A New Quantity Counted by OEIS Sequence A006012

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We prove a conjecture of Callan in [1] that OEIS sequence A006012 counts a certain kind of permutation. Call this sequence $(a_n)_{n=1}^{\infty}$; then a_n is defined by $a_1 = 1$, $a_2 = 2$, and $a_n = 4a_{n-1} - 2a_{n-2}$ (the actual sequence in the OEIS is offset by one, so $a_0 = 1$, $a_1 = 2$, and the recursion is the same). The conjecture states that a_n is equal to the number of permutations of length n for which no subsequence $abcd$ has the following two properties: $c = b + 1$ and $\max\{a, c\} < \min\{b, d\}$.

We can rewrite this conjecture in the language of pattern avoidance, in particular, using the dashed notation for generalized pattern avoidance introduced in [2]. Therefore, we define a pattern to be a permutation $\pi_1 ... \pi_k$, some (or all) of whose elements may be separated by dashes. We say that a subsequence of a permutation is an occurrence of a pattern if (i) all the elements have the same relative order as the elements of the pattern, and (ii) if there is no dash between the i^{th} and $i+1^{\text{th}}$ elements of the pattern, then the i^{th} and $i+1^{\text{th}}$ element of the subsequence occur consecutively in the permutation. We say that a permutation avoids a pattern if it does not contain any occurrence of the pattern, and a permutation avoids a set of patterns if it does not contain any occurrence of any of them. If A is a set of patterns, we will write $\text{Av}(A)$ for the set of permutations which avoid them, and $\text{Av}_n(A)$ for the set of length-n permutations which avoid them. The following two examples should help clarify these definitions.

Example: The permutation 251346 contains the subsequence 5146 which is an occurrence of the pattern 3-1-24 because the elements of the subsequence occur in the same relative order as 3124, and the 4 and 6 are consecutive in the original permutation (the 5 and 1 are also consecutive - that is allowed but not necessary).

Example: The permutation 251346 avoids 32-1-4 (i.e. $251346 \in \text{Av}_n(\{32\text{-}1\text{-}4\}) \subseteq \text{Av}(\{32\text{-}1\text{-}4\})$).

Using this notation, we rewrite the conjecture as $a_n = |\text{Av}_n(\{1\text{-}32\text{-}4, 1\text{-}42\text{-}3, 2\text{-}31\text{-}4, 2\text{-}41\text{-}3\}|$.

We will prove two propositions. The first is that if $A = \{1\text{-}32\text{-}4, 1\text{-}42\text{-}3, 2\text{-}31\text{-}4, 2\text{-}41\text{-}3\}$ and $B = \{1\text{-}3\text{-}2\text{-}4, 1\text{-}4\text{-}2\text{-}3, 2\text{-}3\text{-}1\text{-}4, 2\text{-}4\text{-}1\text{-}3\}$, then $\text{Av}(A)$ and $\text{Av}(B)$ are the same set. The second proposition is that $|\text{Av}_n(B)|$ follows the defining recurrence of a_n, i.e. $|\text{Av}_1(B)| = 1, |\text{Av}_2(B)| = 2, |\text{Av}_n(B)| = 4|\text{Av}_{n-1}(B)| - 2|\text{Av}_{n-2}(B)|$.

Proposition 1: Let $A = \{1\text{-}32\text{-}4, 1\text{-}42\text{-}3, 2\text{-}31\text{-}4, 2\text{-}41\text{-}3\}$ and $B = \{1\text{-}3\text{-}2\text{-}4, 1\text{-}4\text{-}2\text{-}3, 2\text{-}3\text{-}1\text{-}4, 2\text{-}4\text{-}1\text{-}3\}$. The sets $\text{Av}(A)$ and $\text{Av}(B)$ are the same.

Proof: We will show that any permutation containing an occurrence of an element of B must also contain an occurrence of an element of A (the converse is immediately clear). Let π be a permutation. First, note that a subpermutation $\pi_a\pi_b\pi_c\pi_d$ of π is an occurrence of a pattern in A if and only if $c = b + 1$ and $\max\{\pi_a, \pi_c\} < \min\{\pi_b, \pi_d\}$ (in fact, this is the definition Callan provides in the OEIS). Similarly, a subpermutation $\pi_a\pi_b\pi_c\pi_d$ of π is an occurrence of a pattern in B if and only if $\max\{\pi_a, \pi_c\} < \min\{\pi_b, \pi_d\}$.

