CORRELATION FUNCTIONS OF COLOURED TENSOR MODELS AND THEIR SCHWINGER-DYSON EQUATIONS

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ABSTRACT. We analyze the correlation functions of coloured tensor models and use the Ward-Takahashi identity in order to derive the full tower of exact Schwinger-Dyson equations. We write them explicitly for ranks $D = 3$ and $D = 4$. Throughout, we follow a non-perturbative approach. We propose the extension of this program to the Gurau-Witten model, a holographic tensor model based on the Sachdev-Ye-Kitaev model (SYK model).

1. INTRODUCTION

Inspired by matrix models, and being initially in fact very similar to them, *tensors models* are a natural way to extend the two dimensional random geometry of matrix models [3] to higher dimensions. As an important application, tensor models aim at evaluating the partition function of simplicial quantum gravity [1; 4; 11; 14; 16; 17] and could be seen, under mild assumptions, as a generator of a family of graph-encoded discretizations of the Einstein-Hilbert action, in whose continuum limit smooth geometries are expected to emerge.

One of the initial differences between matrix and tensor models, which turned out to impair the natural development of the latter, was the impossibility to derive their $1/N$-expansion. This problem was solved by Gurău [8], who introduced a unitary symmetry, the "colouring" the tensors, that forbids some unwished contributions in the perturbative expansion (see Sec. 2.1 here and [7]). He showed that the $1/N$-expansion of rank-D tensor models is controlled by an integer called Gurău's degree associated to each Feynman graph. This integer happens to be related to the value of the Einstein-Hilbert action on a D-dimensional equilateral triangulation associated to that graph. The discrete spectrum of Gurău's degree is then the set of values that the Regge discretisation of the general relativity action can take. Additional to the initial motivations of tensor models, new applications to AdS/CFT (also admitting a $1/N$-expansion [10]) via the Sachdev-Ye–Kitaev (SYK) models have been found in [20]; along these lines the Gurău-Witten model [9] has been newly proposed. This sets the foundations for the so-called holographic tensors.

All these new results enliven the physics of random tensors. Yet, the quantum theory of these objects itself deserves a more thorough mathematical scrutiny, and, in this vein, the present paper is a study of the correlation functions of coloured tensors, already begun in [15], and of the equations they obey. There, the partition function $Z[J, \bar{J}]$ of coloured tensor models (CTM) has been shown to satisfy a constraint, namely the Ward-Takahashi identity, is a byproduct of the colouring of the tensors. We ventured to anticipate that this constraint would allow to derive an equation for each correlation function of coloured tensor models, and the aim of this paper is to obtain those for arbitrary rank.

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The connected correlation $2k$-point function of rank-D Tensor Field theories are usually defined by

$$G^{(2k)}(\mathbf{x}_1,\ldots,\mathbf{x}_k;\mathbf{y}_1,\ldots,\mathbf{y}_k) = \Big(\prod_{i=1}^{k} \frac{\delta}{\delta J_{\mathbf{x}_i}} \frac{\delta}{\delta \bar{J}_{\mathbf{y}_i}}\Big) \log(Z[J,\bar{J}])\Big|_{J=\bar{J}=0} \qquad (\mathbf{x}_i,\mathbf{y}_i \in \mathbb{Z}^D). \qquad (*)$$

For CTM, this definition is redundant, when not equivocal (e.g. $G^{(2)}(\mathbf{x};\mathbf{y})$ identically vanishes outside the diagonal $\mathbf{x}=\mathbf{y}$). In [15] we proposed to split each function $G^{(2k)}$ in sectors $G^{(2k)}_{\mathcal{B}}$ that encompass all Feynman graphs indexed by so-called boundary graphs $\mathcal{B}$ (see Sec. 2). Here $2k$ denotes the number of vertices of $\mathcal{B}$ and this integer coincides with the number of external legs of the graphs summed in $G^{(2k)}_{\mathcal{B}}$.

There are two reasons to classify correlation functions by boundary graphs: First, by using these correlation functions one gains a clear geometric interpretation in terms of bordisms. Feynman diagrams in CTMs are coloured graphs, and these represent graph-encoded triangulations of PL-manifolds. The momentum flux between external legs of an open graph $\mathcal{G}$ determines its so-called boundary, $\mathcal{B}=\partial\mathcal{G}$. Boundary graphs are important because they also triangulate a manifold, and this manifold coincides with the boundary, in the usual sense, of the manifold that the original graph triangulates [11]. Also, by fixing a boundary graph $\mathcal{B}$, one can sum all connected Feynman graphs that contribute to $G^{(2k)}_{\mathcal{B}}$, and these are interpreted as bordisms whose boundary is triangulated by $\mathcal{B}$; for instance, the connected components, and of (the graph indexing) $G^{(8)}_{|\,|\,|}$ triangulate a sphere and a torus, respectively, and (connected Feynman) graphs contributing to this correlation function are triangulations of bordisms $\mathbb{S}^2 \to \mathbb{T}^2$ that are compatible with their boundary being "triangulated by" $\sqcup$.

Secondly, one must do the splitting of the correlations in boundary graphs, otherwise the momenta of the sources interfere with one another. The correlation functions that we propose here need only half the arguments of the functions from definition (∗). For $k=1,2,3,4$, the connected correlation functions indexed by connected *boundary* graphs are

$$G^{(2)}, \qquad (1.1a)$$

$$G^{(4)}_{1\;1},\; G^{(4)}_{2\;2},\; G^{(4)}_{3\;3}, \qquad (1.1b)$$

$$G^{(6)},\; G^{(6)}_{1},\; G^{(6)}_{2},\; G^{(6)}_{3},\; G^{(6)}_{2\,3},\; G^{(6)}_{1\,3},\; G^{(6)}_{1\,2}, \qquad (1.1c)$$

$$G^{(8)},\; G^{(8)},\; G^{(8)},\; G^{(8)},\; G^{(8)},\; G^{(8)},\; G^{(8)},\; G^{(8)},\ \text{and } \mathfrak{S}_3\text{-orbits thereof.} \qquad (1.1d)$$

Also, functions like $G^{(6)}_{|\,|\,|}$ and $G^{(8)}_{|\,|\,|}$, indexed by disconnected graphs, need to be considered. None of these graphs is a Feynman graph: in fact we will not deal with them here[1], since we proceed non-perturbatively.

To these two reasons, we add as motivation the success that this treatment gave for matrix models [6]. There, by splitting in boundary components, the matricial Ward identity was exploited and combined with the Schwinger-Dyson equations. This allowed to derive an integral equation for the quartic matrix models and, in the planar sector, finally solve for all correlation functions in terms of the two point function [6] via algebraic recursions. Here, we import these techniques to the CTM setting.

In this article we derive the full tower of equations that correspond to connected boundary graphs. We also obtain the 2-point and some higher-point Schwinger-Dyson equations

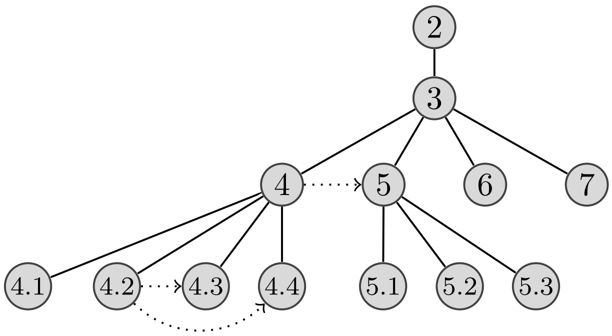
[1]Except two Feynman diagram examples appearing in Sec. 2.1 and in Fig. 7.

(SDE) in an explicit form rank-3 and rank-4 theories. Section 2 recalls the setting of coloured tensor models in a condensed fashion, and the expansion of the free energy in boundary graphs. The Ward-Takahashi Identity (WTI) [15] for coloured tensors, which we recall in Section 2.3, is a fundamental auxiliary and bases on this boundary graph expansion. There, we also introduce language to deal with the proper derivation of the full SDE-tower in Section 3. We continue with the derivation of the SDE-equations for quartic rank-3 theories (Sec. 4) and rank-4 theories (Sec. 5; moreover, rank-5 are shortly addressed in App. B).

In order to derive the SDE for a certain $2k$-point function it is necessary to know, also to order $2k$ in the sources, the form of certain generating functional (for rank 3, Lemma 4.1, with proof located in App. A) which appears in the Ward-Identity. This requires knowledge of the free energy to order $2(k+1)$ (in the sources), which in turn needs information about all the graphs with this number of vertices and their coloured automorphism groups. Later on, in Section 5 we find the SDE for rank-4 theories with melonic quartic vertices. Explicitly, only the two-point functions and 4-point functions are obtained, since the graph theory in four colours is much more complicated. Section 6 presents a model that has simpler SDEs and looks solvable, since, as shown there, it posses a very similar expansion in boundary graphs. It is a tensor model that can be used to study the random geometry of 3-spheres.

A short section before the conclusions analyzes the boundary graphs of the Gurău-Witten (SYK-like) model and sets a plan to extend to it the CTM-methods developed in previous sections.

We motivate a non-linear reading of this article. The dependence of the sections is sketched by means of lines in the next diagram, the dashed ones meaning weak dependence.



Acknowledgments

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Contents

2. Boundary graph expansions

This section rapidly introduces the notation in graph theory and recapitulates previous results that are relevant in our present study. There are few examples in Fig. 1 that are intended as support to rapidly grasp the next definitions. Also the rather panoramic Table 1 organizes the concepts introduced below.

2.1. **Coloured tensors and coloured graphs.** Let N be a (large) integer, thought of as an energy scale, and consider D distinguished representations, $(\mathcal{H}_1, \rho_1), \ldots, (\mathcal{H}_D, \rho_D)$ of $\mathrm{U}(N)$. A coloured tensor model is concerned with the quantum theory of tensor fields $\varphi, \bar{\varphi} : \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_D \to \mathbb{C}$ whose components transform under said D representations as

$$\varphi_{x_1 \ldots x_D} \mapsto \varphi'_{x_1 \ldots x_D} = \sum\nolimits_{y_a} [\rho_a(W_a)]_{x_a y_a} \varphi_{x_1 \ldots y_a \ldots x_D} \,,$$
$$\bar{\varphi}_{x_1 \ldots x_D} \mapsto \bar{\varphi}'_{x_1 \ldots x_D} = \sum\nolimits_{y_a} [\overline{\rho_a(W_a)}]_{x_a y_a} \bar{\varphi}_{x_1 \ldots y_a \ldots x_D} \,,$$

for all $W_a \in \mathrm{U}(N)$ and being each x_a and y_a in suitable index-sets $I_a \subset \mathbb{Z}$, for each integer (or *colour*) $a = 1, \ldots, D$. Usually one sets $\mathcal{H}_a = \mathbb{C}^N$ or $\mathcal{H}_a = \ell^2[-n, n]$ for suitable $n = n(N)$, and $\rho_a = \mathrm{id}_{\mathcal{H}_a}$ for each colour a. However, at the same time, one insists that the representations are distinguished, so that indices are anchored to a spot assigned by its colour. Thus, the indices of the tensors have no symmetries (e.g. $\varphi_{ijk} = \varphi_{ikj}$ is forbidden) and only indices of the same colour can be contracted.

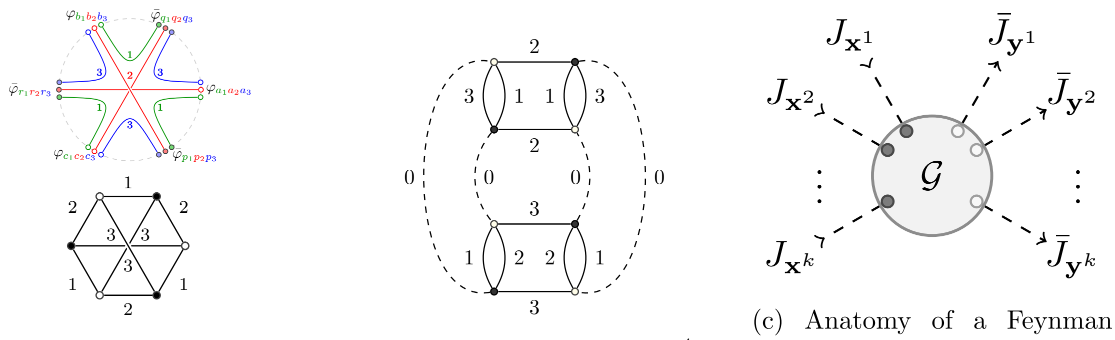
A particular tensor model is specified by two additional data: a finite subset of interaction vertices given by real monomials in φ and $\bar{\varphi}$ that are $\mathrm{U}(N)$-invariant under the chosen D representations; the second data is a quadratic form

$$S_0(\varphi, \bar{\varphi}) = \mathrm{Tr}_2(\bar{\varphi}, E\varphi) = \sum\nolimits_{\mathbf{x}} \bar{\varphi}_{\mathbf{x}} E_{\mathbf{x}} \varphi_{\mathbf{x}}, \quad \text{for certain function } E : I_1 \times \ldots \times I_D \to \mathbb{R}^+ \,,$$

determining the kinetic term S_0 in the classical action. Sums are (implicitly) over the finite lattice $I_1 \times \ldots \times I_D \subset \mathbb{Z}^D$. These I_a sets depend usually on a cutoff scale related to N and we will assume, also implicitly, that throughout they are all $\mathbb{Z}$, keeping in mind that one needs to regularize.

In order to characterize the interaction vertices, one uses vertex-bipartite regularly edge-D-coloured graphs, or, in the sequel, just "*D-coloured graphs*". A graph $\mathcal{G}$ being *vertex-bipartite* means that its vertex-set $\mathcal{G}^{(0)}$ splits into two disjoint sets $\mathcal{G}^{(0)} = \mathcal{G}^{(0)}_{\mathrm{w}} \cup \mathcal{G}^{(0)}_{\mathrm{b}}$. The set $\mathcal{G}^{(0)}_{\mathrm{w}}$ (resp. $\mathcal{G}^{(0)}_{\mathrm{b}}$) consists of *white* (resp. *black*) vertices. The set of edges, denoted by $\mathcal{G}^{(1)}$ is split as $\mathcal{G}^{(1)} = \cup_a \mathcal{G}^{(1)}_a$ into D disjoint sets $\mathcal{G}^{(1)}_a$ of a-coloured edges, $a = 1, \ldots, D$. Given any edge e, the white and black vertices e is attached at, are denoted by $s(e) \in \mathcal{G}^{(0)}_{\mathrm{w}}$ and $t(e) \in \mathcal{G}^{(0)}_{\mathrm{b}}$, respectively. This defines the maps $s, t : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)}$. Regularity of the colouring means that, for each $v \in \mathcal{G}^{(0)}_{\mathrm{w}}$ and each $w \in \mathcal{G}^{(0)}_{\mathrm{b}}$, both preimages $s^{-1}(v)$ and $t^{-1}(w)$ consist of precisely of D edges of different colours. By regularity, the number of white and black vertices is the same and is equal to $k(\mathcal{G}) := \# \mathcal{G}^{(0)}/2$. The set of (closed) D-coloured graphs is denoted by $\mathsf{Grph}^{\mathrm{cl}}_D$.

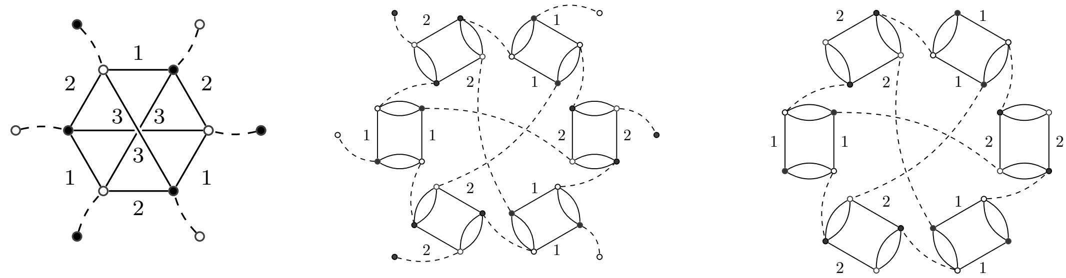
The only way to obtain monomials in the fields φ and $\bar{\varphi}$ that are also invariants, is contracting each coordinate index $\varphi_{\ldots x_c \ldots}$ by a delta $\delta_{x_c y_c}$ with the coordinate $\bar{\varphi}_{\ldots y_c \ldots}$ of the respective colour of the field $\bar{\varphi}$. The imposed $\mathrm{U}(N)$-invariance requires then $D \cdot k(\mathcal{G})$ such coloured deltas. One thus associates to each occurrence of φ a white vertex v and to each occurrence of $\bar{\varphi}$ a black vertex w. For each colour c, to each $\delta_{x_c y_c}$ contracting $\varphi_{\ldots x_c \ldots}$ and $\bar{\varphi}_{\ldots y_c \ldots}$ one draws a c-coloured edge which starts at v, $v = s(e)$,



(a) Example of the relation between traces and monomials. This graph is denoted by $K_{\mathrm{c}}(3,3)$

(b) This graph is in $\mathsf{Feyn}_3^{\mathrm{v}}(\varphi^4)$, i.e. it is a vacuum Feynman graph of the φ^4_3-model, $V(\varphi,\bar\varphi) = \lambda({}_1\boxminus_1 + {}_2\boxminus_2 + {}_3\boxminus_3)$

(c) Anatomy of a Feynman graph and how it determines boundary graph $\mathcal{B}$, which induces the map $\mathcal{B}_* : (\mathbf{x}^1,\ldots,\mathbf{x}^k) \mapsto (\mathbf{y}^1,\ldots,\mathbf{y}^k)$



(d) Open graph with boundary $K_{\mathrm{c}}(3,3)$ (in fact it is the cone of $K_{\mathrm{c}}(3,3)$) but is not in $\mathsf{Feyn}_3(\varphi^4)$

(e) This graph $\mathcal{R}$ is an open graph in $\mathsf{Feyn}_3(\varphi^4) \subset \mathsf{Grph}_{3+1}^{(6)} \subset \mathsf{Grph}_{3+1}$ with $\partial\mathcal{R} = K_{\mathrm{c}}(3,3)$ (see explanation in 1f)

(f) The amputation of $\mathcal{R}$. If one erases the 0-coloured (or dashed) edges, one gets connected components in $\{{}_1\boxminus_1, {}_2\boxminus_2, {}_3\boxminus_3\}$

FIGURE 1. Graph terminology of Sec. 2.1 and concerning examples

and ends at w, $w = t(e)$. Thus, any invariant monomial $\mathrm{Tr}_{\mathcal{B}}$ is fully determined by a coloured graph $\mathcal{B}$, and vice versa. For instance, the trace $\sum_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{p},\mathbf{q},\mathbf{r}}(\bar\varphi_{r_1r_2r_3}\bar\varphi_{q_1q_2q_3}\bar\varphi_{p_1p_2p_3})\cdot(\delta_{a_1p_1}\delta_{a_2r_2}\delta_{a_3q_3}\delta_{b_1q_1}\delta_{b_2p_2}\delta_{b_3r_3}\delta_{c_1r_1}\delta_{c_2q_2}\delta_{c_3p_3})\cdot(\varphi_{a_1a_2a_3}\varphi_{b_1b_2b_3}\varphi_{c_1c_2c_3})$ is depicted in 1a.

Any model is then given by an interaction potential $V(\varphi,\bar\varphi) = \sum_{\mathcal{B}\in\Lambda}\lambda_{\mathcal{B}}\mathrm{Tr}_{\mathcal{B}}(\varphi,\bar\varphi)$, for Λ a finite subset of $\mathsf{Grph}_D^{\mathrm{cl}}$. For a fixed model $S = S_0 + V$, one can write down the corresponding partition function:

$$Z[J,\bar J] = Z_0 \int \mathcal{D}[\varphi,\bar\varphi]\, \mathrm{e}^{\mathrm{Tr}_2(\bar J,\varphi)+\mathrm{Tr}_2(\bar\varphi,J)-N^{D-1}S[\varphi,\bar\varphi]}\,,\ \mathcal{D}[\varphi,\bar\varphi] := \prod_{\mathbf{x}\in\mathbb{Z}^D} N^{D-1}\frac{\mathrm{d}\varphi_{\mathbf{x}}\mathrm{d}\bar\varphi_{\mathbf{x}}}{2\pi\mathrm{i}}\,. \quad (2.1)$$

Here $S_0 = \mathrm{Tr}_2(\bar\varphi,\varphi)$ is the only quadratic invariant, namely $\ominus$. Later on, at the level of propagator, we will allow this invariance to be broken (see Sec. 2).

Using Wick's theorem one evaluates the contributions to the generating functional. Wick's contractions (*propagators*) are assigned a new colour, 0, which one commonly draws as dashed line. For (complex) matrix models ($D = 2$), this 0 colour would be the ribbon line propagator, thus, for tensors, this colour 0 substitutes a cumbersome notation of D parallel lines. It is easy to see that Feynman vacuum graphs of rank-D coloured tensors are vertex-bipartite regularly edge-$(D+1)$-coloured graphs, now the colours being the integers from 0 to D. Vacuum graphs can be connected or disconnected. The set of strictly disconnected graphs is denoted by Ξ_{D+1}^{cl} and $\mathsf{Grph}_{D+1}^{\amalg,\mathrm{cl}}$ denotes the set of possibly disconnected graphs. We assume that any Feynman graph is connected and get rid of *Feynman* graphs

Map of the graph theory of an arbitrary rank-D model $V(\varphi,\bar\varphi)$				
Number of of colours	**Closed** *Connected*	*Disconnected*	**Open** *Connected*	*Disconnected*
D	• Observables • Traces $\mathcal{B}$, $\mathrm{Tr}_{\mathcal{B}}$ in $V=\sum_{\mathcal{B}}\lambda_{\mathcal{B}}\mathrm{Tr}_{\mathcal{B}}(\varphi,\bar\varphi)$ • boundaries $\partial\mathcal{G}$	• generic boundary graph in $\partial(\mathsf{Feyn}_D(V))$	$\emptyset$	$\emptyset$
Notation →	$\mathsf{Grph}^{\mathrm{cl}}_D$	$\Xi^{\mathrm{cl}}_D=\mathsf{Grph}^{\mathrm{II,cl}}_D\setminus\mathsf{Grph}^{\mathrm{cl}}_D$	Grph_D	Ξ_D
$D+1$	vacuum Feynman diagrams, whose set we denote by $\mathsf{Feyn}^{\mathrm{v}}_D(V)$	$\emptyset$ (no contribution), for taking the logarithm of $Z[J,\bar J]$ gets rid of them	the generic Feynman graphs of the model, $\mathsf{Feyn}_D(V)$	$\emptyset$
Notation →	$\mathsf{Grph}^{\mathrm{cl}}_{D+1}$	$\Xi^{\mathrm{cl}}_{D+1}=\mathsf{Grph}^{\mathrm{II,cl}}_{D+1}\setminus\mathsf{Grph}^{\mathrm{cl}}_{D+1}$	Grph_{D+1}	Ξ_{D+1}

Table 1. Terminology of Sec. 2, and why both graphs in D and $D+1$ number of colours appear in a rank-D models, and which their respective roles are. Here "disconnected" strictly means "not connected". The notation $\emptyset$ stands for "no contributions to/no role in the rank-D theory".

in Ξ^{cl}_{D+1} by working with the free energy, $W[J,\bar J]=\log(Z[J,\bar J])$, rather than with the partition function.

Since we are mainly interested in the *connected* correlation functions we have to consider *open* Feynman graphs, i.e. graphs with n external legs, each of which is attached to a tensorial source, J or $\bar J$, that obeys the same transformation rules of the field φ or $\bar\varphi$, respectively. The external legs are exceptional edges of valence-1 white (for the source J) or black (for $\bar J$) vertices. All external legs' edges have colour 0. Clearly, because of bipartiteness, this number has to be even, $n=2k$. We denote by $\mathsf{Grph}^{(2k)}_{D+1}$ the set of Feynman diagrams with $2k$ external legs and further set

$$\mathsf{Grph}_{D+1}=\cup_{k=1}^{\infty}\mathsf{Grph}^{(2k)}_{D+1}\cup\mathsf{Grph}^{\mathrm{cl}}_{D+1}$$

generically for open or closed $(D+1)$-coloured graphs.

Importantly, not every graph in Grph_{D+1} is a Feynman graph. The set of Feynman graphs of a model $V(\varphi,\bar\varphi)=\sum_{\mathcal{B}\in\Lambda}\lambda_{\mathcal{B}}\mathrm{Tr}_{\mathcal{B}}(\varphi,\bar\varphi)$ is denoted by $\mathsf{Feyn}_D(V(\varphi,\bar\varphi))$ or $\mathsf{Feyn}_D(V)$. This set consists of the graphs in Grph_{D+1} that satisfy the following condition: after amputating all external legs and removing all the 0-coloured edges, the remaining graph has connected components in the set of interaction-vertices $\Lambda\subset\mathsf{Grph}^{\mathrm{cl}}_D$ (see Figs. 1e,1f).

2.2. **Boundary graphs.** There is a boundary map $\partial:\mathsf{Grph}_{D+1}\to\mathsf{Grph}^{\mathrm{II,cl}}_D$, which for all $\mathcal{G}\in\mathsf{Grph}_{D+1}$ is given by

$$(\partial\mathcal{G})^{(0)}:=\{\text{external legs of }\mathcal{G}\}\,,$$

$$(\partial\mathcal{G})^{(1)}_a:=\{(0a)\text{-bicoloured paths between external legs in }\mathcal{G}\}\,.$$

The vertex set inherits the bipartiteness from $\mathcal{G}$, to wit a vertex in $(\partial\mathcal{G})^{(0)}$ is *black* if it corresponds from an external line attached to a white vertex, and *white* if it is attached to a black vertex in $\mathcal{G}$. The edge set is regularly D-coloured $(\partial\mathcal{G})^{(1)}=\cup_a(\partial\mathcal{G})^{(1)}_a$.

For a fixed model $V(\varphi,\bar\varphi)$, the image of the restriction $\partial_V:=\partial|_{\mathsf{Feyn}_D(V)}$ of ∂ to $\mathsf{Feyn}_D(V)$ is deemed *boundary sector*, and this set is, of course, model dependent. A graph in the boundary sector is a *boundary graph*. For melonic quartic theories, as a matter of fact [15], this boundary map is surjective, so all (possibly disconnected) D-coloured graphs are boundaries. Thus, all the correlation functions we propose have non-trivial contributions. Incidentally, this means that quartic coloured random tensor models are able to ponder probabilities of triangulation of all bordisms, provided they exist, as in dimension d ($d = D-1 = 2,3$) as classical objects (oriented manifolds); in presence of obstructions, there are pseudo-manifolds yielding those bordisms.

Given a closed coloured graph $\mathcal{B}$, $\mathrm{Aut_c}(\mathcal{B})$ denotes the set of its *coloured automorphisms.* These are graph maps $\mathcal{B} \to \mathcal{B}$ that preserve adjacency, the bipartiteness of $\mathcal{B}^{(0)}$ and also its edge-colouring. Each automorphism of $\mathcal{B}$ arises from a lifting of an element π of $\mathrm{Sym}(\mathcal{B}^{(0)}_{\mathrm{w}}) = \mathfrak{S}_{k(\mathcal{B})}$ to a unique map $\hat{\pi}: \mathcal{B} \to \mathcal{B}$, as one can easily see, determined by the preservation of said structure. Figures 2 and 3 show all the automorphism groups for graphs having up to 8 vertices in $D=3$ and up to 6 vertices for $D=4$, respectively.

We shall assume that both the white vertex-set $\mathcal{B}^{(0)}_{\mathrm{w}} = (v^1, \ldots, v^{k(\mathcal{B})})$ as well as the black vertex-set $\mathcal{B}^{(0)}_{\mathrm{b}} = (w^1, \ldots, w^{k(\mathcal{B})})$ of a boundary graph $\mathcal{B}$ are given an ordering. Then $e_a^{v^\mu}$, the edge of colour a attached to a white vertex $v \in \mathcal{B}^{(0)}_{\mathrm{w}}$, i.e. $s(e_a^{v^\mu}) = v^\mu$, is denoted by e_a^μ.

Let $\mathcal{B}$ be a boundary graph and $k = k(\mathcal{B})$. Then $\mathcal{B}$ induces a map[2] $\mathcal{B}_* : M_{D\times k}(\mathbb{Z}) \to M_{D\times k}(\mathbb{Z})$ by $\mathbf{X} = (\mathbf{x}^1, \ldots, \mathbf{x}^k) \mapsto \mathcal{B}_*(\mathbf{X}) = (\mathbf{y}^1, \ldots, \mathbf{y}^k)$, where $y_a^\alpha = x_a^\nu$ (for $\alpha = 1, \ldots, k$) if and only if there exists an a-coloured edge starting at v^α and ending at w^ν. Regularity of the colouring and bipartiteness of the vertex set ensure that there is exactly one such edge, thus rendering $\mathcal{B}_*$ well-defined. This map $\mathcal{B}_*$ is deduced by momentum transmission inside any graph $\mathcal{G}$ for with $\partial\mathcal{G} = \mathcal{B}$ by following the $a0$-coloured paths in $\mathcal{G}$ between its external vertices. One further associates to $\mathcal{B}$ and $\mathbf{X}$ a cycle of sources

$$\mathbb{J}(\mathcal{B})(\mathbf{X}) = J_{\mathbf{x}^1} \cdots J_{\mathbf{x}^k} \bar{J}_{\mathbf{y}^1} \cdots \bar{J}_{\mathbf{y}^k}\,, \qquad \text{where } \mathcal{B}_*(\mathbf{X}) = (\mathbf{y}^1, \ldots, \mathbf{y}^k)\,, \tag{2.2}$$

which is evidently independent of the ordering given to $\mathcal{B}^{(0)}_{\mathrm{w}}$ and $\mathcal{B}^{(0)}_{\mathrm{b}}$. According to [15], the free energy $W[J,\bar{J}] = \log(Z[J,\bar{J}])$ can be expanded in these cycles indexed by all the boundary graphs of a given model:

$$W[J,\bar{J}] = \sum_{l=1}^{\infty} \sum_{\substack{\mathcal{B} \in \operatorname{im}\partial_V \\ k(\mathcal{B}) = l}} \frac{1}{|\mathrm{Aut_c}(\mathcal{B})|} G^{(2l)}_{\mathcal{B}} \star \mathbb{J}(\mathcal{B}) \tag{2.3}$$

where $\star$ is a pairing between a function $f: M_{D\times k(\mathcal{B})} \to \mathbb{C}$ and a boundary graph $\mathcal{B} \in \operatorname{im}\partial_V \subset \mathsf{Grph}_D^{\mathrm{II,cl}}$ given by $f \star \mathbb{J}(\mathcal{B}) = \sum_{\mathbf{X} \in M_{D\times k(\mathcal{B})}(\mathbb{Z})} f(\mathbf{X}) \cdot \mathbb{J}(\mathcal{B})(\mathbf{X})$. To read off the the correlation functions $G^{(2l)}_{\mathcal{B}}$ from eq. (2.3), one takes graph derivatives, introduced in [15] and recapitulated in the next section.

