z(q)$, which implies that $z(q) = m$ as required. ☐

Definition 3.25. Let $\eta_{\texttt{FPF}} : \mathcal{F}_\mathbb{Z} - \{\Theta\} \to \mathcal{F}_\mathbb{Z}$ be the map $\eta_{\texttt{FPF}} : z \mapsto (q, r)z(q, r)$ where (q, r) is the maximal FPF-visible inversion of z.

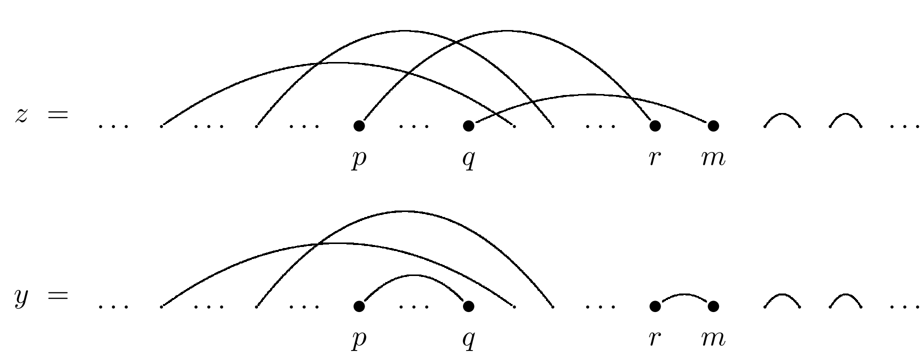
Remark 3.26. Suppose $z \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$ has maximal FPF-visible inversion (q, r). Let $p = z(r)$ and $y = \eta_{\texttt{FPF}}(z) = (q, r)z(q, r)$ and write m for the largest even integer such that $z(m) \neq m - 1$. The two cases of Lemma 3.24 correspond to the following pictures:

(a) If $z(q) < q < r \leq m$ then y and z may be represented as



We have $z(q+1) < z(q+2) < \cdots < z(r) < z(q)$, and if $r < m$ then $z(q) < z(r+1) < z(r+2) < \cdots < z(m) < q$.

(b) If $q < z(q) = r + 1 = m$ then y and z may be represented as



Here, we have $z(q+1) < z(q+2) < \cdots < z(r) = p < q$, so $z(i) < q$ whenever $p < i < q$.

Recall the definition of $\beta_{\min}(z)$ from Lemma 3.6.

Proposition 3.27. If (q,r) is the maximal FPF-visible inversion of $z \in \mathcal{F}_\infty - \{\Theta\}$ and $w = \beta_{\min}(z)$ is the minimal element of $\mathcal{A}_{\texttt{FPF}}(z)$, then $w(q,r) = \beta_{\min}(\eta_{\texttt{FPF}}(z))$ is the minimal atom of $\eta_{\texttt{FPF}}(z)$.

Proof. Let $\mathrm{Cyc}_{\mathbb{P}}(z) = \{(a_i,b_i) : i \in \mathbb{P}\}$ and $\mathrm{Cyc}_{\mathbb{P}}(\eta_{\texttt{FPF}}(z)) = \{(c_i,d_i) : i \in \mathbb{P}\}$ where $a_1 < a_2 < \dots$ and $c_1 < c_2 < \dots$. By Lemma 3.6, it suffices to show that interchanging q and r in the word $a_1b_1a_2b_2\cdots$ gives $c_1d_1c_2d_2\cdots$, which is straightforward from Remark 3.26. ☐

Recall the definition of the sets $\hat\Psi^+(y,r)$ and $\hat\Psi^-(y,r)$ from (2.4).

Lemma 3.28. If $z \in \mathcal{F}_{\mathbb{Z}} - \{\Theta\}$ has maximal FPF-visible inversion (q,r) then $\hat\Psi^+(\eta_{\texttt{FPF}}(z),q) = \{z\}$.

Proof. This holds by Proposition 2.12, Remark 3.26, and the definitions of $\eta_{\texttt{FPF}}(z)$ and $\hat\Psi^+(y,q)$. ☐

For $z \in \mathcal{F}_{\mathbb{Z}}$ let

$$\hat{\mathfrak{T}}_1^{\texttt{FPF}}(z) = \begin{cases} \varnothing & \text{if } z \text{ is FPF-Grassmannian} \\ \hat\Psi^-(y,p) & \text{otherwise} \end{cases}$$

where in the second case, we define $y = \eta_{\texttt{FPF}}(z)$ and $p = y(q)$ where q is the maximal FPF-visible descent of z.

Definition 3.29. The *FPF-involution Lascoux-Schützenberger tree* $\hat{\mathfrak{T}}^{\texttt{FPF}}(z)$ of $z \in \mathcal{F}_{\mathbb{Z}}$ is the tree with root z, in which the children of any vertex $v \in \mathcal{F}_{\mathbb{Z}}$ are the elements of $\hat{\mathfrak{T}}_1^{\texttt{FPF}}(v)$.

Remark 3.30. As the name suggests, our definition is inspired by the classical construction of the *Lascoux-Schützenberger tree* for ordinary Stanley symmetric functions; see [12, 13] or [9, §4.2].

For $z \in \mathcal{F}_n$ we define $\hat{\mathfrak{T}}^{\texttt{FPF}}(z) = \hat{\mathfrak{T}}^{\texttt{FPF}}(\iota(z))$. A given involution is allowed to correspond to more than one vertex in $\hat{\mathfrak{T}}^{\texttt{FPF}}(z)$. All vertices v in $\hat{\mathfrak{T}}^{\texttt{FPF}}(z)$ satisfy $\hat\ell_{\texttt{FPF}}(v) = \hat\ell_{\texttt{FPF}}(z)$ by construction, so if $z \neq \Theta$ then Θ is not a vertex in $\hat{\mathfrak{T}}^{\texttt{FPF}}(z)$. An example tree $\hat{\mathfrak{T}}^{\texttt{FPF}}(z)$ is shown in Figure 1.

Corollary 3.31. Suppose $z \in \mathcal{F}_{\mathbb{Z}}$ is a fixed-point-free involution which is not FPF-Grassmannian, whose maximal FPF-visible descent is $q \in \mathbb{Z}$. The following identities then hold:

(a) $\hat{\mathfrak{S}}^{\texttt{FPF}}_z = (x_p + x_q)\hat{\mathfrak{S}}^{\texttt{FPF}}_y + \sum_{v \in \hat{\mathfrak{T}}_1^{\texttt{FPF}}(z)} \hat{\mathfrak{S}}^{\texttt{FPF}}_v$ where $y = \eta_{\texttt{FPF}}(z)$ and $p = y(q)$.

(b) $\hat F^{\texttt{FPF}}_z = \sum_{v \in \hat{\mathfrak{T}}_1^{\texttt{FPF}}(z)} \hat F^{\texttt{FPF}}_v$.

Proof. The result follows from Theorems 2.14 and 2.16 and Lemma 3.28. ☐

We would like to show that the intervals between the minimal and maximal FPF-visible descents of the vertices in $\hat{\mathfrak{T}}^{\texttt{FPF}}(z)$ form a descending chain as one moves down the tree. This fails, however: a child in the tree may have strictly smaller FPF-visible descents than its parent. A similar property does hold if we instead consider the visible descents of the image of $z \in \mathcal{F}_{\mathbb{Z}}$ under the map $\mathcal{I} : \mathcal{F}_{\mathbb{Z}} \to \mathcal{I}_{\mathbb{Z}}$ from Definition 3.13. Recall that a visible descent for $y \in \mathcal{I}_{\mathbb{Z}}$ is an integer $i \in \mathbb{Z}$ with $z(i+1) \leq \min\{i, z(i)\}$. The following is [9, Lemma 4.24].

Lemma 3.32 (See [9])**.** Let $z \in \mathcal{I}_{\mathbb{Z}} - \{1\}$ and suppose $j \in \mathbb{Z}$ is the smallest integer such that $z(j) < j$. Then $j-1$ is the minimal visible descent of z.