Choose any element of B, and suppose that π contains an occurrence of this element. As noted above, this means that we can find $a < b < c < d$ such that $\max\{\pi_a, \pi_c\} < \min\{\pi_b, \pi_d\}$. Let e be the largest index less than c such that $\pi_e > \max\{\pi_a, \pi_c\}$, i.e. $e = \max\{i : i < c, \pi_i > \max\{\pi_a, \pi_c\}\}$. Because b is an element of $\{i : i < c, \pi_i > \max\{\pi_a, \pi_c\}\}$, it follows that e exists and $a < b \leq e < e+1 \leq c < d$. Now, we claim that $\pi_a \pi_e \pi_{e+1} \pi_d$ is an occurrence of a pattern in A. Obviously $e+1 = e+1$, and so it remains to check that $\max\{\pi_a, \pi_{e+1}\} < \min\{\pi_e, \pi_d\}$. Because $\max\{\pi_a, \pi_c\} < \min\{\pi_b, \pi_d\}$, we conclude that $\pi_a < \pi_d$ and by the choice of e, we also have $\pi_a < \pi_e$. Now, either $e+1 = c$, in which case $\pi_{e+1} = \pi_c$, or else $\pi_{e+1} < \max\{\pi_a, \pi_c\}$ because otherwise we would have chosen $e+1$ as the $\max\{i : i < c, \pi_i > \max\{\pi_a, \pi_c\}\}$ instead of e. It follows that $\pi_{e+1} \leq \max\{\pi_a, \pi_c\} < \pi_d, \pi_e$ for the same reasons as π_a. Therefore, $\max\{\pi_a, \pi_{e+1}\} < \min\{\pi_e, \pi_d\}$ and $\pi_a \pi_e \pi_{e+1} \pi_d$ is an occurrence of a pattern in A. We conclude that the permutations avoiding the patterns of A are the same as the permutations avoiding the patterns of B.

Proposition 2: The number of permutations of length n avoiding all patterns in B (and hence in A) satisfies the recurrence $a_1 = 1, a_2 = 2, a_n = 4a_{n-1} - 2a_{n-2}$.

Proof: Since $\mathrm{Av}_1(B) = \{1\}$ and $\mathrm{Av}_2(B) = \{12, 21\}$, the initial conditions hold. Our strategy will be as follows: given $\mathrm{Av}_n(B)$, define four maps which, when all of them are applied to all the permutations of $\mathrm{Av}_{n-1}(B)$, will generate all of the permutations of $\mathrm{Av}_n(B)$. Then, we will count how many permutations of $\mathrm{Av}_n(B)$ are double counted in this way, and find that there are two for every element of $\mathrm{Av}_{n-2}(B)$, thereby establishing the recurrence.

Note that, for a permutation to avoid all patterns of A, it must be the case that either 1 and 2 occur consecutively (not necessarily in that order) or either 1 or 2 is the last element of the permutation. This observation motivates the following definitions of the four maps $f_{\text{before}}, f_{\text{after}}, f_{\text{end}}, f_{\text{bump}}$. Let f_{before} be the function that inputs a permutation and outputs that permutation with all elements increased by 1 and a 1 inserted immediately before the new 2. Let f_{after} be the function that also inputs a permutation and outputs that permutation with all the elements increased by 1 and a 1 inserted immediately after the new 2. Similarly, let f_{end} be the function that inputs a permutation, increases all its elements by 1 and puts a 1 at the end of it, and let f_{bump} be the function that inputs a permutation, increases all its elements by 1, replaces the new 2 with a one and puts a 2 at the end. The following example gives a concrete illustration of the four functions.

Example: Let $\pi = 31542$. Then $f_{\text{before}}(\pi) = 412653, f_{\text{after}}(\pi) = 421653, f_{\text{end}}(\pi) = 426531$, and $f_{\text{bump}}(\pi) = 416532$. Note that $\pi \in \mathrm{Av}(B)$, and so are all its images.