2.3. **Graph-generated functionals.** We also recall some results from [15]. Let

$$\mathcal{F}_{D,k} := \{(\mathbf{y}^1, \ldots, \mathbf{y}^k) \in M_{D\times k}(\mathbb{Z}) \,|\, y_c^\alpha \neq y_c^\nu \text{ for all } c = 1, \ldots, D \text{ and } \alpha, \nu = 1, \ldots, k, \alpha \neq \nu\}\,.$$

Thus $\mathcal{F}_{D,k}$ is the set of matrices $M_{D\times k}(\mathbb{Z})$ having all different entries on any fixed row. We define the graph derivative of any functional $X[J,\bar{J}]$ with respect to $\mathcal{B}$ at $\mathbf{X} \in \mathcal{F}_{D,k}$ as

$$\frac{\partial X[J,\bar{J}]}{\partial \mathcal{B}(\mathbf{X})} := \frac{\delta^{2k(\mathcal{B})} X[J,\bar{J}]}{\delta(\mathbb{J}(\mathcal{B}))(\mathbf{X})}\bigg|_{J=0=\bar{J}} = \prod_{\alpha=1}^{k} \frac{\delta}{\delta J_{\mathbf{x}^\alpha}} \frac{\delta}{\delta \bar{J}_{\mathbf{y}^\alpha}} X[J,\bar{J}]\bigg|_{J=0=\bar{J}}.$$

Let $(\mathbf{y}^1, \ldots, \mathbf{y}^l) \in M_{D\times l}(\mathbb{Z})$. For closed, coloured graphs $\mathcal{Q}, \mathcal{C} \in \mathsf{Grph}_D^{\mathrm{cl}}$ one has [15]:

$$\frac{\partial \mathcal{Q}(\mathbf{y}^1, \ldots, \mathbf{y}^l)}{\partial \mathcal{C}(\mathbf{x}^1, \ldots, \mathbf{x}^k)} = \left\{ \begin{matrix} \displaystyle\sum_{\hat{\sigma} \in \mathrm{Aut_c}(\mathcal{C})} \delta_{lk} \cdot \delta^{\mathbf{y}^{\sigma(1)}, \ldots, \mathbf{y}^{\sigma(k)}}_{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k} & \text{if } \mathcal{C} \cong \mathcal{Q} \\ 0 & \text{otherwise} \end{matrix} \right\} = \sum_{\sigma \in \mathfrak{S}_k} \delta^{\mathbf{y}^{\sigma(1)}, \ldots, \mathbf{y}^{\sigma(k)}}_{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^k} \delta(\mathcal{Q}, \mathcal{C}) \tag{2.4}$$

[2] What this paper concerns, the use of matrices $M_{D\times k}(\mathbb{Z})$, instead of plainly $\mathbb{Z}^{kD}$, merely eases the definition of $\mathcal{F}_{D,k}$ below. No matrix multiplication is so far needed.

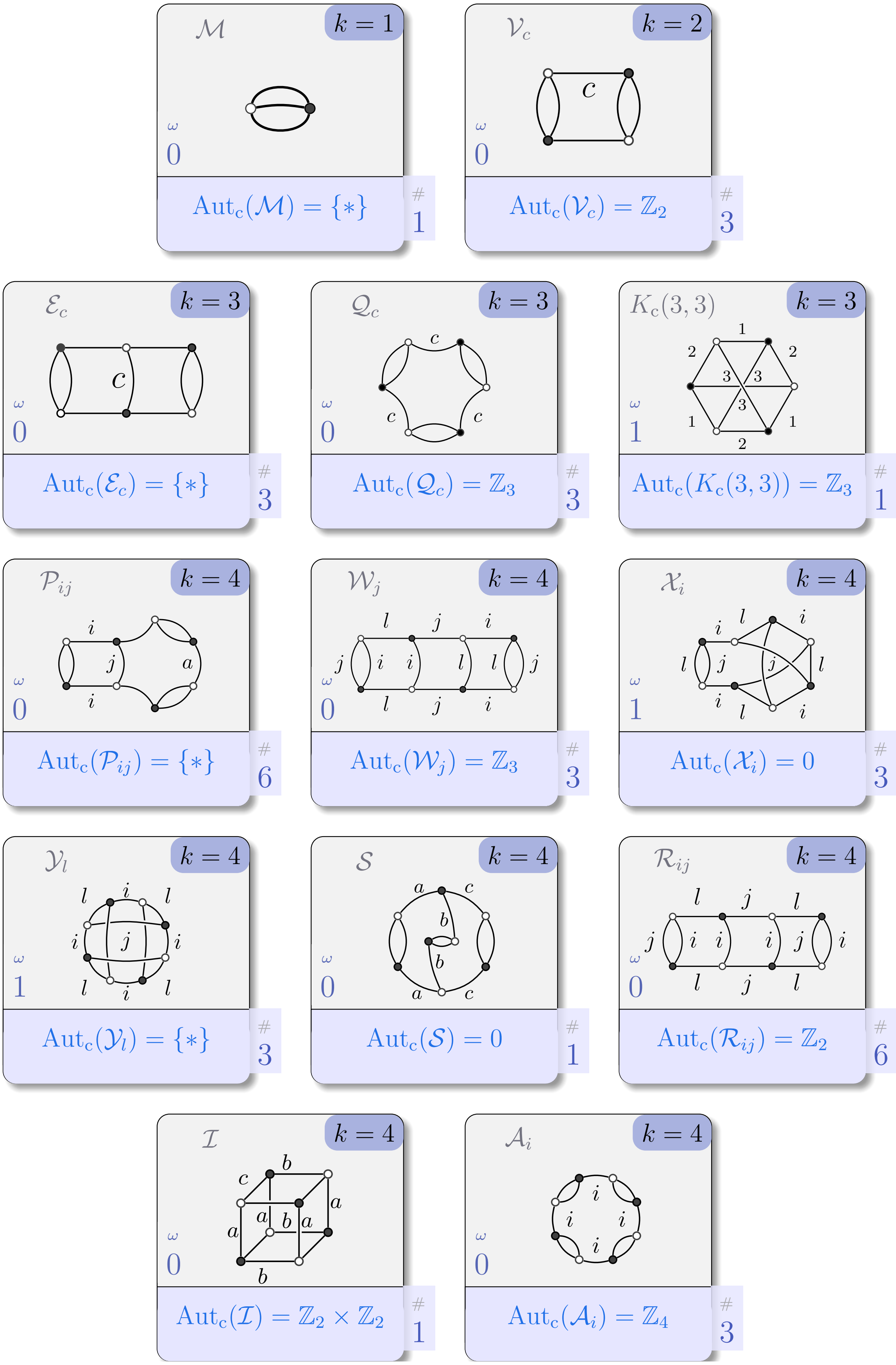


FIGURE 2. Enumeration of 3-coloured graphs with $2, 4, 6$ and 8 vertices and their Gurău-degree ω and coloured automorphism group.

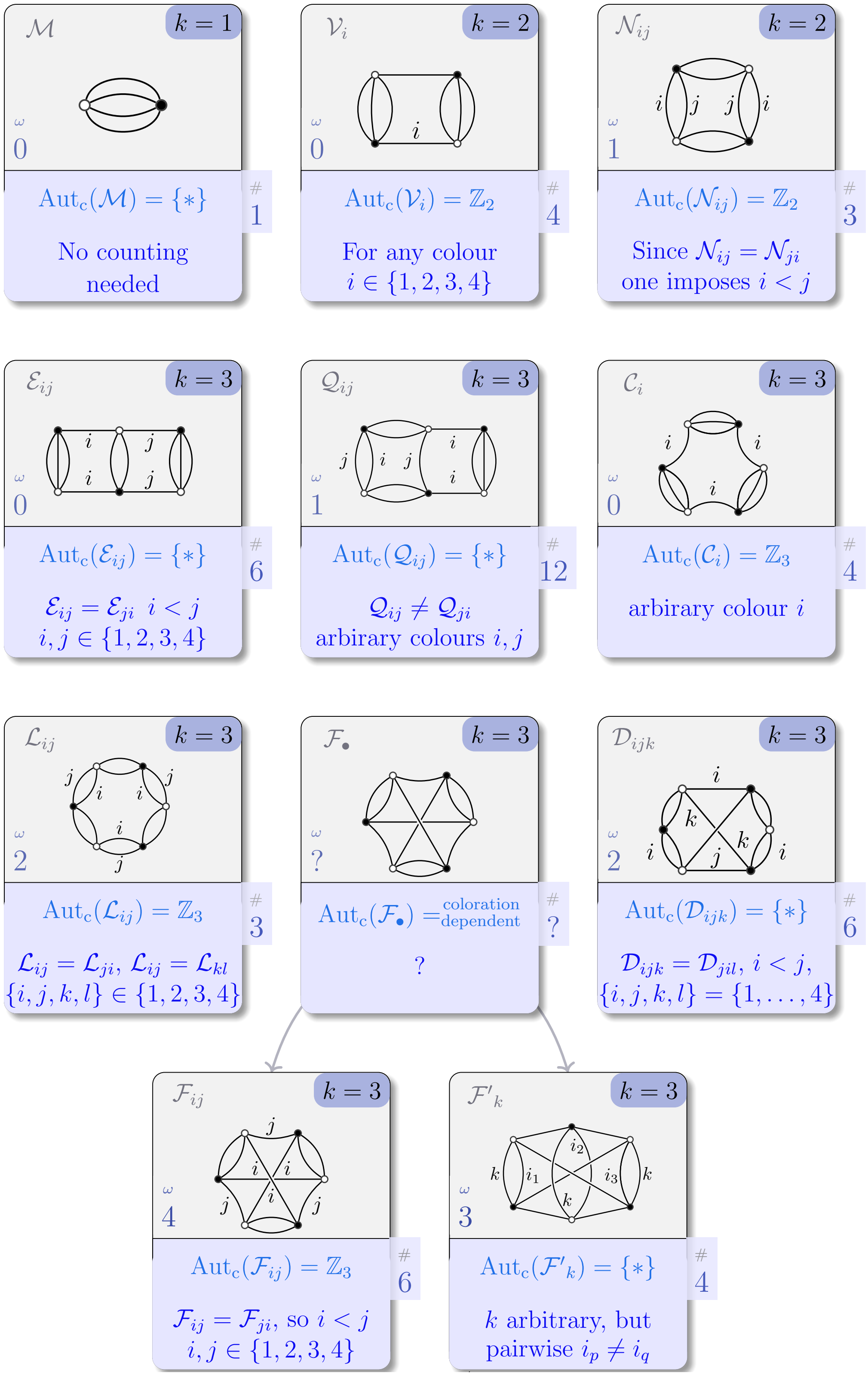


FIGURE 3. This table shows the rank-4 graphs until 6 vertices. As before, # is the number of graphs that are obtained by the action of $\mathfrak{S}_4$ in the edge-colouring and ω is Gurău's degree of the graph in question. For graphs with $k=4$ see [13, Fig. 8] (there, only those marked with a B are bipartite). The graphs displayed there are neither given a colouration nor classified by $\mathfrak{S}_4$-orbits, though (Klebanov and Tarnopolsky treat them as vacuum Feynman graphs; here our graphs are boundaries and we need to count them and their Aut_c-groups)

where $\delta(\mathcal{Q},\mathcal{C}) = 1$ if the graphs $\mathcal{Q}$ and $\mathcal{C}$ are isomorphic, and 0 otherwise. We consider functionals generated by a given family of closed D-coloured (non-isomorphic) graphs, $\Upsilon \subset \mathsf{Grph}_D^{\mathrm{II,cl}}$. That means that if

$$X[J,\bar{J}] = \sum_{\mathcal{C}\in\Upsilon} \mathfrak{l}_{\mathcal{C}} \star \mathbb{J}(\mathcal{C}), \qquad \text{for } \mathfrak{l}_{\mathcal{C}} : (\mathbb{Z}^D)^{\times k(\mathcal{C})} \to \mathbb{C},\ \mathcal{C} \in \Upsilon\,, \tag{2.5}$$

is known, we want to know the graph derivatives of $X[J,\bar{J}]$ with respect to connected graphs. Here $k(\mathcal{C})$ denotes the number $\#(\mathcal{C}_{\mathrm{w}}^{(0)})$ of white (or black) vertices of $\mathcal{C}$.

Proposition 2.1. *Let X be as in eq. (2.5). Then, for all $\mathcal{C} \in \Upsilon \cap \mathsf{Grph}_D^{\mathrm{cl}}$, the functions $\mathfrak{l}_{\mathcal{C}}$ satisfy*

$$\frac{\partial X[J,\bar{J}]}{\partial\, \mathcal{C}(\mathbf{X})} = \sum_{\hat{\sigma}\in \mathrm{Aut}_{\mathrm{c}}(\mathcal{C})} (\sigma^*\mathfrak{l}_{\mathcal{C}})(\mathbf{X}), \text{ where } (\sigma^*\mathfrak{l}_{\mathcal{C}})(\mathbf{x}^1,\ldots,\mathbf{x}^{k(\mathcal{C})}) := \mathfrak{l}_{\mathcal{C}}(\mathbf{x}^{\sigma^{-1}(1)},\ldots,\mathbf{x}^{\sigma^{-1}(k(\mathcal{C}))})\,,$$

for all $\mathbf{X} = (\mathbf{x}^1,\ldots,\mathbf{x}^{k(\mathcal{C})}) \in \mathcal{F}_{D,k(\mathcal{C})}$.

Proof. From formula (2.4), one has

$$\begin{aligned}
\frac{\partial X[J,\bar{J}]}{\partial(\mathcal{C}(\mathbf{X}))} &= \frac{\partial}{\partial\,\mathcal{C}(\mathbf{X})}\sum_{\mathcal{Q}\in\Upsilon} \mathfrak{l}_{\mathcal{Q}} \star \mathbb{J}(\mathcal{Q}) = \sum_{\mathcal{Q}\in\Upsilon}\sum_{\mathbf{Y}\in(\mathbb{Z}^D)^k} \mathfrak{l}_{\mathcal{Q}}(\mathbf{Y}) \frac{\partial\,\mathcal{Q}(\mathbf{Y})}{\partial\,\mathcal{C}(\mathbf{X})} \\
&= \sum_{\mathcal{Q}\in\Upsilon}\sum_{\mathbf{Y}\in(\mathbb{Z}^D)^{\times k(\mathcal{Q})}} \mathfrak{l}_{\mathcal{Q}}(\mathbf{Y}) \sum_{\sigma\in\mathfrak{S}_{k(\mathcal{Q})}} \delta(\mathcal{Q},\mathcal{C}) \prod_{i=1}^{k(\mathcal{Q})} \delta(\mathbf{y}^{\sigma(i)},\mathbf{x}^i) \\
&= \sum_{\mathcal{Q}\in\Upsilon}\sum_{\mathbf{Y}\in(\mathbb{Z}^D)^{\times k(\mathcal{C})}} \mathfrak{l}_{\mathcal{Q}}(\mathbf{Y}) \sum_{\sigma\in\mathfrak{S}_{k(\mathcal{Q})}} \delta(\mathcal{Q},\mathcal{C}) \prod_{i=1}^{k(\mathcal{Q})} \delta(\mathbf{y}^{i},\mathbf{x}^{\sigma^{-1}(i)}) \\
&= \sum_{\mathcal{Q}\in\Upsilon}\sum_{\sigma\in\mathfrak{S}_{k(\mathcal{Q})}} \mathfrak{l}_{\mathcal{C}}(\mathbf{x}^{\sigma^{-1}(1)},\ldots,\mathbf{x}^{\sigma^{-1}(k)})\delta(\mathcal{Q},\mathcal{C})\,.
\end{aligned}$$

Since Υ consists only of graphs that are not isomorphic, the sum over $\mathcal{Q}$ yields, because of the delta $\delta(\mathcal{Q},\mathcal{C})$, only one term. Hence, the last expression is precisely the sum over automorphisms of $\mathcal{C}$. □

As a consequence of this, one can recover the correlation functions via

$$G_{\mathcal{B}}^{(2k(\mathcal{B}))}(\mathbf{X}) = \frac{\partial W[J,\bar{J}]}{\partial\mathcal{B}(\mathbf{X})}\,.$$

Notice that $\mathbf{X} = (\mathbf{x}^1,\ldots,\mathbf{x}^k) \in \mathcal{F}_{D,k}$ if and only if $\mathcal{B}_*(\mathbf{X}) \in \mathcal{F}_{D,k}$. Since $W[J,\bar{J}]$ is real-valued, one has the relation

$$\overline{G_{\mathcal{B}}^{(2k(\mathcal{B}))}}(\mathbf{X}) = \prod_{\alpha=1}^{k} \frac{\delta}{\delta \bar{J}_{\mathbf{x}^\alpha}} \frac{\delta}{\delta J_{\mathbf{y}^\alpha}} \overline{W[J,\bar{J}]}\Big|_{J=0=\bar{J}} = G_{\bar{\mathcal{B}}}^{(2k(\mathcal{B}))}(\mathcal{B}_*(\mathbf{X})) \qquad (\mathbf{X} \in \mathcal{F}_{D,k(\mathcal{B})})\,, \tag{2.6}$$

where $\bar{\mathcal{B}}$ is essentially the graph $\mathcal{B}$ after inverting vertex-colouration, $\bar{\mathcal{B}}_{\mathrm{w}}^{(0)} = \mathcal{B}_{\mathrm{b}}^{(0)}$ and $\bar{\mathcal{B}}_{\mathrm{b}}^{(0)} = \mathcal{B}_{\mathrm{w}}^{(0)}$, but otherwise with the same adjacency and edge-colouration.

We now explain how this graph derivatives are relevant in the WTI. The WTI is rather a set of equations, one for each colour $a = 1, 2, \ldots, D$, in which a new generating functional of the form

$$Y_{s_a}^{(a)}[J,\bar{J}] = \textstyle\sum_{\mathcal{C}\in\Omega_V} \mathfrak{f}_{\mathcal{C},s_a}^{(a)} \star \mathbb{J}(\mathcal{C}) \qquad (s_a \in I_a \subset \mathbb{Z}) \tag{2.7}$$

appears. Here, $\partial_V : \mathsf{Feyn}_D(V) \to \mathsf{Grph}_D^{\mathrm{II,cl}}$ denotes the boundary map in terms of which we describe the graph family Ω_V as follows: If e_a^v is the a-coloured edge at the white

(a) Locally, edge and vertex labelling in $\mathcal{B}$ before forming $\mathcal{B} \ominus e_a^r$

(b) Locally $\mathcal{B} \ominus e_a^r$

FIGURE 4. Some notation concerning the definition of $\Delta^{\mathcal{B}}_{s_a,r}$.

vertex $v \in \mathcal{B}^{(0)}_{\mathrm{w}}$, then the graph $\mathcal{B} \ominus e_a^v$ denotes the graph that is obtained by the next steps: first, remove the two end-vertices, $v = s(e_a^v)$ and $t(e_a^v)$, of e_a^v; then, remove all their common edges $I(e_a^v) := s^{-1}(s(e_a^v)) \cap t^{-1}(t(e_a^v))$; finally, glue colourwise the broken edges, i.e. the each broken edge of the set $s^{-1}(v) \setminus I(e_a^v)$ with the respective broken edge in $t^{-1}(t(e_a^v)) \setminus I(e_a^v)$. Then Ω is defined by

$$\Omega_V := \{\mathcal{B} \ominus e_a^v \,|\, \mathcal{B} \in \operatorname{im} \partial_V, v \in \mathcal{B}^{(0)}_{\mathrm{w}}\} \,.$$

Definition 2.2. Let a be a colour, $F : (\mathbb{Z}^D)^k \to \mathbb{C}$ a function and $\mathcal{B} \in \mathsf{Grph}^{\mathrm{II,cl}}_D$. For any integer r, $1 \le r \le k(\mathcal{B})$, we define the function $\Delta^{\mathcal{B}}_{s_a,r} F : (\mathbb{Z}^D)^{k-1} \to \mathbb{C}$ by

$$(\Delta^{\mathcal{B}}_{s_a,r} F)(\mathbf{Y}) = \sum_{q_h} F(\mathbf{y}^1, \ldots, \mathbf{y}^{r-1}, \mathbf{z}^r(s_a, \mathbf{q}, \mathbf{Y}), \mathbf{y}^r, \ldots, \mathbf{y}^{k-1}) \,,$$

for each $\mathbf{Y} = (\mathbf{y}^1, \ldots, \mathbf{y}^{k-1}) \in (\mathbb{Z}^D)^{k-1}$, where the sum is over a dummy variable q_h for each element of the set $h \in I(e_a^r) \setminus \{a\}$. Before specifying $\mathbf{z}^r(s_a, \mathbf{q}, \mathbf{Y})$, we stress that this sum can be empty, in which case

$$(\Delta^{\mathcal{B}}_{s_a,r} F)(\mathbf{Y}) = F(\mathbf{y}^1, \ldots, \mathbf{y}^{r-1}, \mathbf{z}^r(s_a, \mathbf{Y}), \mathbf{y}^r, \ldots, \mathbf{y}^k) \,.$$

The momentum $\mathbf{z}^r \in \mathbb{Z}^D$ has entries defined by:

$$z_i^r(s_a, \mathbf{q}, \mathbf{Y}) = \begin{cases} s_a & \text{if } \ i = a \,, \\ q_i & \text{if } \ i \in I(e_a^r) \setminus \{a\} \,, \\ y_i^{\kappa(r,i,a)} & \text{if } \ i \in \text{colours of } A_{t(e_a^r)} = \{1, \ldots, D\} \setminus I(e_a^r) \,, \end{cases}$$

where $\mathbf{y}^{\kappa(r,i,a)}$ $(1 \le \kappa(r,i,a) < k)$ is the white vertex $\mathcal{B} \ominus e_a^r$ defined by

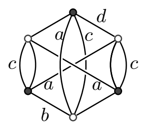
$$\kappa(r,i,a) = \begin{cases} \xi(r,i,a) & \text{if } \xi(r,i,a) < r \,, \\ \xi(r,i,a) - 1 & \text{if } \xi(r,i,a) > r \,. \end{cases} \tag{2.8}$$

(see also Fig. 4). This definition depends on the labeling of the vertices. However, the pairing $\langle\!\langle G^{(2k)}_{\mathcal{B}}, \mathcal{B} \rangle\!\rangle_{s_a}$ defined as follows does not, for it is a sum over over graphs after removal of *all* a-coloured edges:

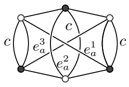
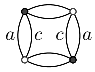
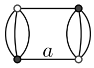
$$\langle\!\langle G^{(2k)}_{\mathcal{B}}, \mathcal{B} \rangle\!\rangle_{s_a} := \sum_{r=1}^{k} \left(\Delta^{\mathcal{B}}_{s_a,r} G^{(2k)}_{\mathcal{B}} \right) \star \mathbb{J}(\mathcal{B} \ominus e_a^r) \,. \tag{2.9}$$

Remark 2.3. Unless otherwise stated, we set the convention of ordering the white-vertex-set $\mathcal{B}^{(0)}_{\mathrm{w}}$ in appearance from left to right.

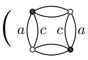
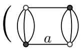
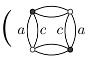
Example 2.4. Let $\{a, b, c, d\} = \{1, 2, 3, 4\}$ and

$\mathcal{F}'_c =$ 

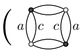
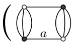
(see also Fig. 2). For a fixed colour a and $s_a \in I_a \subset \mathbb{Z}$, we obtain $\langle\!\langle G^{(2k)}_{\mathcal{F}'_c}, \mathcal{F}'_c \rangle\!\rangle_{s_a}$. According to remark 2.3, the first white vertex is the left upper left white vertex, the second is the lowermost, the third is the upper right. This orders the a-coloured edges $\{e_a^1, e_a^2, e_a^3\}$. Explicitly

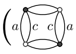
$\mathcal{F}'_c =$  , thus, $\mathcal{F}'_c \ominus e_a^1 = \mathcal{F}'_c \ominus e_a^3 =$  , $\mathcal{F}'_c \ominus e_a^2 =$ 

hence

$\langle\!\langle G^{(6)}_{\mathcal{F}'_c}, \mathcal{F}'_c \rangle\!\rangle_{s_a} = \Delta_{s_a,1} G^{(6)}_{\mathcal{F}'_c} \star \mathbb{J}\big($  $\big) + \Delta_{s_a,2} G^{(6)}_{\mathcal{F}'_c} \star \mathbb{J}\big($  $\big) + \Delta_{s_a,3} G^{(6)}_{\mathcal{F}'_c} \star \mathbb{J}\big($  $\big)$

which, in turn, equals

$\sum_{\mathbf{y},\mathbf{z}} \big\{ \Delta_{s_a,1} G^{(6)}_{\mathcal{F}'_c}(\mathbf{y},\mathbf{z}) \mathbb{J}\big($  $\big)(\mathbf{y},\mathbf{z}) + \Delta_{s_a,2} G^{(6)}_{\mathcal{F}'_c}(\mathbf{y},\mathbf{z}) \mathbb{J}\big($  $\big)(\mathbf{y},\mathbf{z})$

$+ \Delta_{s_a,3} G^{(6)}_{\mathcal{F}'_c}(\mathbf{y},\mathbf{z}) \mathbb{J}\big($  $\big)(\mathbf{y},\mathbf{z}) \big\}$

$$= \sum_{\mathbf{y},\mathbf{z}} \big\{ G^{(6)}_{\mathcal{F}'_c}(s_a, z_b, z_c, y_d, \mathbf{y}, \mathbf{z}) \bar{J}_{y_a z_b y_c z_d} \bar{J}_{z_a y_b z_c y_d} J_{\mathbf{y}} J_{\mathbf{z}} + \big(\sum_{q_c} G^{(6)}_{\mathcal{F}'_c}(\mathbf{y}, s_a, y_b, q_c, z_d, \mathbf{z})$$
$$\times \bar{J}_{y_a z_b z_c z_d} \bar{J}_{z_a y_b y_c y_d} J_{\mathbf{y}} J_{\mathbf{z}} \big) + G^{(6)}_{\mathcal{F}'_c}(\mathbf{y}, \mathbf{z}, s_a, z_b, y_c, y_d) \bar{J}_{y_a z_b y_c z_d} \bar{J}_{z_a y_b z_c y_d} J_{\mathbf{y}} J_{\mathbf{z}} \big\} \,.$$

We assume all the entries of momenta in $\mathbb{Z}^4$ are ordered by colour, e.g. $(z_1 y_4 z_3 y_2)$ really means $(z_1 y_2 z_3 y_4)$.

We now recall the full Ward-Takahashi Identity, proven in [15].

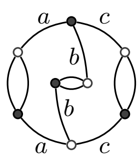
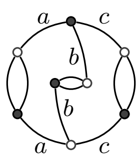
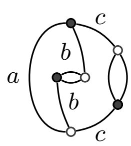
Theorem 2.5. *Consider a rank-D tensor model, $S = S_0 + V$, with a kinetic form* $\mathrm{Tr}_2(\bar{\varphi}, E\varphi)$ *such that the difference of propagators $E_{p_1 \ldots p_{a-1} m_a p_{a+1} \ldots p_D} - E_{p_1 \ldots p_{a-1} n_a p_{a+1} \ldots p_D} = E(m_a, n_a)$ is independent of the momenta $\mathbf{p}_{\hat{a}} = (p_1, \ldots, \widehat{p_a}, \ldots, p_D)$. Then that model has a partition function $Z[J, \bar{J}]$ that satisfies*

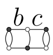
$$\sum_{\mathbf{p}_{\hat{a}}} \frac{\delta^2 Z[J,\bar{J}]}{\delta J_{p_1 \ldots p_{a-1} m_a p_{a+1} \ldots p_D} \delta \bar{J}_{p_1 \ldots p_{a-1} n_a p_{a+1} \ldots p_D}} - \Big(\delta_{m_a n_a} Y^{(a)}_{m_a}[J,\bar{J}] \Big) \cdot Z[J,\bar{J}] \tag{2.10}$$
$$= \sum_{\mathbf{p}_{\hat{a}}} \frac{1}{E_{p_1 \ldots m_a \ldots p_D} - E_{p_1 \ldots n_a \ldots p_D}} \left(\bar{J}_{p_1 \ldots m_a \ldots p_D} \frac{\delta}{\delta \bar{J}_{p_1 \ldots n_a \ldots p_D}} - J_{p_1 \ldots n_a \ldots p_D} \frac{\delta}{\delta J_{p_1 \ldots m_a \ldots p_D}} \right) Z[J,\bar{J}]$$

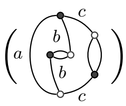
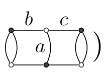
where

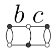
$$Y^{(a)}_{m_a}[J,\bar{J}] := \sum_{l=1}^{\infty} \sum_{\substack{\mathcal{B} \in \mathrm{im}\, \partial_V \\ k(\mathcal{B}) = l}} \langle\!\langle G^{(2l)}_{\mathcal{B}}, \mathcal{B} \rangle\!\rangle_{m_a} \,. \tag{2.11}$$

There is a subtlety regarding the ordering of the vertices. We associate an ordering of the white vertices of a graph $\mathcal{B}$ in $G^{(2k)}_{\mathcal{B}}$. The k $\mathbb{Z}^D$-arguments of this function match this vertex-ordering. But the edge-removal sometimes will yield a graph which should be reoriented. To illustrate this, for $D = 3$, consider for instance the next graph $\mathcal{S}$. The edge contraction yields, for any $i = 1, 2, 3$, the following:

If $\mathcal{S} =$  then $\mathcal{S} \ominus e_i^1 =$  $\ominus e_i^1 =$  (2.12)

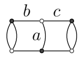
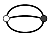
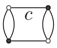
As a graph, $\mathcal{S} \ominus e_i^1$ is just , but when one considers $f \star \mathbb{J}(\mathcal{S} \ominus e_i^1)$, for some function $f : (\mathbb{Z}^3)^{\times 3} \to \mathbb{C}$, the order of the vertices does matter:

$f \star \mathbb{J}\Big($  $\Big) = ((12)^* f) \star \mathbb{J}($  $)$

In going from the graph 2.12 to , one permuted the first and second white vertices. Accordingly, one 'corrects' f and replaces it by $(12)^*(f)$. Notice that the cycle $(12) \in \mathfrak{S}_3$ does not lift to a coloured automorphism. If this was the case, we could just as well ignore the correction.