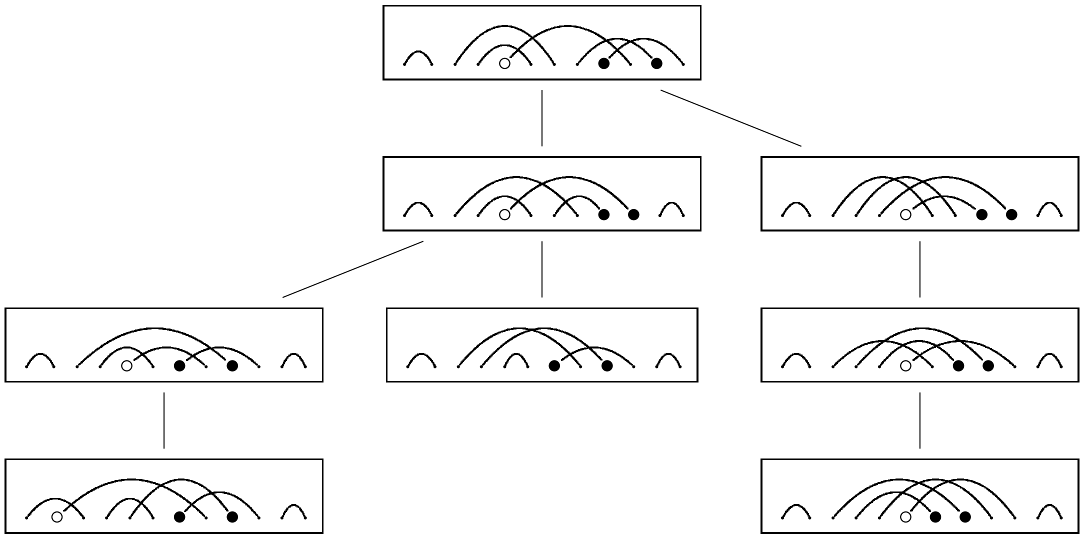


Figure 1: The tree $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$ for $z = (1,2)(3,7)(4,6)(5,10)(8,11)(9,12) \in \mathcal{F}_{12} \hookrightarrow \mathcal{F}_{\mathbb{Z}}$. We draw all vertices as elements of $\mathcal{F}_{12} \subset \mathcal{I}_{12}$ for convenience. The maximal FPF-visible inversion of each vertex is marked with $\bullet$, and the minimal FPF-visible descent is marked with $\circ$ (when this is not also maximal). By Theorem 3.19 and Corollary 3.31, we have $\hat{F}^{\mathsf{FPF}}_z = P_{(5,2)} + P_{(4,3)} + P_{(4,2,1)}$.

Lemma 3.33. Let $z \in \mathcal{F}_{\mathbb{Z}} - \{\Theta\}$ and suppose $(i,j) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$ is the cycle with j minimal such that $i < b < j$ for some $(a,b) \in \mathrm{Cyc}_{\mathbb{Z}}(z)$. Then $j-1$ is the minimal visible descent of $\mathcal{I}(z)$.

Proof. The claim follows by the preceding lemma since j is minimal such that $\mathcal{I}(z)(j) < j$. $\square$

Lemma 3.34. If $z \in \mathcal{F}_{\mathbb{Z}}$ then $i \in \mathbb{Z}$ is a visible descent of $\mathcal{I}(z)$ if and only if one of these holds:

(a) $z(i+1) < z(i) < i$.

(b) $z(i) < z(i+1) < i$ and $\{t \in \mathbb{Z} : z(i) < t < i\} \subset \{z(t) : i < t\}$.

(c) $z(i+1) < i < z(i)$ and $\{t \in \mathbb{Z} : z(i+1) < t < i+1\} \not\subset \{z(t) : i+1 < t\}$.

Proof. It is straightforward to check that $i \in \mathbb{Z}$ is a visible descent of $\mathcal{I}(z)$ if and only if either (a) $z(i+1) < z(i) < i$; (b) $z(i) < z(i+1) < i$ and i is a fixed point of $\mathcal{I}(z)$; or (c) $z(i+1) < i < z(i)$ and $i+1$ is not a fixed point of $\mathcal{I}(z)$. The given conditions are equivalent to these statements. $\square$

Corollary 3.35. Let $y, z \in \mathcal{F}_{\mathbb{Z}}$ and let $i, j \in \mathbb{Z}$ be integers with $i < j$. Suppose $y(t) = z(t)$ for all integers $t > i$. Then j is a visible descent of $\mathcal{I}(y)$ if and only if j is a visible descent of $\mathcal{I}(z)$.

Proof. By Lemma 3.34, whether or not j is a visible descent of $\mathcal{I}(z)$ depends only on the action of z on integers greater than or equal to j. $\square$

Corollary 3.36. Let $z \in \mathcal{F}_{\mathbb{Z}}$ and suppose i is a visible descent of $\mathcal{I}(z)$. Then either i or $i-1$ is an FPF-visible descent of z. Therefore, if j is the maximal FPF-visible descent of z, then $i \leq j+1$.

Proof. It follows from Lemma 3.34 that i is an FPF-visible descent of z unless $z(i) < z(i+1) < i$ and $\{t \in \mathbb{Z} : z(i) < t < i\} \subset \{z(t) : i < t\}$, in which case $i-1$ is an FPF-visible descent of z. $\square$

The following statement is the first of two key technical lemmas in this section.

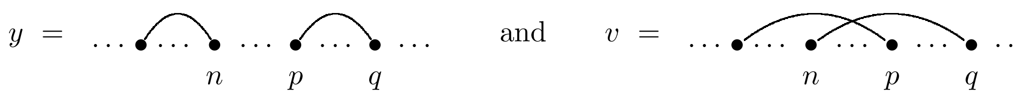
Lemma 3.37. Let $y \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$ and $(p,q) \in \mathrm{Cyc}_\mathbb{Z}(y)$ and suppose $v = (n,p)y(n,p) \in \hat\Psi^-(y,p)$.

(a) If $i \in \mathbb{Z} \setminus \{n, y(n), p, q\}$ is such that $\mathcal{I}(y)(i) = i$, then $\mathcal{I}(v)(i) = i$.

(b) If j and k are the minimal visible descents of $\mathcal{I}(y)$ and $\mathcal{I}(v)$ and $j \leq q-1$, then $j \leq k$.

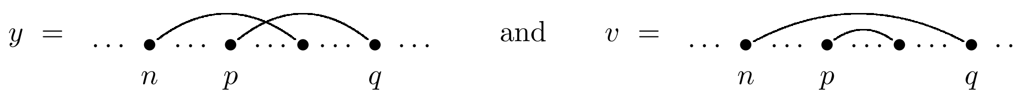
Remark 3.38. Part (b) is false if $j \geq q$: consider $y = (6,7)\Theta(6,7)$ and $(n,p,q) = (2,3,4)$. There is no analogous inequality governing the minimal FPF-visible descents of y and v.

Proof. Since $y \lessdot_\mathcal{F} v = (n,p)y(n,p) \in \hat\Psi^-(y,p)$, it follows from Proposition 2.12 that either $y(n) < n < p < q$, in which case $n < p < v(p) < q = v(n)$ and y and v correspond to the diagrams



(3.1)

or $n < p < y(n) < q$, in which case $n < p < v(p) < q = v(n)$ and we instead have



(3.2)

Let $A = \{n, y(n), p, q\} = \{n, p, v(p), q\}$ and note that $y(i) = v(i)$ for all $i \in \mathbb{Z} \setminus A$. Suppose $(a,b) \in \mathrm{Cyc}_\mathbb{Z}(y)$ is such that $b \notin A$ and $b < y(i)$ for all $a < i < b$, so that a and b are both fixed points of $\mathcal{I}(y)$. Then (a,b) is also a cycle of v, and to prove part (a) it suffices to check that $b < v(i)$ for all $i \in A$ with $a < i < b$. This holds if $i \in \{n, y(n)\}$ since then $y(i) < v(i)$, and we cannot have $a < q < b$ since $y(q) < q$. Suppose $a < p < b$; it remains to show that $b < v(p)$. Since $b < y(i)$ for all $a < i < b$ by hypothesis, it follows that if y and v are as in (3.1) then $n < a < p < b < q$, and that if y are v are as in (3.2) then $a < p < b < y(n)$. The first of these cases cannot occur in view of Proposition 2.12(a), since $y \lessdot_\mathcal{F} v$. In the second case $y(n) = v(p)$ so $b < v(p)$ as needed.