We claim that (i) these four functions all map elements of $\mathrm{Av}_{n-1}(B)$ to elements of $\mathrm{Av}_n(B)$ and (ii) $f_{\text{before}}(\mathrm{Av}_{n-1}(B)) \cup f_{\text{after}}(\mathrm{Av}_{n-1}(B)) \cup f_{\text{end}}(\mathrm{Av}_{n-1}(B)) \cup f_{\text{bump}}(\mathrm{Av}_{n-1}(B)) \supseteq \mathrm{Av}_n(B)$ (by claim (i), we could replace the '$\supseteq$' in claim (ii) with '$=$'). To verify the first claim, choose some $\sigma \in \mathrm{Av}_{n-1}(B)$, and consider each function in turn. If $f_{\text{before}}(\sigma)$ or $f_{\text{after}}(\sigma)$ contains an occurrence of a pattern in A, then this occurrence must use no more than 1 of the elements 1 and 2 (because they are consecutive in both $f_{\text{before}}(\sigma)$ and $f_{\text{after}}(\sigma)$ but can't be in any pattern in A). Therefore, either this occurrence fails to use 1 and would have already been an occurrence of the pattern in σ, or else it fails to use 2, in which case it could have used

2 instead of 1 and been an occurrence of the pattern in σ. Thus, no such occurrence is possible in $f_{\text{before}}(\sigma)$ or $f_{\text{after}}(\sigma)$. In addition, if $f_{\text{end}}(\sigma)$ or $f_{\text{bump}}(\sigma)$ contains an occurrence of a pattern in A, then this occurrence cannot use the last element because that element is either a 1 or a 2, and patterns in A only end with 3 or 4. So, this occurrence would already be an occurrence of the pattern in σ, and therefore cannot exist.

To verify the second claim, chose some $\pi \in \text{Av}_n(B)$. As previously noted, either 1 and 2 occur consecutively in π, or else either 1 or 2 is the final element of π. Let π' be π with the 1 removed and each element decreased by 1. We have introduced no new patterns, and so $\pi' \in \text{Av}_{n-1}(B)$. Suppose that 1 occurs immediately before 2 in π, then $f_{\text{before}}(\pi') = \pi$. If the 1 occurs immediately after 2 in π, then $f_{\text{after}}(\pi') = \pi$. If the 1 occurs at the end of π, then $f_{\text{end}}(\pi') = \pi$. If the 2 occurs at the end of π, then we will need to define π'' which is π with the 1 removed, the 2 moved the position where the 1 used to be, and each element decreased by 1. Again, we have introduced no new patterns, and so $\pi'' \in \text{Av}_{n-1}(B)$, and $f_{\text{bump}}(\pi'') = \pi$.

If these four functions all had disjoint ranges, we could conclude that $a_n = 4a_{n-1}$. Unfortunately, some permutations are counted twice. Each f outputs a certain kind of permutation: f_{before} outputs permutations where 1 immediately precedes 2, f_{after} outputs permutations where 2 immediately precedes 1, f_{end} outputs permutations where 1 occurs at the end, and f_{bump} outputs permutations where 2 occurs at the end. If a permutation fulfills two of these criteria it will be double-counted. Such permutations must be counted once by either f_{before} or f_{after} and again by either f_{end} or f_{bump} because no permutation can be counted by both f_{before} and f_{after} or both f_{end} and f_{bump}. Thus, the final two elements of such permutations are 1 and 2 (not necessarily in that order). Let $g : \text{Av}_n(B) \to \text{Av}_{n-2}(B)$ be defined as the function which takes a permutation, removes from it the elements 1 and 2, and reduces all other elements by 2. If we restrict g to those permutations which end in either 12 or 21, g becomes a 2-to-1 map from the double-counted permutations of $\text{Av}_n(B)$ to the permutations of $\text{Av}_{n-2}(B)$, and so the number of double-counted permutations is twice a_{n-2}. It follows that $a_n = 4a_{n-1} - 2a_{n-2}$.

The author would like to thank his (intended) advisor Doron Zeilberger for introducing him to the conjecture and fixing typos in the original draft.

References

[1] OEIS Foundation Inc. (2017), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A006012

[2] E. Babson and E. Steingrímsson (2000). "Generalized permutation patterns and a classification of the Mahonian statistics," *Sém. Lothar. Combin.* **44**, Art: B44b