Universal Peculiar Linear Mean Relationships in All Polynomials

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Abstract. In any cubic polynomial, the average of the slopes at the 3 roots is the negation of the slope at the average of the roots. In any quartic, the average of the slopes at the 4 roots is twice the negation of the slope at the average of the roots. We generalize such situations and present a procedure for determining all such relationships for polynomials of any degree. E.g., in any septic f, letting $\overline{f}_n$ denote the mean f value over all zeroes of the derivative $f^{(n)}$, it holds that $37\ \overline{f}_1 - 150\ \overline{f}_3 + 200\,\overline{f}_4 - 135\,\overline{f}_5 + 48\,\overline{f}_6 = 0$; and in any quartic it holds that $5\ \overline{f}_1 - 6\ \overline{f}_2 + 1\,\overline{f}_3 = 0$. Having calculated such relationships in all dimensions up to 40, in all even dimensions there is a single relationship, in all odd dimensions there is a two-dimensional family of relationships. We come upon connections to Tchebyshev, Bernoulli, & Euler polynomials, and Stirling numbers.

This paper is a spin-off from, and is material to, our recent *Insights Via Representational Naturality: New Surprises Intertwining Statistics, Cardano's Cubic Formula, Triangle Geometry, Polynomial Graphs* [**13**]. This paper started with (a) the realization that the quadratic formula essentially provides the roots to be $x = \mu \pm \sigma$ where μ is the mean of the two roots and where σ is the standard deviation of the two roots considered equally likely, (b) the realization that the slopes at the roots are $\pm\sqrt{\mathit{Discriminant}} = \pm 2\,a\sigma$, and (c) the trivial observation that the average of the slopes at the roots equals the slope at the average of the roots. Curiosity took us next to cubics, to find that the Cardano-Tartaglia cubic formula can be perceived as providing the roots to be $r_{_k} = \mathbb{E} + \omega^k\,\overrightarrow{T}_{_+} + \omega^{-\,k}\,\overrightarrow{T}_{_-}$ where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is a primitive cube root of unity, $k \in \{0,1,2\}$, and where $\overrightarrow{T}_{\pm} := \sqrt[3]{\frac{\mathbb{W}}{2} \pm \sqrt{\left(\frac{\mathbb{W}}{2}\right)^2 - \left(\frac{\mathbb{V}}{2}\right)^3}}$ with $\mathbb{V}$ being the variance of the 3 roots, $\mathbb{W}$ being the 3rd central moment (sometimes referred to as the *unscaled skewness*), and $\mathbb{E} \equiv \mu$ being the expectation of the roots, considered as being equally likely. Moreover, the coordinates of the cubic's inflection point are $(\mathbb{E},\,-a\,\mathbb{W})$, and the slope at the inflection point is the negation of the mean slope at the 3 roots. Indeed, the inflection point slope will be $-\frac{3}{2}\,a\mathbb{V}$. Next we saw that the mean slope at the 4 roots of a quartic is the negation of twice the slope at the point $(\mathbb{E},\,f\,(\mathbb{E}))\ldots$ and the hunt was on to find all such relationships! It was not just mean slope relationships either– e.g., one can of course note that the mean of a cubic's function values at the roots of the first derivative equals the 'mean' of the function's (sole) value at the root of the second derivative, viz. $-a\mathbb{W}$. What other relationships lie waiting to be discovered? Have we only seen the tip of an iceberg? Perhaps the reader is starting to suspect that such relationships are really typical.

1. THE GENERAL QUESTION We establish some notations. For any degree D polynomial f let $\mathcal{R} \equiv \mathcal{R}_{_f}$ denote the family ($\equiv$ multiset) of D roots, and let $\mathcal{R}^{(\rho)} \equiv \mathcal{R}^{(\rho)}_{_f}$ denote the family of $(D-\rho)$ roots of the ρ^{th} derivative $f^{(\rho)}$ (sometimes using primemarks $[\,']$ for lower orders); we also write $|\mathcal{R}_f| = D$ (even if some root(s) have multiplicity greater than 1). Denote the average value of the function on the ρ^{th} derivative roots family as $\overline{f\left(\mathcal{R}^{(\rho)}\right)}$, and similarly use $\overline{f^{(\delta)}\left(\mathcal{R}^{(\rho)}\right)}$ for the average value of the $\delta^{\text{ th}}$ derivative of f over the same roots family. Sometimes we will emphasize the degree of f by explicitly writing $f_{_D}$, and we put $\varphi_{_{D\delta\rho}} := \overline{f_{_D}^{(\delta)}\left(\mathcal{R}^{(\rho)}\right)}$.

Our initial results mentioned above suggest that we seek to determine $\varphi_{_{D\delta\rho}}$ for all D, δ, & ρ, and that we seek relationships among various $\varphi_{_{D\delta\rho}}$, of the form $\sum_\rho \alpha_{_\rho}\, \varphi_{_{D\delta\rho}} = 0$. (Other linear combinations of the $\varphi_{_{D\delta\rho}}$ that sum over D or over δ are unnatural because of dimensionality considerations.)

Proof Routes
In cases where the cardinality of the roots family $\mathcal{R}^{(\rho)}$ is 2 or 3, we can compute by brute force since the values of the roots will be given by either the quadratic or cubic formula. Perhaps this method could even be pushed to the case of a family of 4 roots via the Ferarri quartic formula, but the level of complexity is greatly increased. In any case, for root families of size 5 or greater, we need some alternative route.