To prove part (b), note that $\Theta \notin \{y, v\}$ so neither $\mathcal{I}(y)$ nor $\mathcal{I}(v)$ is the identity. Let j and k be the minimal visible descents of $\mathcal{I}(y)$ and $\mathcal{I}(v)$ and assume $j \leq q-1$. Write S_y for the set of integers $i \in \mathbb{Z} \setminus A$ such that $\mathcal{I}(y)(i) < i$, and let $T_y = S_y \setminus A$ and $U_y = S_y \cap A$. Define S_v, T_v, and U_v similarly. Lemma 3.32 implies that $j \leq k$ if and only if $\min S_y \leq \min S_v$. Since $j \leq q-1$ we have $\min S_y \leq q$. It follows from part (a) that $T_v \subset T_y$, so $\min T_y \leq \min T_v$.

There are two cases to consider. First suppose $y(n) < n < p < q$ and $v(p) < n < p < q = v(n)$. It is then evident from (3.1) that $\{q\} \subset U_v \subset \{p, q\}$. Since $\min S_y \leq q$ by hypothesis, to prove that $\min S_y \leq \min S_v$ it suffices to show that if $p \in U_v$ then $\min S_y < p$. Since $y \lessdot_\mathcal{F} v$, neither y nor v can have any cycles (a,b) with $y(n) < a < p$ and $n < b < p$. It follows that if $p \in U_v$ then y and v share a cycle (a,b) with either (i) $a < b$ and $y(n) < b < n$, or (ii) $a < y(n) < n < b < p$. If (i) occurs then $n \in U_y$ while if (ii) occurs then $\min T_y < p$, so $\min S_y < p$ as desired.

Suppose instead that $n < p < y(n) < q$ and $n < p < v(p) < q = v(n)$. In view of (3.2), we then have $\{q\} \subset U_v \subset \{y(n), q\}$. As $\min S_y \leq q$, to prove that $\min S_y \leq \min S_v$ it now suffices to show that if $y(n) \in U_v$ then $y(n) \in U_y$. This implication is clear from (3.2), since if $y(n) = v(p) \in U_v$ then y and v must share a cycle (a,b) with $a < b$ and $p < b < y(n)$. ☐

Lemma 3.39. Let $y \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$ and $(p,q) \in \mathrm{Cyc}_\mathbb{Z}(y)$ and suppose $z = (q,r)y(q,r) \in \hat\Psi^+(y,q)$. The involution $\mathcal{I}(y)$ has a visible descent less than $q-1$ if and only if $\mathcal{I}(z)$ does, and in this case the minimal visible descents of $\mathcal{I}(y)$ and $\mathcal{I}(z)$ are equal.

Proof. Let $\mathcal{C}_w$ for $w \in \mathcal{F}_\mathbb{Z}$ be the set of cycles $(a,b) \in \mathrm{Cyc}_\mathbb{Z}(w)$ with $b < q$. By Lemma 3.33, the set $\mathcal{C}_w$ determines whether or not $\mathcal{I}(w)$ has a visible descent less than $q-1$ and, when this occurs, the value of $\mathcal{I}(w)$'s smallest visible descent. Since $q < r$ we have $\mathcal{C}_y = \mathcal{C}_z$, so the result follows. $\square$

Our second key technical lemma is the following.

Lemma 3.40. Suppose $z \in \mathcal{F}_\mathbb{Z}$ is not FPF-Grassmannian, so that $\eta_{\mathtt{FPF}}(z) \neq \Theta$. Let (q,r) be the maximal FPF-visible inversion of z and define $y = \eta_{\mathtt{FPF}}(z) = (q,r)z(q,r)$.

(a) The maximal visible descent of $\mathcal{I}(z)$ is q or $q+1$.

(b) The maximal visible descent of $\mathcal{I}(y)$ is at most q.

(c) The minimal visible descent of $\mathcal{I}(y)$ is equal to that of $\mathcal{I}(z)$, and is at most $q-1$.

Proof. Adopt the notation of Remark 3.26. To prove the first two parts, let j and k be the maximal visible descents of $\mathcal{I}(y)$ and $\mathcal{I}(z)$, respectively. In case (a) of Remark 3.26, it follows by inspection that $j \leq q = k$, with equality unless $r = q+1$ and there exists at least one cycle $(a,b) \in \mathrm{Cyc}_\mathbb{Z}(z)$ such that $p < b < q$. In case (b) of Remark 3.26, one of the following occurs:

- If $p = q-1 = r-2$, then $j < q-1 < k = q+1$.
- If $p = q-1 < r-2$, then $j = q$ and $k \in \{q, q+1\}$.
- If $p < q-1$, then $j = k = q$.

We conclude that $j \leq q$ and $k \in \{q, q+1\}$ as required.

Let j and k now be the minimal visible descents of $\mathcal{I}(y)$ and $\mathcal{I}(z)$, respectively. Part (c) is immediate from Lemmas 3.28 and 3.39 if $j < q-1$ or $k < q-1$, so assume that j and k are both at least $q-1$. Suppose $z(q) < q < r \leq m$ so that we are in case (a) of Remark 3.26, when q is the maximal visible descent of $\mathcal{I}(z)$. Since z is not FPF-Grassmannian, we must have $k = q-1$, so by Lemma 3.33 there exists $(a,b) \in \mathrm{Cyc}_\mathbb{Z}(z)$ with $z(q) < b < q$. Since $y(q) = p < z(q)$, it follows that $j \leq q-1$; as the reverse inequality holds by hypothesis, we get $j = k = q-1$ as desired.

Suppose instead that we are in case (b) of Remark 3.26. Since $q < z(q)$, it cannot hold that $q-1$ is a visible descent of $\mathcal{I}(z)$, so we must have $k \geq q$. As z is not FPF-Grassmannian, it follows from part (a) that $k = q$ and that $q+1$ is the maximal visible descent of $\mathcal{I}(z)$. This is impossible, however, since we can only have $k = q$ if there exists $(a,b) \in \mathrm{Cyc}_\mathbb{Z}(z)$ with $z(q+1) < b < q+1$, while $q+1$ can only be a visible descent of $\mathcal{I}(z)$ if no such cycle exists. $\square$

Lemma 3.41. Suppose $z \in \mathcal{F}_\mathbb{Z}$ is not FPF-Grassmannian and $v \in \hat{\mathfrak{T}}^{\mathtt{FPF}}_1(z)$. Let i and j be the minimal and maximal visible descents of $\mathcal{I}(z)$. If d is a visible descent of $\mathcal{I}(v)$, then $i \leq d \leq j$.

Proof. Let (q,r) be the maximal FPF-visible descent of z, set $y = (q,r)z(q,r) = \eta_{\mathtt{FPF}}(z)$ and $p = y(q) = z(r)$, and let $n < p < q$ be the unique integer such that $v = (n,p)y(n,p)$. Since $y \lessdot_\mathcal{F} v$, it must hold that $y(n) < q$, so $v(t) = y(t)$ for all $t > q$. The maximal visible descent of $\mathcal{I}(y)$ is at most $q \leq j$ by Lemma 3.40, so the same is true of the maximal visible descent of $\mathcal{I}(v)$ by Corollary 3.35. On the other hand, the minimal visible descent of $\mathcal{I}(y)$ is $i \leq q-1$ by Lemma 3.40, so by Lemma 3.37 the minimal visible descent of $\mathcal{I}(v)$ is at least i. $\square$

For any $z \in \mathcal{F}_\mathbb{Z}$, let $\hat{\mathfrak{T}}^{\mathtt{FPF}}_0(z) = \{z\}$ and define $\hat{\mathfrak{T}}^{\mathtt{FPF}}_n(z) = \bigcup_{v \in \hat{\mathfrak{T}}^{\mathtt{FPF}}_{n-1}(z)} \hat{\mathfrak{T}}^{\mathtt{FPF}}_1(v)$ for $n \geq 1$.

Lemma 3.42. Suppose $z \in \mathcal{F}_{\mathbb{Z}}$ and $v \in \hat{\mathfrak{T}}^{\mathsf{FPF}}_1(z)$. Let (q,r) be the maximal FPF-visible inversion of z, and let (q_1, r_1) be any FPF-visible inversion of v. Then $q_1 < q$ or $r_1 < r$. Hence, if $n \geq r - q$ then the maximal FPF-visible descent of every element of $\hat{\mathfrak{T}}^{\mathsf{FPF}}_n(z)$ is strictly less than q.