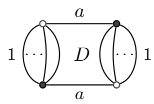
The next definition is needed in order to describe some terms appearing in the SDEs.

Definition 2.6. Let $\mathcal{B} \in \mathsf{Grph}_D^{\mathrm{cl}}$ and let v, w be vertices of the same colour (either both black $v, w \in \mathcal{B}_{\mathrm{b}}^{(0)}$ or both white $v, w \in \mathcal{B}_{\mathrm{w}}^{(0)}$). We define the graph $\varsigma_a(\mathcal{B}; v, w)$ as the coloured graph obtained from $\mathcal{B}$ by swapping the a-coloured edges at v and w. Usually, vertices in boundary graphs are indexed by numbered momenta $v = \mathbf{x}^\alpha, w = \mathbf{x}^\gamma \in \mathbb{Z}^D$, in which case we write $\varsigma_a(\mathcal{B}; \mathbf{x}^\alpha, \mathbf{x}^\gamma)$ or just $\varsigma_a(\mathcal{B}; \alpha, \gamma)$. These graphs are, generally, disconnected.

Example 2.7. For any colour $a = 1, 2, 3$, one has $\varsigma_a($$; u, v) =$  $=: \mathcal{E}_a$ for two black (or white) vertices u, v of . If x and y are the leftmost black vertices of $\mathcal{E}_a$, then $\varsigma_b(\mathcal{E}_a; x, y) =$  $\sqcup$ .

3. The Schwinger-Dyson equation tower in arbitrary rank

We pick the following quartic model $S = S_0 + V$, with interaction vertices $V[\varphi, \bar\varphi] = \lambda \sum_{a=1}^D \mathrm{Tr}_{\mathcal{V}_a}(\varphi, \bar\varphi)$, being each vertex $\mathcal{V}_a$ the melonic vertex of colour a,

$\mathcal{V}_a =$ 

Moreover assume that the propagator obeys that, for each colour a, the following difference

$$E(t_a, s_a) = E_{p_1 \dots t_a \dots p_D} - E_{p_1 \dots s_a \dots p_D}$$

does not depend on p_i, for each $i \neq a$. Such is the case for Tensor Group Field theories, say with group $\mathrm{U}(1)$, being the origin of E is the Laplacian operator on $\mathrm{U}(1)^D$ after taking Fourier transform, and the tensors the Fourier modes [19]. We call this model the $\varphi^4_{D,\mathrm{m}}$-theory[3]. Here, the subindex m denotes melonicity.

One observes that, if $\delta(\mathcal{V}_a)(\mathbf{b}, \mathbf{c}, \mathbf{x}, \mathbf{y})$ is the invariant of the trace, that is

$$\mathrm{Tr}_{\mathcal{V}_a}(\varphi, \bar\varphi) = \lambda \sum_{\mathbf{b},\mathbf{c},\mathbf{x},\mathbf{y}} \bar\varphi_{\mathbf{b}} \bar\varphi_{\mathbf{c}} \delta(\mathcal{V}_a)(\mathbf{b}, \mathbf{c}, \mathbf{x}, \mathbf{y}) \varphi_{\mathbf{y}} \varphi_{\mathbf{x}} \,,$$

one gets for $\mathbf{s} = (s_1, \dots, s_D) \in \mathbb{Z}^D$, the following expression:

$$\left(\frac{\partial V(\varphi, \bar\varphi)}{\partial \bar\varphi_{\mathbf{s}}}\right)_{\varphi^\flat \to \delta/\delta J^\sharp} = 2\lambda \Bigg\{ \sum_a \Bigg(\sum_{b_a} \frac{\delta}{\delta \bar J_{s_1 \dots s_{a-1} b_a s_{a+1} \dots s_D}} \sum_{\mathbf{b}_{\hat a}} \frac{\delta}{\delta J_{b_1 \dots b_D}} \frac{\delta}{\delta \bar J_{b_1 \dots b_{a-1} s_a b_{a+1} \dots b_D}} \Bigg) \Bigg\}, \tag{3.1}$$

where $\mathbf{b}_{\hat a} = (b_1, \dots \widehat{b}_a, \dots, b_D) = (b_1 \dots, b_{a-1}, b_{a+1} \dots, b_D) \in \mathbb{Z}^{D-1}$ and $\sharp$ can either act trivially on a variable or be complex conjugation, and $\varphi^\flat = \bar\varphi$ or $\varphi^\flat = \varphi$ according to whether $J^\sharp = \bar J$ or $J^\sharp = J$, respectively. The term $\big(\partial V(\varphi, \bar\varphi)/\partial \bar\varphi_{\mathbf{s}}\big|_{\varphi^\flat \to \delta/\delta J^\sharp}\big) Z[J, \bar J]$ can be computed with aid of the WTI. We depart from the formally integrated form of the partition function

$$Z[J, \bar J] \propto \exp\left(-V(\varphi, \bar\varphi)\right)\big|_{\varphi^\flat \to \delta/\delta J^\sharp} \exp\Big(\textstyle\sum_{\mathbf{q} \in \mathbb{Z}^D} \bar J_{\mathbf{q}} E_{\mathbf{q}}^{-1} J_{\mathbf{q}}\Big) \,,$$

[3]For $D = 3$ all quartic invariants are melonic, so we refer to it only as φ^4_3-theory.

where we will ignore a (possibly infinite) constant and write equality and derive its logarithm:

$$\begin{aligned}\frac{\delta W[J,\bar J]}{\delta \bar J_{\mathbf{s}}} &= \frac{1}{Z[J,\bar J]}\exp\left(-V(\delta/\delta\bar J,\delta/\delta J)\right)J_{\mathbf{s}}E_{\mathbf{s}}^{-1}\mathrm{e}^{\sum_{\mathbf{q}\in\mathbb{Z}^D}\bar J_{\mathbf{q}}E_{\mathbf{q}}^{-1}J_{\mathbf{q}}} \\ &= \frac{1}{Z[J,\bar J]}\Bigg\{\frac{1}{E_{\mathbf{s}}}J_{\mathbf{s}}\exp\left(-V(\delta/\delta\bar J,\delta/\delta J)\right)\mathrm{e}^{\sum_{\mathbf{q}\in\mathbb{Z}^D}\bar J_{\mathbf{q}}E_{\mathbf{q}}^{-1}J_{\mathbf{q}}} \\ &\qquad + \left(\frac{\partial V(\varphi,\bar\varphi)}{\partial\bar\varphi_{\mathbf{s}}}\right)\bigg|_{\varphi^\flat\to\delta/\delta J^\sharp}\exp\left(-V(\varphi,\bar\varphi)\right)\big|_{\varphi^\flat\to\delta/\delta J^\sharp}\mathrm{e}^{\sum_{\mathbf{q}\in\mathbb{Z}^D}\bar J_{\mathbf{q}}E_{\mathbf{q}}^{-1}J_{\mathbf{q}}}\Bigg\} \\ &= \frac{1}{E_{\mathbf{s}}}\Bigg\{J_{\mathbf{s}} - \frac{1}{Z[J,\bar J]}\left(\frac{\partial V(\varphi,\bar\varphi)}{\partial\bar\varphi_{\mathbf{s}}}\right)\bigg|_{\varphi^\flat\to\delta/\delta J^\sharp}Z[J,\bar J]\Bigg\}.\end{aligned} \tag{3.2}$$

For sake of notation, we introduce the shorthands $\mathbf{b}_{\hat a}s_a = (b_1\ldots,b_{a-1},s_a,b_{a+1}\ldots,b_D)$ and, similarly, $\mathbf{s}_{\hat a}b_a = (s_1\ldots,s_{a-1},b_a,s_{a+1}\ldots,s_D)$, for any $a=1,\ldots,D$. By applying the colour-a-WTI to the rightmost double derivative term appearing in (3.1), the following:

$$\begin{aligned}\left(\frac{\partial V(\varphi,\bar\varphi)}{\partial\bar\varphi_{\mathbf{s}}}\right)\bigg|_{\varphi^\flat\to\,\delta/\delta J^\sharp} Z[J,\bar J] &= 2\lambda\sum_a\Bigg\{\sum_{b_a}\frac{\delta}{\delta\bar J_{\mathbf{s}_{\hat a}b_a}}\Big(\delta_{s_ab_a}Y^{(a)}_{s_a}[J,\bar J]\cdot \\ &\qquad + \sum\nolimits_{\mathbf{b}_{\hat a}}\frac{1}{E(b_a,s_a)}\big(\bar J_{\mathbf{b}}\frac{\delta}{\delta\bar J_{\mathbf{b}_{\hat a}s_a}} - J_{\mathbf{b}_{\hat a}s_a}\frac{\delta}{\delta J_{\mathbf{b}}}\big)\Big)Z[J,\bar J]\Bigg\} \\ &= 2\lambda\sum_a\Bigg\{\frac{\delta Y^{(a)}_{s_a}[J,\bar J]}{\delta\bar J_{\mathbf{s}}}\cdot Z[J,\bar J] + Y^{(a)}_{s_a}[J,\bar J]\cdot\frac{\delta Z[J,\bar J]}{\delta\bar J_{\mathbf{s}}} \\ &\qquad + \sum\nolimits_{\mathbf{b}}\frac{1}{E(b_a,s_a)}\frac{\delta}{\delta\bar J_{\mathbf{s}_{\hat a}b_a}}\big(\bar J_{\mathbf{b}}\frac{\delta}{\delta\bar J_{\mathbf{b}_{\hat a}s_a}} - J_{\mathbf{b}_{\hat a}s_a}\frac{\delta}{\delta J_{\mathbf{b}}}\big)Z[J,\bar J]\Bigg\} \\ &= 2\lambda\sum_a\Bigg\{\frac{\delta Y^{(a)}_{s_a}[J,\bar J]}{\delta\bar J_{\mathbf{s}}}\cdot Z[J,\bar J] + Y^{(a)}_{s_a}[J,\bar J]\cdot\frac{\delta Z[J,\bar J]}{\delta\bar J_{\mathbf{s}}} \\ &\qquad + \sum_{b_a}\frac{1}{E(b_a,s_a)}\frac{\delta Z[J,\bar J]}{\delta\bar J_{\mathbf{s}}} + \sum_{\mathbf{b}}\frac{\bar J_{\mathbf{b}}}{E(b_a,s_a)}\frac{\delta^2 Z[J,\bar J]}{\delta\bar J_{\mathbf{s}_{\hat a}b_a}\delta\bar J_{\mathbf{b}_{\hat a}s_a}} \\ &\qquad - \sum\nolimits_{\mathbf{b}}\frac{1}{E(b_a,s_a)}J_{\mathbf{b}_{\hat a}s_a}\frac{\delta^2 Z[J,\bar J]}{\delta\bar J_{\mathbf{s}_{\hat a}b_a}\delta J_{\mathbf{b}}}\Bigg\} \\ &= 2\lambda\sum\nolimits_a\left(A_a(\mathbf{s}) - B_a(\mathbf{s}) + C_a(\mathbf{s}) + D_a(\mathbf{s}) + F_a(\mathbf{s})\right),\end{aligned} \tag{3.3}$$

with

$$A_a(\mathbf{s}) = Y^{(a)}_{s_a}[J,\bar J]\cdot\frac{\delta Z[J,\bar J]}{\delta\bar J_{\mathbf{s}}}, \qquad B_a(\mathbf{s}) = \sum\nolimits_{\mathbf{b}}\frac{1}{E(b_a,s_a)}J_{\mathbf{b}_{\hat a}s_a}\frac{\delta^2 Z[J,\bar J]}{\delta\bar J_{\mathbf{s}_{\hat a}b_a}\delta J_{\mathbf{b}}},$$

$$C_a(\mathbf{s}) = \sum_{b_a}\frac{1}{E(b_a,s_a)}\frac{\delta Z[J,\bar J]}{\delta\bar J_{\mathbf{s}}}, \qquad D_a(\mathbf{s}) = \sum_{\mathbf{b}}\frac{\bar J_{\mathbf{b}}}{E(b_a,s_a)}\frac{\delta^2 Z[J,\bar J]}{\delta\bar J_{\mathbf{s}_{\hat a}b_a}\delta\bar J_{\mathbf{b}_{\hat a}s_a}},$$

$$F_a(\mathbf{s}) = \frac{\delta Y^{(a)}_{s_a}[J,\bar J]}{\delta\bar J_{\mathbf{s}}}\cdot Z[J,\bar J].$$

One shall be interested in derivatives of $W[J,\bar J]$ of the following form:

$$\Big(\prod_{i=1}^{k}\frac{\delta}{\delta J_{\mathbf{x}_i}}\frac{\delta}{\delta\bar J_{\mathbf{y}_i}}\Big)W[J,\bar J]\bigg|_{J=\bar J=0}, \qquad \mathbf{X} = (\mathbf{x}^1,\ldots,\mathbf{x}^k)\in\mathcal{F}_{D,k},\ \ \mathcal{B}_*(\mathbf{X}) = (\mathbf{y}^1,\ldots,\mathbf{y}^k), \tag{3.4}$$

for $\mathcal{B}\in\mathsf{Grph}^{\mathrm{cl}}_D$, and then use formula (3.3) with, say, the vertex $\mathbf{s}=\mathbf{y}^1$. As said in the introduction, deltas of the interaction vertices and the propagators (proportional to deltas) inside each Feynman diagrams render the definition of the $2k$-multipoint function based on (3.4) redundant, if one treats the $\mathbf{x}$-variables and the $\mathbf{y}$-variables as independent. In fact, all the $\mathbf{y}$'s can be expressed in terms of coordinates of $\mathbf{X}=(\mathbf{x}^1,\ldots,\mathbf{x}^k)$ of the same colour, the combinatorics of which uniquely determines a so-called boundary graph $\mathcal{B}$ with $2k$ vertices; moreover, non-vanishing terms in the formula above are precisely a *graph derivative of* $W[J,\bar{J}]$ with respect to $\mathcal{B}$ at $\mathbf{X}$.

For the time being we pick only a *connected* boundary graph graph $\mathcal{B}$ and we want to know what the rest of the derivatives $\delta/\delta J_{\mathbf{x}^i}$, $\delta/\delta \bar{J}_{\mathbf{y}^\alpha(\{\mathbf{x}^i\}_i)}$, $(\alpha=2,\ldots,D)$ do to the expression (3.2). By using (3.3) with $\mathbf{s}=\mathbf{y}^1$ we analyze the five summands in the (lowermost) RHS:

$$\mathfrak{m}_a(\mathbf{X};\mathbf{s};\mathcal{B}) := \frac{1}{Z_0}\prod_{\substack{\alpha>1\\ \nu=1,\ldots,k}} \frac{\delta}{\delta \bar{J}_{\mathbf{y}^\alpha}}\frac{\delta}{\delta J_{\mathbf{x}^\nu}} M_a(\mathbf{s})\bigg|_{J=\bar{J}=0}$$

for

$$(\mathfrak{m},M)\in\{(\mathfrak{a},A),(\mathfrak{b},B),(\mathfrak{c},C),(\mathfrak{d},D),(\mathfrak{f},F)\}\,.$$

Actually $\mathbf{s}$ is a function of $\mathbf{X}$ —and so is any other $\mathbf{y}^\alpha$— but the dependence of $\mathfrak{m}_a$ on it only shows that $\mathbf{s}$ is the variable respect to which we firstly[4] derived $W[J,\bar{J}]$. Each $\mathfrak{m}_a$ depends on the boundary graph $\mathcal{B}$ through $\{\mathbf{y}^\alpha\}_{\alpha=1}^k$ given by (2.2). Ignoring the common $(-2\lambda/E_{\mathbf{s}})$ prefactor:

- $\mathfrak{a}_a(\mathbf{X};\mathbf{s};\mathcal{B})$ be easily be seen to yield $Y^{(a)}_{s_a}[0,0]\cdot G^{(2k)}_{\mathcal{B}}(\mathbf{X})$
- also, derivatives on $C_a(\mathbf{s})$, $\mathfrak{c}_a(\mathbf{X};\mathbf{s};\mathcal{B})$, readily give $\sum_{b_a} E(b_a,s_a)^{-1}G^{(2k)}_{\mathcal{B}}(\mathbf{X})$
- the term $\mathfrak{f}_a(\mathbf{X};\mathbf{s};\mathcal{B})$ is, according to Proposition 2.1, $\sum_{\hat{\pi}\in\mathrm{Aut}_c(\mathcal{B})}\pi^*\mathfrak{f}^{(a)}_{\mathcal{B}}(\mathbf{X})$

The remaining two terms, $\mathfrak{b}_a$ and $\mathfrak{d}_a$, need a more detailed inspection, though:

$$\begin{aligned}
\mathcal{O}(\bar{J}) + \prod_{\substack{\alpha>1\\ \nu=1,\ldots,k}} \frac{\delta}{\delta \bar{J}_{\mathbf{y}^\alpha}}\frac{\delta}{\delta J_{\mathbf{x}^\nu}} D_a(\mathbf{s}) &= \prod_{\alpha>1;\nu}\frac{\delta}{\delta \bar{J}_{\mathbf{y}^\alpha}}\frac{\delta}{\delta J_{\mathbf{x}^\nu}}\bigg[\sum\nolimits_{\mathbf{b}}\frac{1}{E(b_a,s_a)}\bar{J}_{\mathbf{b}}\frac{\delta^2 Z[J,\bar{J}]}{\delta \bar{J}_{\mathbf{s}_{\hat{a}}b_a}\delta \bar{J}_{\mathbf{b}_{\hat{a}}s_a}}\bigg]\\
&= \sum_{\rho=2}^k \prod_{\substack{\alpha>1(\alpha\neq\rho)\\ \nu=1,\ldots,k}} \frac{\delta}{\delta \bar{J}_{\mathbf{y}^\alpha}}\frac{\delta}{\delta J_{\mathbf{x}^\nu}}\bigg[\sum\nolimits_{\mathbf{b}}\frac{\delta^{\mathbf{b}}_{\mathbf{y}^\rho}}{E(b_a,s_a)}\frac{\delta^2 Z[J,\bar{J}]}{\delta \bar{J}_{\mathbf{s}_{\hat{a}}b_a}\delta \bar{J}_{\mathbf{b}_{\hat{a}}s_a}}\bigg]\\
&= \sum_{\rho=2}^k \prod_{\substack{\alpha;(1\neq\alpha\neq\rho)\\ \nu=1,\ldots,k}} \frac{\delta}{\delta \bar{J}_{\mathbf{y}^\alpha}}\frac{\delta}{\delta J_{\mathbf{x}^\nu}}\bigg[\frac{1}{E(y^\rho_a,s_a)}\frac{\delta^2 Z[J,\bar{J}]}{\delta \bar{J}_{\mathbf{s}_{\hat{a}}y^\rho_a}\delta \bar{J}_{\mathbf{y}^\rho_{\hat{a}}s_a}}\bigg]\,.
\end{aligned}\tag{3.5}$$

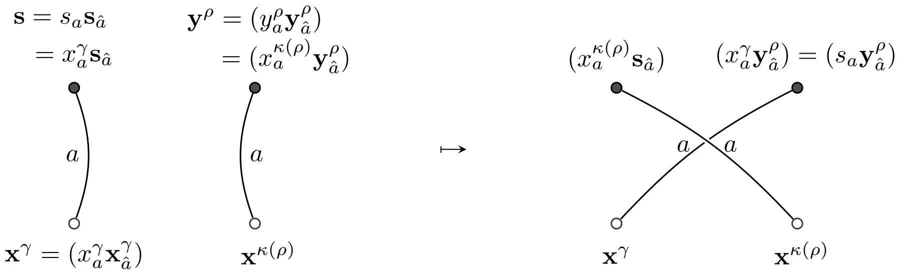
As for the derivatives on $B_a(\mathbf{s})$,

$$\begin{aligned}
\mathcal{O}(J) + \prod_{\substack{\alpha>1\\ \nu=1,\ldots,k}} \frac{\delta}{\delta \bar{J}_{\mathbf{y}^\alpha}}\frac{\delta}{\delta J_{\mathbf{x}^\nu}} B_a(\mathbf{s}) &= \prod_{\alpha>1;\nu}\frac{\delta}{\delta \bar{J}_{\mathbf{y}^\alpha}}\frac{\delta}{\delta J_{\mathbf{x}^\nu}}\bigg[\sum\nolimits_{\mathbf{b}}\frac{1}{E(b_a,s_a)}J_{\mathbf{b}_{\hat{a}}s_a}\frac{\delta^2 Z[J,\bar{J}]}{\delta \bar{J}_{\mathbf{s}_{\hat{a}}b_a}\delta J_{\mathbf{b}}}\bigg]\\
&= \sum_{\beta=1}^k \prod_{\alpha>1;\nu\neq\beta} \frac{\delta}{\delta \bar{J}_{\mathbf{y}^\alpha}}\frac{\delta}{\delta J_{\mathbf{x}^\nu}}\bigg[\sum\nolimits_{\mathbf{b}}\frac{1}{E(b_a,s_a)}\delta^{s_a}_{x^\beta_a}\delta^{\mathbf{b}_{\hat{a}}}_{\mathbf{x}^\beta_{\hat{a}}}\frac{\delta^2 Z[J,\bar{J}]}{\delta \bar{J}_{\mathbf{s}_{\hat{a}}b_a}\delta J_{\mathbf{b}}}\bigg]\\
&= \sum_{\beta=1}^k \prod_{\alpha>1;\nu\neq\beta} \frac{\delta}{\delta \bar{J}_{\mathbf{y}^\alpha}}\frac{\delta}{\delta J_{\mathbf{x}^\nu}}\bigg[\sum_{b_a}\frac{1}{E(b_a,s_a)}\delta^{s_a}_{x^\beta_a}\frac{\delta^2 Z[J,\bar{J}]}{\delta \bar{J}_{\mathbf{s}_{\hat{a}}b_a}\delta J_{\mathbf{x}^\beta_{\hat{a}}b_a}}\bigg]\\
&= \prod_{\alpha>1;\nu\neq\gamma} \frac{\delta}{\delta \bar{J}_{\mathbf{y}^\alpha}}\frac{\delta}{\delta J_{\mathbf{x}^\nu}}\bigg[\sum_{b_a}\frac{1}{E(b_a,x^\gamma_a)}\frac{\delta^2 Z[J,\bar{J}]}{\delta \bar{J}_{\mathbf{s}_{\hat{a}}b_a}\delta J_{\mathbf{x}^\gamma_{\hat{a}}b_a}}\bigg]\,.
\end{aligned}\tag{3.6}$$

[4]The order should play no role when one obtains closed equations for a single correlation function in each sector of common Gurău-degree

For the last equality, one uses the fact that $\mathcal{B}$ is regular. Thus, there exists precisely one white vertex $\mathbf{x}^\gamma$, $\gamma = \gamma(a)$, such that $x_a^\gamma = s_a$. In turn, this means that $\delta^{s_a}_{x_a^\beta} = \delta^{s_a}_{x_a^\beta}\delta^\gamma_\beta$.

- as evident in eq. (3.5), the derivatives on the D_a-term give, after setting the sources to zero, all the (coloured) graphs obtained from $\mathcal{B}$ by a swapping of the following form (only a-colour and only the four implied vertices visible):



  for ρ running over the black vertices which are not $\bar{J}_{\mathbf{s}} = \bar{J}_{\mathbf{y}^1}$. Hence the contribution of this term is

$$\sum_{\rho>1} \frac{1}{E(y_a^\rho, s_a)} Z_0^{-1} \frac{\partial Z[J,J]}{\partial \varsigma_a(\mathcal{B};1,\rho)(\mathbf{X})} \text{ for } \rho = 2,\ldots,k.$$

  Since $\mathbf{y}^1 = \mathbf{s}$, we also write $\varsigma_a(\mathcal{B};\mathbf{y}^1,\mathbf{y}^\rho) = \varsigma_a(\mathcal{B};1,\rho)$ for this new indexing graph ($\rho > 1$).
- concerning the derivatives of B_a above in eq. (3.6), the only surviving term is $\delta^2 Z[J,\bar{J}]/\delta\bar{J}_{\mathbf{s}_{\hat a} b_a}\delta J_{\mathbf{x}^\gamma_{\hat a} b_a}$, and is selected by $\delta^{s_a}_{x_a^\beta}$, which is just, after taking into account the rest of the derivatives, the graph derivative $\partial Z/\partial\mathcal{B}(\mathbf{X})|_{x_a^\gamma \to b_a}$ the single coordinate x_a^γ substituted by (the running) b_a. If $\mathcal{B}$ is connected (as we assumed), after setting the sources to zero, one has

$$\mathfrak{b}_a(\mathbf{X};\mathbf{s};\mathcal{B}) = \sum_{b_a} \frac{1}{E(b_a, x_a^\gamma)} G^{(2k)}_{\mathcal{B}}(\mathbf{x}^1;\ldots;\mathbf{x}^{\gamma-1}; x_1^\gamma,\ldots,x_{a-1}^\gamma, b_a, x_{a+1}^\gamma;\mathbf{x}^{\gamma+1};\ldots x_D^\gamma;\ldots;\mathbf{x}^k)\,.$$

From eqs. (3.3) one has

$$\begin{aligned}\frac{\partial W[J,\bar{J}]}{\partial\mathcal{B}(\mathbf{X})} &= \prod_{\substack{\alpha>1\\ \nu=1,\ldots,k}} \frac{\delta}{\delta\bar{J}_{\mathbf{y}^\alpha}}\frac{\delta}{\delta J_{\mathbf{x}^\nu}}\left(\frac{-2\lambda E_{\mathbf{s}}^{-1}}{Z[J,\bar{J}]}\sum\nolimits_a (A_a(\mathbf{s}) + C_a(\mathbf{s}) + D_a(\mathbf{s}) + F_a(\mathbf{s}) - B_a(\mathbf{s}))\right)\Bigg|_{\substack{\bar{J}=0\\ J=0}} \\ &= \frac{(-2\lambda)}{E_{\mathbf{s}}}\sum\nolimits_a (\mathfrak{a}_a(\mathbf{X};\mathbf{s};\mathcal{B}) + \mathfrak{c}_a(\mathbf{X};\mathbf{s};\mathcal{B}) + \mathfrak{d}_a(\mathbf{X};\mathbf{s};\mathcal{B}) + \mathfrak{f}_a(\mathbf{X};\mathbf{s};\mathcal{B}) - \mathfrak{b}_a(\mathbf{X};\mathbf{s};\mathcal{B}))\,,\end{aligned}$$

where each summand is now known. Because $Y^{(a)}_{s_a}[0,0] = \Delta_{s_a,1}G^{(2)}_{\ominus}$ we have proven:

Proposition 3.1 (Schwinger-Dyson equations)**.** *Let $D \geq 3$ and let $\mathcal{B}$ be a connected boundary graph of the quartic melonic model, $\mathcal{B} \in \mathsf{Feyn}_D(\varphi^4_{\mathrm{m},D}) = \mathit{Grph}^{\mathrm{II,cl}}_D$. Suppose that $\mathcal{B}$ has $2k$ vertices, $k > 1$. Let $\mathbf{s} = \mathbf{y}^1$, where $\mathcal{B}_*(\mathbf{X}) = (\mathbf{y}^1,\ldots,\mathbf{y}^k)$ for any $\mathbf{X} \in \mathcal{F}_{k(\mathcal{B}),D}$. The*

$(2k)$-point Schwinger-Dyson equation corresponding to $\mathcal{B}$ is

$$\Big(1 + \frac{2\lambda}{E_{\mathbf{s}}} \sum_{a=1}^{D} \sum_{\mathbf{q}_{\hat a}} G^{(2)}(s_a, \mathbf{q}_{\hat a})\Big) G^{(2k)}_{\mathcal{B}}(\mathbf{X}) \tag{3.7}$$

$$= \frac{(-2\lambda)}{E_{\mathbf{s}}} \sum_{a=1}^{D} \Bigg\{ \sum_{\hat\sigma \in \mathrm{Aut_c}(\mathcal{B})} \sigma^* \mathfrak{f}^{(a)}_{\mathcal{B},s_a}(\mathbf{X}) + \sum_{\rho>1} \frac{1}{E(y^\rho_a, s_a)} Z_0^{-1} \frac{\partial Z[J,J]}{\partial \varsigma_a(\mathcal{B};1,\rho)(\mathbf{X})}$$

$$- \sum_{b_a} \frac{1}{E(s_a, b_a)} \big[G^{(2k)}_{\mathcal{B}}(\mathbf{X}) - G^{(2k)}_{\mathcal{B}}(\mathbf{X}|_{x^\gamma_a \to b_a}) \big] \Bigg\}$$

for all $\mathbf{X} \in \mathcal{F}_{D,k(\mathcal{B})}$. Here $\gamma \in \{1, \dots, k\}$ is uniquely determined by $s_a = x^\gamma_a$ and $(s_a, \mathbf{q}_{\hat a})$ is abuse of notation for $(q_1, q_2, \dots, q_{a-1}, s_a, q_{a+1}, \dots, q_D)$. Also $E(u_a, v_a) = E_{u_a \mathbf{q}_{\hat a}} - E_{v_a \mathbf{q}_{\hat a}}$.