Again we benefit from establishing notations. We shall use what we refer to as the quasi-binomial representation of polynomials, expressing coefficients in terms of averaged symmetric polynomials in the roots. For example, a general cubic is represented as $f\,(x) = a\left(x^3 - 3\,\overline{r}\,x^2 + 3\,\overline{\overline{r}}\,x - \overline{\overline{\overline{r}}}\right)$ where it turns out that $\overline{r}$ is the average of the $\binom{3}{1}$ roots, $\overline{\overline{r}}$ is the average of the $\binom{3}{2}$ products of pairs of roots, and $\overline{\overline{\overline{r}}}$ is the 'average' of the $\binom{3}{3}$ product of triplets of roots. With these, the earlier-mentioned Cardano-Tartaglia cubic formula can be computed easily via $\mathbb{E} = \overline{r}$, $\mathbb{V} = 2\left(\overline{r}^{\,2} - \overline{\overline{r}}\right)$, and $\mathbb{W} = 2\,\overline{r}^{\,3} - 3\,\overline{r}\,\overline{\overline{r}} + \overline{\overline{\overline{r}}}$. In general we write $f_D(x) = \sum_{\substack{i\,+\,j\,=\,D \\ 0\,\leq\, i,\,j\,\leq\, D}} (-1)^i \binom{\;D\;}{i\quad j} \overset{i}{\overline{r}}\, x^j$ where $\overset{i}{\overline{r}}$ denotes that quasi-binomial parameter with i-many bars, $\binom{\;D\;}{i\quad j} = \frac{D!}{i!\,j!}$, and we write $\overrightarrow{\mathcal{R}}$ for the *ordered* family $\left(\overline{r},\, \overline{\overline{r}},\, \overline{\overline{\overline{r}}},\, \ldots\right)$. Notice that $f\,'\,(x) = 3 \cdot a\left(x^2 - 2\,\overline{r}\,x + \overline{\overline{r}}\right)$ is just 3 times the quasi-binomial representation of a generic quadratic. This is typical: with quasi-binomial representations of polynomials, $f_{_D}' = D \cdot f_{_{D\,-\,1}}$, i.e. taking the derivative coincides with truncating the highest order term from the quasi-binomial family $\overrightarrow{\mathcal{R}}$; in other words, for a given quasi-binomial parameter family $\overrightarrow{\mathcal{R}}$, we obtain a finite Appell sequence of derived functions. Also, in terms of the quasi-binomial parameters, the statistically-presented version of the quadratic formula is simply $x = \overline{r} \pm \sqrt{\overline{r}^{\;2} - \overline{\overline{r}}}$.

It remains instructive to establish that the slope of a cubic at its inflection point is the negated mean of the slopes at the three roots, via the route of brute force computation. We want to evaluate f' at the three roots $r_{_k} = \mathbb{E} + \omega^k\,\overrightarrow{T}_{_+} + \omega^{-\,k}\,\overrightarrow{T}_{_-}$(for $k \in \{0,\,1,\,2\}$): $\;f'\left(r_{_k}\right) = 3 \cdot a\left(r_{_k}^2 - 2\,\overline{r}\,r_{_k} + \overline{\overline{r}}\right)$. For this we will use

$$r_{_k}^2 = \left(\mathbb{E} + \omega^k\,\overrightarrow{T}_{_+} + \omega^{-\,k}\,\overrightarrow{T}_{_-}\right)^2 =$$

$$\overline{r}^{\,2} + \omega^{2\,k}\,\overrightarrow{T}_{_+}^{\;2} + \omega^{-2\,k}\,\overrightarrow{T}_{_-}^{\;2} + 2\left(\overline{r}\left(\omega^k\,\overrightarrow{T}_{_+} + \omega^{-\,k}\,\overrightarrow{T}_{_-}\right) + \omega^k\,\overrightarrow{T}_{_+} \cdot \omega^{-\,k}\,\overrightarrow{T}_{_-}\right).$$

Now averaging over $k \in \{0,1,2\}$ we encounter convenient terms such as $(\omega^{2\cdot 0}+\omega^{2\cdot 1}+\omega^{2\cdot 2})\,\vec{T}_+^{\,2} = (\omega^0+\omega^2+\omega^1)\,\vec{T}_+^{\,2} = 0$, and $(\,\omega^0+\omega^1+\omega^2)\,\vec{T}_+ = 0$, and three copies of $1\cdot\vec{T}_+\vec{T}_-$. Hence we obtain $\overline{r_k^2} = \overline{r}^{\,2} + 2\,\vec{T}_+\vec{T}_-$. From the definitions, we further simplify $2\vec{T}_+\vec{T}_- = \mathbb{V}$. Of course $\overline{r_k} = \mathbb{E} = \overline{r}$. Together we now have $\overline{f'(r_k)} = 3\cdot a\left(\mathbb{E}^2+\mathbb{V}-2\overline{r}\,\overline{r}+\overline{\overline{r}}\right) = 3\cdot a\left(-\overline{r}^{\,2}+2\left(\overline{r}^{\,2}-\overline{\overline{r}}\right)+\overline{\overline{r}}\right) = 3\cdot a\left(\overline{r}^{\,2}-\overline{\overline{r}}\right) = \frac{3}{2}\,a\mathbb{V}$. Whereas the inflection point occurs at $x=\overline{r}$, we also compute $f'(\overline{r}) = 3\cdot a\left(\overline{r}^{\,2}-2\,\overline{r}\;\overline{r}+\overline{\overline{r}}\right) = -\frac{3}{2}\,a\mathbb{V}$, thus establishing our goal.

Looking back, our efforts were aided here by the fact that $\frac{\omega^0+\omega^1+\omega^2}{3}=0$, something particular to the case of cube roots– something that we won't have available for general cases. What we *will* have in the general case is computing means of powers of the roots, and here is our key to higher degree proofs: we want to be able to express powers of the roots in terms of the quasi-binomial parameters, i.e. in terms of the mean elementary symmetric polynomials in the roots. Exemplifying this in the case of 3 roots, consider:

$$\left(\frac{1}{3}\sum r_k\right)^2 = \frac{1}{3}\left\{\frac{1}{3}\sum r_k^{\,2} + \frac{2}{3}\sum r_k r_{k'}\right\} \text{ whence}$$

$$\overline{r_k^2} \equiv \frac{1}{3}\sum r_k^{\,2} = 3\,\overline{r}^{\,2} - 2\,\overline{\overline{r}} \; = \mathbb{V}+\mathbb{E}^2\,. \tag{1}$$