Proof. It is considerably easier to track the FPF-visible inversions of z and v than the visible inversions of $\mathcal{I}(z)$ and $\mathcal{I}(v)$, and this result follows essentially by inspecting Remark 3.26. In more detail, let $y = \eta_{\mathsf{FPF}}(z) = (q,r)z(q,r)$ and $p = z(r) = y(q)$. Since $y \lessdot_{\mathcal{F}} v = (n,p)y(n,p)$ for some $n < p$, we must have $v(i) = y(i)$ for all $i > q$, and so it is apparent from Remark 3.26 that $q_1 \leq q$. If $q_1 = q$, then necessarily $v(q) < p < v(i)$ for all $i \geq r$, and it follows that $r_1 < r$. ☐

Theorem 3.43. The FPF-involution Lascoux-Schützenberger tree $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$ is finite for all $z \in \mathcal{F}_{\mathbb{Z}}$, and it holds that $\hat{F}^{\mathsf{FPF}}_z = \sum_v \hat{F}^{\mathsf{FPF}}_v$ where the sum is over the finite set of leaf vertices v in $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$.

Proof. By induction, Corollary 3.36, and Lemmas 3.41 and 3.42, we deduce that for a sufficiently large n either $\hat{\mathfrak{T}}^{\mathsf{FPF}}_n(z) = \varnothing$ or all elements of $\hat{\mathfrak{T}}^{\mathsf{FPF}}_n(z)$ are FPF-Grassmannian, whence $\hat{\mathfrak{T}}^{\mathsf{FPF}}_{n+1}(z) = \varnothing$. The tree $\hat{\mathfrak{T}}^{\mathsf{FPF}}(z)$ is therefore finite, so the identity $\hat{F}^{\mathsf{FPF}}_z = \sum_v \hat{F}^{\mathsf{FPF}}_v$ holds by Corollary 3.31. ☐

As a corollary, we recover Theorem 1.4 from the introduction.

Corollary 3.44. If $z \in \mathcal{F}_{\mathbb{Z}}$ then $\hat{F}^{\mathsf{FPF}}_z \in \mathbb{N}\text{-span}\left\{\hat{F}^{\mathsf{FPF}}_y : y \in \mathcal{F}_{\mathbb{Z}} \text{ is FPF-Grassmannian}\right\}$ and this symmetric function is consequently Schur P-positive.

3.3 Triangularity

We can show that the expansion of $\hat{F}^{\mathsf{FPF}}_z$ into Schur P-functions not only has positive coefficients, but is unitriangular with respect to the dominance order on (strict) partitions, which denote by $<$. Recall the definition of $\hat{c}_{\mathsf{FPF}}(z)$ for $z \in \mathcal{F}_\infty$ from Section 3.1.

Definition 3.45. Let $\nu(z)$ for $z \in \mathcal{F}_\infty$ be the transpose of the partition given by sorting $\hat{c}_{\mathsf{FPF}}(z)$.

One can show that this construction is consistent with our earlier definition of $\nu(z)$ when $z \in \mathcal{F}_\infty$ is FPF-Grassmannian. Define $<_{\mathcal{A}_{\mathsf{FPF}}}$ on S_∞ as the transitive relation generated by setting $v <_{\mathcal{A}_{\mathsf{FPF}}} w$ when the one-line representation of v^{-1} can be transformed to that of w^{-1} by replacing a consecutive subsequence starting at an odd index of the form $adbc$ with $a < b < c < d$ by $bcad$, or equivalently when $s_i v > v > s_{i+1} v > s_{i+2}s_{i+1}v = s_i s_{i+1} w < s_{i+1} w < w < s_i w$ for an odd number $i \in \mathbb{P}$. For example, $235164 = (412635)^{-1} <_{\mathcal{A}_{\mathsf{FPF}}} (413526)^{-1} = 253146$, but $(12534)^{-1} \not<_{\mathcal{A}_{\mathsf{FPF}}} (13425)^{-1}$. Recall the definition of $\beta_{\min}(z)$ from Lemma 3.6. In our earlier work, we showed [7, Theorem 6.22] that $<_{\mathcal{A}_{\mathsf{FPF}}}$ is a partial order and that $\mathcal{A}_{\mathsf{FPF}}(z) = \{w \in S_\infty : \beta_{\min}(z) \leq_{\mathcal{A}_{\mathsf{FPF}}} w\}$ for all $z \in \mathcal{F}_\infty$.

Write λ^T for the transpose of a partition λ. Recall that $\lambda \leq \mu$ if and only if $\mu^T \leq \lambda^T$ [15, Eq. (1.11), §I.1]. The *shape* of $w \in S_\infty$ is the partition $\lambda(w)$ given by sorting $c(w)$.

Lemma 3.46. Let $z \in \mathcal{F}_\infty$. If $v, w \in \mathcal{A}_{\mathsf{FPF}}(z)$ and $v <_{\mathcal{A}_{\mathsf{FPF}}} w$, then $\lambda(v) < \lambda(w)$.

Proof. Suppose $v, w \in \mathcal{A}_{\mathsf{FPF}}(z)$ are such that $s_i v > v > s_{i+1} v > s_{i+2}s_{i+1}v = s_i s_{i+1} w < s_{i+1} w < w < s_i w$ for an odd number $i \in \mathbb{P}$, so that $v <_{\mathcal{A}_{\mathsf{FPF}}} w$. Define $a = w^{-1}(i+2)$, $b = w^{-1}(i)$, $c = w^{-1}(i+1)$, and $d = w^{-1}(i+3)$ so that $a < b < c < d$. The diagram $D(v^{-1})$ is then given by permuting rows i, $i+1$, $i+2$, and $i+3$ of $D(w^{-1}) \cup \{(i+3, b), (i+3, c)\} - \{(i, a), (i+1, a)\}$, and so $\lambda(v)$ is given by sorting $\lambda(w) - 2e_j + e_k + e_l$ for some indices $j < k < l$ with $\lambda(w)_j - 2 \geq \lambda(w)_k \geq \lambda(w)_l$. One checks in this case that $\lambda(v) < \lambda(w)$, as desired. ☐

Theorem 3.47. Let $z \in \mathcal{F}_\infty$ and $\nu = \nu(z)$. Then $\nu^T \leq \nu$. If $\nu^T = \nu$ then $\hat{F}^{\texttt{FPF}}_z = s_\nu$ and otherwise $\hat{F}^{\texttt{FPF}}_z \in s_{\nu^T} + s_\nu + \mathbb{N}\text{-span}\left\{s_\lambda : \nu^T < \lambda < \nu\right\}$.

Proof. It follows from [21, Theorem 4.1] that if $w \in S_\infty$ then $\lambda(w) \leq \lambda(w^{-1})^T$, and if equality holds then $F_w = s_{\lambda(w)}$ while otherwise $F_w \in s_{\lambda(w)} + s_{\lambda(w^{-1})^T} + \mathbb{N}\text{-span}\{s_\nu : \lambda(w) < \nu < \lambda(w^{-1})^T\}$. Lemma 3.8 implies that $\nu(z)^T = \lambda(\beta_{\min}(z))$, so by Lemma 3.46 we have $\hat{F}^{\texttt{FPF}}_z = \sum_{w\in\mathcal{A}_{\texttt{FPF}}(z)} F_w \in s_{\nu(z)^T} + \mathbb{N}\text{-span}\{s_\mu : \nu(z)^T < \mu\}$. The result follows since $\hat{F}^{\texttt{FPF}}_z$ is Schur P-positive and each P_μ is fixed by the linear map $\omega : \Lambda \to \Lambda$ with $\omega(s_\mu) = s_{\mu^T}$ for partitions μ [15, Example 3(a), §III.8]. □

As a corollary, we obtain Theorem 1.7 from the introduction.

Corollary 3.48. If $z \in \mathcal{F}_\infty$ then $\nu(z)$ is strict and $\hat{F}^{\texttt{FPF}}_z \in P_{\nu(z)} + \mathbb{N}\text{-span}\left\{P_\lambda : \lambda < \nu(z)\right\}$.

Proof. One has $P_\lambda \in s_\lambda + \mathbb{N}\text{-span}\{s_\nu : \nu < \lambda\}$ for any strict partition λ [15, Eq. (8.17)(ii), §III.8]. Since $\hat{F}^{\texttt{FPF}}_z$ is Schur P-positive, the result follows by Theorem 3.47. □

Strangely, we do not know of an easy way to show directly that $\nu(z)$ is a strict partition.