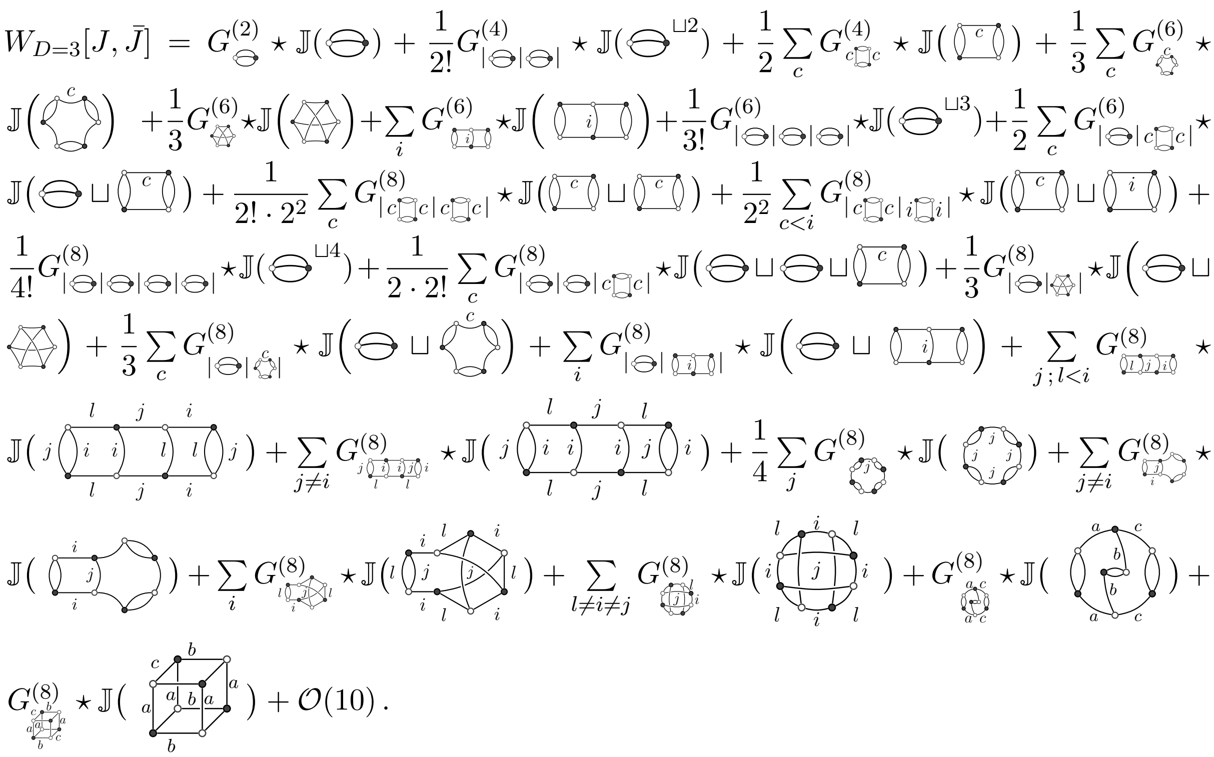
We will ease the notation $\mathfrak{f}^{(a)}_{\mathcal{B},s_a} = \mathfrak{f}^{(a)}_{\mathcal{B}}$, when no risk of confusion arises, keeping in mind the dependence of this function on s_a. Notice that if the graph $\varsigma_a(\mathcal{B};1,\rho)$ is connected, then the respective derivative on $Z[J,\bar J]$ is just

$$\frac{1}{Z_0} \frac{\partial Z[J,J]}{\partial \varsigma_a(\mathcal{B};1,\rho)(\mathbf{X})} = G^{(2k)}_{\varsigma_a(\mathcal{B};1,\rho)}(\mathbf{X})\,;$$

otherwise, the RHS of this expression contains, on top of $G^{(2k)}_{\varsigma_a(\mathcal{B};1,\rho)}$, also a product of correlation functions indexed by the connected components of $\varsigma_a(\mathcal{B};1,\rho)$ with a number of points which add up to $2k$ (see Sec. 4.1). Observe that the equation still depends at this stage on the choice of the vertex $\bar J_{\mathbf{s}}$, with respect to which we first derived. This dependence is also a feature in matrix models that disappears when one splits the equations in genus-sectors.

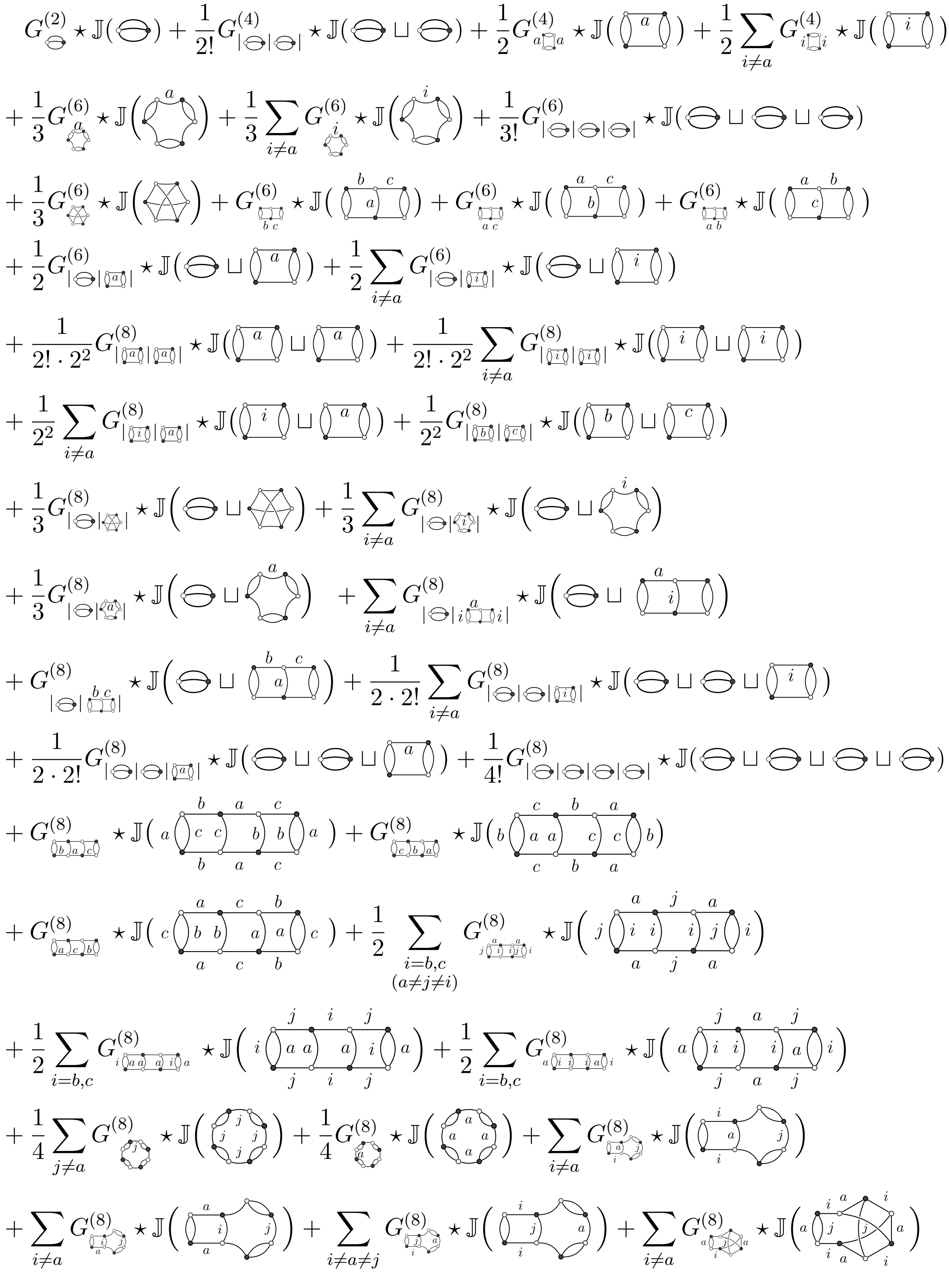
4. Schwinger-Dyson equations for rank-3 theories

According to [15], the boundary sector $\operatorname{im}\partial$ of the φ^4_3-theory is all of $\partial(\mathsf{Feyn}_3(\varphi^4)) = \mathsf{Grph}^{\mathrm{II,cl}}_3$. Therefore $W[J,\bar J] = \log Z[J,\bar J]$ can be expanded in boundary graphs as:



The WTI will be used for each colour $a = 1, 2, 3$, and it will be convenient to single out a in this last expression. From here on[5] $b = b(a) = \min(\{1, 2, 3\} \setminus \{a\})$ and $c = c(a) = \max(\{1, 2, 3\} \setminus \{a\})$:

$$W_{D=3}[J, \bar{J}] =$$



[5]Beware this is only a notation for rank-3 theories; for rank 4 another notation shall be used

$$+\sum_{i\neq a} G^{(8)} \star \mathbb{J}\Big(\Big) + G^{(8)} \star \mathbb{J}\Big(\Big) + \frac{1}{4} G^{(8)} \star \mathbb{J}\Big[\Big]$$

$$+\frac{1}{4}\sum_{i\neq j} G^{(8)} \star \mathbb{J}\Big(\Big) + \frac{1}{4} G^{(8)} \star \mathbb{J}\Big(\Big) + \mathcal{O}(10)$$

In [15], the term $Y^{(a)}_{s_a}[J,\bar{J}]$ for the φ^4_3-theory to $\mathcal{O}(4)$ has been found. This expansion is enough for deriving any of the 4-point SDEs. However, since we want the explicit 6-point SDEs, we need to compute $Y^{(a)}_{s_a}[J,\bar{J}]$ to $\mathcal{O}(6)$ in the sources, i.e. consider the free energy to order $\mathcal{O}(J^4,\bar{J}^4)$, to be precise.

Lemma 4.1. *To order-6, $Y^{(a)}_{s_a}[J,\bar{J}]$ is given by:*

$$Y^{(a)}_{s_a}[J,\bar{J}]$$

$$=\sum_{q_b,q_c} G^{(2)}(s_a,q_b,q_c) + \frac{1}{2}\sum_{r=1}^{2}\big(\Delta_{m_a,r}G^{(4)} + \Delta_{s_a,r}G^{(4)} + \Delta_{s_a,r}G^{(4)} + \Delta_{s_a,r}G^{(4)}\big)\star\mathbb{J}(\,)$$

$$+\Big(\frac{1}{3}\sum\nolimits_{r=1}^{3}(\Delta_{s_a,r}G^{(6)} + \Delta_{s_a,r}G^{(6)}) + \Delta_{s_a,3}G^{(6)} + \Delta_{s_a,3}G^{(6)} + \frac{1}{2}\Delta_{s_a,1}G^{(6)}\Big)\star\mathbb{J}(\,)$$

$$+\Big(\frac{1}{3}\sum\nolimits_{r=1}^{3}\Delta_{s_a,r}G^{(6)} + \Delta_{s_a,3}G^{(6)} + \sum\nolimits_{r=1}^{2}\Delta_{s_a,r}G^{(6)} + \frac{1}{2}\Delta_{s_a,1}G^{(6)}\Big)\star\mathbb{J}(\,)$$

$$+\Big(\frac{1}{3}\sum\nolimits_{r=1}^{3}\Delta_{s_a,r}G^{(6)} + \Delta_{s_a,2}G^{(6)} + \sum\nolimits_{r=1}^{2}\Delta_{s_a,r}G^{(6)} + \frac{1}{2}\Delta_{s_a,1}G^{(6)}\Big)\star\mathbb{J}(\,)$$

$$+\Big(\frac{1}{3!}\sum\nolimits_{r=1}^{3}\Delta_{s_a,r}G^{(6)} + \Delta_{s_a,2}G^{(6)} + \sum\nolimits_{r=2,3}\sum\nolimits_{i=1}^{3}\Delta_{s_a,r}G^{(6)}\Big)\star\mathbb{J}(\,\sqcup\,)$$

$$+\Big\{\frac{1}{4}\sum_{p=3,4}(\Delta_{s_a,p}G^{(8)} + \Delta_{s_a,p}G^{(8)})$$

$$+\frac{1}{4!}\sum_{u=1}^{4}(\Delta_{s_a,u}G^{(8)}) + \Delta_{s_a,3}G^{(8)}\Big\}\star\mathbb{J}(\,\sqcup\,\sqcup\,)$$

$$+\Big\{\frac{1}{8}\Big(\sum_{r=1,2}\Delta_{s_a,r}G^{(8)} + \sum_{r=3,4}(123)^*(\Delta_{s_a,r}G^{(8)})\Big) + \frac{1}{4}\sum_{i\neq a}\Big(\sum_{r=1,2}\Delta_{s_a,r}G^{(8)}\Big)$$

$$+\frac{1}{3}\sum_{q=2,3,4}\Delta_{s_a,q}G^{(8)} + \frac{1}{3}\sum_{q=2,3,4}\Delta_{s_a,q}G^{(8)} + \Delta_{s_a,4}G^{(8)} + \Delta_{s_a,4}G^{(8)}$$

$$+\frac{1}{4}\sum_{r=1,2}\Delta_{s_a,r}G^{(8)} + \Delta_{s_a,2}G^{(8)} + (123)^*(\Delta_{s_a,3}G^{(8)})\Big\}\star\mathbb{J}(\,\sqcup\,)$$

$$+\Big\{\frac{1}{8}\Big(\sum_{r=1,2}\Delta_{s_a,r}G^{(8)} + \sum_{r=3,4}(123)^*\Delta_{s_a,r}G^{(8)}\Big) + \frac{1}{4}\sum_{p=3,4}(123)^*\Delta_{s_a,p}G^{(8)}$$

$$+\frac{1}{4}\sum_{p=3,4}(123)^*\Delta_{s_a,p}G^{(8)} + \sum_{h=2,3}\Delta_{s_a,h}G^{(8)} + \Delta_{s_a,4}G^{(8)} + \frac{1}{4}\sum_{r=1,2}\Delta_{s_a,r}G^{(8)}$$

$$+\frac{1}{2}\big(\Delta_{s_a,2}G^{(8)} + (123)^*\big(\Delta_{s_a,3}G^{(8)}\big)\big) + \Delta_{s_a,2}G^{(8)}\Big\}\star\mathbb{J}(\,\sqcup\,)$$

$$+\Big\{\frac{1}{8}\Big(\sum_{r=1,2}\Delta_{s_a,r}G^{(8)} + \sum_{r=3,4}(123)^*\Delta_{s_a,r}G^{(8)} + \frac{1}{4}\sum_{p=3,4}(123)^*\Delta_{s_a,p}G^{(8)}\Big)$$

$$+\frac{1}{4}\Big(\sum_{r=1,2}\Delta_{s_a,r}G^{(8)}\Big)+\sum_{h=2,3}\Delta_{s_a,h}G^{(8)}+\Delta_{s_a,2}G^{(8)}+\frac{1}{4}\sum_{r=1,2}\Delta_{s_a,r}G^{(8)}$$

$$+\frac{1}{2}\big(\Delta_{s_a,2}G^{(8)}+(123)^*(\Delta_{s_a,3}G^{(8)})\big)+\Delta_{s_a,2}G^{(8)}\Big\}\star\mathbb{J}(\;\sqcup\;)$$

$$+\Big\{\frac{1}{3}\Delta_{s_a,1}G^{(8)}+\Delta_{s_a,1}G^{(8)}+\Delta_{s_a,1}G^{(8)}$$

$$+\sum_{p=3,4}\Delta_{s_a,p}G^{(8)}+\frac{1}{4}\sum_{u=1}^{4}\Delta_{s_a,u}G^{(8)}\Big\}\star\mathbb{J}(\;)$$

$$+\Big\{\frac{1}{3}\Delta_{s_a,1}G^{(8)}+\Delta_{s_a,1}G^{(8)}+\Delta_{s_a,1}G^{(8)}+$$

$$+\frac{1}{4}\sum_{r=1}^{4}\Delta_{s_a,r}G^{(8)}+\sum_{r=1,2}\Delta_{s_a,r}G^{(8)}\Big\}\star\mathbb{J}(\;)$$

$$+\Big\{\frac{1}{3}\Delta_{s_a,1}G^{(8)}+\frac{1}{2}\sum_{h=2,3}\Delta_{s_a,h}G^{(8)}$$

$$+\frac{1}{4}\sum_{r=1}^{4}\Delta_{s_a,r}G^{(8)}+\Delta_{s_a,1}G^{(8)}+\sum_{r=1,2}\Delta_{s_a,r}G^{(8)}\Big\}\star\mathbb{J}(\;)$$

$$+\Big\{\frac{1}{3}\Delta_{s_a,1}G^{(8)}+\frac{1}{2}\sum_{h=2,3}\Delta_{s_a,h}G^{(8)}$$

$$+\frac{1}{4}\sum_{r=1}^{4}\Delta_{s_a,r}G^{(8)}+\Delta_{s_a,1}G^{(8)}+\sum_{r=1,2}\Delta_{s_a,r}G^{(8)}\Big\}\star\mathbb{J}(\;)$$

$$+\Big\{\Delta_{s_a,1}G^{(8)}+\sum_{h=2,3}\Delta_{s_a,h}G^{(8)}+(13)^*(\Delta_{s_a,3}G^{(8)})+(13)^*(\Delta_{s_a,4}G^{(8)})+$$

$$+\sum_{p=3,4}\big(\Delta_{s_a,p}G^{(8)}+(13)^*\Delta_{s_a,p}G^{(8)}\big)+(13)^*(\Delta_{s_a,1}G^{(8)})+(13)^*(\Delta_{s_a,2}G^{(8)})$$

$$+\frac{1}{2}\Big(\Delta_{s_a,1}G^{(8)}+(13)^*(\Delta_{s_a,1}G^{(8)})+(13)^*(\Delta_{s_a,4}G^{(8)})+\Delta_{s_a,4}G^{(8)}\Big)$$

$$+\sum_{r=1,2}(13)^*\big(\Delta_{s_a,r}G^{(8)}\big)+\frac{1}{4}\big((23)^*\Delta_{s_a,1}G^{(8)}+\Delta_{s_a,2}G^{(8)}$$

$$+(13)^*\Delta_{s_a,3}G^{(8)}+(123)^*\Delta_{s_a,4}G^{(8)}\big)\Big\}\star\mathbb{J}(\;)$$

$$+\Big\{\Delta_{s_a,1}G^{(8)}+\Delta_{s_a,1}G^{(8)}+\Delta_{s_a,4}G^{(8)}+\frac{1}{2}\Big(\sum_{r=1,2}(13)^*(\Delta_{s_a,r}G^{(8)})$$

$$+\sum_{p=3,4}\Delta_{s_a,p}G^{(8)}+\Delta_{s_a,1}G^{(8)}+(13)^*\Delta_{s_a,4}G^{(8)}\Big)+\sum_{p=3,4}\Delta_{s_a,p}G^{(8)}$$

$$+\sum_{q=2,3,4}(13)^*\Delta_{s_a,q}G^{(8)}+(13)^*\Delta_{s_a,q}G^{(8)}+\Delta_{s_a,3}G^{(8)}$$

$$+\frac{1}{4}\big(\Delta_{s_a,1}G^{(8)}+(123)^*\Delta_{s_a,2}G^{(8)}+(23)^*\Delta_{s_a,3}G^{(8)}+(13)^*\Delta_{s_a,4}G^{(8)}\big)\Big\}\star\mathbb{J}(\;)$$

$$+\Big\{\Delta_{s_a,1}G^{(8)}+(13)^*(\Delta_{s_a,4}G^{(8)})+(13)^*(\Delta_{s_a,1}G^{(8)})$$

$$+\frac{1}{2}\Big(\sum_{r=1,2}(13)^*(\Delta_{s_a,r}G^{(8)}_{\square}) + \sum_{p=3,4}\Delta_{s_a,p}G^{(8)}_{\square} + \Delta_{s_a,1}G^{(8)}_{\square}$$

$$+ (13)^*\Delta_{s_a,4}G^{(8)}_{\square}\Big) + \sum_{p=3,4}\Delta_{s_a,p}G^{(8)}_{\square} + \sum_{q=2,3,4}(13)^*\Delta_{s_a,q}G^{(8)}_{\square}$$

$$+\sum\nolimits_{q=2,3,4}(13)^*\Delta_{s_a,q}G^{(8)}_{\square} + \sum_{p=3,4}(13)^*\big(\Delta_{s_a,p}G^{(8)}_{\square}\big)$$

$$+\frac{1}{4}\big(\Delta_{s_a,1}G^{(8)}_{\square} + (123)^*\Delta_{s_a,2}G^{(8)}_{\square} + (23)^*\Delta_{s_a,3}G^{(8)}_{\square} + (13)^*\Delta_{s_a,4}G^{(8)}_{\square}\big)\Big\} \star \mathbb{J}(\square)\,.$$

Proof. See Appendix A. $\square$

Since we already derived the 2-point equation in [15], we immediately proceed with the higher-point functions.

4.1. **Four-point function SDEs for the φ^4_3-theory.** We can use the colour symmetry in order to write down the equations for $G^{(4)}_{2\square 2}$ and $G^{(4)}_{3\square 3}$ from that for $G^{(4)}_{1\square 1}$, which we now compute. We will obtain, as stated by the proposition of previous section, the SDE for $G^{(4)}_{1\square 1}$.

We need first, to compute the functions $\mathfrak{f}^{(a)}_{1\square 1}$ for each colour a. To this end, Proposition 4.1 is used:

$$\mathfrak{f}^{(1)}_{1\square 1} = \frac{1}{3}\sum\nolimits_{r=1}^{3}(\Delta_{x_1,r}G^{(6)}_{\square}) + \frac{1}{3}\sum\nolimits_{r=1}^{3}(\Delta_{x_1,r}G^{(6)}_{\square}) + (\Delta_{x_1,3}G^{(6)}_{\square_{13}} + \Delta_{x_1,3}G^{(6)}_{\square_{12}}) + \Delta_{x_1,1}G^{(6)}_{|\ominus|1\square 1|}\,,$$

$$\mathfrak{f}^{(2)}_{1\square 1} = \frac{1}{3}\sum\nolimits_{r=1}^{3}(\Delta_{y_2,r}G^{(6)}_{\square}) + \Delta_{y_2,3}G^{(6)}_{\square_{13}} + \sum\nolimits_{r=1}^{2}(\Delta_{y_2,r}G^{(6)}_{\square_{12}}) + \frac{1}{2}(\Delta_{y_2,1}G^{(6)}_{|\ominus|1\square 1|})\,,$$

$$\mathfrak{f}^{(3)}_{1\square 1} = \frac{1}{3}\sum\nolimits_{r=1}^{3}(\Delta_{y_3,r}G^{(6)}_{\square}) + \Delta_{y_2,3}G^{(6)}_{\square_{12}} + \sum\nolimits_{r=1}^{2}(\Delta_{y_3,r}G^{(6)}_{\square_{13}}) + \frac{1}{2}(\Delta_{y_3,1}G^{(6)}_{|\ominus|1\square 1|})\,.$$

Also notice that

$$\varsigma_1(1\square 1;1,2) = \ominus \sqcup \ominus, \quad \varsigma_2(1\square 1;1,2) = 2\square 2, \quad \varsigma_3(1\square 1;1,2) = 3\square 3.$$

The derivatives with respect to these, evaluated in $\mathbf{X} = (\mathbf{x},\mathbf{y})$ are then

$$G^{(2)}_{\ominus}(\mathbf{x})\cdot G^{(2)}_{\ominus}(\mathbf{y}) + G^{(4)}_{|\ominus|\ominus|}(\mathbf{x},\mathbf{y})\,, \quad G^{(4)}_{3\square 3}(\mathbf{x},\mathbf{y})\,, \quad \text{and } G^{(4)}_{2\square 2}(\mathbf{x},\mathbf{y})\,.$$

respectively. Letting $\mathbf{s} = (x_1, y_2, y_3)$ and $\mathbf{t} = (y_1, x_2, x_3)$ and using Proposition 3.1, one obtains

$$\Big(1 + \frac{2\lambda}{E_{\mathbf{s}}}\sum\nolimits_a\sum\nolimits_{\mathbf{q}_{\hat a}} G^{(2)}_{\ominus}(s_a,\mathbf{q}_{\hat a})\Big)\cdot G^{(4)}_{1\square 1}(\mathbf{x},\mathbf{y}) \tag{4.2}$$

$$= \frac{(-2\lambda)}{E_{\mathbf{s}}}\sum_{a=1}^{3}\Bigg\{\sum_{\hat\sigma\in \mathrm{Aut}_c(1\square 1)}\sigma^*\mathfrak{f}^{(a)}_{1\square 1}(\mathbf{X}) + \sum_{\rho>1}\frac{Z_0^{-1}}{E(y_a^\rho, s_a)}\frac{\partial Z[J,J]}{\partial\varsigma_a(1\square 1;1,\rho)(\mathbf{X})}$$

$$-\sum_{b_a}\frac{1}{E(s_a,b_a)}\big[G^{(4)}_{1\square 1}(\mathbf{X}) - G^{(4)}_{1\square 1}(\mathbf{X}|_{s_a\to b_a})\big]\Bigg\}$$

$$= \frac{(-2\lambda)}{E_{x_1y_2y_3}}\Big\{(\delta^{\mathbf{x}}_{\mathbf{u}}\delta^{\mathbf{y}}_{\mathbf{v}} + \delta^{\mathbf{y}}_{\mathbf{u}}\delta^{\mathbf{x}}_{\mathbf{v}})\cdot\Big(\frac{1}{3}\sum_{r=1}^{3}(\Delta_{x_1,r}G^{(6)}_{\square}) + \frac{1}{3}\sum_{r=1}^{3}(\Delta_{x_1,r}G^{(6)}_{\square})$$

$$+ (\Delta_{x_1,3}G^{(6)}_{\square_{13}} + \Delta_{x_1,3}G^{(6)}_{\square_{12}}) + \Delta_{x_1,1}G^{(6)}_{|\ominus|1\square 1|}$$

$$+\frac{1}{3}\sum_{r=1}^{3}(\Delta_{y_2,r}G^{(6)}_{\square}) + \Delta_{y_2,3}G^{(6)}_{\square_{13}} + \sum_{r=1}^{2}(\Delta_{y_2,r}G^{(6)}_{\square_{12}}) + \frac{1}{2}(\Delta_{y_2,1}G^{(6)}_{|\ominus|1\square 1|})$$

$$
\begin{aligned}
&+\frac{1}{3}\sum_{r=1}^{3}(\Delta_{y_3,r}G^{(6)}_{1}) + \Delta_{y_2,3}G^{(6)}_{12} + \sum_{r=1}^{2}(\Delta_{y_3,r}G^{(6)}_{13}) + \frac{1}{2}(\Delta_{y_3,1}G^{(6)}_{|\ominus|_1 \, {}_1|}\Big)(\mathbf{u},\mathbf{v})\\
&+\frac{1}{E(y_1,x_1)}\big(G^{(2)}_{\ominus}(\mathbf{x})\cdot G^{(2)}_{\ominus}(\mathbf{y}) + G^{(4)}_{|\ominus|\ominus|}(\mathbf{x},\mathbf{y})\big)\\
&+\frac{1}{E(x_2,y_2)}G^{(4)}_{3\,3}(\mathbf{x},\mathbf{y}) + \frac{1}{E(x_3,y_3)}G^{(4)}_{2\,2}(\mathbf{x},\mathbf{y})\\
&-\sum\nolimits_{b_1}\frac{1}{E(x_1,b_1)}\big(G^{(4)}_{1\,1}(\mathbf{x},\mathbf{y}) - G^{(4)}_{1\,1}(b_1,x_2,x_3,\mathbf{y})\big)\\
&-\sum\nolimits_{b_2}\frac{1}{E(y_2,b_2)}\big(G^{(4)}_{1\,1}(\mathbf{x},\mathbf{y}) - G^{(4)}_{1\,1}(\mathbf{x},y_1,b_2,y_3)\big)\\
&-\sum\nolimits_{b_3}\frac{1}{E(y_3,b_3)}\big(G^{(4)}_{1\,1}(\mathbf{x},\mathbf{y}) - G^{(4)}_{1\,1}(\mathbf{x},y_1,y_2,b_3)\big)\Big\}
\end{aligned}
$$

4.2. **The Schwinger-Dyson equation for $G^{(6)}$.** We now derive the whole set of six-point function equations for the φ^4_3-theory. They hold for any model for which the boundary sector is the whole of $\mathsf{Grph}_3^{\mathrm{II,cl}}$. From Prop. 4.1, one can read off the $\mathfrak{f}^{(a)}_{\mathcal{B}}$ functions.