For any (multi)set $\mathcal{Z}$ of numbers, the elementary symmetric polynomials on $\mathcal{Z}$ are defined as $e_i(\mathcal{Z}) \equiv e_i := \sum\limits_{\substack{A\subseteq\mathcal{Z}\\|A|=i}}\prod\limits_{z\in A} z$ (for $i=0\ldots|\mathcal{Z}|;\, e_0=1$); e.g. if $\mathcal{Z}=\mathcal{R}$ as above with $|\mathcal{R}|=3$, then e_2 is the sum of all $\binom{3}{2}$ products of taking 2 roots at a time, and e_1 is the sum of all $\binom{3}{1}$ 'products' of taking 1 root at a time. Also define the power sums on $\mathcal{Z}$ per $p_i(\mathcal{Z}) \equiv p_i := \sum\limits_{z\in\mathcal{Z}} z^i$; $p_0 := |\mathcal{Z}|$. With these definitions, equation (1) above is equivalent to $p_2 = \sum r_k^{\,2} = e_1^2 - 2e_2$. Such effort to determine the sum of powers (or average of powers) is an established result, the Girard (1629) and Waring (1762) formula [**2**] which is very closely connected to Newton's identities (ca. 1687) [**10**]. Both are based upon the fact that for a (multi)set $\mathcal{Z}$ with $|\mathcal{Z}|=n$ it holds that $\sum\limits_{\substack{i+j=n\\0\le i,j\le n}}(-1)^j\,p_j(\mathcal{Z})e_i(\mathcal{Z})=0$. (Note that $e_i(\mathcal{R}) = \binom{n}{i}\,\overset{i}{\overline{r}}$.) This relationship can either be solved for the e_i or for the p_j. Solving for e_i one obtains Newton's recursive identities $e_i = \frac{1}{i}\sum\limits_{\substack{i+j=n\\0\le i,j\le n}}(-1)^{j+1}\,p_j e_i$. Solving instead for p_j, one obtains the Girard-Waring formula. To state the Girard-Waring formula compactly, Put $\mathfrak{n} := \{1,2,\ldots,n\}$ and consider the n-tuples of natural numbers, $\vec{\kappa} \equiv (k_i)_{i\in\mathfrak{n}} \in \mathbb{N}^n$ $(0\in\mathbb{N})$, and define $\mathcal{K}_{\mathfrak{n}j} := \left\{\vec{\kappa}\in\mathbb{N}^n \;\middle|\; \left\|\vec{\kappa}\right\| = j\right\}$ with $\left\|\vec{\kappa}\right\| := \sum\limits_{i\in\mathfrak{n}} i\,k_i$, with n being the number of data (or root) values. Note that for $n\ge j$, $\mathcal{K}_{\mathfrak{n}j}$ is isomorphic to $\mathcal{K}_{\mathbf{j}j} =: \mathcal{K}_j$ which is the set of integer partitions of j; the only difference (which is insignificant) is that elements of $\mathcal{K}_{\mathfrak{n}j}$ may have trailing zeroes in the partition. E.g., $(3,0,0,1,0,0,0)\in\mathcal{K}_7$ denotes the partition $1+1+1+4$ of 7. The Girard-Waring formula is that $p_j = \sum_{\vec{\kappa} \in \mathcal{K}_j} \gamma_{\vec{\kappa}} \prod_{i \in \mathfrak{n}} e_i^{\,k_i}$, where $\gamma_{\vec{\kappa}} := j\,(-1)^j \frac{(-1)^{|\vec{\kappa}|}}{|\vec{\kappa}|} \binom{|\vec{\kappa}|}{\vec{\kappa}}$, where $|\vec{\kappa}| := \sum_{i \in \mathfrak{n}} k_i$ and where $\binom{|\vec{\kappa}|}{\vec{\kappa}} = \frac{|\vec{\kappa}|!}{\prod_{i \in \mathfrak{n}} k_i!}$ is the multinomial coefficient over the family of subindices k_i in $\vec{\kappa}$. (See Gould [**2**]). (The $\gamma_{\vec{\kappa}}$ are given as sequence A210258 in Sloane's OEIS [**12**].) From our perspectives it will become more natural to represent this relationship by replacing p_j and e_i by their normalized forms $\overline{p_j} = p_j/n$ and $\overline{e_i} = \overset{i}{\overline{r}} = e_i/\binom{n}{i}$, admittedly superficial changes to Girard-Waring. We thus obtain $\overline{p_j} = \sum_{\vec{\kappa} \in \mathcal{K}_j} c_{\vec{\kappa}} \prod_{i \in \mathfrak{n}} \overset{i}{\overline{r}}{}^{k_i}$ where

$$c_{\vec{\kappa}} := \frac{j\,(-1)^j}{n} \frac{(-1)^{|\vec{\kappa}|}}{|\vec{\kappa}|} \binom{|\vec{\kappa}|}{\vec{\kappa}} \prod_{i \in \mathfrak{n}} \binom{n}{i}^{k_i}. \tag{2}$$

We emphasize that for any partition of the power j, i.e. $\forall\,(k_i)_{i\in\mathfrak{n}} \in \mathcal{K}_j$, the coefficient of the corresponding term $\prod_{i\in\mathfrak{n}} \overset{i}{\overline{r}}{}^{k_i}$ is given by this formula for $c_{\vec{\kappa}}$. By holding fixed all but one of the many parameters in equation (2), one obtains many sequences of coefficients. E.g., the coefficients of $\overline{t}^{\,\ell}\, \overline{\overline{\overline{t}}}^{\,2}$ for a family of $n = 3$ gives sequence $(1, 7, 36, 162, \ldots)$ (OEIS A080420 [**12**]) with formula $\frac{j(j-5)}{18} 3^{j-5}$ (degree $j = \ell + 6$). E.g., the coefficients of $\overline{u}^{\,\ell}\, \overline{\overline{u}}$ for a family of $n = 4$ gives sequence $(3, 18, 96, 480, \ldots)$ (not in OEIS) with formula $6\,j\,4^{j-3}$ (degree $j = \ell + 2$). Tabulated results are in Tables $\overline{\text{GW}}.\mathfrak{n}$=2 through $\overline{\text{GW}}.\mathfrak{n}$=7 and in Tables $\overline{\text{GW}}.\deg$=2 through $\overline{\text{GW}}.\deg$=7, at the end of the paper. We particularly note that the coefficients in Table $\overline{\text{GW}}.\mathfrak{n}$=2 are exactly the coefficients of the Tchebyshev polynomials of the first kind. Thus our Tables $\overline{\text{GW}}.\mathfrak{n}$ are generalizations of these Tchebyshev polynomials.