3.4 FPF-vexillary involutions

Define an element z of $\mathcal{F}_n$ or $\mathcal{F}_\mathbb{Z}$ to be *FPF-vexillary* if $\hat{F}^{\texttt{FPF}}_z = P_\mu$ for a strict partition μ. In this section, we derive a pattern avoidance condition classifying such involutions.

Remark 3.49. All FPF-Grassmannian involutions, as well as all elements of $\mathcal{F}_n$ for $n \in \{2, 4, 6\}$, are FPF-vexillary. The sequence $(v^{\texttt{FPF}}_{2n})_{n\geq1} = (1, 3, 15, 92, 617, 4354, \dots)$, with $v^{\texttt{FPF}}_n$ counting the FPF-vexillary elements of $\mathcal{F}_n$, again seems unrelated to any existing entry in [20].

Recall that if $E \subset \mathbb{Z}$ is a finite set of size n then we write ψ_E and ϕ_E for the order-preserving bijections $E \to [n]$ and $[n] \to E$, and define $[z]_E = \psi_{z(E)} \circ z \circ \phi_E \in S_n$ for permutations z of $\mathbb{Z}$. In this section, let $[[z]]_E = \iota([z]_E) \in \mathcal{F}_\infty$ for $z \in \mathcal{F}_\mathbb{Z}$ and finite sets $E \subset \mathbb{Z}$ with $z(E) = E$.

Lemma 3.50. If $z \in \mathcal{F}_\mathbb{Z}$ is FPF-Grassmannian and $E \subset \mathbb{Z}$ is a finite set with $z(E) = E$, then the fixed-point-free involution $[[z]]_E$ is also FPF-Grassmannian.

Proof. Suppose $z \in \mathcal{F}_\mathbb{Z}$ is FPF-Grassmannian and $E \subset \mathbb{Z}$ is finite and z-invariant. We may assume that $z \in \mathcal{F}_\infty$ and $E \subset \mathbb{P}$. Fix a set $F = \{1, 2, \dots, 2n\}$ where $n \in \mathbb{P}$ is large enough that $E \subset F$ and $[[z]]_F = z$. Note that for any z-invariant set $D \subset E$ we have $[[z]]_D = [[z']]_{D'}$ for $z' = [[z]]_E$ and $D' = \psi_E(D)$. Inductively applying this property, we see that it suffices to show that $[[z]]_E$ is FPF-Grassmannian when $E = F \setminus \{a, b\}$ with $\{a, b\} \subset F$ a nontrivial cycle of z. In this special case, it is a straightforward exercise to check that $\mathcal{I}([[z]]_E)$ is either $[\mathcal{I}(z)]_E$ or the involution formed by replacing the leftmost cycle of $[\mathcal{I}(z)]_E$ by two fixed points. In either case it is easy to see that $\mathcal{I}([[z]]_E)$ is I-Grassmannian, so $[[z]]_E$ is FPF-Grassmannian as needed. □

We fix the following notation in Lemmas 3.51, 3.53, and 3.54. Let $z \in \mathcal{F}_\mathbb{Z} - \{\Theta\}$ and write $(q, r) \in \mathbb{Z} \times \mathbb{Z}$ for the maximal FPF-visible inversion of z. Set $y = \eta_{\texttt{FPF}}(z) = (q, r)z(q, r) \in \mathcal{F}_\mathbb{Z}$ and define $p = y(q) < q$ so that $\hat{\mathfrak{T}}^{\texttt{FPF}}_1(z) = \hat{\Psi}^-(y, p)$ if z is not FPF-Grassmannian.

Lemma 3.51. Let $E \subset \mathbb{Z}$ be a finite set with $\{q, r\} \subset E$ and $z(E) = E$. Then $(\psi_E(q), \psi_E(r))$ is the maximal FPF-visible inversion of $[[z]]_E$. Moreover, it holds that $[[\eta_{\texttt{FPF}}(z)]]_E = \eta_{\texttt{FPF}}([[z]]_E)$.

Proof. The first assertion holds since the set of FPF-visible inversions of z contained in $E \times E$ and the set of all FPF-visible inversions of $[[z]]_E$ are in bijection via the order-preserving map $\psi_E \times \psi_E$. The second claim follows from the definition of $\eta_{\mathtt{FPF}}$ since $\{q, r, z(q), z(r)\} \subset E$. $\square$

We write $L^{\mathtt{FPF}}(z)$ for the set of integers $i < p$ with $(i,p)y(i,p) \in \hat{\Psi}^-(y,p)$. For any $E \subset \mathbb{Z}$ we define $\mathfrak{C}^{\mathtt{FPF}}(z, E) = \left\{(i,p)y(i,p) : i \in E \cap L^{\mathtt{FPF}}(z)\right\}$. Also let $\mathfrak{C}^{\mathtt{FPF}}(z) = \mathfrak{C}^{\mathtt{FPF}}(z, \mathbb{Z})$, so that $\mathfrak{C}^{\mathtt{FPF}}(z) = \hat{\mathfrak{T}}_1^{\mathtt{FPF}}(z)$ if z is not FPF-Grassmannian. The following shows that $\mathfrak{C}^{\mathtt{FPF}}(z)$ is always nonempty.

Lemma 3.52. If $z \in \mathcal{F}_{\mathbb{Z}} - \{\Theta\}$ is FPF-Grassmannian, then $|\mathfrak{C}^{\mathtt{FPF}}(z)| = 1$.

Proof. Assume $z \in \mathcal{F}_{\mathbb{Z}} - \{\Theta\}$ is FPF-Grassmannian. By Proposition 3.15 we have $z = \mathcal{F}(g)$ for an I-Grassmannian involution $g \in \mathcal{I}_{\mathbb{Z}}$. Using this fact and the observations in Remark 3.26, one checks that $\mathfrak{C}^{\mathtt{FPF}}(z) = \{(i,p)y(i,p)\}$ where i is the greatest integer less than p such that $y(i) < q$. $\square$

Lemma 3.53. Let $E \subset \mathbb{Z}$ be a finite set such that $\{q, r\} \subset E$ and $z(E) = E$.

(a) The operation $v \mapsto [[v]]_E$ restricts to an injective map $\mathfrak{C}^{\mathtt{FPF}}(z, E) \to \mathfrak{C}^{\mathtt{FPF}}([[z]]_E)$.

(b) If E contains $L^{\mathtt{FPF}}(z)$, then the injective map in (a) is a bijection.

Proof. Part (a) is straightforward from the definition of $\mathfrak{C}^{\mathtt{FPF}}(z)$ given Lemma 3.51. We prove the contrapositive of part (b). Suppose $a < b = \psi_E(p)$ and $(a,b)[[y]]_E(a,b)$ belongs to $\mathfrak{C}^{\mathtt{FPF}}([[z]]_E)$ but is not in the image of $\mathfrak{C}^{\mathtt{FPF}}(z, E)$ under the map $v \mapsto [[v]]_E$. Suppose $a = \psi_E(i)$ for $i \in E$. Then $(a,b)[[y]]_E(a,b) = [[(i,p)y(i,p)]]_E$, and it follows from Proposition 2.12 that $[[y]]_E(a) < [[y]]_E(b)$, so we likewise have $y(i) < y(p)$. Since $(i,p)y(i,p) \notin \mathfrak{C}^{\mathtt{FPF}}(z, E)$, there must exist an integer j with $i < j < p$ and $y(i) < y(j) < y(p)$. Let j be maximal with this property and set $k = z(j)$. One can check using Proposition 2.12 that either j or k belongs to $L^{\mathtt{FPF}}(z)$ but not E, so $E \not\supset L^{\mathtt{FPF}}(z)$. $\square$

We say that $z \in \mathcal{F}_{\mathbb{Z}}$ *contains a bad FPF-pattern* if there exists a finite set $E \subset \mathbb{Z}$ with $z(E) = E$ and $|E| \leq 12$, such that $[[z]]_E$ is not FPF-vexillary. We refer to E as a *bad FPF-pattern* for z.

Lemma 3.54. If $z \in \mathcal{F}_{\mathbb{Z}}$ is such that $|\hat{\mathfrak{T}}_1^{\mathtt{FPF}}(z)| \geq 2$, then z contains a bad FPF-pattern.