For the boundary graph, one has, for each colour $a=1,2,3$,

$$
\mathfrak{f}^{(a)} = \frac{1}{3}\Delta_{s_a,1}G^{(8)}_{|\ominus|\,|} + \Delta_{s_a,1}G^{(8)}_{a\,c\,b\,a} + \Delta_{s_a,1}G^{(8)}_{a\,b\,c\,a} + \sum_{p=3,4}\Delta_{s_a,p}G^{(8)}_{i\,j\,a\,i} + \frac{1}{4}\sum\nolimits_{u=1}^{4}\Delta_{s_a,u}G^{(8)}_{a\,j\,i}\,.
$$

Departing from Proposition 3.1, this last very expression allows now for an explicit derivation of the equation for $G^{(6)}$. Namely, for $\mathbf{X}=(\mathbf{x}^1,\mathbf{x}^2,\mathbf{x}^3)=(\mathbf{x},\mathbf{y},\mathbf{z})$, and choosing $\mathbf{s}=(x_1,y_2,z_3)$,

$$
\begin{aligned}
&\Big(1+\frac{2\lambda}{E_{x_1y_2z_3}}\sum_{a=1}^{3}\sum_{\mathbf{q}_{\hat a}}G^{(2)}_{\ominus}(s_a,\mathbf{q}_{\hat a})\Big)G^{(6)}(\mathbf{X})\\
&=\frac{(-2\lambda)}{E_{x_1y_2z_3}}\sum_{a=1}^{3}\Big\{\sum_{\hat\sigma\in\mathrm{Aut_c}()}\sigma^*\mathfrak{f}^{(a)}(\mathbf{X}) + \sum_{\rho>1}\frac{Z_0^{-1}}{E(y_a^\rho,s_a)}\frac{\partial Z[J,J]}{\partial\varsigma_a(\,;1,\rho)(\mathbf{X})}\\
&\qquad -\sum_{b_a}\frac{1}{E(s_a,b_a)}\big[G^{(6)}(\mathbf{X}) - G^{(6)}(\mathbf{X}|_{s_a\to b_a})\big]\Big\}
\end{aligned}
\tag{4.3}
$$

One finds:

$$
\begin{aligned}
&\sum_{a=1}^{3}\sum_{\rho>1}\frac{Z_0^{-1}}{E(y_a^\rho,s_a)}\frac{\partial Z[J,J]}{\partial\varsigma_a(\,;1,\rho)(\mathbf{X})}\\
&=\Big(\frac{1}{E(y_1,x_1)}(23)^*G^{(6)}_{23} + \frac{1}{E(z_1,x_1)}(13)^*G^{(6)}_{23} + \frac{1}{E(x_2,y_2)}(123)^*G^{(6)}_{13}\\
&\quad+\frac{1}{E(z_2,y_2)}(132)^*G^{(6)}_{13} + \frac{1}{E(x_3,z_3)}(13)^*G^{(6)}_{12} + \frac{1}{E(y_3,z_3)}(12)^*G^{(6)}_{12}\Big)(\mathbf{X})\,.
\end{aligned}
$$

The meaning of the $\mathfrak{f}^{(a)}$ summed over colours a and over the automorphism group is

$$
\sum_a\sum_{\hat\pi\in\mathbb{Z}_3}\pi^*\Big\{\frac{1}{3}G^{(8)}_{|\ominus|\,|} + \Delta_{s_a,1}G^{(8)}_{a\,c\,b\,a} + \Delta_{s_a,1}G^{(8)}_{a\,b\,c\,a}
$$

$$\sum_{p=3,4}\Delta_{s_a,p} + G^{(8)}_{i\,j\,i\,a} + \frac{1}{4}\sum\nolimits_{u=1}^{4}\Delta_{s_a,u}G^{(8)}_{j\,a\,i}\Big\}$$

$$= \sum_{\hat\pi\in\mathbb{Z}_3}\pi^*\Big\{\frac{1}{3}\sum\nolimits_a \Delta_{s_a,1}G^{(8)}_{|\ominus|\,|} + \big(\Delta_{x_1,1}G^{(8)}_{1\,3\,1\,2} + \Delta_{x_1,1}G^{(8)}_{2\,1\,2\,3} + \Delta_{y_2,1}G^{(8)}_{3\,2\,3\,1} + \Delta_{y_2,1}G^{(8)}_{2\,1\,2\,3}$$

$$+ \Delta_{z_3,1}G^{(8)}_{1\,3\,1\,2} + \Delta_{z_3,1}G^{(8)}_{2\,1\,2\,3} + \sum_{p=3,4}\Delta_{x_1,p}G^{(8)}_{3\,2\,3\,1} + \Delta_{y_2,p}G^{(8)}_{1\,3\,1\,2} + \Delta_{z_3,p}G^{(8)}_{2\,1\,2\,3}\big)$$

$$+ \frac{1}{4}\sum_{u=1}^{4}\big(\Delta_{x_1,u}G^{(8)}_{3\,1\,2} + \Delta_{y_2,u}G^{(8)}_{1\,2\,3} + \Delta_{z_3,u}G^{(8)}_{2\,3\,1}\big)\Big\}$$

where $\mathbb{Z}_3$ is generated rotation of by $2\pi/3$, that is $\hat\pi$ is the liftings of the identity, of (123) and (132) in $\mathfrak{S}_3$. Finally, the difference-term is

$$\sum_a\sum_{b_a}\frac{1}{E(s_a,b_a)}\big[G^{(6)}(\mathbf{X}) - G^{(6)}(\mathbf{X}\big|_{s_a\to b_a})\big] = \sum_{b_1}\frac{1}{E(x_1,b_1)}\big[G^{(6)}(\mathbf{X}) - G^{(6)}(b_1,x_2,x_3;\mathbf{y};\mathbf{z})\big]$$

$$+ \sum_{b_2}\frac{1}{E(y_2,b_2)}\big[G^{(6)}(\mathbf{X}) - G^{(6)}(\mathbf{x};y_1,b_2,y_3;\mathbf{z})\big]$$

$$+ \sum_{b_3}\frac{1}{E(z_3,b_3)}\big[G^{(6)}(\mathbf{X}) - G^{(6)}(\mathbf{x};\mathbf{y};z_1,z_2,b_3)\big]$$

Explicitly

$$-\bigg(\frac{E_{x_1y_2z_3}}{2\lambda} + \sum_{m,n}\big[G^{(2)}_{\ominus}(x_1,m,n) + G^{(2)}_{\ominus}(m,y_2,n) + G^{(2)}_{\ominus}(m,n,z_3)\big]\bigg)\cdot G^{(6)}(\mathbf{X})$$

$$= \bigg(\frac{1}{E(y_1,x_1)}(23)^*G^{(6)}_{2\,3} + \frac{1}{E(z_1,x_1)}(13)^*G^{(6)}_{2\,3} + \frac{1}{E(x_2,y_2)}(123)^*G^{(6)}_{1\,3}$$

$$+ \frac{1}{E(z_2,y_2)}(132)^*G^{(6)}_{1\,3} + \frac{1}{E(x_3,z_3)}(13)^*G^{(6)}_{1\,2} + \frac{1}{E(y_3,z_3)}(12)^*G^{(6)}_{1\,2}\bigg)(\mathbf{X})$$

$$+ \sum_{\hat\pi\in\mathbb{Z}_3}\pi^*\Big[\frac{1}{3}\sum\nolimits_a \Delta_{s_a,1}G^{(8)}_{|\ominus|\,|} \tag{4.4}$$

$$+ \big(\Delta_{x_1,1}G^{(8)}_{1\,3\,1\,2} + \Delta_{x_1,1}G^{(8)}_{2\,1\,2\,3} + \Delta_{y_2,1}G^{(8)}_{3\,2\,3\,1} + \Delta_{y_2,1}G^{(8)}_{2\,1\,2\,3} + \Delta_{z_3,1}G^{(8)}_{1\,3\,1\,2} + \Delta_{z_3,1}G^{(8)}_{2\,1\,2\,3}$$

$$+ \sum_{p=3,4}\Delta_{x_1,p}G^{(8)}_{3\,2\,3\,1} + \Delta_{y_2,p}G^{(8)}_{1\,3\,1\,2} + \Delta_{z_3,p}G^{(8)}_{2\,1\,2\,3}\big) + \frac{1}{4}\sum_{u=1}^{4}\big(\Delta_{x_1,u}G^{(8)}_{3\,1\,2}$$

$$+ \Delta_{y_2,u}G^{(8)}_{1\,2\,3} + \Delta_{z_3,u}G^{(8)}_{2\,3\,1}\big)\Big](\mathbf{X}) - \sum_{b_1}\frac{1}{E(x_1,b_1)}\big[G^{(6)}(\mathbf{X}) - G^{(6)}(b_1,x_2,x_3;\mathbf{y};\mathbf{z})\big]$$

$$- \sum_{b_2}\frac{1}{E(y_2,b_2)}\big[G^{(6)}(\mathbf{X}) - G^{(6)}(\mathbf{x};y_1,b_2,y_3;\mathbf{z})\big]$$

$$- \sum_{b_3}\frac{1}{E(z_3,b_3)}\big[G^{(6)}(\mathbf{X}) - G^{(6)}(\mathbf{x};\mathbf{y};z_1,z_2,b_3)\big]\,.$$

4.3. **The Schwinger-Dyson equation for $G^{(6)}_{a}$.** First, we compute $\mathfrak{b}_a$-terms for $\mathcal{Q}_a$, one by one y are:

$$\frac{1}{Z_0}\frac{\partial Z[J,J]}{\partial\varsigma_1(1;1,2)} = G^{(4)}_{1\,1}(\mathbf{x},\mathbf{z})\cdot G^{(2)}_{\ominus}(\mathbf{y}) + (12)^*G^{(6)}_{|\ominus|\,1\,|}(\mathbf{X})\,, \tag{4.5a}$$

$$\frac{1}{Z_0}\frac{\partial Z[J,J]}{\partial\varsigma_1(\;;1,3)} = G^{(4)}_{1\;1}(\mathbf{y},\mathbf{z})\cdot G^{(2)}_{\ominus}(\mathbf{x}) + G^{(6)}_{|\ominus|\;1\;|}(\mathbf{X})\,, \tag{4.5b}$$

$$\frac{1}{Z_0}\frac{\partial Z[J,J]}{\partial\varsigma_2(\;;1,2)} = (23)^*G^{(6)}_{1\,3}(\mathbf{X})\,, \qquad \frac{1}{Z_0}\frac{\partial Z[J,J]}{\partial\varsigma_2(\;;1,2)} = (13)^*G^{(6)}_{1\,3}(\mathbf{X})\,, \tag{4.5c,d}$$

$$\frac{1}{Z_0}\frac{\partial Z[J,J]}{\partial\varsigma_3(\;;1,3)} = (23)^*G^{(6)}_{1\,2}(\mathbf{X})\,, \qquad \frac{1}{Z_0}\frac{\partial Z[J,J]}{\partial\varsigma_3(\;;1,3)} = (13)^*G^{(6)}_{1\,2}(\mathbf{X})\,. \tag{4.5e,f}$$

We stepwise collect the $\mathfrak{f}^{(a)}$-terms from the expansion in Prop. 4.1:

$$\mathfrak{f}^{(1)} \text{ is the coefficient of } \mathbb{J}(\;^{a}\;) \text{ in } Y^{(1)}_{x_1}[J,\bar{J}] \text{ with } a=1, b=2, c=2, \tag{4.6}$$

$$\mathfrak{f}^{(2)} \text{ is the coefficient of } \mathbb{J}(\;^{b}\;) \text{ in } Y^{(2)}_{y_2}[J,\bar{J}] \text{ with } a=2, b=1, c=3, \text{ and} \tag{4.7}$$

$$\mathfrak{f}^{(3)} \text{ is the coefficient of } \mathbb{J}(\;^{b}\;) \text{ in } Y^{(3)}_{y_3}[J,\bar{J}] \text{ with } a=3, b=1, c=2, \tag{4.8}$$

namely

$$\mathfrak{f}^{(1)} = \frac{1}{3}\Delta_{x_1,1}G^{(8)}_{|\ominus|\;1\;|} + \Delta_{x_1,1}G^{(8)}_{2\;1\;3} + \Delta_{x_1,1}G^{(8)}_{3\;1\;2} + \frac{1}{4}\sum_{r=1}^{4}\Delta_{x_1,r}G^{(8)}_{1} + \sum_{r=1,2}\Delta_{x_1,r}G^{(8)}_{3\,2\,3\,1}\,,$$

$$\begin{aligned}\mathfrak{f}^{(2)} = \frac{1}{3}\Delta_{y_2,1}G^{(8)}_{|\ominus|\;1\;|} + \frac{1}{2}\sum_{h=2,3}\Delta_{y_2,h}G^{(8)}_{2\,3\,3\;3\,2\,3} + \frac{1}{4}\sum_{r=1}^{4}\Delta_{y_2,r}G^{(8)}_{1} + \Delta_{y_2,1}G^{(8)}_{2\;1\;3}\\ + \sum_{r=1,2}\Delta_{y_2,r}G^{(8)}_{3\;1\;2}\,,\end{aligned}$$

$$\begin{aligned}\mathfrak{f}^{(3)} = \frac{1}{3}\Delta_{y_3,1}G^{(8)}_{|\ominus|\;1\;|} + \frac{1}{2}\sum_{h=2,3}\Delta_{y_3,h}G^{(8)}_{3\,2\,2\;2\,3\,2} + \frac{1}{4}\sum_{r=1}^{4}\Delta_{y_3,r}G^{(8)}_{1} + \Delta_{y_3,1}G^{(8)}_{3\;1\;2}\\ + \sum_{r=1,2}\Delta_{y_3,r}G^{(8)}_{2\;1\;3}\,.\end{aligned}$$

Explicitly

$$\begin{aligned}&-\Big(\frac{E_{x_1y_2y_3}}{2\lambda} + \sum_{m,n}\big[G^{(2)}_{\ominus}(x_1,m,n) + G^{(2)}_{\ominus}(m,y_2,n) + G^{(2)}_{\ominus}(m,n,y_3)\big]\Big)\cdot G^{(6)}_{1}(\mathbf{X})\\
&= \frac{1}{E(z_1,x_1)}G^{(4)}_{1\;1}(\mathbf{x},\mathbf{z})\cdot G^{(2)}_{\ominus}(\mathbf{y}) + \frac{1}{E(y_1,x_1)}G^{(4)}_{1\;1}(\mathbf{y},\mathbf{z})\cdot G^{(2)}_{\ominus}(\mathbf{x})\\
&+\Big[\frac{1}{E(z_1,x_1)}(12)^*G^{(6)}_{|\ominus|\;1\;|} + \frac{1}{E(y_1,x_1)}G^{(6)}_{|\ominus|\;1\;|} + \frac{1}{E(z_2,y_2)}(23)^*G^{(6)}_{1\,3}\\
&+\frac{1}{E(x_2,y_2)}(13)^*G^{(6)}_{1\,3} + \frac{1}{E(z_3,y_3)}(23)^*G^{(6)}_{1\,2} + \frac{1}{E(x_3,y_3)}(13)^*G^{(6)}_{1\,2}\Big](\mathbf{X})\\
&+\sum_{\hat{\pi}\in\mathbb{Z}_3}\pi^*\Big[\frac{1}{3}\Delta_{x_1,1}G^{(8)}_{|\ominus|\;1\;|} + \Delta_{x_1,1}G^{(8)}_{2\;1\;3} + \Delta_{x_1,1}G^{(8)}_{3\;1\;2}\\
&+\frac{1}{4}\sum_{r=1}^{4}\Delta_{x_1,r}G^{(8)}_{1} + \sum_{r=1,2}\Delta_{x_1,r}G^{(8)}_{3\,2\,3\,1} + \frac{1}{3}\Delta_{y_2,1}G^{(8)}_{|\ominus|\;1\;|} + \frac{1}{2}\sum_{h=2,3}\Delta_{y_2,h}G^{(8)}_{2\,3\,3\;3\,2\,3}\\
&+\frac{1}{4}\sum_{r=1}^{4}\Delta_{y_2,r}G^{(8)}_{1} + \Delta_{y_2,1}G^{(8)}_{2\;1\;3} + \sum_{r=1,2}\Delta_{y_2,r}G^{(8)}_{3\;1\;2} + \frac{1}{3}\Delta_{y_3,1}G^{(8)}_{|\ominus|\;1\;|}\end{aligned} \tag{4.9}$$

$$+\frac{1}{2}\sum_{h=2,3}\Delta_{y_3,h}G^{(8)}_{3\,22\,23\,2}+\frac{1}{4}\sum_{r=1}^{4}\Delta_{y_3,r}G^{(8)}+\Delta_{y_3,1}G^{(8)}_{3\,1\,2}+\sum_{r=1,2}\Delta_{y_3,r}G^{(8)}_{2\,1\,3}\Big](\mathbf{X})$$

$$-\sum_{b_1}\frac{1}{E(x_1,b_1)}\big[G^{(6)}_{1}(\mathbf{X})-G^{(6)}_{1}(b_1,x_2,x_3;\mathbf{y};\mathbf{z})\big]$$

$$-\sum_{b_2}\frac{1}{E(y_2,b_2)}\big[G^{(6)}_{1}(\mathbf{X})-G^{(6)}_{1}(\mathbf{x};y_1,b_2,y_3;\mathbf{z})\big]$$

$$-\sum_{b_3}\frac{1}{E(y_3,b_3)}\big[G^{(6)}_{1}(\mathbf{X})-G^{(6)}_{1}(\mathbf{x};y_1,y_2,b_3;\mathbf{z})\big]\,.$$

4.4. **The Schwinger-Dyson equation for $G^{(6)}_{i}$.** Concerning the correlation function $G^{(6)}_{23}$, the terms with swapping black vertices are

$$G^{(6)}_{3},G^{(6)},G^{(6)}_{|\ominus|\,3\,|},G^{(2)}_{\ominus}\cdot G^{(4)}_{3\,3},(132)^*G^{(6)}_{12},G^{(6)}_{13},(12)^*G^{(6)}_{12}\,, \tag{4.10}$$

which need to be divided by differences of propagators. We now find the rest of the terms. Since $\mathrm{Aut}_{\mathrm{c}}(\,{}^{2\,3}\,)$ is trivial, the contribution of the ${}^{2\,3}$-derivative on $\sum_a Y^{(a)}_{s_a}[J,\bar J]$ is given by the sum $\sum_a \mathfrak{f}^{(a)}_{23}$ where, for each colour a:

$$\mathfrak{f}^{(1)}_{23}=\Delta_{x_1,1}G^{(8)}_{|\ominus|\,23\,|}+\sum_{h=2,3}\Delta_{x_1,h}G^{(8)}_{2\,1\,3}+(13)^*(\Delta_{x_1,3}G^{(8)}_{3\,2\,1})+(13)^*(\Delta_{x_1,4}G^{(8)}_{3\,2\,1})$$
$$+\sum_{p=3,4}\big(\Delta_{x_1,p}G^{(8)}_{1\,3\,2}+(13)^*\Delta_{x_1,p}G^{(8)}_{1\,2\,3}\big)+(13)^*(\Delta_{x_1,1}G^{(8)}_{1\,3\,2})+(13)^*(\Delta_{x_1,2}G^{(8)}_{1\,3\,2})$$
$$+\frac{1}{2}\Big(\Delta_{x_1,1}G^{(8)}_{2\,11\,12\,1}+(13)^*(\Delta_{x_1,1}G^{(8)}_{3\,11\,13\,1})+(13)^*(\Delta_{x_1,4}G^{(8)}_{3\,11\,13\,1})+\Delta_{x_1,4}G^{(8)}_{3\,11\,13\,1}\Big)$$
$$+\sum_{r=1,2}(13)^*\big(\Delta_{x_1,r}G^{(8)}_{13\,13}\big)+\frac{1}{4}\Big((23)^*\Delta_{x_1,1}G^{(8)}_{32\,11\,1\,23}+\Delta_{x_1,2}G^{(8)}_{32\,11\,1\,23}+(13)^*\Delta_{x_1,3}G^{(8)}_{32\,11\,1\,23}$$
$$+(123)^*\Delta_{x_1,4}G^{(8)}_{32\,11\,1\,23}\Big)$$

$$\mathfrak{f}^{(2)}_{23}=\Delta_{y_2,1}G^{(8)}_{|\ominus|\,23\,|}+\Delta_{y_2,1}G^{(8)}_{1\,3\,2}+\Delta_{y_2,4}G^{(8)}_{1\,3\,2}+\frac{1}{2}\Big(\sum_{r=1,2}(13)^*(\Delta_{y_2,r}G^{(8)}_{3\,11\,13\,1})$$
$$+\sum_{p=3,4}\Delta_{y_2,p}G^{(8)}_{3\,11\,13\,1}+\Delta_{y_2,1}G^{(8)}_{2\,11\,12\,1}+(13)^*\Delta_{y_2,4}G^{(8)}_{2\,11\,12\,1}\Big)$$
$$+\sum_{p=3,4}\Delta_{y_2,p}G^{(8)}_{1\,3\,2}+\sum_{q=2,3,4}(13)^*\Delta_{y_2,q}G^{(8)}_{1\,2\,3}+(13)^*\Delta_{y_2,q}G^{(8)}_{2\,1\,2\,3}$$
$$+\Delta_{y_2,3}G^{(8)}_{13\,13}+\frac{1}{4}\big(\Delta_{y_2,1}G^{(8)}_{3\,2\,1}+(123)^*\Delta_{y_2,2}G^{(8)}_{3\,2\,1}+(23)^*\Delta_{y_2,3}G^{(8)}_{3\,2\,1}+(13)^*\Delta_{y_2,4}G^{(8)}_{3\,2\,1}\big)$$

$$\mathfrak{f}^{(3)}_{32}=(13)^*\Delta_{x_3,1}G^{(8)}_{|\ominus|\,23\,|}+\Delta_{x_3,1}G^{(8)}_{1\,3\,2}+\Delta_{x_3,4}G^{(8)}_{3\,2\,1}+\frac{1}{2}\Big(\sum_{r=1,2}(13)^*(\Delta_{x_3,r}G^{(8)}_{2\,11\,12\,1})$$
$$+\sum_{p=3,4}\Delta_{x_3,p}G^{(8)}_{2\,11\,12\,1}+\Delta_{x_3,1}G^{(8)}_{3\,11\,13\,1}+(13)^*\Delta_{x_3,4}G^{(8)}_{3\,11\,13\,1}\Big)+\sum_{p=3,4}\Delta_{x_3,p}G^{(8)}_{1\,3\,2}$$
$$+\sum_{q=2,3,4}(13)^*\Delta_{x_3,q}G^{(8)}_{1\,2\,3}+(13)^*\Delta_{x_3,q}G^{(8)}_{1\,3\,2\,1}+\Delta_{x_3,3}G^{(8)}_{13\,13}$$

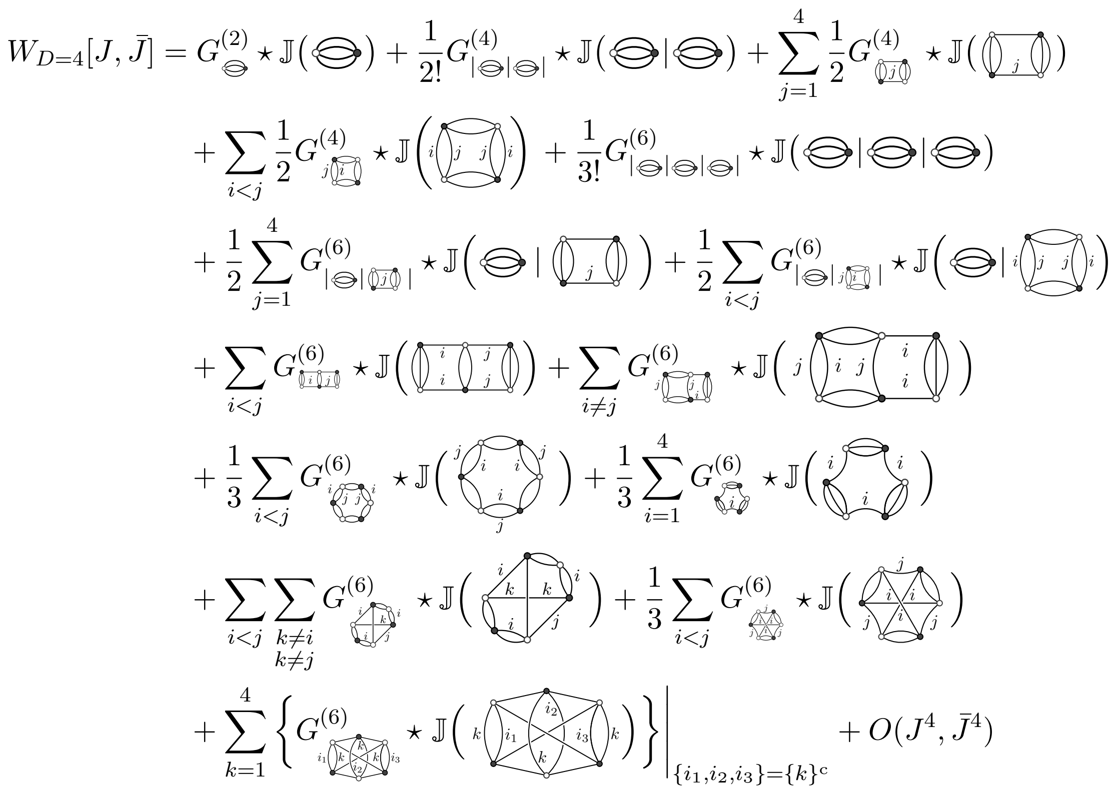
$$+\frac{1}{4}\big(\Delta_{x_3,1}G^{(8)}_{\boxed{1}23} + (123)^*\Delta_{x_3,2}G^{(8)}_{\boxed{1}23} + (23)^*\Delta_{x_3,3}G^{(8)}_{\boxed{1}23} + (13)^*\Delta_{x_3,4}G^{(8)}_{\boxed{1}23}\big)$$

Here $\mathbf{s} = (x_1, y_2, x_3)$. Therefore the explicit equation is

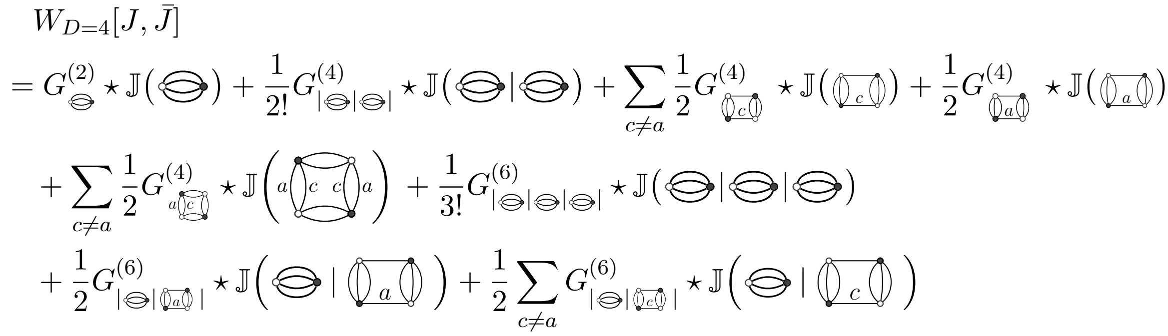
$$
\begin{aligned}
&-\bigg(\frac{E_{x_1y_2x_3}}{2\lambda} + \sum_{m,n}\big[G^{(2)}(x_1,m,n) + G^{(2)}(m,y_2,n) + G^{(2)}(m,n,x_3)\big]\bigg)\cdot G^{(6)}_{23}(\mathbf{X}) \\
&= \bigg\{\frac{1}{E(y_1,x_1)}G^{(6)}_{3} + \frac{1}{E(y_1,x_1)}G^{(6)} + \frac{1}{E(x_2,y_2)}G^{(6)}_{|\ominus|\boxed{3}|} + \frac{1}{E(z_2,y_2)}(132)^*G^{(6)}_{12} \qquad (4.11)\\
&+ \frac{1}{E(z_3,x_3)}G^{(6)}_{13} + \frac{1}{E(y_3,x_3)}(12)^*G^{(6)}_{12}\bigg\}(\mathbf{X}) + \frac{1}{E(x_2,y_2)}\big(G^{(2)}(\mathbf{x})\cdot G^{(4)}_{33}(\mathbf{y},\mathbf{z})\big) \\
&+ \bigg\{\Delta_{x_1,1}G^{(8)}_{|\ominus|\,23\,|} + \sum_{h=2,3}\Delta_{x_1,h}G^{(8)}_{213} + (13)^*(\Delta_{x_1,3}G^{(8)}_{321}) + (13)^*(\Delta_{x_1,4}G^{(8)}_{321}) \\
&+ \sum_{p=3,4}\big(\Delta_{x_1,p}G^{(8)}_{123} + (13)^*\Delta_{x_1,p}G^{(8)}_{132}\big) + (13)^*(\Delta_{x_1,1}G^{(8)}_{132}) + (13)^*(\Delta_{x_1,2}G^{(8)}_{132}) \\
&+ \frac{1}{2}\Big(\Delta_{x_1,1}G^{(8)}_{2\,11\,12\,1} + (13)^*(\Delta_{x_1,1}G^{(8)}_{3\,11\,13\,1}) + (13)^*(\Delta_{x_1,4}G^{(8)}_{3\,11\,13\,1}) + \Delta_{x_1,4}G^{(8)}_{3\,11\,13\,1}\Big) \\
&+ \sum_{r=1,2}(13)^*\big(\Delta_{x_1,r}G^{(8)}_{13\,13}\big) + \Delta_{y_2,1}G^{(8)}_{|\ominus|\,23\,|} + \Delta_{y_2,1}G^{(8)}_{132} + \Delta_{y_2,4}G^{(8)}_{132} \\
&+ \frac{1}{4}\big((23)^*\Delta_{x_1,1}G^{(8)}_{3\,2\,11\,1\,23} + \Delta_{x_1,2}G^{(8)}_{3\,2\,11\,1\,23} + (13)^*\Delta_{x_1,3}G^{(8)}_{3\,2\,11\,1\,23} + (123)^*\Delta_{x_1,4}G^{(8)}_{3\,2\,11\,1\,23}\big) \\
&+ \frac{1}{2}\Big(\sum_{r=1,2}(13)^*(\Delta_{y_2,r}G^{(8)}_{3\,11\,13\,1}) + \sum_{p=3,4}\Delta_{y_2,p}G^{(8)}_{3\,11\,13\,1} + \Delta_{y_2,1}G^{(8)}_{2\,11\,12\,1} \\
&+ (13)^*\Delta_{y_2,4}G^{(8)}_{2\,11\,12\,1}\Big) + \sum_{p=3,4}\Delta_{y_2,p}G^{(8)}_{123} + \sum_{q=2,3,4}(13)^*\Delta_{y_2,q}G^{(8)}_{132} + (13)^*\Delta_{y_2,q}G^{(8)}_{2\,1\,3\,2} \\
&+ \Delta_{y_2,3}G^{(8)}_{13\,13} + \frac{1}{4}\big(\Delta_{y_2,1}G^{(8)}_{\boxed{2}31} + (123)^*\Delta_{y_2,2}G^{(8)}_{\boxed{2}31} + (23)^*\Delta_{y_2,3}G^{(8)}_{\boxed{2}31} + (13)^*\Delta_{y_2,4}G^{(8)}_{\boxed{2}31}\big) \\
&+ (13)^*\Big[(13)^*\Delta_{x_3,1}G^{(8)}_{|\ominus|\,23\,|} + \Delta_{x_3,1}G^{(8)}_{132} + \Delta_{x_3,4}G^{(8)}_{321} + \frac{1}{2}\Big(\sum_{r=1,2}(13)^*(\Delta_{x_3,r}G^{(8)}_{2\,11\,12\,1}) \\
&+ \sum_{p=3,4}\Delta_{x_3,p}G^{(8)}_{2\,11\,12\,1} + \Delta_{x_3,1}G^{(8)}_{3\,11\,13\,1} + (13)^*\Delta_{x_3,4}G^{(8)}_{3\,11\,13\,1}\Big) \\
&+ \sum_{p=3,4}\Delta_{x_3,p}G^{(8)}_{132} + \sum_{q=2,3,4}(13)^*\Delta_{x_3,q}G^{(8)}_{123} + (13)^*\Delta_{x_3,q}G^{(8)}_{1\,3\,2\,1} + \Delta_{x_3,3}G^{(8)}_{13\,13} \\
&+ \frac{1}{4}\big(\Delta_{x_3,1}G^{(8)}_{\boxed{1}23} + (123)^*\Delta_{x_3,2}G^{(8)}_{\boxed{1}23} + (23)^*\Delta_{x_3,3}G^{(8)}_{\boxed{1}23} + (13)^*\Delta_{x_3,4}G^{(8)}_{\boxed{1}23}\big)\Big]\bigg\}(\mathbf{X}) \\
&- \sum_{b_1}\frac{1}{E(x_1,b_1)}\big[G^{(6)}_{23}(\mathbf{X}) - G^{(6)}_{23}(b_1,x_2,x_3;\mathbf{y};\mathbf{z})\big] \\
&- \sum_{b_2}\frac{1}{E(y_2,b_2)}\big[G^{(6)}_{23}(\mathbf{X}) - G^{(6)}_{23}(\mathbf{x};y_1,b_2,y_3;\mathbf{z})\big] \\
&- \sum_{b_3}\frac{1}{E(x_3,b_3)}\big[G^{(6)}_{23}(\mathbf{X}) - G^{(6)}_{23}(x_1,x_2,b_3;\mathbf{y};\mathbf{z})\big]\,.
\end{aligned}
$$

We count the graphs with $2k$ vertices for $k < 4$ in order to obtain free energy expansion until $\mathcal{O}(J^4, \bar{J}^4)$.