2. EXAMPLE An illustrative example is to consider the long-term goal of determining relationships among $\varphi_{4,0,\rho} \equiv \overline{f_4\left(\mathcal{R}^{(\rho)}\right)}$ for $\rho \in \{1, 2, 3\}$. A better example would address $\varphi_{6,0,\rho} \equiv \overline{f_6\left(\mathcal{R}^{(\rho)}\right)}$ for $\rho \in \{1, 2, 3, 4, 5\}$, since with f' being a quintic we have no hope of algebraically knowing those roots, but the larger example demands much lengthier efforts. So begin with $f_4(x) = a\left(x^4 - 4\,\overline{r}\,x^3 + 6\,\overline{\overline{r}}\,x^2 - 4\,\overline{\overline{\overline{r}}}\,x + \overline{\overline{\overline{\overline{r}}}}\right)$. For $\rho = 3$, we have the cardinality 1 family $\mathcal{R}''' = (\overline{r})$, thus quickly we obtain

$$\varphi_{4,0,3} = f_4(\overline{r}) = -a\left(3\,\overline{r}^{\,4} - 6\,\overline{\overline{r}}\,\overline{r}^{\,2} + 4\,\overline{\overline{\overline{r}}}\,\overline{r} - \overline{\overline{\overline{\overline{r}}}}\right).$$

For $\rho = 2$, let the family of the roots of the 2nd derivative be $\mathcal{R}'' = (s_1, s_2)$, and toward determining $\varphi_{4,0,2} \equiv \overline{f_4\left(\mathcal{R}^{(2)}\right)}$ let us consider $f_4(s_i) = a\left(s_i^4 - 4\,\overline{r}\,s_i^3 + 6\,\overline{\overline{r}}\,s_i^2 - 4\,\overline{\overline{\overline{r}}}\,s_i + \overline{\overline{\overline{\overline{r}}}}\right)$. From our mean versions of the Girard-Waring formula (presented as Table $\overline{\text{GW}}.\mathfrak{n}$=2 at the end of the paper) we have:

$$\overline{s^2} = 2\,\overline{s}^{\,2} - 1\,\overline{\overline{s}},\quad \overline{s^3} = 4\,\overline{s}^{\,3} - 3\,\overline{s}\,\overline{\overline{s}},\ \text{and}\ \overline{s^4} = 8\,\overline{s}^{\,4} - 8\,\overline{s}^{\,2}\,\overline{\overline{s}} + 1\,\overline{\overline{s}}^{\,2}.$$

These give us

$$\overline{f_4\left(s_i\right)} = a\Bigg(\Big(8\,\overline{s}^{\,4} - 8\,\overline{s}^{\,2}\,\overline{\overline{s}} + 1\,\overline{\overline{s}}^{\,2}\Big) - 4\,\overline{r}\,\left(4\,\overline{s}^{\,3} - 3\,\overline{s}\,\overline{\overline{s}}\right) + 6\,\overline{\overline{r}}\left(2\,\overline{s}^{\,2} - 1\,\overline{\overline{s}}\right) - 4\,\overline{\overline{\overline{r}}}\,\overline{s} + \overline{\overline{\overline{\overline{r}}}}\Bigg).$$

We would be stuck here were it not for the fact that the s_i are roots of a derivative of f, and thus we are blessed with the facts that $\overline{s} = \overline{r}$ and $\overline{\overline{s}} = \overline{\overline{r}}$. This is the key issue enabling general degree success. Thus we substantively extend Girard-Waring by considering $\overline{p_j}\left(\mathcal{R}_f^{(m)}\right)$ and $\overline{e_i}\left(\mathcal{R}_f\right)$ where $\mathcal{R}_f^{(m)}$ is the roots family of the derivative $f^{(m)}$. Simplifying our current degree 4 expression, we now have

$$\varphi_{4,0,2} = a\left(-8\,\overline{r}^{\,4} + 16\,\overline{r}^{\,2}\,\overline{\overline{r}} - 4\,\overline{r}\,\overline{\overline{\overline{r}}} - 5\,\overline{\overline{r}}^{\,2} + 1\,\overline{\overline{\overline{\overline{r}}}}\right)\,.$$

For $\rho = 1$, let the family of the roots of the 1st derivative be $\mathcal{R}' = (t_1, t_2, t_3)$, and toward determining $\varphi_{4,0,1} \equiv \overline{f_4\left(\mathcal{R}^{(1)}\right)}$ let us consider $f_4\left(t_i\right) = a\left(t_i^4 - 4\,\overline{r}\,t_i^3 + 6\,\overline{\overline{r}}\,t_i^2 - 4\,\overline{\overline{\overline{r}}}\,t_i + \overline{\overline{\overline{\overline{r}}}}\right)$. From our mean versions of the Girard-Waring formula (Table $\overline{\text{GW}}.\mathfrak{n}$=3) we have $\overline{t^2} = 3\,\overline{t}^{\,2} - 2\,\overline{\overline{t}}$, $\overline{t^3} = 9\,\overline{t}^{\,3} - 9\,\overline{t}\,\overline{\overline{t}} + \overline{\overline{\overline{t}}}$, and $\overline{t^4} = 27\,\overline{t}^{\,4} - 36\,\overline{t}^{\,2}\,\overline{\overline{t}} + 4\,\overline{t}\,\overline{\overline{\overline{t}}} + 6\,\overline{\overline{t}}^{\,2}$. These give us

$$\overline{f_4\left(t_i\right)} = a\Bigg(\Big(27\,\overline{t}^{\,4} - 36\,\overline{t}^{\,2}\,\overline{\overline{t}} + 4\,\overline{t}\,\overline{\overline{\overline{t}}} + 6\,\overline{\overline{t}}^{\,2}\Big) - 4\,\overline{r}\left(9\,\overline{t}^{\,3} - 9\,\overline{t}\,\overline{\overline{t}} + \overline{\overline{\overline{t}}}\right) + 6\,\overline{\overline{r}}\left(3\,\overline{t}^{\,2} - 2\,\overline{\overline{t}}\right) - 4\,\overline{\overline{\overline{r}}}\,\overline{t} + \overline{\overline{\overline{\overline{r}}}}\Bigg).$$

We would again be stuck here were it not for the fact that the t_i are roots of a derivative of f, and thus we are blessed with the facts that $\overline{t} = \overline{r}$, $\overline{\overline{t}} = \overline{\overline{r}}$, and $\overline{\overline{\overline{t}}} = \overline{\overline{\overline{r}}}$. Simplifying, we now have

$$\varphi_{4,0,1} = a\left(-9\,\overline{r}^{\,4} + 18\,\overline{r}^{\,2}\,\overline{\overline{r}} - 4\,\overline{r}\,\overline{\overline{\overline{r}}} - 6\,\overline{\overline{r}}^{\,2} + 1\,\overline{\overline{\overline{\overline{r}}}}\right)\,.$$

Henceforward we shall assume that the leading coefficient is $a = 1$. We have summarized our $\left(\varphi_{D,0,\rho}\right)_{\rho\in\{1,2,\ldots,\,D-1\}}$ results for other degrees in Tables $\boldsymbol{\varphi}.\mathbf{2}$ through $\boldsymbol{\varphi}.\mathbf{7}$ at the end of the paper.