Proof. If $u \neq v$ and $\{u, v\} \subset \hat{\mathfrak{T}}_1^{\mathtt{FPF}}(z)$, then u, v, and z agree outside a set $E \subset \mathbb{Z}$ of size 8 with $z(E) = E$. It follows by Lemmas 3.52 and 3.53 that E is a bad FPF-pattern for z. $\square$

Lemma 3.55. Suppose $z \in \mathcal{F}_{\mathbb{Z}}$ is such that $\hat{\mathfrak{T}}_1^{\mathtt{FPF}}(z) = \{v\}$ is a singleton set. Then z contains no bad FPF-patterns if and only if v contains no bad FPF-patterns.

Proof. By definition, z and v agree outside a set $A \subset \mathbb{Z}$ of size 6 with $v(A) = z(A) = A$. If z (respectively, v) contains a bad FPF-pattern which is disjoint from A, then the other involution clearly does also. If z contains a bad FPF-pattern B which intersects A, then $E = A \cup B$ has size at most 16 since $|B| \leq 12$ and both A and B are z-invariant. In this case, $[[z]]_E$ contains a bad FPF-pattern and Lemma 3.53(b) shows that $\mathfrak{C}^{\mathtt{FPF}}([[z]]_E) = \{[[v]]_E\}$, and if $[[v]]_E$ contains a bad FPF-pattern then v does also. By similar arguments, it follows that if v contains a bad FPF-pattern B that intersects A, then $E = A \cup B$ has size at most 16, $[[v]]_E$ contains a bad FPF-pattern, $\mathfrak{C}^{\mathtt{FPF}}([[z]]_E) = \{[[v]]_E\}$, and v contains a bad FPF-pattern if $[[v]]_E$ does.

These observations show that to prove the lemma, it suffices to consider the case when z belongs to the image of $\iota : \mathcal{F}_{16} \hookrightarrow \mathcal{F}_{\mathbb{Z}}$. Using a computer, we have checked that if z is such an involution and $\mathfrak{C}^{\mathtt{FPF}}(z) = \{v\}$ is a singleton set, then z contains no bad FPF-patterns if and only if v contains no bad FPF-patterns. There are 940,482 possibilities for z, a sizeable but still tractable number. $\square$

Theorem 3.56. An involution $z \in \mathcal{F}_\mathbb{Z}$ is FPF-vexillary if and only if $[[z]]_E$ is FPF-vexillary for all sets $E \subset \mathbb{Z}$ with $z(E) = E$ and $|E| = 12$.

Proof. Let $\mathcal{X} \subset \mathcal{F}_\mathbb{Z}$ be the set which contains $z \in \mathcal{F}_\mathbb{Z}$ if and only if z is FPF-Grassmannian or $\hat{\mathfrak{T}}_1^{\mathtt{FPF}}(z) = \{v\}$ and $v \in \mathcal{X}$. It follows from Corollary 3.31(b) that $\mathcal{X}$ is the set of all FPF-vexillary involutions in $\mathcal{F}_\mathbb{Z}$. On the other hand, Lemmas 3.50, 3.54, and 3.55 show that $\mathcal{X}$ is the set of involutions $z \in \mathcal{F}_\mathbb{Z}$ which contain no bad FPF-patterns. We deduce that $z \in \mathcal{F}_\mathbb{Z}$ is FPF-vexillary if and only if z has no bad FPF-patterns, which is equivalent to the theorem statement. □

Corollary 3.57. An involution $z \in \mathcal{F}_\mathbb{Z}$ is FPF-vexillary if and only if for all finite sets $E \subset \mathbb{Z}$ with $z(E) = E$ the involution $[z]_E$ is not any of the following sixteen permutations:

$$\begin{array}{lll}
(1,3)(2,4)(5,8)(6,7), & (1,5)(2,3)(4,7)(6,8), & (1,6)(2,4)(3,8)(5,7), \\
(1,3)(2,5)(4,7)(6,8), & (1,5)(2,3)(4,8)(6,7), & (1,6)(2,5)(3,8)(4,7), \\
(1,3)(2,5)(4,8)(6,7), & (1,5)(2,4)(3,7)(6,8), & (1,3)(2,4)(5,7)(6,9)(8,10), \\
(1,3)(2,6)(4,8)(5,7), & (1,5)(2,4)(3,8)(6,7), & (1,3)(2,5)(4,6)(7,9)(8,10), \\
(1,4)(2,3)(5,7)(6,8), & (1,6)(2,3)(4,8)(5,7), & (1,3)(2,4)(5,7)(6,8)(9,11)(10,12). \\
(1,4)(2,3)(5,8)(6,7), & &
\end{array}$$

Proof. It follows by a computer calculation using the formulas in Theorems 3.19 and 3.43 that $z \in \iota(\mathcal{F}_{12}) \subset \mathcal{F}_\infty$ is not FPF-vexillary if and only if there exists a z-invariant subset $E \subset \mathbb{Z}$ such that $[z]_E$ is one of the given involutions. The corollary follows from this fact by Theorem 3.56. □

3.5 Pfaffian formulas

The *Pfaffian* of a skew-symmetric $n \times n$ matrix A is

$$\operatorname{pf} A = \sum_{z \in \mathcal{F}_n} (-1)^{\hat{\ell}_{\mathtt{FPF}}(z)} \prod_{z(i) < i \in [n]} A_{z(i),i}. \tag{3.3}$$

It is a classical fact that $\det A = (\operatorname{pf} A)^2$. Since $\det A = 0$ when A is skew-symmetric but n is odd, the definition (3.3) is consistent with the fact that the set $\mathcal{F}_n$ of fixed-point-free involutions in S_n is nonempty only if n is even.

Example 3.58. If $A = (a_{ij})$ is a 2×2 skew-symmetric matrix then $\operatorname{pf} A = a_{12} = -a_{21}$. If $A = (a_{ij})$ is a 4×4 skew-symmetric matrix then $\operatorname{pf} A = a_{21}a_{43} - a_{31}a_{42} + a_{41}a_{32}$.

Both $\hat{\mathfrak{S}}_z^{\mathtt{FPF}}$ and $\hat{F}_z^{\mathtt{FPF}}$ can be expressed by certain Pfaffian formulas when z is FPF-Grassmannian. We fix the following notation for the duration of this section: first, let

$$n, r \in \mathbb{P} \qquad \text{and} \qquad \phi \in \mathbb{P}^r \text{ with } 0 < \phi_1 < \phi_2 < \cdots < \phi_r < n. \tag{3.4}$$

Set $\phi_i = 0$ for $i > r$. Define $y = (\phi_1, n+1)(\phi_2, n+2)\cdots(\phi_r, n+r) \in \mathcal{I}_\infty$ and $z = \mathcal{F}(y)$, and let

$$\hat{\mathfrak{S}}^{\mathtt{FPF}}[\phi_1, \phi_2, \ldots, \phi_r; n] = \hat{\mathfrak{S}}_z^{\mathtt{FPF}} \qquad \text{and} \qquad \hat{F}^{\mathtt{FPF}}[\phi_1, \phi_2, \ldots, \phi_r; n] = \hat{F}_z^{\mathtt{FPF}}.$$

When r is odd, we also set $\hat{\mathfrak{S}}^{\mathtt{FPF}}[\phi_1, \phi_2, \ldots, \phi_r, 0; n] = \hat{\mathfrak{S}}_z^{\mathtt{FPF}}$ and $\hat{F}^{\mathtt{FPF}}[\phi_1, \phi_2, \ldots, \phi_r, 0; n] = \hat{F}_z^{\mathtt{FPF}}$.

Proposition 3.59. In the notation just given, $z \in \mathcal{F}_\infty$ is FPF-Grassmannian with shape $\nu(z) = (n - \phi_1, n - \phi_2, \ldots, n - \phi_r)$. Moreover, each FPF-Grassmannian element of $\mathcal{F}_\infty - \{\Theta\}$ occurs as such an involution z for a unique choice of $n, r \in \mathbb{P}$ and $\phi \in \mathbb{P}^r$ as in (3.4).