Figure 3 summarizes some properties of these graphs that the free energy expansion depends on. Although they had been enumerated, neither had they been identified nor their symmetry factors (the order of the coloured automorphism groups) found. We can now expand the free energy until sixth oder, which would in theory allow the computation of 4-point function's equations starting from (2.3). For the φ^4_{m}-theory, the sum is over all $\partial\,\mathsf{Feyn}_D(\varphi^4_{\mathsf{m}}) = \mathsf{Grph}^{\mathrm{II,cl}}_D$, as shown in [15]. For that model (and also for any other model containing those interaction vertices and thus the same boundary sector), the free energy $W_{D=4}[J, \bar{J}]$ to $\mathcal{O}(6)$ is then given by the following expansion, where $b = b_{a,c} = \min(\{1,2,3,4\} \setminus \{a,c\})$ and $d = d_{a,c} = \max(\{1,2,3,4\} \setminus \{a,c\})$ and $(i_1(a), i_2(a), i_3(a))$ is the ordered set of $\{1,2,3,4\} \setminus \{a\}$:



It is convenient to single a particular colour a we want to use the WTI for. Care has been taken in order to colour the graph's edges in non-redundant, but univocal way. In particular, edges are labelled strictly by the closest letter next to them.



$$+\frac{1}{2}\sum_{c\neq a} G^{(6)} \star \mathbb{J}\big(\big) + \sum_{c\neq a} G^{(6)} \star \mathbb{J}\big(\big)$$

$$+\sum_{c\neq a} G^{(6)} \star \mathbb{J}\big(\big) + \sum_{c\neq a} G^{(6)} \star \mathbb{J}\big(\big)$$

$$+\sum_{c\neq a} G^{(6)} \star \mathbb{J}\big(\big) + \sum_{c\neq a}\sum_{f=b,d} G^{(6)} \star \mathbb{J}\big(\big)$$

$$+\frac{1}{3}\sum_{c\neq a} G^{(6)} \star \mathbb{J}\big(\big) + \frac{1}{3} G^{(6)} \star \mathbb{J}\big(\big) + \frac{1}{3}\sum_{c\neq a} G^{(6)} \star \mathbb{J}\big(\big)$$

$$+\sum_{c\neq a}\sum_{f=b,d} G^{(6)} \star \mathbb{J}\big(\big) + \frac{1}{3}\sum_{c\neq a} G^{(6)} \star \mathbb{J}\big(\big) + \frac{1}{3}\sum_{c\neq a} G^{(6)} \star \mathbb{J}\big(\big)$$

$$+\sum_{c\neq a}\Big\{ G^{(6)} \star \mathbb{J}\big(\big)\Big\} + G^{(6)} \star \mathbb{J}\big(\big) + O(J^4, \bar{J}^4)$$

For the sequel, we adopt the notation of writing entries of $\mathbb{Z}^D$ as unordered sets, even though we mean them having a colour-ordering (by the subindices). Hence, the D-tuple $(q_{p_1}, q_{p_2}, \ldots, q_{p_{D-1}}, q_{p_D})$ actually means $(q_r, q_s, \ldots, q_t, q_u)$ where $r < s < \ldots < t < u$, being $\{r, s, \ldots, t, u\} = \{p_i\}_{i=1}^D$ as sets.

The simplified $Y^{(a)}_{m_a}$-term given by (2.7) and by the Ward Takahashi identity after taking the $(m_a n_a)$-entry of a generator of the a-th summand of $\mathsf{Lie}(\mathrm{U}(N)^D)$, reads then:

$$\sum\nolimits_{q_{i_1}, q_{i_2}, q_{i_3}} G^{(2)}(m_a, q_{i_1}, q_{i_2}, q_{i_3})$$

$$+\frac{1}{2}\Big\{ \sum_{s=1,2} \Delta_{m_a,s} G^{(4)} + \sum_{i=1}^{4}\sum_{s=1,2} \Delta_{m_a,s} G^{(4)} + \sum_{c\neq a}\sum_{s=1,2} \Delta_{m_a,s} G^{(4)} \Big\} \star \mathbb{J}( ) \tag{5.1}$$

$$+\Big\{ \frac{1}{3!}\sum_{r=1}^{3} \Delta_{m_a,r} G^{(6)} + \frac{1}{2}\sum_{i=1}^{4}\sum_{s=2,3} \Delta_{m_a,s} G^{(6)} + \frac{1}{2}\sum_{c\neq a}\Big[\sum_{s=2,3} \Delta_{m_a,s} G^{(6)} \tag{5.2}$$

$$+ \Delta_{m_a,2} G^{(6)} + \Delta_{m_a,2} G^{(6)} \Big]\Big\} \star \mathbb{J}( )$$

$$+\Big\{ \frac{1}{2}\Delta_{m_a,1} G^{(6)} + \frac{1}{3}\sum_{r=1}^{3} \Delta_{m_a,r} G^{(6)} + \sum_{c\neq a}\Big[\Delta_{m_a,1} G^{(6)} + \frac{1}{3}\sum_{r=1}^{3} \Delta_{m_a,r} G^{(6)}$$

$$+ \Delta_{m_a,2} G^{(6)} + \Delta_{m_a,3} G^{(6)} \Big]\Big\} \star \mathbb{J}( )$$

$$+\sum_{\alpha=1}^{3}\Big\{ \frac{1}{2}\Delta_{m_a,1} G^{(6)} + \Delta_{m_a,\alpha} G^{(6)} \Big\} \star \mathbb{J}( )$$

$$+\sum_{c\neq a}\Big\{\Big[\sum_{s=1,2} \Delta_{m_a,s} G^{(6)} + \Delta_{m_a,1} G^{(6)} + \frac{1}{3}\sum_{r=1}^{3} \Delta_{m_a,r} G^{(6)}$$

$$+ \Delta_{m_a,1} G^{(6)} + \Delta_{m_a,1} G^{(6)} \Big] \star \mathbb{J}( )$$

$$+\Big[\Delta_{m_a,3}G^{(6)} + \sum_{s=1,2}\Delta_{m_a,s}G^{(6)}\Big]\star\mathbb{J}\big(\ \big)$$

$$+\Big[\Delta_{m_a,1}G^{(6)} + \sum_{s=1,2}\Delta_{m_a,s}G^{(6)}\Big]\star\mathbb{J}\big(\ \big)\Big\}$$

$$+\sum_{c\neq a}\Big\{\Big[\frac{1}{2}\Delta_{m_a,1}G^{(6)} + \sum_{s=2,3}\Delta_{m_a,s}G^{(6)} + \Delta_{m_a,3}G^{(6)}$$

$$+\frac{1}{3}\sum_{r=1}^{3}\Delta_{m_a,r}G^{(6)} + \frac{1}{3}\sum_{r=1}^{3}\Delta_{m_a,r}G^{(6)} + \sum_{\ell=1,3}\Delta_{m_a,\ell}G^{(6)}\Big]\star\mathbb{J}\big(\ \big)$$

$$+\Big[\Delta_{m_a,3}G^{(6)} + \Delta_{m_a,2}G^{(6)} + \Delta_{m_a,3}G^{(6)}\Big]\star\mathbb{J}\big(\ \big)$$

$$+\Big[\Delta_{m_a,3}G^{(6)} + \Delta_{m_a,3}G^{(6)} + \Delta_{m_a,2}G^{(6)}\Big]\star\mathbb{J}\big(\ \big)\Big\} + O(J^3,\bar{J}^3)\ .$$

5.1. **Two-point equation for rank-4 theories.** We think it is instructive to derive directly, without using Proposition 3.1 the SDE for the 2-point function:

$$\begin{aligned}
G^{(2)}(\mathbf{a}) &= \frac{1}{Z_0}\left\{\frac{\delta}{\delta J_{\mathbf{a}}}\left[\exp\big(-V(\delta/\delta\bar{J},\delta/\delta J)\big)\,\frac{1}{E_{\mathbf{a}}}J_{\mathbf{a}}\mathrm{e}^{\sum_{\mathbf{q}}\bar{J}_{\mathbf{q}}E_{\mathbf{q}}^{-1}J_{\mathbf{q}}}\right]\right\}_{J=\bar{J}=0} \\
&= \frac{1}{Z_0E_{\mathbf{a}}}\left[\exp\big(-V(\delta/\delta\bar{J},\delta/\delta J)\big)\,\mathrm{e}^{\sum_{\mathbf{q}}\bar{J}_{\mathbf{q}}E_{\mathbf{q}}^{-1}J_{\mathbf{q}}}\right]_{J=\bar{J}=0} \\
&\quad+\frac{1}{Z_0E_{\mathbf{a}}}\left(\exp\big(-V(\delta/\delta\bar{J},\delta/\delta J)\big)\,J_{\mathbf{a}}\frac{\delta}{\delta J_{\mathbf{a}}}\mathrm{e}^{\sum_{\mathbf{q}}\bar{J}_{\mathbf{q}}E_{\mathbf{q}}^{-1}J_{\mathbf{q}}}\right)_{J=\bar{J}=0} \\
&= \frac{1}{E_{\mathbf{a}}} + \frac{1}{Z_0}\frac{1}{E_{\mathbf{a}}}\left(\bar{\varphi}_{\mathbf{a}}\frac{\partial}{\partial\bar{\varphi}_{\mathbf{a}}}\big(-V(\varphi,\bar{\varphi})\big)\right)_{\varphi^\flat\to\delta/\delta J^\sharp}Z[J,\bar{J}]\Bigg|_{J=\bar{J}=0}\ ,
\end{aligned}\tag{5.3}$$

$$\begin{aligned}
&\bar{\varphi}_{\mathbf{x}}\frac{\partial(-V(\varphi,\bar{\varphi}))}{\partial\bar{\varphi}_{\mathbf{x}}}\Bigg|_{\varphi^\flat\to\delta/\delta J^\sharp} \\
&= -2\lambda\Bigg\{\frac{\delta}{\delta J_{x_1x_2x_3x_4}}\sum_{y_1}\frac{\delta}{\delta\bar{J}_{y_1x_2x_3x_4}}\sum_{y_2,y_3,y_4}\frac{\delta}{\delta J_{y_1y_2y_3y_4}}\frac{\delta}{\delta\bar{J}_{x_1y_2y_3y_4}} \qquad (\mathbf{x}\in\mathbb{Z}^4) \\
&\quad+\frac{\delta}{\delta J_{x_1x_2x_3x_4}}\sum_{y_2}\frac{\delta}{\delta\bar{J}_{x_1y_2x_3x_4}}\sum_{y_1,y_3,y_4}\frac{\delta}{\delta J_{y_1y_2y_3y_4}}\frac{\delta}{\delta\bar{J}_{y_1x_2y_3y_4}} \\
&\quad+\frac{\delta}{\delta J_{x_1x_2x_3x_4}}\sum_{y_3}\frac{\delta}{\delta\bar{J}_{x_1x_2y_3x_4}}\sum_{y_1,y_2,y_4}\frac{\delta}{\delta J_{y_1y_2y_3y_4}}\frac{\delta}{\delta\bar{J}_{y_1y_2x_3y_4}} \\
&\quad+\frac{\delta}{\delta J_{x_1x_2x_3x_4}}\sum_{y_4}\frac{\delta}{\delta\bar{J}_{x_1x_2x_3y_4}}\sum_{y_1,y_2,y_3}\frac{\delta}{\delta J_{y_1y_2y_3y_4}}\frac{\delta}{\delta\bar{J}_{y_1y_2y_3x_4}}\Bigg\}Z[J,\bar{J}]\Bigg|_{J=\bar{J}=0}\ .
\end{aligned}$$

One uses the WTI for the double derivatives of the form

$$\sum_{y_2,y_3,y_4}\frac{\delta^2Z[J,\bar{J}]}{\delta J_{y_1y_2y_3y_4}\delta\bar{J}_{x_1y_2y_3y_4}}\ ,\ \dots\ ,\ \sum_{y_1,y_2,y_3}\frac{\delta^2Z[J,\bar{J}]}{\delta J_{y_1y_2y_3y_4}\delta\bar{J}_{y_1y_2y_3x_4}}\ .$$

Then

$$\bar{\varphi}_{\mathbf{x}}\frac{\partial(-V(\varphi,\bar{\varphi}))}{\partial\bar{\varphi}_{\mathbf{x}}}\Bigg|_{\varphi^\flat\to\delta/\delta J^\sharp}$$

$$
= -2\lambda Z_0 \Bigg\{ \sum_{a=1}^{4} \Bigg[\frac{\delta^2 Y^{(a)}_{x_a}[J,\bar J]}{\delta J_{\mathbf{x}} \delta \bar J_{\mathbf{x}}} \Bigg|_{J=\bar J=0} + Y^{(a)}_{x_a}[0,0] \cdot G^{(2)}_{\ominus}(\mathbf{x}) \tag{5.4}
$$
$$
- \sum_{y_a} \frac{1}{|x_a|^2 - |y_a|^2} \big(G^{(2)}_{\ominus}(\mathbf{x}) - G^{(2)}_{\ominus}(y_a, x_{i_1(a)}, x_{i_2(a)}, x_{i_3(a)}) \big) \Bigg] \Bigg\} .
$$

Recall that (q_i, q_j, q_k, q_l) implies an ordering of the entries, that is, reordering so that q_s appears to the left of q_r if and only if $s < r$, $s, r \in \{i, j, k, l\} = \{1, 2, 3, 4\}$. Twice the double derivative appearing there, $2\delta^2 Y^{(a)}_{x_a}[J, \bar J]/\delta J_{\mathbf{x}} \delta \bar J_{\mathbf{x}}$, is given by

$$
\sum_{q_{i_1(a)}, q_{i_2(a)}, q_{i_3(a)}} \Big(G^{(4)}_{|\ominus|\ominus|}(x_a, q_{i_1(a)}, q_{i_2(a)}, q_{i_3(a)}; \mathbf{x}) + G^{(4)}_{|\ominus|\ominus|}(\mathbf{x}; x_a, q_{i_1(a)}, q_{i_2(a)}, q_{i_3(a)}) \Big)
$$
$$
+ \sum_{c \neq a} \sum_{q_{b(a,c)}, q_{d(a,c)}} \Big(G^{(4)}_{\boxed{c}}(x_a, x_c, q_b, q_d; \mathbf{x}) + G^{(4)}_{\boxed{c}}(\mathbf{x}; x_a, x_c, q_b, q_d) \Big)
$$
$$
+ 2G^{(4)}_{\boxed{a}}(\mathbf{x}; \mathbf{x}) + \sum_{c \neq a} \sum_{q_c} \Big(G^{(4)}_{{}_a\boxed{c}}(x_a, x_b, q_c, x_d; \mathbf{x}) + G^{(4)}_{{}_a\boxed{c}}(\mathbf{x}; x_a, x_b, q_c, x_d) \Big) .
$$

Thus, since $Y^{(a)}_{m_a}[0, \bar 0] = \sum_{q_{i_1}, q_{i_2}, q_{i_3}} G^{(2)}_{\ominus}(m_a, q_{i_1}, q_{i_2}, q_{i_3})$, one has

$$
G^{(2)}_{\ominus}(\mathbf{x}) = \frac{1}{E_{\mathbf{x}}} + \frac{1}{Z_0} \frac{1}{E_{\mathbf{x}}} \left(\bar\varphi_{\mathbf{x}} \frac{\partial}{\partial \bar\varphi_{\mathbf{x}}} (-V(\varphi, \bar\varphi)) \right)_{\varphi^\flat \to \delta/\delta J^\sharp} Z[J, \bar J] \Bigg|_{J=\bar J=0}
$$
$$
= \frac{1}{E_{\mathbf{x}}} + \frac{(-\lambda)}{E_{\mathbf{x}}} \Bigg\{ \sum_{a=1}^{4} \Bigg[2 \cdot G^{(2)}_{\ominus}(\mathbf{x}) \cdot \Big(\sum_{q_{i_1(a)}} \sum_{q_{i_2(a)}} \sum_{q_{i_3(a)}} G^{(2)}_{\ominus}(x_a, q_{i_1(a)}, q_{i_2(a)}, q_{i_3(a)}) \Big)
$$
$$
+ \sum_{q_{i_1(a)}} \sum_{q_{i_2(a)}} \sum_{q_{i_3(a)}} \Big(G^{(4)}_{|\ominus|\ominus|}(x_a, q_{i_1(a)}, q_{i_2(a)}, q_{i_3(a)}; \mathbf{x})
$$
$$
+ G^{(4)}_{|\ominus|\ominus|}(\mathbf{x}; x_a, q_{i_1(a)}, q_{i_2(a)}, q_{i_3(a)}) \Big)
$$
$$
+ \sum_{c \neq a} \sum_{q_{b(a,c)}} \sum_{q_{d(a,c)}} \Big(G^{(4)}_{\boxed{c}}(x_a, x_c, q_b, q_d; \mathbf{x}) + G^{(4)}_{\boxed{c}}(\mathbf{x}; x_a, x_c, q_b, q_d) \Big)
$$
$$
+ \sum_{c \neq a} \sum_{q_c} \Big(G^{(4)}_{{}_a\boxed{c}}(x_a, x_b, q_c, x_d; \mathbf{x}) + G^{(4)}_{{}_a\boxed{c}}(\mathbf{x}; x_a, x_b, q_c, x_d) \Big) + 2G^{(4)}_{\boxed{a}}(\mathbf{x}; \mathbf{x})
$$
$$
- \sum_{y_a} \frac{2}{|x_a|^2 - |y_a|^2} \big(G^{(2)}_{\ominus}(\mathbf{x}) - G^{(2)}_{\ominus}(y_a, x_{i_1(a)}, x_{i_2(a)}, x_{i_3(a)}) \big) \Bigg] \Bigg\}
$$

5.2. **Four-point equation for $G^{(4)}_{{}_1\boxed{}_1}$ in rank-4 theories.** Since $\mathcal{V}_1$ has $\mathbb{Z}_2$ as automorphism group, according to Proposition 3.1, the equation satisfied by $G^{(4)}_{{}_1\boxed{}_1}$ is the following:

$$
\Big(1 + \frac{2\lambda}{E_{x_1, y_2, y_3, y_4}} \sum_{a=1}^{4} \sum_{\mathbf{q}_{\hat a}} G^{(2)}_{\ominus}(s_a, \mathbf{q}_{\hat a}) \Big) G^{(4)}_{{}_1\boxed{}_1}(\mathbf{X}) \tag{5.5}
$$
$$
= \frac{(-2\lambda)}{E_{\mathbf{s}}} \sum_{a=1}^{4} \Bigg\{ \sum_{\hat\sigma \in \mathbb{Z}_2} \sigma^* \mathfrak{f}^{(a)}_{{}_1\boxed{}_1}(\mathbf{X}) + \sum_{\rho > 1} \frac{Z_0^{-1}}{E(y_a^\rho, s_a)} \frac{\partial Z[J, J]}{\partial \varsigma_a({}_1\boxed{}_1; 1, \rho)(\mathbf{X})}
$$
$$
- \sum_{b_a} \frac{1}{E(s_a, b_a)} \big[G^{(4)}_{{}_1\boxed{}_1}(\mathbf{X}) - G^{(4)}_{{}_1\boxed{}_1}(\mathbf{X}|_{s_a \to b_a}) \big] \Bigg\}
$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^4$, $\mathbf{X} = (\mathbf{x}, \mathbf{y})$, and $\mathbf{s} = (x_1, y_2, y_3, y_4)$. We write down the terms in the first line of the RHS. The trivial part is

$$\sum_{a=1}^{4} \frac{Z_0^{-1}}{E(y_a^2, s_a)} \frac{\partial Z[J,J]}{\partial \varsigma_a(\;;1,2)(\mathbf{X})} = \frac{1}{E(y_1,x_1)}\big(G^{(4)}(\mathbf{X}) + G^{(2)}(\mathbf{x}) \cdot G^{(2)}(\mathbf{y})\big) + \sum_{c\neq 1} \frac{1}{E(x_c, y_c)} G^{(4)}(\mathbf{X})$$

Less so is to find $\sum_a \mathfrak{f}^{(a)}$. The contributions to $\mathfrak{f}^{(a)}$, for fixed colour a, are all functions occurring in front of a $\mathbb{J}(\;)$-source term. These functions come from coefficients of the following source terms in (5.1) $\mathbb{J}(\;)$ and $\mathbb{J}(\;)$ for the values[6] when $a = 1$ or $c = 1$ (in the sum over c), but also form the following values:

$$\mathbb{J}(\;),\ \text{for } a \in \{2,3,4\};\ \mathbb{J}(\;),\ \text{for } (a,c) \in \{(2,3),(2,4),(3,2),(3,4),(4,2),(4,3)\};$$
$$\text{none from } \mathbb{J}(\;)\,.$$

Hence

$$\mathfrak{f}^{(1)} = \Big\{\frac{1}{2}\Delta_{x_1,1}G^{(6)} + \frac{1}{3}\sum_{r=1}^{3}\Delta_{x_1,r}G^{(6)} + \sum_{c\neq a}\Big[\Delta_{x_1,1}G^{(6)} + \frac{1}{3}\sum_{r=1}^{3}\Delta_{x_1,r}G^{(6)} + \Delta_{x_1,2}G^{(6)} + \Delta_{x_1,3}G^{(6)}\Big]\Big\}, \tag{5.6}$$

$$\mathfrak{f}^{(2)} = \sum_{s=1,2}\Delta_{y_2,s}G^{(6)} + \Delta_{y_2,1}G^{(6)} + \frac{1}{3}\sum_{r=1}^{3}\Delta_{y_2,r}G^{(6)} + \Delta_{y_2,1}G^{(6)} + \Delta_{y_2,1}G^{(6)} + \frac{1}{2}\Delta_{y_2,1}G^{(6)} + \Delta_{y_2,\alpha}G^{(6)} + \sum_{c=3,4}\Big[\Delta_{y_2,3}G^{(6)} + \sum_{s=1,2}\Delta_{y_2,s}G^{(6)}\Big], \tag{5.7}$$

$$\mathfrak{f}^{(3)} = \sum_{s=1,2}\Delta_{y_3,s}G^{(6)} + \Delta_{y_3,1}G^{(6)} + \frac{1}{3}\sum_{r=1}^{3}\Delta_{y_3,r}G^{(6)} + \Delta_{y_3,1}G^{(6)} + \Delta_{y_3,1}G^{(6)} + \frac{1}{2}\Delta_{y_3,1}G^{(6)} + \Delta_{y_3,\alpha}G^{(6)} + \sum_{c=2,4}\Big[\Delta_{y_3,3}G^{(6)} + \sum_{s=1,2}\Delta_{y_3,s}G^{(6)}\Big], \tag{5.8}$$

$$\mathfrak{f}^{(4)} = \sum_{s=1,2}\Delta_{y_4,s}G^{(6)} + \Delta_{y_4,1}G^{(6)} + \frac{1}{3}\sum_{r=1}^{3}\Delta_{y_4,r}G^{(6)} + \Delta_{y_4,1}G^{(6)} + \Delta_{y_4,1}G^{(6)} + \frac{1}{2}\Delta_{y_4,1}G^{(6)} + \Delta_{y_4,\alpha}G^{(6)} + \sum_{c=2,3}\Big[\Delta_{y_4,3}G^{(6)} + \sum_{s=1,2}\Delta_{y_4,s}G^{(6)}\Big]. \tag{5.9}$$

One inserts the sum of these four terms in equation (5.5).

[6] Recall that c ($c \neq a$) in that expansion (5.1) is seen as running variable, while $b < d$ are defined in terms of a and c by $\{a,b,c,d\} = \{1,2,3,4\}$. Also $i_1(a) < i_2(a) < i_3(a)$, and $\{i_1, i_2, i_3, a\} = \{1,2,3,4\}$.

5.3. **Four-point equation for $G^{(4)}_{1\,2}$ in rank-4 theories.** In order to get the equation for $G^{(4)}_{1\,2}$, we calculate first $\sum_a \mathfrak{f}^{(a)}_{1\,2}$.

$$\mathfrak{f}^{(1)}_{1\,2} = \Big[\frac{1}{2}\Delta_{x_1,1} G^{(6)}_{|\ominus|\,1\,2\,|} + \sum_{s=2,3}\Delta_{x_1,s} G^{(6)}_{2\,1} + \Delta_{x_1,3} G^{(6)}_{1\,1\,2} \tag{5.10a}$$

$$+ \frac{1}{3}\sum_{r=1}^{3}\Delta_{x_1,r} G^{(6)}_{1\,2\,2\,1} + \frac{1}{3}\sum_{r=1}^{3}\Delta_{x_1,r} G^{(6)}_{2\,2\,2} + \sum_{\ell=1,3}\Delta_{x_1,\ell} G^{(6)}_{2\,1\,1\,2} \Big]$$

$$+ \sum\nolimits_{c=3,4}\Big[\Delta_{x_1,3} G^{(6)}_{c\,1\,c\,d\,1} + \Delta_{x_1,2} G^{(6)}_{1\,d\,1\,1\,c} + \Delta_{x_1,3} G^{(6)}_{1\,b\,1\,1\,c}\Big],$$

$$\mathfrak{f}^{(2)}_{1\,2} = \Big[\frac{1}{2}\Delta_{x_2,1} G^{(6)}_{|\ominus|\,1\,2\,|} + \sum_{s=2,3}\Delta_{x_2,s} G^{(6)}_{1\,2} + \Delta_{x_2,3} G^{(6)}_{2\,2\,1} \tag{5.10b}$$

$$+ \frac{1}{3}\sum_{r=1}^{3}\Delta_{x_2,r} G^{(6)}_{1\,2\,2\,1} + \frac{1}{3}\sum_{r=1}^{3}\Delta_{x_2,r} G^{(6)}_{1\,1\,1} + \sum_{\ell=1,3}\Delta_{x_2,\ell} G^{(6)}_{1\,2\,2\,1} \Big]$$

$$+ \sum\nolimits_{c=3,4}\Big[\Delta_{x_2,3} G^{(6)}_{c\,2\,c\,d\,2} + \Delta_{x_2,2} G^{(6)}_{2\,d\,2\,2\,c} + \Delta_{x_2,3} G^{(6)}_{2\,b\,2\,2\,c}\Big],$$

$$\mathfrak{f}^{(3)}_{1\,2} = \Big[\frac{1}{2}\Delta_{y_3,1} G^{(6)}_{|\ominus|\,3\,4\,|} + \sum_{s=2,3}\Delta_{y_3,s} G^{(6)}_{4\,3} + \Delta_{y_3,3} G^{(6)}_{3\,3\,4} \tag{5.10c}$$

$$+ \frac{1}{3}\sum_{r=1}^{3}\Delta_{y_3,r} G^{(6)}_{3\,4\,4\,3} + \frac{1}{3}\sum_{r=1}^{3}\Delta_{y_3,r} G^{(6)}_{4\,4\,4} + \sum_{\ell=1,3}\Delta_{y_3,\ell} G^{(6)}_{3\,4\,4\,3} \Big]$$

$$+ \sum\nolimits_{c=2,4}\Big[\Delta_{y_3,3} G^{(6)}_{c\,3\,c\,b\,3} + \Delta_{y_3,3} G^{(6)}_{3\,d\,3\,3\,c} + \Delta_{y_3,2} G^{(6)}_{3\,b\,3\,3\,c}\Big],$$

$$\mathfrak{f}^{(4)}_{1\,2} = \Big[\frac{1}{2}\Delta_{y_4,1} G^{(6)}_{|\ominus|\,3\,4\,|} + \sum_{s=2,3}\Delta_{y_4,s} G^{(6)}_{3\,4} + \Delta_{y_4,3} G^{(6)}_{4\,4\,3} \tag{5.10d}$$

$$+ \frac{1}{3}\sum_{r=1}^{3}\Delta_{y_4,r} G^{(6)}_{3\,4\,4\,3} + \frac{1}{3}\sum_{r=1}^{3}\Delta_{y_4,r} G^{(6)}_{3\,3\,3} + \sum_{\ell=1,3}\Delta_{y_4,\ell} G^{(6)}_{4\,3\,3\,4} \Big].$$

$$+ \sum\nolimits_{c=2,3}\Big[\Delta_{y_4,3} G^{(6)}_{c\,4\,c\,b\,4} + \Delta_{y_4,3} G^{(6)}_{4\,d\,4\,4\,c} + \Delta_{y_4,2} G^{(6)}_{4\,b\,4\,4\,c}\Big]$$

The remaining terms from swapping edges are:

$$\sum_{a=1}^{4}\frac{Z_0^{-1}}{E(y_a^2,s_a)}\frac{\partial Z[J,\bar J]}{\partial\varsigma_a(1\,2\,;1,2)(\mathbf{X})} = \frac{1}{E(y_1,x_1)}G^{(4)}_{2\,2}(\mathbf{x},\mathbf{y}) + \frac{1}{E(y_2,x_2)}G^{(4)}_{1\,1}(\mathbf{x},\mathbf{y})$$
$$+ \frac{1}{E(x_3,y_3)}G^{(4)}_{4\,4}(\mathbf{x},\mathbf{y}) + \frac{1}{E(x_4,y_4)}G^{(4)}_{3\,3}(\mathbf{x},\mathbf{y}),$$

whence the SDE for $G^{(4)}_{1\,2}$ is

$$\Big(1+\frac{2\lambda}{E_{\mathbf{s}}}\sum_{a=1}^{4}\sum_{\mathbf{q}_{\hat a}} G^{(2)}_{\ominus}(s_a,\mathbf{q}_{\hat a})\Big)\cdot G^{(4)}_{1\,2}(\mathbf{X})$$

$$= \frac{(-2\lambda)}{E_{\mathbf{s}}}\sum_{a=1}^{4}\Bigg\{\sum_{\hat\sigma\in\mathbb{Z}_2}\sigma^*\mathfrak{f}^{(a)}_{1\,2}(\mathbf{X}) + \frac{1}{E(y_1,x_1)}G^{(4)}_{2\,2}(\mathbf{x},\mathbf{y}) + \frac{1}{E(y_2,x_2)}G^{(4)}_{1\,1}(\mathbf{x},\mathbf{y}) \tag{5.11}$$

$$+\frac{1}{E(x_3,y_3)}G^{(4)}_{4\square 4}(\mathbf{x},\mathbf{y})+\frac{1}{E(x_4,y_4)}G^{(4)}_{3\square 3}(\mathbf{x},\mathbf{y})-\sum_{b_a}\frac{1}{E(s_a,b_a)}\big[G^{(4)}_{1\square 2}(\mathbf{X})-G^{(4)}_{1\square 2}(\mathbf{X}|_{s_a\to b_a})\big]\Bigg\}$$

with $\mathbf{X}=(\mathbf{x},\mathbf{y})$ and $\mathbf{s}=(x_1,x_2,y_3,y_4)$ with the functions $\mathfrak{f}^{(a)}_{1\square 2}$ given by eqs. (5.10).