We summarize the above procedure. For a given order ρ of derivative, consider the $(D-\rho)$ roots of the derivative, and express the normalized power sum means in terms of the derivative family's quasi-binomial parameters, making use of our normalized variant of the Girard-Waring formulas. Then to evaluate the mean value of the original degree D function over the derivative roots family, appropriately substitute these mean power sum expressions in where the argument of the function occurs; this takes advantage of the fact that averaging is a linear process, to wit, the mean value of the polynomial function is the sum of the mean values of its monomial components. Next, taking advantage of the fact that the quasi-binomial parameters of a family of derivative roots is the same (albeit truncated) as the quasi-binomial parameters of the degree D function, we are able to simplify the expression of the $\varphi_{D,\delta,\rho}$ to be entirely in terms of the degree D function's quasi-binomial (i.e. normalized symmetric function) parameters.

With such procedure in hand, we are enabled to determine general degree D results *ad libitum*. Tabulated results are in Tables $\boldsymbol{\varphi.D}$ for $D = 2$ through $D = 7$, at the end of the paper.

Let us look a bit more closely at the results thus far obtained in the above example. We have:

$$\varphi_{4,0,1} = -9\,\overline{r}^{\,4} + 18\,\overline{r}^{\,2}\,\overline{\overline{r}} - 4\,\overline{r}\,\overline{\overline{\overline{r}}} - 6\,\overline{\overline{r}}^{\,2} + 1\,\overline{\overline{\overline{\overline{r}}}},$$

$$\varphi_{4,0,2} = -8\,\overline{r}^{\,4} + 16\,\overline{r}^{\,2}\,\overline{\overline{r}} - 4\,\overline{r}\,\overline{\overline{\overline{r}}} - 5\,\overline{\overline{r}}^{\,2} + 1\,\overline{\overline{\overline{\overline{r}}}},$$

$$\varphi_{4,0,3} = f_4\left(\overline{r}\right) = -3\,\overline{r}^{\,4} + 6\,\overline{\overline{r}}\,\overline{r}^{\,2} - 4\,\overline{\overline{\overline{r}}}\,\overline{r} + 1\,\overline{\overline{\overline{\overline{r}}}}.$$

Observe that all of these have common terms $-4\,\overline{\overline{\overline{r}}}\,\overline{r} + 1\,\overline{\overline{\overline{\overline{r}}}}$, so there is more structure here than what we have put our finger on. After sufficiently inspired inspection we might realize the following unexpected relationship:

$$5\,\varphi_{4,0,1} - 6\,\varphi_{4,0,2} + 1\,\varphi_{4,0,3} = 0\,,$$

i.e.

$$5\,\overline{f_4\left(\mathcal{R}'\right)} - 6\,\overline{f_4\left(\mathcal{R}''\right)} + 1\,\overline{f_4\left(\mathcal{R}'''\right)} = 0 \quad !$$

Remarks. (1) We desire a more systematic way, with less inspiration required, to obtain such relationships. In effect we are striving to solve $\alpha_1\,\varphi_{4,0,1} + \alpha_2\,\varphi_{4,0,2} + \alpha_3\,\varphi_{4,0,3} = 0$, where the $\varphi_{4,0,\rho}$ are "vectors" of linear combinations of the five "basis elements" $\left(\overline{r}^{\,4},\ \overline{r}^{\,2}\,\overline{\overline{r}},\ \overline{r}\,\overline{\overline{\overline{r}}},\ \overline{\overline{r}}^{\,2},\ \overline{\overline{\overline{\overline{r}}}}\right)$. Thus the matter of finding all triplet $(\alpha_1, \alpha_2, \alpha_3)$ solutions is a simple linear algebra issue involving row reduction.

(2) At first glance, trying to solve $\alpha_1\,\varphi_{4,0,1} + \alpha_2\,\varphi_{4,0,2} + \alpha_3\,\varphi_{4,0,3} = 0$ for the α_ρ seems like a hopeless venture– for this degree 4 case, with 5 basis elements we are essentially trying to solve 5 equations with only 3 degrees of freedom in our αs. But the fact that all three of the $\varphi_{4,0,\rho}$ have common terms $-4\,\overline{\overline{\overline{r}}}\,\overline{r} + 1\,\overline{\overline{\overline{\overline{r}}}}$ saves us: if we can happen to satisfy the other three basis elements with αs such that $\sum_{\rho\in\{1,2,3\}} \alpha_\rho = 0$, then the $\overline{\overline{\overline{r}}}\,\overline{r}$ and $\overline{\overline{\overline{\overline{r}}}}$ constraints will automatically be satisfied. Besides that, there is further structure within our $\varphi_{4,0,\rho}$ results– observe that the coefficients of both $\overline{r}^{\,4}$ and $\overline{\overline{r}}\,\overline{r}^{\,2}$ are in the proportion $\varphi_{4,0,1} : \varphi_{4,0,2} : \varphi_{4,0,3} :: 9 : 8 : 3$. This again increases our hope of finding at least one $(\alpha_1, \alpha_2, \alpha_3)$ solution triplet.

(3) Observe that: (a) the coefficients within the $\varphi_{D,\delta,\rho}$ are somewhat larger than the coefficients in the eventual relationship $5\,\varphi_{4,0,1} - 6\,\varphi_{4,0,2} + 1\,\varphi_{4,0,3} = 0$; and (b) the number of basis elements inside each $\varphi_{D,\delta,\rho}$ is greater than the number of means $\varphi_{D,\delta,\rho}$ in our eventual relationship. These are typical situations. Indeed, for relationships involving degrees up to 7, we sometimes see coefficients within some $\varphi_{D,\delta,\rho}$ in the tens of thousands, with as many as 15 basis elements, yet the eventual relationships remain small, with coefficients near 100, and with 2 to 6 $\varphi_{D,\delta,\rho}$ means involved. The number

of basis elements for different degree polynomials is the number of integer partitions of the degree (see OEIS A000041 [**12**]):

Degree	2	3	4	5	6	7	8	9	10	11	12	13
# Basis Elements	2	3	5	7	11	15	22	30	42	56	77	101

This all strongly suggests that our proof procedure is unnecessarily convoluted, and that some more natural and simple proof path is yet to be found. Streamlined proofs still elude us.

(4) Having noted in the chart above how quickly the number of basis elements increases with increasing degree, we should be doubtful about the prospects of finding $\left(\alpha_1, \alpha_2, \ldots, \alpha_{D-1}\right)$ solution tuples for higher degree D cases. Our hope rests in there being "special circumstances" similar to those noted in Remark (2) above. We shall find that sometimes such circumstances do hold.