Proof. Let $X = [n] \setminus \{\phi_1, \phi_2, \dots, \phi_r\}$ and note that $n \in X$. If $|X|$ is even then $\mathcal{I}(z) = y$. If $|X|$ is odd and at least 3, then $\mathcal{I}(z) = y \cdot (n, n+r+1)$. If $|X| = 1$, finally, then $\phi = (1, 2, \dots, n-1)$ and $\mathcal{I}(z) = (2, n+2)(3, n+3) \cdots (n, 2n)$. In each case, we have $\nu(z) = (n-\phi_1, n-\phi_2, \dots, n-\phi_r)$ as desired. The second assertion holds since an FPF-Grassmannian element of $\mathcal{F}_\infty$ is uniquely determined by its image under $\mathcal{I} : \mathcal{F}_\infty \to \mathcal{I}_\infty$, which must be I-Grassmannian with an even number of fixed points in $[n]$ and not equal to $(i+1, n+1)(i+2, n+2) \cdots (n, 2n-i)$ for any $i \in [n]$. $\square$

Let $\ell^+(\phi)$ be whichever of r or $r+1$ is even, and let $[a_{ij}]_{1 \le i < j \le n}$ denote the skew-symmetric matrix with a_{ij} in position (i,j) and $-a_{ij}$ in position (j,i) for $i < j$ (and zeros on the diagonal).

Corollary 3.60. In the setup of (3.4), $\hat{F}^{\mathsf{FPF}}[\phi_1, \phi_2, \dots, \phi_r; n] = \mathrm{pf}\left[\hat{F}^{\mathsf{FPF}}[\phi_i, \phi_j; n]\right]_{1 \le i < j \le \ell^+(\phi)}$.

Proof. Macdonald proves that if λ is a strict partition then $P_\lambda = \mathrm{pf}[P_{\lambda_i \lambda_j}]_{1 \le i < j \le \ell^+(\lambda)}$ [15, Eq. (8.11), §III.8]. Given this fact and the preceding proposition, the result follows from Theorem 3.19. $\square$

Our goal is to prove that the identity in this corollary holds with $\hat{F}^{\mathsf{FPF}}[\cdots; n]$ replaced by $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\cdots; n]$. In the following lemmas, we let $\mathfrak{M}^{\mathsf{FPF}}[\phi; n] = \mathfrak{M}^{\mathsf{FPF}}[\phi_1, \phi_2, \dots, \phi_r; n]$ denote the skew-symmetric matrix $\left[\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_i, \phi_j; n]\right]_{1 \le i < j \le \ell^+(\phi)}$.

Lemma 3.61. Maintain the notation of (3.4), and suppose $p \in [n-1]$. Then

$$\partial_p \left(\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[\phi; n]\right) = \begin{cases} \mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[\phi + e_i; n] & \text{if } p = \phi_i \notin \{\phi_2 - 1, \dots, \phi_r - 1\} \text{ for some } i \in [r] \\ 0 & \text{otherwise} \end{cases}$$

where $e_i = (0, \dots, 0, 1, 0, 0, \dots)$ is the standard basis vector whose ith coordinate is 1.

Proof. Let $\mathfrak{M} = \mathfrak{M}^{\mathsf{FPF}}[\phi; n]$. If $1 \le i < j \le \ell^+(\phi)$ then (2.3) implies that $\partial_p \mathfrak{M}_{ij} = \partial_p \hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_i, \phi_j; n]$ is $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_i + 1, \phi_j]$ if $p = \phi_i \neq \phi_j - 1$, $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_i, \phi_j + 1]$ if $p = \phi_j$, and 0 otherwise. Thus if $p \notin \{\phi_1, \phi_2, \dots, \phi_r\}$ then $\partial_p (\mathrm{pf}\, \mathfrak{M}) = 0$. Suppose $p = \phi_k$. Then $\partial_p \mathfrak{M}_{ij} = 0$ unless $i = k$ or $j = k$, so $\partial_p (\mathrm{pf}\, \mathfrak{M}) = \mathrm{pf}\, \mathfrak{N}$ where $\mathfrak{N}$ is the matrix formed by applying ∂_p to the entries in the kth row and kth column of $\mathfrak{M}$. If $k < r$ and $\phi_k = \phi_{k+1} - 1$, then columns k and $k+1$ of $\mathfrak{N}$ are identical, so $\mathrm{pf}\, \mathfrak{M} = \mathrm{pf}\, \mathfrak{N} = 0$. If $k = r$ or if $k < r$ and $\phi_k \neq \phi_{k+1} - 1$, then $\mathfrak{N} = \mathfrak{M}^{\mathsf{FPF}}[\phi + e_k; n]$. $\square$

Lemma 3.62. Let $n \ge 2$ and $D = (x_1 + x_2)(x_1 + x_3) \cdots (x_1 + x_n)$. Then $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1; n] = D$, and if $b \in \mathbb{P}$ is such that $1 < b < n$, then $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1, b; n]$ is divisible by D.

Proof. Theorem 3.5 implies that $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1; n] = D$ and, when $n > 2$, that $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1, 2; n] = (x_2 + x_3) \cdots (x_2 + x_n) D$. If $2 < b < n$ then $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1, b; n] = \partial_{b-1}(\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1, b-1; n])$ by the previous lemma. Since D is symmetric in x_{b-1} and x_b, the desired property holds by induction. $\square$

If $i : \mathbb{P} \to \mathbb{N}$ is a map with $i^{-1}(\mathbb{P}) \subset [n]$, then we define $x^i = x_1^{i(1)} x_2^{i(2)} \cdots x_n^{i(n)}$. Given a nonzero polynomial $f = \sum_{i : \mathbb{P} \to \mathbb{N}} c_i x^i \in \mathbb{Z}[x_1, x_2, \dots]$, let $j : \mathbb{P} \to \mathbb{N}$ be the lexicographically minimal index such that $c_j \neq 0$ and define $\mathrm{lt}(f) = c_j x^j$. We refer to $\mathrm{lt}(f)$ as the *least term* of f. Set $\mathrm{lt}(0) = 0$, so that $\mathrm{lt}(fg) = \mathrm{lt}(f)\,\mathrm{lt}(g)$ for any polynomials f, g. The following is [6, Proposition 3.14].

Lemma 3.63 (See [6]). If $z \in \mathcal{F}_\infty$ then $\mathrm{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}_z) = x^{\hat{c}_{\mathsf{FPF}}(z)} = \prod_{(i,j) \in \hat{D}_{\mathsf{FPF}}(z)} x_i$.

Let $\mathscr{M}$ denote the set of monomials $x^i = x_1^{i(1)} x_2^{i(2)} \cdots$ for maps $i : \mathbb{P} \to \mathbb{N}$ with $i^{-1}(\mathbb{P})$ finite. Define $\prec$ as the "lexicographic" order on $\mathscr{M}$, that is, the order with $x^i \prec x^j$ when there exists $n \in \mathbb{P}$ such that $i(t) = j(t)$ for $1 \leq t < n$ and $i(n) < j(n)$. Note that $\mathrm{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}_z) \in \mathscr{M}$. Also, observe that if $a, b, c, d \in \mathscr{M}$ and $a \preceq c$ and $b \preceq d$, then $ab \preceq cd$ with equality if and only if $a = c$ and $b = d$.

Lemma 3.64. Let $i, j, n \in \mathbb{P}$. The following identities then hold:

(a) If $i < n$ then $\mathrm{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}[i; n]) \succeq x_{i+1} x_{i+2} \cdots x_n$, with equality if and only if i is odd.

(b) If $i < j < n$ then $\mathrm{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}[i, j; n]) \succeq (x_{i+1} x_{i+2} \cdots x_n)(x_{j+1} x_{j+2} \cdots x_n)$, with equality if and only if i is odd and j is even.

Proof. The result follows by routine calculations using Lemma 3.63. For example, suppose $i < j < n$ and let $y = (i, n+1)(j, n+2)$ and $z = \mathcal{F}(y)$, so that $\hat{\mathfrak{S}}^{\mathsf{FPF}}[i, j; n] = \hat{\mathfrak{S}}^{\mathsf{FPF}}_z$. If i is even and $j = i+1$, then $\hat{D}_{\mathsf{FPF}}(z) = \{(i, i-1), (i+1, i-1)\} \cup \{(i+1, i), (i+3, i), \ldots, (n, i)\} \cup \{(i+3, i+1), \ldots, (n, i+1)\}$ so $\mathrm{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}[i, j; n]) = (x_i x_{i+1} x_{i+3} \cdots x_n)(x_j x_{j+2} \cdots x_n)$. The other cases follow by similar analysis. □

Lemma 3.65. If $n \in \mathbb{P}$ and $r \in [n-1]$ then $\hat{\mathfrak{S}}^{\mathsf{FPF}}[1, 2, \ldots, r; n] = \mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[1, 2, \ldots, r; n]$.