6. A simple quartic model

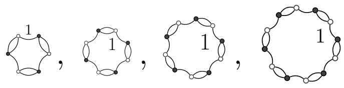
In order to obtain a simpler set of SDE equations, we consider a model which has less correlation functions. Its probability theory is expected to ponder only spherical geometries. Nevertheless it is interesting because its equations are particularly simple. We consider the rank-3 tensor model with action $S[\varphi,\bar\varphi]=S_0[\varphi,\bar\varphi]+V[\varphi,\bar\varphi]$ where

$$S_0[\varphi,\bar\varphi]=\mathrm{Tr}_2(\bar\varphi,E\varphi)=\sum\nolimits_{\mathbf{x}\in\mathbb{Z}^3}\bar\varphi_{\mathbf{x}}(m^2+|\mathbf{x}|^2)\varphi_{\mathbf{x}}\quad\text{and}\quad V[\varphi,\bar\varphi]=\lambda\cdot{}_1\square_1\,. \tag{6.1}$$

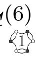
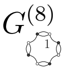
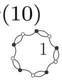
Here $|\mathbf{x}|^2=x_1^2+x_2^2+x_3^2$, $\mathbf{x}=(x_1,x_2,x_3)\in\mathbb{Z}^3$. In particular all the bordisms that this theory triangulates are null-bordisms and bordisms between spheres. Notice that the boundary graphs are all graphs having the following property: two edges are connected by a 2-coloured edge, if and only if they are connected by a 3-coloured edge. We denote by Θ ($\Theta\subset\mathsf{Grph}_3$) the set of *connected* graphs with this property. Thus the boundary sector $\mathrm{im}\partial_{{}_1\square_1}$ of this model is determined by

$$\partial\,\mathsf{Feyn}_3({}_1\square_1)=\{\mathcal{B}\in\mathsf{Grph}_3^{\amalg}:\mathcal{B}\text{ has connected components in }\Theta\}$$

being

$$\Theta=\big\{\ominus,\ {}_1\square_1\,,$$

$$,\dots\big\}\,.$$

Let $\mathcal{X}_{2k}$ be the graph in Θ with $2k$ vertices. That is to say, the set of correlation functions is precisely indexed by Θ and we set $G^{(2k)}:=G^{(2k)}_{\mathcal{X}_{2k}}$, i.e.

$$G^{(2)}=G^{(2)}_{\ominus},\ G^{(4)}=G^{(4)}_{{}_1\square_1},\ G^{(6)}=G^{(6)}_{\ \ },\ G^{(8)}=G^{(8)}_{\ \ },\ G^{(10)}=G^{(10)}_{\ \ }\,.$$




Any $(2k)$-point function with disconnected components can be labelled by integer partitions $(n_1,\dots,n_\ell)$ such that

$$\mathcal{B}=\mathcal{X}_2^{n_1}\sqcup\mathcal{X}_4^{n_1}\sqcup\ldots\sqcup\mathcal{X}_{2\ell}^{n_\ell}\,, \tag{6.2}$$

being ℓ the maximum number of vertices that a connected component of $\mathcal{B}$ has. These numbers n_i satisfy

$$k=\sum\nolimits_{i=1}^{\ell}i\cdot n_i\quad\text{and}\quad B=\sum\nolimits_{i=1}^{\ell}n_i\,, \tag{6.3}$$

where B is the number of connected components of $\mathcal{B}$. Then the free energy boils down to the expression

$$W[J,\bar J]=\sum_{l=1}^{\infty}\ \sideset{}{'}\sum_{\substack{\mathcal{B}\in\partial(\mathsf{Feyn}_3({}_1\square_1))\\ k(\mathcal{B})=l}}G^{(2l)}_{\mathcal{B}}\star\mathbb{J}(\mathcal{B})\,, \tag{6.4}$$

where the prime in the sum means that it is performed with the restrictions (6.3). More concretely, writing any graph $\mathcal{B}$ as in eq. (6.2), one can rephrase the sum rather over ℓ, the largest number of black (or white) vertices found in a connected component of $\mathcal{B}$. This modification readily yields

$$W[J,\bar J]=\sum_{\ell=1}^{\infty}\Bigg(\prod_{j=1}^{\ell}\frac{1}{j^{n_j}\cdot n_j!}\Bigg)G^{(2k)}_{|\mathcal{X}_2^{\sqcup n_1}|\ldots|\mathcal{X}_{2i}^{\sqcup n_i}\ldots|\mathcal{X}_{2\ell}^{\sqcup n_\ell}|}\star\mathbb{J}(\mathcal{X}_2^{n_1}\sqcup\mathcal{X}_4^{n_1}\sqcup\ldots\sqcup\mathcal{X}_{2\ell}^{n_\ell})\,.$$

To obtain the last line one observes that $\mathrm{Aut_c}(\mathcal{X}_{2k})=\langle\text{rotation by }2\pi/k\rangle=\mathbb{Z}_k$, and $|\mathrm{Aut_c}(\mathcal{B})|=n_1!\ldots n_\ell!\cdot|\mathrm{Aut_c}(\mathcal{X}_2)|^{n_1}\cdots|\mathrm{Aut_c}(\mathcal{X}_{2\ell})|^{n_\ell}$. It should be noticed that this form has already been found in the free energy expansion of (real) matrix models, here with twice the number of sources of each monomial with respect to that [6, Sec. 2.3]. It is also noteworthy that the Grosse-Wulkenhar model ($\varphi_4^{\star 4}$ self-dual theory) [6] was shown to be solvable by using matrix techniques. Here we have shown that the $_1\square_1$-model obeys the very same expansion of the free energy and that the number of $(2k)$-point functions of both theories is the same for any k.

The growth, as function of the number of vertices, of the number of correlation functions of this model is milder than that of the models with full boundary sector. We further simplify the notation and set $\mathfrak{f}_{2k,s_1} = \mathfrak{f}^{(1)}_{\mathcal{X}_{2k},s_1}$. With this notation, the Schwinger-Dyson equations in Section 3 can be derived for the connected boundary graphs of the $_1\square_1$-model.

Proposition 6.1 (Schwinger-Dyson equations for the $_1\square_1$-model)**.** *Let $\mathcal{B}$ be a connected boundary graph of the quartic model with $2k$ vertices ($k \geq 1$), $\mathcal{B} \in \mathsf{Feyn}_3(_1\square_1)$. Let $\mathbf{s} = \mathbf{y}^1$, where $(\mathcal{X}_{2k})_*(\mathbf{X}) = (\mathbf{y}^1, \dots, \mathbf{y}^k)$ for any $\mathbf{X} \in \mathcal{F}_{3,k}$. The $(2k)$-point Schwinger-Dyson equation corresponding to $\mathcal{B}$ is*

$$\begin{aligned}\Big(1 + \frac{2\lambda}{m^2 + |\mathbf{s}|^2} \sum_{q,p\,\in\mathbb{Z}} G^{(2)}(s_1,q,p)\Big) \cdot G^{(2k)}(\mathbf{X}) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6.5)\\ = \frac{2\lambda}{m^2+|\mathbf{s}|^2}\Bigg\{ \frac{\delta_{1,k}}{2\lambda} - \sum_{\hat\sigma\in\mathbb{Z}_k} \sigma^*\mathfrak{f}_{2k,s_1}(\mathbf{X}) - \sum_{\rho>1} \frac{Z_0^{-1}}{[(y_1^\rho)^2 - s_1^2]} \cdot \frac{\partial Z[J,J]}{\partial \varsigma_1(\mathcal{X}_{2k};1,\rho)(\mathbf{X})} \\ + \sum_{q\in\mathbb{Z}} \frac{1}{s_1^2 - q^2}\big[G^{(2k)}(\mathbf{X}) - G^{(2k)}(\mathbf{X}|_{s_1\to q})\big]\Bigg\}.\end{aligned}$$

Proof. For $k > 1$, it is immediate by setting $D = 3$ and by cutting the sums over the number of colours to only $a = 1$, since one does no longer have the vertices $_2\square_2$ and $_3\square_3$ in the action. After using $\mathrm{Aut_c}(\mathcal{X}_{2k}) = \mathbb{Z}_k$, and after inserting the form of the difference of propagators, as given by (6.1), the result follows. If $k = 1$, one additionally obtains the pure propagator term (the $\delta_{1,k}$-term) that would be otherwise annihilated by fourth or higher derivatives. For $k = 1$, the sum over ρ is empty (thus equal to zero). $\square$

One can still work out the functions $\mathfrak{f}_{2k}$ and give the correlation functions implied in the $\varsigma_1(\mathcal{X}_{2k};1,\rho)$-derivatives in eq. (6.5). Notice that the expansion of the term $Y^{(1)}_{s_1}[J,\bar J]$ is

$$Y^{(1)}_{s_1}[J,\bar J] = \sum_{k=0}^{\infty} \mathfrak{f}_{2k,s_1} \star \mathbb{J}(\mathcal{X}_{2k}) + \sum_{\mathcal{C}\text{ disconnected}} \mathfrak{f}^{(1)}_{\mathcal{C},s_1} \star \mathbb{J}(\mathcal{C}) \tag{6.6}$$

In order to determine $\mathfrak{f}_{2k,s_1}$ we find the graphs $\mathcal{B}$ such that $\mathcal{B} \ominus e_1^r = \mathcal{X}_{2k}$ for certain (say, the r-th) vertex of $\mathcal{B}$. The restrictions (6.3) with $B \geq 2$ and the connectedness of $\mathcal{B}$ after edge-removal imply that either

$$n_1 = n_k = 1 \quad \text{ and } \quad n_i = 0, \quad \text{if } i \neq 1, k\,,$$

or

$$n_{k+1} = 1 \quad \text{ and } \quad n_i = 0 \quad \text{ if } i \neq k+1\,.$$

That is to say, any such $\mathcal{B}$ has $2(k+1)$ vertices and, concretely, they might only be either $\ominus \sqcup \mathcal{X}_{2k}$ or $\mathcal{X}_{2k+2}$, when $k \geq 2$. Adding the obvious case when $k = 1$, one has:

$$\mathfrak{f}_{2,s_1} = \frac{1}{2}\sum\nolimits_{r=1}^{2} \big(\Delta_{s_1,r} G^{(4)}_{|\ominus|\ominus|} + \Delta_{s_1,r} G^{(4)}\big) \tag{6.7a}$$

$$\mathfrak{f}_{2k,s_1} = \frac{1}{k}\Delta_{s_1,1} G^{(2k+2)}_{|\ominus|\mathcal{X}_{2k}|} + \frac{1}{k+1}\sum_{r=1}^{k} \Delta_{s_1,r} G^{(2k+2)}, \text{ for } k \geq 2\,. \tag{6.7b}$$

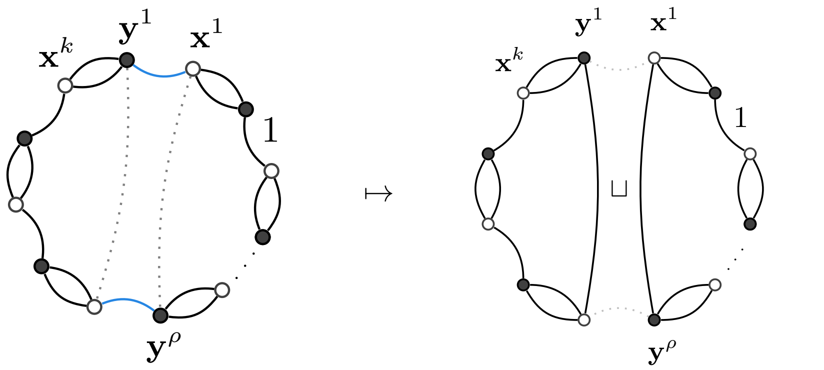


FIGURE 5. Shows the splitting of $\mathcal{X}_{2k}$ into the two components of $\varsigma_1(\mathcal{X}_{2k}; 1, \rho)$, $\rho > 0$

Notice that $\varsigma_1(\mathcal{X}_{2k}; 1, \rho) = \mathcal{X}_{2\rho-2} \sqcup \mathcal{X}_{2k-2\rho+2}$, whence (see Fig. 5)

$$\frac{1}{Z_0}\frac{\partial Z[J,J]}{\partial\varsigma_1(\mathcal{X}_{2k};1,\rho)(\mathbf{X})} = G^{(2\rho-2)}(\mathbf{x}^1,\ldots,\mathbf{x}^{\rho-1})\cdot G^{(2k-2\rho+2)}(\mathbf{x}^\rho,\ldots,\mathbf{x}^k) + G^{(2k)}_{|\mathcal{X}_{2(\rho-1)}|\mathcal{X}_{2k-2(\rho-1)}|}(\mathbf{X})\,. \tag{6.8}$$

Using the last four equations one can easily prove

Corollary 6.2. *The exact 2-point equation for the* $1\ominus1$*-model is given, for any* $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$, *by*

$$\Big(1 + \frac{2\lambda}{m^2+|\mathbf{x}|^2}\sum_{q,p\,\in\mathbb{Z}} G^{(2)}(x_1,q,p)\Big)\cdot G^{(2)}(\mathbf{x}) \tag{6.9}$$

$$= \frac{1}{m^2+|\mathbf{x}|^2} + \frac{(-2\lambda)}{m^2+|\mathbf{x}|^2}\Bigg\{\sum_{p,q\in\mathbb{Z}} G^{(4)}_{|\ominus|\ominus|}(x_1,q,p,\mathbf{x}) + G^{(4)}(\mathbf{x},\mathbf{x})$$

$$-\sum_{q\in\mathbb{Z}}\frac{1}{x_1^2-q^2}\big[G^{(2)}(x_1,x_2,x_3) - G^{(2)}(q,x_2,x_3)\big]\Bigg\}.$$

For $k \geq 2$, *the multipoint equation for* $G^{(2k)}$, *the single correlation function of connected boundary graph, is given by*

$$\Big(1 + \frac{2\lambda}{m^2+|\mathbf{s}|^2}\sum_{q,p\,\in\mathbb{Z}} G^{(2)}(x_1^1,q,p)\Big)\cdot G^{(2k)}(\mathbf{x}^1,\ldots,\mathbf{x}^k)$$

$$= \frac{(-2\lambda)}{m^2+|\mathbf{s}|^2}\Bigg\{\sum_{l=1}^k\Bigg[\frac{1}{k}\sum_{p,q\in\mathbb{Z}} G^{(2k+2)}_{|\ominus|\mathcal{X}_{2k}|}(x_1^1,q,p;\mathbf{x}^{1+l},\ldots,\mathbf{x}^{k+l}) \tag{6.10}$$

$$+\frac{1}{k+1}\sum_{r=1}^k G^{(2k+2)}(\mathbf{x}^{1+l},\mathbf{x}^{2+l},\ldots,\mathbf{x}^{r+l-1},x_1^1,x_2^{r+l-1},x_2^{r+l-1},\mathbf{x}^{r+l},\ldots,\mathbf{x}^{k+l})\Bigg]$$

$$+\sum_{\rho=2}^k\frac{1}{[(x_1^\rho)^2-(x_1^1)^2]}\Big(G^{(2\rho-2)}(\mathbf{x}^1,\ldots,\mathbf{x}^{\rho-1})\cdot G^{(2k-2\rho+2)}(\mathbf{x}^\rho,\ldots,\mathbf{x}^k)\Big)$$

$$-\sum_{q\in\mathbb{Z}}\frac{G^{(2k)}(x_1^1,x_2^1,x_3^1,\mathbf{x}^2,\ldots,\mathbf{x}^k) - G^{(2k)}(q,x_2^1,x_3^1,\mathbf{x}^2,\ldots,\mathbf{x}^k)}{(x_1^1)^2-q^2}\Bigg\}.$$

for $(\mathbf{x}^1,\ldots,\mathbf{x}^k) \in \mathcal{F}_{3,k}$, $\mathbf{s} := (x_1^1, x_2^r, x_3^r)$, *and* $\mathbf{x}^i = (x_1^i, x_2^i, x_3^i)$ *for all* $i \in \{1,\ldots,k\}$. *Moreover* $\mathbf{x}^j = \mathbf{x}^i \mod k$, *for and* $j \in \mathbb{N}$ *with* $i \in \{1,\ldots,k\}$.

It is pertinent to stress that $\mathbf{s} = \mathbf{y}^1$ is a “chosen” black vertex, and this equation holds for any other choice $\mathbf{s} = \mathbf{y}^i$, $i \neq 1$, $(\mathcal{X}_{2k})_*(\mathbf{X}) = (\mathbf{y}^1, \ldots, \mathbf{y}^k)$, after the pertinent changes (e.g. the sum over ρ excludes not 1 but i).

Proof. One uses the equations (6.7) and (6.8), the triviality of the automorphisms group $\mathrm{Aut_c}(\ominus)$, and the invariance of $G^{(4)}_{|\ominus|\ominus|}$ and $G^{(4)}$:

$$G^{(4)}_{|\ominus|\ominus|}(\mathbf{z}, \mathbf{y}) = G^{(4)}_{|\ominus|\ominus|}(\mathbf{y}, \mathbf{z}) \qquad \text{and} \qquad G^{(4)}(\mathbf{z}, \mathbf{y}) = G^{(4)}(\mathbf{y}, \mathbf{z}) \,.$$

This is enough to obtain the 2-point equation. For $k \geq 2$, on top of using (6.7) and (6.8) one explicitly writes the action of $\sigma \in \mathbb{Z}_k$. This is rotation by $2\pi l/k$, $1 \leq l \leq k$, so

$$\sigma^* f(\mathbf{X}) = f(\mathbf{x}^{\sigma^{-1}(1)}, \ldots, \mathbf{x}^{\sigma^{-1}(k)}) = f(\mathbf{x}^{1+l}, \ldots, \mathbf{x}^{k+l})$$

where $\mathbf{x}^j = \mathbf{x}^i \bmod k$, for $i \in \{1, \ldots, k\}$ and $j \in \mathbb{N}$ and f any (appropriate) function. $\square$

Remark 6.3. An analysis on the divergence degree as function of Gurău’s degree, and the boundary components was done in [2; 19] for group field theories. It turns out that graphs with a disconnected boundary are suppressed and therefore any graph contributing to $G^{(4)}_{|\ominus|\ominus|}$ is expected to be suppressed at least by N^{-1}, with respect to those summed in $G^{(4)}$. Also, by results of matrix theory [6], the term $G^{(4)}(\mathbf{x}, \mathbf{x})$ is expected to be analogously suppressed. Hence, conjecturally, for the 1 1-model, the leading order $G^{(2)}_{\mathrm{mel}}$ of the two-point function (6.9) should satisfy the clearly more simple closed equation

$$\Big(m^2 + |\mathbf{x}|^2 + 2\lambda \sum_{q,p\,\in\mathbb{Z}} G^{(2)}_{\mathrm{mel}}(x_1, q, p) \Big) \cdot G^{(2)}_{\mathrm{mel}}(\mathbf{x}) \tag{6.11}$$

$$= 1 + 2\lambda \sum_{q\in\mathbb{Z}} \frac{1}{x_1^2 - q^2} \big[G^{(2)}_{\mathrm{mel}}(x_1, x_2, x_3) - G^{(2)}_{\mathrm{mel}}(q, x_2, x_3) \big] \,.$$

and by the same token, one could truncate the equation for the $2k$-point function (6.10) to the following one, where the equally suppressed terms $\mathfrak{f}_{2k,s_1}$ also are suppressed:

$$\Big(1 + \frac{2\lambda}{m^2 + |\mathbf{s}|^2} \sum_{q,p\,\in\mathbb{Z}} G^{(2)}_{\mathrm{mel}}(x_1^1, q, p) \Big) \cdot G^{(2k)}_{\mathrm{mel}}(\mathbf{x}^1, \ldots, \mathbf{x}^k) \tag{6.12}$$

$$= \frac{(-2\lambda)}{m^2 + |\mathbf{s}|^2} \Bigg[\sum_{\rho=2}^{k} \frac{1}{[(x_1^\rho)^2 - (x_1^1)^2]} \Big(G^{(2\rho-2)}_{\mathrm{mel}}(\mathbf{x}^1, \ldots, \mathbf{x}^{\rho-1}) \cdot G^{(2k-2\rho+2)}_{\mathrm{mel}}(\mathbf{x}^\rho, \ldots, \mathbf{x}^k) \Big)$$

$$- \sum_{q\in\mathbb{Z}} \frac{G^{(2k)}_{\mathrm{mel}}(x_1^1, x_2^1, x_3^1, \mathbf{x}^2, \ldots, \mathbf{x}^k) - G^{(2k)}_{\mathrm{mel}}(q, x_2^1, x_3^1, \mathbf{x}^2, \ldots, \mathbf{x}^k)}{(x_1^1)^2 - q^2} \Bigg] \,.$$

We warn the reader that these relations —what we could call the *melonic limit* and corresponds to the planar limit in matrix models [6]— still must be carefully proven (see Sec. 8). It is very encouraging to see, though, that after determining $G^{(2)}_{\mathrm{mel}}$, the “melonic $2k$-point SDE” (6.12) for any $k > 1$ can now be entirely expressed in terms of already known functions $G^{(2)}_{\mathrm{mel}}, G^{(4)}_{\mathrm{mel}}, \ldots, G^{(2k-2)}_{\mathrm{mel}}$ and constitutes an equation only for $G^{(2k)}_{\mathrm{mel}}$, which would decouple the tower.

7. Outlook: Gurau-Witten SYK-like model

We believe that some of the present methods can be extended to the so-called Gurău-Witten model(s) based on work of Sachdev, Ye and Kitaev. We sketch here how.

Gurău-Witten model consists of fermions ψ^a that are tensorial of rank 3, transforming in the trifundamental representation of $G_{ab} \times G_{ac} \times G_{ad}$, where $\{a, b, c, d\} = \{0, 1, 2, 3\}$ and

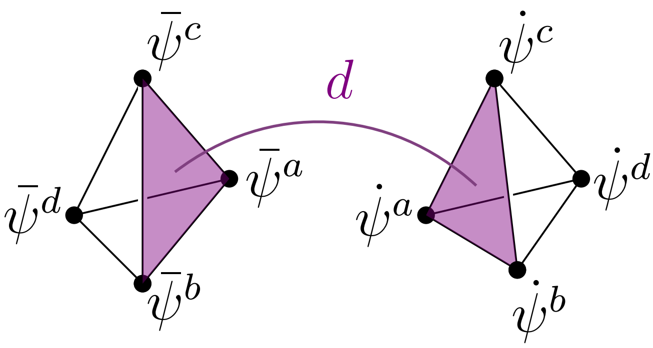


FIGURE 6. Gluing of tetrahedra caused by the Wick contraction $\bar{\psi}^d \dot{\psi}^d$, $\dot{\psi} = \mathrm{d}\bar{\psi}/\mathrm{d}\tau$

each $G_{ij} = G_{ji}$ is a copy of a Lie group, e.g. $\mathrm{O}(n)$. That is, in $(\psi^a)_{bcd}$, each subindex e, independently transforms under the fundamental representation of G_{ae}, $e \neq a$. The new integer n is related to the old number of sites N (fermions in the original SYK-model [12; 18]) by $N = 4n^3$. The quartic monomial in Witten's model is given by the $\mathrm{O}(n)$-*invariant*

$$S_{\text{int.}}[\,\{\psi^b\}_{b=0}^3\,] = -\eta(n) \sum_{\mu_i^j=1}^{n} (\psi^0)_{\mu_1^0\mu_2^0\mu_3^0}(\psi^1)_{\mu_0^1\mu_2^1\mu_3^1}(\psi^2)_{\mu_0^2\mu_1^2\mu_3^2}(\psi^3)_{\mu_0^3\mu_1^3\mu_2^3} \prod_{i\neq j} \delta_{\mu_j^i,\mu_i^j}\,, \tag{7.1}$$

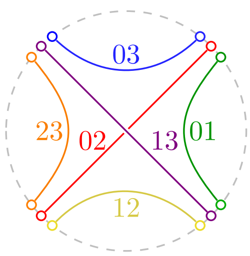
which is also abbreviated as $-\eta(n)\psi^0\psi^1\psi^2\psi^3$, being $\eta(n) = \eta_0 n^{-3/2}, \eta_0 \in \mathbb{R}$. With his action, the partition function is

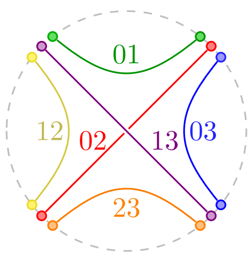
$$Z^{\mathbb{R}}_{\text{Gu.Wi.}}[\{J^a\}] = Z_0^{-1}\int \mathcal{D}\psi \exp\bigg(-\int \mathrm{d}\tau \sum\nolimits_b \frac{1}{2}\psi^b \frac{\mathrm{d}\psi^b}{\mathrm{d}\tau} + \eta(n)\psi^0\psi^1\psi^2\psi^3 + \sum\nolimits_b \psi^b J^b \bigg).$$

Here the propagator is the sum of the four quadratic $\mathrm{O}(n)$-invariants, thus, for each b $\psi^b\mathrm{d}\psi^b/\mathrm{d}\tau$ stands actually for $\sum_{\rho,\mu,\sigma=1}^n (\psi^b)_{\alpha\gamma\sigma}(\mathrm{d}\psi^b/\mathrm{d}\tau)_{\alpha\gamma\sigma}$. We use its complex version, and in that case the partition function $Z^{\mathbb{C}}_{\text{Gu.Wi.}}[\{J^a, \bar{J}^a\}]$ reads

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\bigg(-\int \mathrm{d}\tau \sum\nolimits_b \bar{\psi}_b \frac{\mathrm{d}\psi^b}{\mathrm{d}\tau} + \eta(n)(\psi^0\psi^1\psi^2\psi^3 + \bar{\psi}^3\bar{\psi}^2\bar{\psi}^1\bar{\psi}^0) + \sum\nolimits_b \bar{\psi}^b J^b + \bar{J}^b\psi^b \bigg).$$

The interaction-vertices of the last theory are graphically represented by

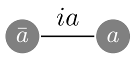
$\psi^0\psi^1\psi^2\psi^3 =$ 

and $\bar{\psi}^3\bar{\psi}^2\bar{\psi}^1\bar{\psi}^0 =$ 

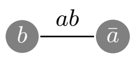
Triples of empty (resp. filled) dots marked by edges having the colour a in the bicolouration represent a the field ψ^a (resp. $\bar{\psi}^a$). These edges are the deltas in eq. (7.1). Also, there are four terms in the propagator (the four summands in the quadratic part; see the Feynman diagram shown in Fig. 7)

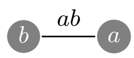
Each quartic interaction vertex can be seen as a tetrahedron with fields ψ^d at their vertices and with marked (coloured) faces, being ψ^d opposite to the face with colour d for each $d = 0, \dots, 3$. Thus, for the complex Gurău-Witten model, the Wick's contraction of $\dot{\psi}^d$ with $\bar{\psi}^d$ is, as in Figure 6, gluing the face coloured d (opposite to the vertex ψ^d in that figure), and any Feynman diagram results in certain simplicial complex, which, in case of having external legs, has a boundary consisting of coloured triangles. If the number of triangles of colour $A \in \{0, \dots, 3, \bar{0}, \dots, \bar{3}\}$ at the boundary is denoted by κ_A, a boundary graph consists then of $2k$ triangles and is specified not only by $(\kappa_0, \dots, \kappa_3, \kappa_{\bar{0}} \dots, \kappa_{\bar{3}}) \in \mathbb{Z}^8_{\geq 0}$ such that $2k = \sum_A \kappa_A$, but also by momentum transmission. To wit, *connected* boundary graphs with $2k$ external legs have the following properties:

- the vertex set is octo-partite in colours in the set $\mathbb{O} = \{0, 1, 2, 3, \bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. One writes $p : \mathbb{O} \to \{0, 1, 2, 3\}$, $p(A) = a$ if either $A = a$ or $A = \bar{a}$ We impose on the 'bar' operation $\bar{\cdot} : \mathbb{O} \to \mathbb{O}$ the property $\bar{\bar{A}} = A$ for each $A \in \mathbb{O}$.
- there are κ_a vertices of colour a and these numbers satisfy $2k = \sum_{A \in \mathbb{O}} \kappa_A$
- the edge set is bicoloured, that is every edge is labelled by one of the six elements in S, the set of unordered pairs of different colours in $\{0, 1, 2, 3\}$
- Let $A, B \in \mathbb{O}$ label two vertices of a graph. Either there is no edge joining them or they are connected by an edge which, according to previous point, bears a bi-colouration $\ell \in S$ and one constrained to the following possibilities:
  (i) $p(A) = p(B) =: a$ and $A = \bar{B}$, an in this case $\ell \in S$ can be any of $\{(ai) \,|\, i \neq a\}$



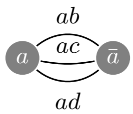
  (ii) $a := p(A) \neq p(B) =: b$. In this case, $\ell = (ab)$, that is



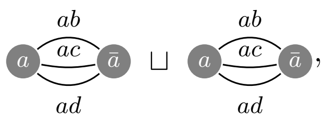
or


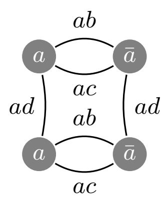
Disconnected boundary graphs have components that enjoy from all these properties. This gives the following classification of correlation functions.

- two-point functions: for $2k = 2$, the only possibilities is $\kappa_a = \kappa_{\bar{a}} = 1$ and $\kappa_B = 0$ for $B \in \mathbb{O} \setminus \{a, \bar{a}\}$. Thus there are four 2-point functions:



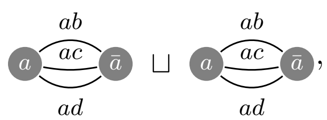
- four-point functions. One can have:
  – $\kappa_a = \kappa_{\bar{a}} = 2$ for certain a, otherwise $\kappa_B = 0$, $B \in \mathbb{O} \setminus p^{-1}(a)$;

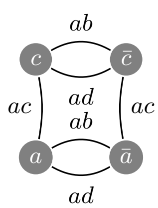


,


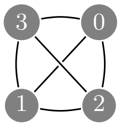
$(d \neq a)$,

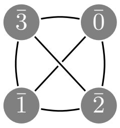
  which have been called 'broken' and 'unbroken' in [10].
  – or $\kappa_a = \kappa_c = \kappa_{\bar{a}} = \kappa_{\bar{c}} = 1$ and $\kappa_B = 0$ $B \in (\mathbb{O} \setminus p^{-1}\{a, c\})$,



,


– $\kappa_0 = \ldots = \kappa_3 = 1$ or $\kappa_{\bar{0}} = \ldots = \kappa_{\bar{3}} = 1$,

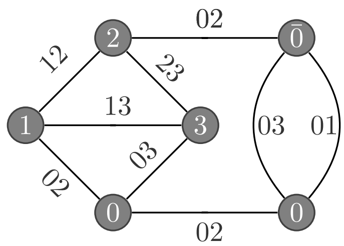


or


, respectively, called 'exceptional' in [10]

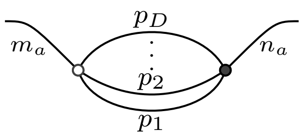
This matches for $k = 1, 2$ the description given in [10] for the real case, in a somehow more different notation than that of Fig. 3 there.