Before we give a resumé of our complete results, we feel compelled to present the strikingly 'weird' degree 6 result: There is a unique solution to $\sum\limits_{\rho\in\{1,\ldots,5\}} \alpha_\rho\, \varphi_{6,0,\rho} = 0$, namely

$$77\,\varphi_{6,0,1} - 120\,\varphi_{6,0,2} + 60\,\varphi_{6,0,3} - 20\,\varphi_{6,0,4} + 3\,\varphi_{6,0,5} = 0\,.$$

Perhaps equally surprising is that in degree 7, even though constraints from 15 basis elements must be satisfied, we enjoy a 2-dimensional family of solution relationships.

3. GRAPHICAL EXAMPLE IN DEGREE 4 See Figure 1 at end of paper.

4. RESULTS The graphical example had us consider $\overline{f'\left(\mathcal{R}\right)}$ which is the average of the slopes at the roots, and this quantity is equal to the negation of twice the slope at the average of the roots. Note that this latter quantity is invariant w.r.t. vertical translations, hence the average of the slopes at the roots does not depend upon the polynomial's constant term. This is typical:

Proposition 1. *If a horizontal line cuts a polynomial graph at as many points as the degree of the polynomial, the average of the slopes at the points of intersection is invariant w.r.t. modest vertical translations of the line.*

Proof. (Sketch) Given a vertical translation Δh, one can compute changes in the various slopes, ignoring terms that are higher order in Δh. Straightforward algebra shows that the sum of all such slope changes is 0 for degrees 2 through 7, and we are confident that the same method works in all degrees. ∎

The interpretation of "modest" in the proposition is that the vertical translation should not cross an extremum. In fact when one accounts for multiplicities and computes, if necessary, complex-valued derivatives for complex-value roots, it is seen that the modesty condition is not a requirement.

Our attempts at an inductive proof of the general case above led to the following confident conjecture (supported by strong numerical evidence). The general proof of proposition 1 would follow as a corollary of the $k = 2$ case of the following. Note that the conjecture gives statements in $(\text{reciprocal root units})^{k-1}$:

Conjecture 1. *In any polynomial of degree ≥ 2 with roots set $\mathcal{R}$ having no repeated roots, it holds that*

$$\forall k \geq 2, \qquad \sum_{r\in\mathcal{R}} \frac{f^{(k)}(r)}{f'(r)} = 0\,.$$

Note that the above can be reinterpreted as: $\forall \ell \geq 1$, the sum of relative rates is $\sum\limits_{s\in\mathcal{S}_c} \frac{g^{(\ell)}(s)}{g(s)} = 0$ where $\mathcal{S}_c$ is the roots family of any antiderivative $\left(\int_0^x g\right) + c$ provided that $\mathcal{S}_c$ has no repeated roots and that $\deg(g) \geq 1$.

We now present some of our results. More completely see Tables $\boldsymbol{\alpha}.\mathbf{4}$ through $\boldsymbol{\alpha}.\mathbf{6}$, as well as Tables $\boldsymbol{\varphi}.\mathbf{2}$ through $\boldsymbol{\varphi}.\mathbf{7}$, at the end of the paper.

Proposition 2. *The following is an exhaustive list of fundamental linear relationships for degrees up to 7, among means of a polynomial's values when evaluated at roots of derivatives.*

A. Degree$(f) = 2$ *N/A*

B. Degree$(f) = 3$

$$1\,\varphi_{3,0,1} - 1\,\varphi_{3,0,2} = 0 \qquad \text{(a)}$$

C. Degree$(f) = 4$

$$5\,\varphi_{4,0,1} - 6\,\varphi_{4,0,2} + 1\,\varphi_{4,0,3} = 0 \qquad \text{(b)}$$

D. Degree$(f) = 5$

$$1\,\varphi_{5,0,1} - 3\,\varphi_{5,0,3} + 2\,\varphi_{5,0,4} = 0 \qquad \text{(c)}$$
$$2\,\varphi_{5,0,2} - 5\,\varphi_{5,0,3} + 3\,\varphi_{5,0,4} = 0 \qquad \text{(d)}$$
$$3\,\varphi_{5,0,1} - 4\,\varphi_{5,0,2} + 1\,\varphi_{5,0,3} = 0$$
$$5\,\varphi_{5,0,1} - 6\,\varphi_{5,0,2} + 1\,\varphi_{5,0,4} = 0$$
$$1\,\varphi_{5,0,1} - 2\varphi_{5,0,2} + 2\,\varphi_{5,0,3} - 1\,\varphi_{5,0,4} = 0$$

(any 3 of the above are linearly dependent; any 2 are independent)

E. Degree$(f) = 6$

$$77\,\varphi_{6,0,1} - 120\,\varphi_{6,0,2} + 60\,\varphi_{6,0,3} - 20\,\varphi_{6,0,4} + 3\,\varphi_{6,0,5} = 0$$

F. Degree$(f) = 7$

$$85\,\varphi_{7,0,1} - 144\,\varphi_{7,0,2} + 90\,\varphi_{7,0,3} - 40\,\varphi_{7,0,4} + 9\,\varphi_{7,0,5} = 0$$
$$82\,\varphi_{7,0,1} - 135\,\varphi_{7,0,2} + 75\,\varphi_{7,0,3} - 25\,\varphi_{7,0,4} + 3\,\varphi_{7,0,6} = 0$$
$$77\,\varphi_{7,0,1} - 120\,\varphi_{7,0,2} + 50\,\varphi_{7,0,3} - 15\,\varphi_{7,0,5} + 8\,\varphi_{7,0,6} = 0$$
$$67\,\varphi_{7,0,1} - 90\,\varphi_{7,0,2} + 50\,\varphi_{7,0,4} - 45\,\varphi_{7,0,5} + 18\,\varphi_{7,0,6} = 0$$
$$37\,\varphi_{7,0,1} - 150\,\varphi_{7,0,3} + 200\,\varphi_{7,0,4} - 135\,\varphi_{7,0,5} + 48\,\varphi_{7,0,6} = 0$$
$$111\,\varphi_{7,0,2} - 335\,\varphi_{7,0,3} + 385\,\varphi_{7,0,4} - 246\,\varphi_{7,0,5} + 85\,\varphi_{7,0,6} = 0$$
$$1\,\varphi_{7,0,1} - 3\,\varphi_{7,0,2} + 5\,\varphi_{7,0,3} - 5\,\varphi_{7,0,4} + 3\,\varphi_{7,0,5} - 1\,\varphi_{7,0,6} = 0$$

(any 3 of the above are linearly dependent; any 2 are independent)

Notes: In all the cases in the above list, the sum of all positive coefficients equals the sum of all negative coefficients. We have computed such relations up to degree 40, observing that (1) in all even degrees (beyond 2) there is a single such fundamental linear relationship, and (2) in all odd degrees beyond 3 there is a 2-dimensional family of fundamental linear relationships.