Proof. The proof is similar to that of [9, Lemma 4.77]. Let $D_i = (x_i + x_{i+1})(x_i + x_{i+2}) \cdots (x_i + x_n)$ for $i \in [n-1]$ and $\mathfrak{M} = \mathfrak{M}^{\mathsf{FPF}}[1, 2, \ldots, r; n]$. Theorem 3.5 implies that $\hat{\mathfrak{S}}^{\mathsf{FPF}}[1, 2, \ldots, r; n] = D_1 D_2 \cdots D_r$. Lemma 3.61 implies that $\mathrm{pf}\, \mathfrak{M}$ is symmetric in $x_1, x_2, \ldots, x_r$. Lemma 3.62 implies that every entry in the first column of $\mathfrak{M}$, and therefore also $\mathrm{pf}\, \mathfrak{M}$, is divisible by D_1. Since $s_i(D_i)$ is divisible by D_{i+1}, it follows that $\mathrm{pf}\, \mathfrak{M}$ is divisible by $\hat{\mathfrak{S}}^{\mathsf{FPF}}[1, 2, \ldots, r; n]$. To prove the lemma, it suffices to show that $\mathrm{pf}\, \mathfrak{M}$ and $\hat{\mathfrak{S}}^{\mathsf{FPF}}[1, 2, \ldots, r; n]$ have the same least term.

Let $m \in \mathbb{P}$ be whichever of r or $r+1$ is even and choose $z \in \mathcal{F}_m$. It follows by Lemma 3.64 that $\mathrm{lt}\left(\prod_{z(i) < i \in [m]} \mathfrak{M}_{z(i), i}\right) \succeq (x_2 x_3 \cdots x_n)(x_3 x_4 \cdots x_n) \cdots (x_{r+1} x_{r+2} \cdots x_n) = \mathrm{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}[1, 2, \ldots, r; n])$, with equality if and only if i is odd and j is even whenever $i < j = z(i)$. The only element $z \in \mathcal{F}_m$ with the latter property is the involution $z = (1,2)(3,4) \cdots (m-1, m) = \Theta_m$, so we deduce from (3.3) that $\mathrm{lt}(\mathrm{pf}\, \mathfrak{M}) = \mathrm{lt}(\hat{\mathfrak{S}}^{\mathsf{FPF}}[1, 2, \ldots, r; n])$ as needed. □

Theorem 3.66. In the setup of (3.4), $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_1, \phi_2, \ldots, \phi_r; n] = \mathrm{pf} \left[\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_i, \phi_j; n]\right]_{1 \leq i < j \leq \ell^+(\phi)}$.

Proof. Let $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi; n] = \hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi_1, \phi_2, \ldots, \phi_r; n]$. It $\phi = (1, 2, \ldots, r)$ then $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi; n] = \mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[\phi; n]$ by the previous lemma. Otherwise, there exists a smallest $i \in [r]$ such that $i < \phi_i$. If $p = \phi_i - 1$ then $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi; n] = \partial_p \hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi - e_i; n]$ by (2.3) and $\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[\phi; n] = \partial_p(\mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[\phi - e_i; n])$ by Lemma 3.61. We may assume that $\hat{\mathfrak{S}}^{\mathsf{FPF}}[\phi - e_i; n] = \mathrm{pf}\, \mathfrak{M}^{\mathsf{FPF}}[\phi - e_i; n]$ by induction, so the result follows. □

Example 3.67. For $\phi = (1, 2, 3)$ and $n = 4$ the theorem reduces to the identity

$$\hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,5)(2,6)(3,7)(4,8)} = \mathrm{pf} \begin{pmatrix} 0 & \hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,5)(2,6)(3,4)} & \hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,5)(2,4)(3,6)} & \hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,5)(2,3)(4,6)} \\ -\hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,5)(2,6)(3,4)} & 0 & \hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,4)(2,5)(3,6)} & \hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,3)(2,5)(4,6)} \\ -\hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,5)(3,6)(2,4)} & -\hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,4)(2,5)(3,6)} & 0 & \hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,2)(3,5)(4,6)} \\ -\hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,5)(2,3)(4,6)} & -\hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,3)(2,5)(4,6)} & -\hat{\mathfrak{S}}^{\mathsf{FPF}}_{(1,2)(3,5)(4,6)} & 0 \end{pmatrix}$$

where for $z \in \mathcal{F}_n$ we define $\hat{\mathfrak{S}}^{\mathsf{FPF}}_z = \hat{\mathfrak{S}}^{\mathsf{FPF}}_{\iota(z)}$. By Theorem 3.5, both of these expressions evaluate to $(x_1 + x_2)(x_1 + x_3)(x_1 + x_4)(x_2 + x_3)(x_2 + x_4)(x_3 + x_4)$.

It is an open problem to find a simple, closed formula for $\hat{\mathfrak{S}}^{\mathsf{FPF}}[i, j; n]$.

References

[1] A. Bertiger, The orbits of the symplectic group on the flag manifold, preprint (2014), `arXiv:1411.2302`.

[2] S. Billey and M. Haiman, Schubert polynomials for the classical groups, *J. Amer. Math. Soc.* **8** (1995), 443–482.

[3] S. C. Billey, W. Jockusch, and R. P. Stanley, Some Combinatorial Properties of Schubert Polynomials, *J. Algebr. Combin.* **2** (1993), 345–374.

[4] M. B. Can, M. Joyce, and B. Wyser, Chains in Weak Order Posets Associated to Involutions, *J. Combin. Theory Ser. A* **137** (2016), 207–225.

[5] P. Edelman and C. Greene, Balanced tableaux, *Adv. Math.* **63** (1987), 42–99.

[6] Z. Hamaker, E. Marberg, and B. Pawlowski, Involution words: counting problems and connections to Schubert calculus for symmetric orbit closures, preprint (2015), `arXiv:1508.01823`.

[7] Z. Hamaker, E. Marberg, and B. Pawlowski, Involution words II: braid relations and atomic structures, *J. Algebr. Comb.* **45** (2017), 701–743.

[8] Z. Hamaker, E. Marberg, and B. Pawlowski, Transition formulas for involution Schubert polynomials, preprint (2016), `arXiv:1609.09625`.

[9] Z. Hamaker, E. Marberg, and B. Pawlowski, Schur P-positivity and involution Stanley symmetric functions, preprint (2017), `arXiv:1701.02824`.

[10] T. Józefiak, Schur Q-functions and cohomology of isotropic Grassmannians, *Math. Proc. Camb. Phil. Soc.* **109**, 471–478.

[11] A. Knutson, Schubert polynomials and symmetric functions, notes for the Lisbon Combinatorics Summer School (2012), available online at `http://www.math.cornell.edu/~allenk/`.

[12] A. Lascoux and M.-P. Schützenberger, Schubert polynomials and the Littlewood-Richardson rule, *Lett. Math. Phys.* **10** (1985), no. 2, 111–124.

[13] D. P. Little, Combinatorial aspects of the Lascoux-Schützenberger tree, *Adv. Math.*, **174** (2003), no. 2, 236–253.

[14] I. G. Macdonald, Notes on Schubert Polynomials, Laboratoire de combinatoire et d'informatique mathématique (LACIM), Université du Québec a Montréal, Montreal, 1991.

[15] I. G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., Oxford University Press, New York, 1999.

[16] L. Manivel, Symmetric Functions, Schubert Polynomials, and Degeneracy Loci, American Mathematical Society, 2001.

[17] P. Pragacz, Algebro-geometric applications of Schur S- and Q-polynomials, *Séminaire d'Algèbre Dubreil-Malliavin* 1989–90, Springer Lecture Notes **1478**, 130–191.

[18] E. M. Rains and M. J. Vazirani, Deformations of permutation representations of Coxeter groups, *J. Algebr. Comb.* **37** (2013), 455–502.

[19] I. Schur, Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen, *J. Reine Angew. Math.* **139** (1911), 155–250.

[20] N. J. A. Sloane, editor (2003), The On-Line Encyclopedia of Integer Sequences, published electronically at `http://oeis.org/`.

[21] R. P. Stanley, On the number of reduced decompositions of elements of Coxeter groups. *European J. Combin.* **5** (1984), 359–372.

[22] J. R. Stembridge, Shifted tableaux and the projective representations of symmetric groups, *Adv. Math.* **74** (1989), 87–134.

[23] B. J. Wyser and A. Yong, Polynomials for symmetric orbit closures in the flag variety, *Transform. Groups*, to appear.