- six-point functions will be either classified by a disconnected boundary with components in the graphs that classify the 2-point and/or 4-point functions adding up to 6-vertices or they will be connected. This latter case needs a more complicated analysis to be fully classified. Examples of connected boundary graphs are the known ones for coloured tensor models with all the possible vertex-octo-colourations of the *melonic* graphs in six edges, but also many new graphs are possible, e.g.



A program to extend the present methods to the Gurău-Witten model begins

- with the association of a cycle of sources $J^c, \bar{J}^c$ for each boundary graph
- to expand the free energy of the model, $\log(Z^{\mathbb{C}}_{\text{Gu.Wi.}}[\{J^b, \bar{J}^b\}])$ in cycles of sources for each graph $\mathcal{B} \in \operatorname{im}\partial^{\mathbb{C}}_{\text{Gu.Wi.}}$ in the boundary sector of this model. Developing a graph-calculus and classify, modulo colour-orbits, multipoint functions
- the triviality of the Ward-Identity might be overcome by choosing a propagator that does depend on the group-variables, since this is a first order derivative, making "time" τ related to momenta $\boldsymbol{\mu}$ (e.g. replacing the propagator by a momentum-dependent propagator $\int \mathrm{d}\tau \sum_a \sum_{\boldsymbol{\mu}} \bar{\psi}^a(\tau)\dot{\psi}(\tau)\delta(\tau - f(\boldsymbol{\mu}))$, for which the WTI would be non-trivial)
- extend the SDE-techniques to interaction-vertices that, for any colour a, do not have the following subgraph



which was essential for the use of the WTI when deriving the SDEs.

8. Conclusions

We studied the correlation functions of coloured tensor models and, mainly, presented a collection of generating functionals that allowed to derive the exact Schwinger-Dyson equations for CTMs of rank 3 and 4 (and in Appendix B, rank-5 theories). The symmetry of the colours should be exploited in order to obtain a simplified version of them, which shall lead to a solution. The path towards closed equations, i.e. equations where a single unknown correlation function appears, is the analysis of Gurău degree ω sectors:

$$G^{(2k)}_{\mathcal{B}} = G^{(2k)}_{\mathcal{B},\text{mel}} + \sum\nolimits_{\omega=1}^{\infty} G^{(2k,\omega)}_{\mathcal{B}}, \qquad (G^{(2k)}_{\mathcal{B},\text{mel}} = G^{(2k,0)}_{\mathcal{B}}).$$

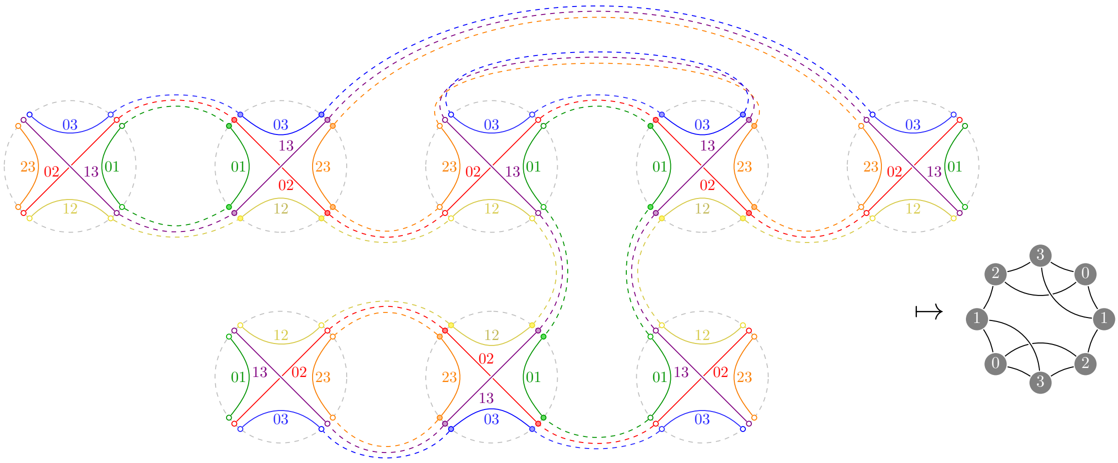


FIGURE 7. Example of a Feynman graph of the (complex) Gurău-Witten model that contributes to the correlation function indexed by the graph in the right, where any edge from a to b coloured vertices is ab-bicoloured (not always the case, as e.g. for $a = \bar{b}$ in another graphs).

This requires a rather deep combinatorial and topological analysis of the contractions of the external lines for tensor models, similar to the one undertaken in [5], and condensed in [6, Prop 3.3] for matrix models. This would prove the claims in remark 6.3 and will allow to find the analogous equations for sectors of any higher value of Gurău's degree $\omega > 1$. That result would provide insight on the solvability of the ${}_1\boxed{\ }_1$-model. We address both problems in a next paper.

Additionally, in Section 7, the boundary sector of the complex Gurău-Witten model has been characterized. This allowed us to write down there a plan that, we hope, would be useful to implement non-perturbative techniques for holographic tensor models.

APPENDIX A. PROOF OF LEMMA 4.1

Proof. The proof is long but straightforward. We compute some terms as a matter of example

$$\frac{1}{2!\cdot 2^2}\langle\!\langle G^{(8)}_{|\boxed{a}|\boxed{a}|}, \boxed{a} \sqcup \boxed{a} \rangle\!\rangle_{s_a} \tag{A.1a}$$

$$= \frac{1}{8}\Big(\sum_{r=1,2} \Delta_{s_a,r} G^{(8)}_{|\boxed{a}|\boxed{a}|} + \sum_{r=3,4} (123)^*(\Delta_{s_a,r} G^{(8)}_{|\boxed{a}|\boxed{a}|})\Big) \star \mathbb{J}(\ominus \sqcup \boxed{a})$$

$$\frac{1}{2!\cdot 2^2}\sum_{i\neq a}\langle\!\langle G^{(8)}_{|\boxed{i}|\boxed{i}|}, \boxed{i} \sqcup \boxed{i} \rangle\!\rangle_{s_a} \tag{A.1b}$$

$$= \frac{1}{8}\sum_{i\neq a}\Big(\sum_{r=1,2} \Delta_{s_a,r} G^{(8)}_{|\boxed{i}|\boxed{i}|} + \sum_{r=3,4} (123)^*\Delta_{s_a,r} G^{(8)}_{|\boxed{i}|\boxed{i}|}\Big) \star \mathbb{J}(\ominus \sqcup \boxed{i})$$

$$\frac{1}{2^2}\sum_{i\neq a}\langle\!\langle G^{(8)}_{|\boxed{i}|\boxed{a}|}, \boxed{i} \sqcup \boxed{a} \rangle\!\rangle_{s_a} \tag{A.1c}$$

$$= \frac{1}{4}\sum_{i\neq a}\Big(\big(\sum_{r=1,2} \Delta_{s_a,r} G^{(8)}_{|\boxed{i}|\boxed{a}|}\big) \star \mathbb{J}(\ominus \sqcup \boxed{a})$$

$$+ \big(\sum_{p=3,4} (123)^*\Delta_{s_a,p} G^{(8)}_{|\boxed{i}|\boxed{a}|}\big) \star \mathbb{J}(\ominus \sqcup \boxed{i})\Big)$$

$$\frac{1}{2^2}\langle\!\langle G^{(8)}_{|b|c|}, \; b \sqcup c \rangle\!\rangle_{s_a} \tag{A.1d}$$

$$= \frac{1}{4}\Big((\sum_{r=1,2} \Delta_{s_a,r} G^{(8)}_{|b|c|}) \star \mathbb{J}(\; \sqcup \; c) + (\sum_{p=3,4} (123)^* \Delta_{s_a,p} G^{(8)}_{|b|c|}) \star \mathbb{J}(\; \sqcup \; b) \Big)$$

$$\frac{1}{3}\Big\langle\!\!\Big\langle G^{(8)}_{|\;|\;|}, \; \sqcup \; \Big\rangle\!\!\Big\rangle_{s_a} \tag{A.1e}$$

$$= \frac{1}{3}\Delta_{s_a,1} G^{(8)}_{|\;|\;|} \star \mathbb{J}(\;) + \frac{1}{3}\sum_{q=2,3,4} \Delta_{s_a,q} G^{(8)}_{|\;|\;|} \star \mathbb{J}(\; \sqcup \; a)$$

$$\frac{1}{3}\sum_{i\neq a}\langle\!\langle G^{(8)}_{|\;|i|}, \; \sqcup \; i \rangle\!\rangle_{s_a} \tag{A.1f}$$

$$= \frac{1}{3}\sum_{i\neq a}\Big\{ \Delta_{s_a,1} G^{(8)}_{|\;|i|} \star \mathbb{J}(i) + \sum_{q=2,3,4} \Delta_{s_a,q} G^{(8)}_{|\;|i|} \star \mathbb{J}(\; \sqcup \; a) \Big\}$$

$$\frac{1}{3}\langle\!\langle G^{(8)}_{|\;|a|}, \; \sqcup \; a \rangle\!\rangle_{s_a} \tag{A.1g}$$

$$= \frac{1}{3}\Big\{ \Delta_{s_a,1} G^{(8)}_{|\;|a|} \star \mathbb{J}(a) + \sum_{q=2,3,4} \Delta_{s_a,q} G^{(8)}_{|\;|a|} \Big\} \star \mathbb{J}(\; \sqcup \; a)$$

$$\sum_{i\neq a}\langle\!\langle G^{(8)}_{|\;|i\,\overset{a}{}\,i|}, \; \sqcup \; \overset{a}{i} \rangle\!\rangle_{s_a} \tag{A.1h}$$

$$= \Delta_{s_a,1} G^{(8)}_{|\;|\underset{a\,b}{}|} \star \mathbb{J}(\overset{a\;\;b}{c}) + \sum_{h=2,3} \Delta_{s_a,h} G^{(8)}_{|\;|\underset{a\,b}{}|} \star \mathbb{J}(\; \sqcup \; b)$$

$$+ \Delta_{s_a,1} G^{(8)}_{|\;|\underset{a\,c}{}|} \star \mathbb{J}(\overset{a\;\;c}{b}) + \sum_{h=2,3} \Delta_{s_a,h} G^{(8)}_{|\;|\underset{a\,c}{}|} \star \mathbb{J}(\; \sqcup \; c)$$

$$+ (\Delta_{s_a,4} G^{(8)}_{|\;|\underset{a\,b}{}|} + \Delta_{s_a,4} G^{(8)}_{|\;|\underset{a\,c}{}|}) \star \mathbb{J}(\; \sqcup \; a)$$

$$\langle\!\langle G^{(8)}_{|\;|\underset{b\,c}{}|}, \; \sqcup \; \overset{b\;\;c}{a} \rangle\!\rangle_{s_a} \tag{A.1i}$$

$$= \Delta_{s_a,1} G^{(8)}_{|\;|\underset{b\,c}{}|} \star \mathbb{J}(\overset{b\;\;c}{a}) + \Delta_{s_a,2} G^{(8)}_{|\;|\underset{b\,c}{}|} \star \mathbb{J}(\; \sqcup \; c)$$

$$+ \Delta_{s_a,3} G^{(8)}_{|\;|\underset{b\,c}{}|} \star \mathbb{J}(\; \sqcup \; \sqcup \;) + \Delta_{s_a,4} G^{(8)}_{|\;|\underset{b\,c}{}|} \star \mathbb{J}(\; \sqcup \; b)$$

$$\frac{1}{2\cdot 2!}\sum_{i\neq a}\langle\!\langle G^{(8)}_{|\;|\;|i|}, \; \sqcup \; \sqcup \; i \rangle\!\rangle_{s_a} \tag{A.1j}$$

$$= \frac{1}{4}\Big\{ \sum\nolimits_{i\neq a} \sum_{r=1,2} \Delta_{s_a,r} G^{(8)}_{|\;|\;|i|} \star \mathbb{J}(\; \sqcup \; i) + \sum_{p=3,4} \Delta_{s_a,p} G^{(8)}_{|\;|\;|i|} \star \mathbb{J}(\;^{\sqcup 3}) \Big\}$$

$$\frac{1}{2\cdot 2!}\sum_{i\neq a}\langle\!\langle G^{(8)}_{|\;|\;|a|}, \; \sqcup \; \sqcup \; a \rangle\!\rangle_{s_a} \tag{A.1k}$$

$$= \frac{1}{4}\Big\{ \sum_{r=1,2} \Delta_{s_a,r} G^{(8)} \star \mathbb{J}(\sqcup) + \sum_{p=3,4} \Delta_{s_a,p} G^{(8)} \star \mathbb{J}(\sqcup \sqcup)\Big\}$$

$$\frac{1}{4!}\langle\!\langle G^{(8)} , \sqcup \sqcup \sqcup \rangle\!\rangle_{s_a} \tag{A.1l}$$

$$= \frac{1}{4!}\sum_{u=1}^{4}(\Delta_{s_a,u} G^{(8)}) \star \mathbb{J}(\sqcup \sqcup)$$

$$\langle\!\langle G^{(8)} , \rangle\!\rangle_{s_a} \tag{A.1m}$$

$$= \Delta_{s_a,1} G^{(8)} \star \mathbb{J}( ) + \Delta_{s_a,2} G^{(8)} \star \mathbb{J}( )$$

$$+ \Delta_{s_a,3} G^{(8)} \star \mathbb{J}( ) + \Delta_{s_a,4} G^{(8)} \star \mathbb{J}( )$$

$$= \Delta_{s_a,1} G^{(8)} \star \mathbb{J}( ) + \Delta_{s_a,2} G^{(8)} \star \mathbb{J}( )$$

$$+ \Delta_{s_a,3} G^{(8)} \star \mathbb{J}( ) + (13)^*(\Delta_{s_a,4} G^{(8)}) \star \mathbb{J}( )$$

$$\langle\!\langle G^{(8)} , \rangle\!\rangle_{s_a} \tag{A.1n}$$

$$= (13)^*(\Delta_{s_a,1} G^{(8)}) \star \mathbb{J}( ) + \Delta_{s_a,2} G^{(8)} \star \mathbb{J}(\sqcup)$$

$$+ [(13)^*(\Delta_{s_a,3} G^{(8)}) + (13)^*(\Delta_{s_a,4} G^{(8)})] \star \mathbb{J}( )$$

$$\langle\!\langle G^{(8)} , \rangle\!\rangle_{s_a}$$

$$= [(13)^*(\Delta_{s_a,1} G^{(8)}) + (13)^*(\Delta_{s_a,2} G^{(8)})] \star \mathbb{J}( )$$

$$+ (123)^*(\Delta_{s_a,3} G^{(8)}) \star \mathbb{J}(\sqcup) + \Delta_{s_a,4} G^{(8)} \star \mathbb{J}( )$$

$$\frac{1}{2}\sum_{\substack{i=b,c\\(a\neq j\neq i)}} \langle\!\langle G^{(8)} , \rangle\!\rangle_{s_a} \tag{A.1o}$$

$$= \frac{1}{2}\sum_{\substack{i=b,c\\(a\neq j\neq i)}} \Big[\sum_{r=1,2} \Delta_{s_a,r} G^{(8)} \star \mathbb{J}( ) + \sum_{p=3,4} \Delta_{s_a,p} G^{(8)} \star \mathbb{J}( )\Big]$$

$$= \frac{1}{2}\sum_{\substack{i=b,c\\(a\neq j\neq i)}} \Big[\Big(\sum_{r=1,2} (13)^*(\Delta_{s_a,r} G^{(8)}) + \sum_{p=3,4} \Delta_{s_a,p} G^{(8)}\Big) \star \mathbb{J}( )\Big]$$

$$= \frac{1}{2}\Big[\Big(\sum_{r=1,2} (13)^*(\Delta_{s_a,r} G^{(8)}) + \sum_{p=3,4} \Delta_{s_a,p} G^{(8)}\Big) \star \mathbb{J}( )$$

$$+\Big(\sum_{r=1,2}(13)^*(\Delta_{s_a,r}G^{(8)}_{\;})+\sum_{p=3,4}\Delta_{s_a,p}G^{(8)}_{\;}\Big)\star\mathbb{J}(\;)\Big]$$

$$\frac{1}{2}\sum_{\substack{i=b,c\\(a\neq j\neq i)}}\langle\!\langle G^{(8)}_{\;},\;\rangle\!\rangle_{s_a} \tag{A.1p}$$

$$=\frac{1}{2}\sum_{\substack{i=b,c\\(a\neq j\neq i)}}\Big[\big(\Delta_{s_a,1}G^{(8)}_{\;}\star\mathbb{J}(\;)+\Delta_{s_a,2}G^{(8)}_{\;}\star\mathbb{J}(\;\sqcup\;)$$

$$+\Delta_{s_a,3}G^{(8)}_{\;}\star\mathbb{J}(\;\sqcup\;)+\Delta_{s_a,4}G^{(8)}_{\;}\star\mathbb{J}(\;)\Big]$$

$$=\frac{1}{2}\Big[\big(\Delta_{s_a,1}G^{(8)}_{\;}\star\mathbb{J}(\;)+\Delta_{s_a,2}G^{(8)}_{\;}\star\mathbb{J}(\;\sqcup\;)$$

$$+\Delta_{s_a,3}G^{(8)}_{\;}\star\mathbb{J}(\;\sqcup\;)+\Delta_{s_a,4}G^{(8)}_{\;}\star\mathbb{J}(\;)\Big]$$

$$+\frac{1}{2}\Big[\big(\Delta_{s_a,1}G^{(8)}_{\;}\star\mathbb{J}(\;)+\Delta_{s_a,2}G^{(8)}_{\;}\star\mathbb{J}(\;\sqcup\;)$$

$$+\Delta_{s_a,3}G^{(8)}_{\;}\star\mathbb{J}(\;\sqcup\;)+\Delta_{s_a,4}G^{(8)}_{\;}\star\mathbb{J}(\;)\Big]$$

$$=\frac{1}{2}\Big[\Big\{\big(\Delta_{s_a,1}G^{(8)}_{\;}+(13)^*(\Delta_{s_a,1}G^{(8)}_{\;})$$

$$+(13)^*(\Delta_{s_a,4}G^{(8)}_{\;})+\Delta_{s_a,4}G^{(8)}_{\;}\Big\}\star\mathbb{J}(\;)$$

$$+\big(\Delta_{s_a,2}G^{(8)}_{\;}+(123)^*(\Delta_{s_a,3}G^{(8)}_{\;})\big)\star\mathbb{J}(\;\sqcup\;)$$

$$+\big(\Delta_{s_a,2}G^{(8)}_{\;}+(123)^*\big(\Delta_{s_a,3}G^{(8)}_{\;}\big)\big)\star\mathbb{J}(\;\sqcup\;)\Big]$$

$$\frac{1}{2}\sum_{\substack{i=b,c\\(a\neq j\neq i)}}\langle\!\langle G^{(8)}_{\;},\;\rangle\!\rangle_{s_a} \tag{A.1q}$$

$$=\frac{1}{2}\Big[\big\{\Delta_{s_a,1}G^{(8)}_{\;}+(13)^*\Delta_{s_a,4}G^{(8)}_{\;}\big\}\star\mathbb{J}(\;)$$

$$+\big\{\Delta_{s_a,1}G^{(8)}_{\;}+(13)^*\Delta_{s_a,4}G^{(8)}_{\;}\big\}\star\mathbb{J}(\;)$$

$$+\sum_{h=2,3}\big(\Delta_{s_a,h}G^{(8)}_{\;}\star\mathbb{J}(\;)+\Delta_{s_a,h}G^{(8)}_{\;}\star\mathbb{J}(\;)\big)\Big]$$

$$\frac{1}{4}\sum_{i\neq a}\langle\!\langle G^{(8)}_{\;},\;\rangle\!\rangle_{s_a}+\frac{1}{4}\langle\!\langle G^{(8)}_{\;},\;\rangle\!\rangle_{s_a} \tag{A.1r}$$

$$=\frac{1}{4}\sum_{i\neq a}\sum_{r=1}^{4}\Delta_{s_a,r}G^{(8)}_{\;}\star\mathbb{J}(\;)+\frac{1}{4}\sum_{r=1}^{4}\Delta_{s_a,r}G^{(8)}_{\;}\star\mathbb{J}(\;)$$

$$\sum_{i\neq a}\langle\!\langle G^{(8)}_{\;},\;\rangle\!\rangle_{s_a} \tag{A.1s}$$

$$= \Delta_{s_a,1} G^{(8)} \star \mathbb{J}(\;) + \Delta_{s_a,2} G^{(8)} \star \mathbb{J}(\; \sqcup \;)$$

$$+ \Delta_{s_a,1} G^{(8)} \star \mathbb{J}(\;) + \Delta_{s_a,2} G^{(8)} \star \mathbb{J}(\; \sqcup \;)$$

$$+ \sum_{p=3,4} \big(\Delta_{s_a,p} G^{(8)} + (13)^* \Delta_{s_a,p} G^{(8)} \big) \star \mathbb{J}(\;)$$

$$\sum_{i \neq a} \langle\!\langle G^{(8)} , \; \rangle\!\rangle_{s_a} \tag{A.1t}$$

$$= \sum_{r=1,2} \Delta_{s_a,r} G^{(8)} \star \mathbb{J}(\;) + \sum_{p=3,4} \Delta_{s_a,p} G^{(8)} \star \mathbb{J}(\;)$$

$$+ \sum_{r=1,2} \Delta_{s_a,r} G^{(8)} \star \mathbb{J}(\;) + \sum_{p=3,4} \Delta_{s_a,p} G^{(8)} \star \mathbb{J}(\;)$$

$$\textstyle\sum_{i,j \neq a; i \neq j} \langle\!\langle G^{(8)} , \; \rangle\!\rangle_{s_a} \tag{A.1u}$$

$$= (\Delta_{s_a,1} G^{(8)} + \Delta_{s_a,1} G^{(8)}) \star \mathbb{J}(\;)$$

$$+ \sum_{q=2,3,4} \big\{ (13)^* \Delta_{s_a,q} G^{(8)} \star \mathbb{J}(\;) + (13)^* \Delta_{s_a,q} G^{(8)} \star \mathbb{J}(\;) \big\}$$

$$\sum_{i \neq a} \langle\!\langle G^{(8)} , \; \rangle\!\rangle_{s_a} \tag{A.1v}$$

$$= (\Delta_{s_a,1} G^{(8)} + \Delta_{s_a,1} G^{(8)}) \star \mathbb{J}(\;)$$

$$+ \textstyle\sum_{q=2,3,4} (13)^* \Delta_{s_a,q} G^{(8)} \star \mathbb{J}(\;) + (13)^* \Delta_{s_a,q} G^{(8)} \star \mathbb{J}(\;)$$

$$\langle\!\langle G^{(8)} , \; \rangle\!\rangle_{s_a} \qquad \text{(symmetric in } b, c)$$

$$= \sum_{r=1,2} \Delta_{s_a,r} G^{(8)} \star \mathbb{J}(\;) + \sum_{p=3,4} \Delta_{s_a,p} G^{(8)} \star \mathbb{J}(\;) \tag{A.1w}$$

$$\left\langle\!\!\left\langle G^{(8)} , \; \right\rangle\!\!\right\rangle_{s_a} \tag{A.1x}$$

$$= \sum_{r=1,2} \Delta_{s_a,r} G^{(8)} \star \mathbb{J}\Big(\;\Big)$$

$$+ \Delta_{s_a,3} G^{(8)} \star \mathbb{J}(\;) + \Delta_{s_a,4} G^{(8)} \star \mathbb{J}\Big(\;\Big)$$

$$= \sum_{r=1,2} (13)^* \big(\Delta_{s_a,r} G^{(8)} \big) \star \mathbb{J}(\;) + \Delta_{s_a,3} G^{(8)} \star \mathbb{J}(\;)$$

$$+\sum_{p=3,4}(13)^*\big(\Delta_{s_a,p}G^{(8)}\big)\star\mathbb{J}(\ )$$

$$\frac{1}{4}\langle\!\langle G^{(8)}\ ,\ \ \rangle\!\rangle_{s_a} \tag{A.1y}$$

$$=\frac{1}{4}\big\{(23)^*\Delta_{s_a,1}G^{(8)}+\Delta_{s_a,2}G^{(8)}+(13)^*\Delta_{s_a,3}G^{(8)}$$

$$+(123)^*\Delta_{s_a,4}G^{(8)}\big\}\star\mathbb{J}(\ )$$

$$\frac{1}{4}\sum_{\substack{i=b,c;\\(i\neq j\neq a)}}\langle\!\langle G^{(8)}\ ,\ \ \rangle\!\rangle_{s_a} \tag{A.1z}$$

$$=\frac{1}{4}\Big[\big\{\Delta_{s_a,1}G^{(8)}+(123)^*\Delta_{s_a,2}G^{(8)}+(23)^*\Delta_{s_a,3}G^{(8)}$$

$$+(13)^*\Delta_{s_a,4}G^{(8)}\big\}\star\mathbb{J}(\ )$$

$$+\big\{\Delta_{s_a,1}G^{(8)}+(123)^*\Delta_{s_a,2}G^{(8)}+(23)^*\Delta_{s_a,3}G^{(8)}$$

$$+(13)^*\Delta_{s_a,4}G^{(8)}\big\}\star\mathbb{J}(\ )\Big]$$

$$\frac{1}{4}\langle\!\langle G^{(8)}\ ,\ \ \rangle\!\rangle_{s_a}=\frac{1}{4}\sum_{r=1}^{4}G^{(8)}\star\mathbb{J}(\ )\,. \tag{A.1aa}$$

One adds all the previous equations and associates by $\mathbb{J}(\mathcal{B})$, for $\mathcal{B}$ one of the 11 graphs with 6 vertices. □

Appendix B. Rank-five quartic theories

The generating function that enumerates the rank-5 connected boundary graphs (and interaction vertices) is the OEIS **A057007**:

$$Z_{\text{conn.},5}(x)=x+15x^2+235x^3+14120x^4+1712845x^5+371515454x^6+\dots$$

We will not classify the 235 connected graphs with six vertices, but, aiming only at obtaining the 2-point function's equation, we will compute the free energy up to $\mathcal{O}(J^3,\bar{J}^3)$:

$$W_{D=5}[J,\bar{J}]=G^{(2)}\star\mathbb{J}(\ )+\frac{1}{2!}G^{(4)}_{|\ |\ |}\star\mathbb{J}(\ |\ )$$

$$+\sum_{j=1}^{5}\frac{1}{2}G^{(4)}\star\mathbb{J}(\ )+\sum_{i<j}\frac{1}{2}G^{(4)}\star\mathbb{J}\Big(\ \Big)+O(J^3,\bar{J}^3)$$

$$=G^{(2)}\star\mathbb{J}(\ )+\frac{1}{2!}G^{(4)}_{|\ |\ |}\star\mathbb{J}(\ |\ )+\frac{1}{2}G^{(4)}\star\mathbb{J}(\ )$$

$$+\sum_{c\neq a}\frac{1}{2}G^{(4)}\star\mathbb{J}(\ )+\sum_{c\neq a}\frac{1}{2}G^{(4)}\star\mathbb{J}\Big(\ \Big)$$

$$+\sum_{\substack{c,d\neq a\\ c<d}} \frac{1}{2} G^{(4)} \star \mathbb{J}\left(\right) + O(J^3, \bar{J}^3)\,.$$

For an arbitrary model in rank 5 one has, up to $\mathcal{O}(J^2, \bar{J}^2)$-terms,

$$Y^{(a)}_{m_a}[J,\bar{J}] \sim \sum_{q_{i_1},\dots,q_{i_4}} G^{(2)}(m_a, q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4}) + \frac{1}{2}\Bigg\{ \sum_{s=1,2} \Delta_{m_a,s} G^{(4)}_{|\ |\ |} + \sum_{c\neq a}\sum_{s=1,2} \Delta_{m_a,s} G^{(4)}$$
$$+ \sum_{\substack{c,d\neq a\\ c<d}} \sum_{s=1,2} \Delta_{m_a,s} G^{(4)} + \sum_{c\neq a}\sum_{s=1,2} \Delta_{m_a,s} G^{(4)} + \sum_{s=1,2} \Delta_{m_a,s} G^{(4)} \Bigg\} \star \mathbb{J}(\ )\,.$$

One straightforwardly gets

$$G^{(2)}(\mathbf{x}) = \frac{1}{E_{\mathbf{x}}} + \frac{(-\lambda)}{E_{\mathbf{x}}} \Bigg\{ \sum_{a=1}^{5} \Bigg[2\cdot G^{(2)}(\mathbf{x}) \cdot \Big(\sum_{q_{i_1(a)}} \sum_{q_{i_2(a)}} \sum_{q_{i_3(a)}} \sum_{q_{i_4(a)}} G^{(2)}(x_a, q_{i_1(a)}, q_{i_2(a)}, q_{i_3(a)})\Big)$$
$$+ \sum_{q_{i_1(a)}} \sum_{q_{i_2(a)}} \sum_{q_{i_3(a)}} \sum_{q_{i_4(a)}} \Big(G^{(4)}_{|\ |\ |}(x_a, q_{i_1(a)}, q_{i_2(a)}, q_{i_3(a)}, q_{i_4(a)}; \mathbf{x})$$
$$+ G^{(4)}_{|\ |\ |}(\mathbf{x}; x_a, q_{i_1(a)}, q_{i_2(a)}, q_{i_3(a)}, q_{i_4(a)})\Big)$$
$$+ \sum_{c\neq a} \sum_{q_{b(a,c)}} \sum_{q_{d(a,c)}} \sum_{q_{e(a,c)}} \Big(G^{(4)}(x_a, x_c, q_b, q_d, q_e; \mathbf{x}) + G^{(4)}(\mathbf{x}; x_a, x_c, q_b, q_d, q_e)\Big)$$
$$+ \sum_{\substack{d,e\neq a\\ d<e}} \sum_{q_c} \sum_{q_b} \Big(G^{(4)}(x_a, x_b, x_c, q_d, q_e; \mathbf{x}) + G^{(4)}(\mathbf{x}; x_a, x_b, x_c, q_d, q_e)\Big)$$
$$+ \sum_{c\neq a} \sum_{q_c} \Big(G^{(4)}(x_a, x_b, q_c, x_d, x_e; \mathbf{x}) + G^{(4)}(\mathbf{x}; x_a, x_b, q_c, x_d, x_e)\Big) + 2G^{(4)}(\mathbf{x};\mathbf{x})$$
$$+ \sum_{y_a} \frac{2}{|x_a|^2 - |y_a|^2} \big(G^{(2)}(\mathbf{x}) - G^{(2)}(y_a, x_{i_1(a)}, x_{i_2(a)}, x_{i_3(a)}, x_{i_4(a)})\big) \Bigg] \Bigg\}\,.$$

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