Also, the alphabetical labels at the right margins in the preceding & following propositions indicate "derivative inheritance" relationships across different degrees.

There are two ways to augment the above fundamental relationships with more relationships: (a) instead of considering values of the function f we can consider values of its derivatives $f^{(n)}$ or its (repeated) antiderivatives, which we denote as $f^{(-n)}$; or (b) we can expand the list of root families available by considering the root familes of antiderivatives $f^{(-n)}$. When using antiderivatives, it turns out that there is no dependence on the constant of integration; this is a consequence of proposition 1.

Proposition 3. *Expressing polynomials as their quasi-binomial representations, $\int f_D = \frac{1}{D+1} f_{D+1}$ where the constant term of f_{D+1} is the constant of integration c. In other words, $\overrightarrow{\mathcal{R}}^{(-1)} = \overrightarrow{\mathcal{R}} \uplus (c)$. Restated yet again, if $\overrightarrow{\mathcal{S}}$ is the quasi-binomial parameter vector of any antiderivative of $f_{\mathcal{R}}$, then $\overrightarrow{\mathcal{R}} \leq \overrightarrow{\mathcal{S}}$, i.e. $\overrightarrow{\mathcal{R}}$ is a (strict) subvector of $\overrightarrow{\mathcal{S}}$ via extension by the constant(s) of integration. Further, if the domain variable of the polynomial carries dimensional units, then the constant c is $(D+1)$-dimensional.*

Dually, if $\overrightarrow{\mathcal{S}}$ is the quasi-binomial parameter vector of any derivative of $f_{\mathcal{R}}$, then $\overrightarrow{\mathcal{S}} \leq \overrightarrow{\mathcal{R}}$, i.e. $\overrightarrow{\mathcal{S}}$ is a (strict) subvector of $\overrightarrow{\mathcal{R}}$ via truncation.

Example: Consider any polynomial f_D, e.g. $f_3(x) = x^3 - 3\,x^2\,\overline{r} + 3\,x^1\,\overline{\overline{r}} - \overline{\overline{\overline{r}}}$ with roots family $\mathcal{R}$, and consider averaging the values of the function over some set $\mathcal{Z}$, $x \in \mathcal{Z}$. We obtain $\overline{f_3(\mathcal{Z})} = \overline{x^3} - 3\,\overline{x^2}\,\overline{r} + 3\,\overline{x^1}\,\overline{\overline{r}} - \overline{\overline{\overline{r}}}$. The average values of the powers $\overline{x^m}$ are obtained by our mean-value modification of the Girard-Waring formulas. The case of $\mathcal{Z}$ being the roots family of a derivative of f is straightforward, as noted in the dual statement in the proposition, with $\overrightarrow{\mathcal{Z}}$ being a subvector of $\overrightarrow{\mathcal{R}}$ via truncation. The case of $\mathcal{Z}$ being the roots family of an antiderivative of f requires greater attention: In the example here, the highest power term present in $f_3(x)$ is order 3 (& dimensionality 3 if x bears units), but the constant of integration in $\int f$ is of dimensionality 4; hence the constant of integration cannot enter into the computation of $\overline{x^3}$, etc., with similar behavior in the general situation. Indeed, the values of $\overline{x^3}$, etc., only depend upon the entries in $\overrightarrow{\mathcal{Z}}$ that were already present in $\overrightarrow{\mathcal{R}}$. In the detail of the present example we have, following our notation for a family of 4, $\mathcal{Z} = \mathcal{F}_4 \equiv \mathcal{U}$, $\overline{u} = \overline{r}$, $\overline{\overline{u}} = \overline{\overline{r}}$ and $\overline{\overline{\overline{u}}} = \overline{\overline{\overline{r}}}$; thus

$$
\begin{aligned}
\overline{f_3(\mathcal{Z})} &= 1\,\overline{u^3} - 3\,\overline{u^2}\,\overline{r} + 3\,\overline{u^1}\,\overline{\overline{r}} - 1\,\overline{\overline{\overline{r}}} \\
&1\left\{16\,\overline{u}^{\,3} - 18\,\overline{u}\,\overline{\overline{u}} + 3\overline{\overline{\overline{u}}}\right\} - 3\left\{4\,\overline{u}^{\,2} - 3\overline{\overline{u}}\right\}\overline{r} + 3\,\left\{\overline{u}^1\right\}\overline{\overline{r}} - \overline{\overline{\overline{r}}} \\
&= \overline{r}^{\,3}\{16 - 12\} + \overline{r}\,\overline{\overline{r}}\,\{-18 + 9 + 3\} + \overline{\overline{\overline{r}}}\,\{3 - 1\} \\
&= 4\,\overline{r}^{\,3} - 6\,\overline{r}\,\overline{\overline{r}} + 2\,\overline{\overline{\overline{r}}} \;=\; 2\,\mathbb{W}.
\end{aligned}
$$

Note, in particular, that the average value $\varphi_{3,0,-1} \equiv \overline{f_3\left(\mathcal{R}^{(-1)}\right)}$ does not at all involve the constant of integration that affects the family $\mathcal{R}^{(-1)}$. This behavior is typical. Let us emphasize: in the computation of mean function values $\overline{f_D\left(\mathcal{R}^{(-m)}\right)}$, individual mean powers such as $\overline{u^k}$ only involve the quasi-binomial parameters of the original $\vec{\mathcal{R}}$ and never involve the constant(s) of integration of $\mathcal{R}^{(-m)}$. Thus we have:

Proposition 4. *The mean quantities* $\varphi_{D,0,-m} \equiv \overline{f_D\left(\mathcal{R}^{(-m)}\right)}$ *and* $\varphi_{D,\delta,-m} \equiv \overline{f_D^{(\delta)}\left(\mathcal{R}^{(-m)}\right)}$ *(with* $\delta, m > 0$*) do not depend on constant(s) of integration.*

Proposition 5. $\varphi_{D,1,0} = D \cdot \varphi_{(D-1),0,-1}$

$$\varphi_{D,\delta,0} = \frac{D!}{(D-\delta)!} \cdot \varphi_{(D-\delta),0,-\delta}$$

Of course, the above proposition can be read in reverse as giving $\varphi_{D,0,-1} = \frac{1}{(D+1)} \cdot \varphi_{(D+1),1,0}$ and $\varphi_{D,0,-\rho} = \frac{D!}{(D+\rho)!} \cdot \varphi_{(D+\rho),\rho,0}$ (with $\rho > 0$). These facts are substantiated in our tables of $\varphi_{D,0,\rho}$ expressions by the fact that whenever $\rho < 0$, the highest order quasi-binomial parameter $\overset{i}{r}$ is always only order D, never an order in excess of D.

For the following five propositioons, see also Tables $\boldsymbol{\alpha}.\mathbf{4}$ through $\boldsymbol{\alpha}.\mathbf{6}$.

Proposition 6. *The following is a partial list of linear relationships among means of a polynomial's derivative values when evaluated at roots of derivatives.*

A. Degree$(f) = 2$ *N/A*

B. Degree$(f) = 3$

$$1\ \varphi_{3,1,0} + 1\ \varphi_{3,1,2} = 0 \qquad \text{(e)}$$

C. Degree$(f) = 4$

$$1\ \varphi_{4,1,0} + 2\,\varphi_{4,1,2} = 0$$
$$1\ \varphi_{4,1,0} + 2\,\varphi_{4,1,3} = 0$$
$$1\ \varphi_{4,1,2} - 1\,\varphi_{4,1,3} = 0$$

D. Degree$(f) = 5$

$$5\ \varphi_{5,1,2} - 6\,\varphi_{5,1,3} + 1\,\varphi_{5,1,4} = 0$$

E. Degree$(f) = 6$

$$1\ \varphi_{6,1,2} - 3\,\varphi_{6,1,4} + 2\,\varphi_{6,1,5} = 0$$
$$2\ \varphi_{6,1,3} - 5\,\varphi_{6,1,4} + 3\,\varphi_{6,1,5} = 0$$
$$3\ \varphi_{6,1,2} - 4\,\varphi_{6,1,3} + 1\,\varphi_{6,1,4} = 0$$
$$5\ \varphi_{6,1,2} - 6\,\varphi_{6,1,3} + 1\,\varphi_{6,1,5} = 0$$

$$1\,\varphi_{6,1,2} - 2\,\varphi_{6,1,3} + 2\,\varphi_{6,1,4} - 1\,\varphi_{6,1,5} = 0$$

(any 3 of the above are linearly dependent; any 2 are independent)

Proposition 7. *The following is a partial list of linear relationships among means of a polynomial's* 2^{nd} *derivative values when evaluated at roots of derivatives.*

A. Degree$(f) = 2$ *N/A*
B. Degree$(f) = 3$ *N/A*

C. *Degree*$(f) = 4$

$$1\,\varphi_{4,2,0} - 2\,\varphi_{4,2,1} = 0$$
$$1\,\varphi_{4,2,1} + 1\,\varphi_{4,2,3} = 0$$
$$1\,\varphi_{4,2,0} + 2\,\varphi_{4,2,3} = 0$$
$$1\,\varphi_{4,2,0} - 1\,\varphi_{4,2,1} + 1\,\varphi_{4,2,3} = 0$$

D. *Degree*$(f) = 5$

$$1\,\varphi_{5,2,0} + 5\,\varphi_{5,2,3} = 0$$
$$1\,\varphi_{5,2,1} + 2\,\varphi_{5,2,3} = 0$$
$$1\,\varphi_{5,2,3} - 1\,\varphi_{5,2,4} = 0$$
$$1\,\varphi_{5,2,0} - 2\,\varphi_{5,2,1} + 2\,\varphi_{5,2,3} - 1\,\varphi_{5,2,4} = 0$$

E. *Degree*$(f) = 6$

$$5\,\varphi_{6,2,3} - 6\,\varphi_{6,2,4} + 1\,\varphi_{6,2,5} = 0$$

Proposition 8. *The following is a partial list of linear relationships among means of a polynomial's* 3^{rd} *derivative values when evaluated at roots of derivatives.*

A. *Degree*$(f) = 2$ *N/A*
B. *Degree*$(f) = 3$ *N/A*
C. *Degree*$(f) = 4$ *N/A*

D. *Degree*$(f) = 5$

$$1\,\varphi_{5,3,0} - 3\,\varphi_{5,3,2} = 0$$

E. *Degree*$(f) = 6$

$$1\,\varphi_{6,3,0} + 9\,\varphi_{6,3,4} = 0$$
$$1\,\varphi_{6,3,1} + 5\,\varphi_{6,3,3} = 0$$
$$1\,\varphi_{6,3,2} + 2\,\varphi_{6,3,4} = 0$$
$$1\,\varphi_{6,3,4} - 1\,\varphi_{6,3,5} = 0$$

Proposition 9. *The following is a partial list of linear relationships among means of a polynomial's* 4^{th} *derivative values when evaluated at roots of derivatives.*

A. *Degree*$(f) = 2$ *N/A*
B. *Degree*$(f) = 3$ *N/A*
C. *Degree*$(f) = 4$ *N/A*
D. *Degree*$(f) = 5$ *N/A*

E. *Degree*$(f) = 6$

$$3\,\varphi_{6,4,0} - 4\,\varphi_{6,4,1} = 0$$
$$1\,\varphi_{6,4,1} - 3\,\varphi_{6,4,3} = 0$$
$$1\,\varphi_{6,4,2} - 2\,\varphi_{6,4,3} = 0$$
$$1\,\varphi_{6,4,3} + 1\,\varphi_{6,4,5} = 0$$

Proposition 10. *The following is a partial list of linear relationships among means of a cubic polynomial's function, derivatives, and/or antiderivatives values when evaluated at roots of derivatives and antiderivatives.*

Degree$(f) = 3$:

$$2\ \varphi_{3,-2,0}\ -5\ \varphi_{3,-2,1}\ +3\ \varphi_{3,-2,2}\ = 0$$
$$1\ \varphi_{3,-2,-1}\ -3\ \varphi_{3,-2,1}\ +2\ \varphi_{3,-2,2}\ = 0$$
$$5\ \varphi_{3,-1,0}\ -6\ \varphi_{3,-1,1}\ +1\ \varphi_{3,-1,2}\ = 0$$
$$1\ \varphi_{3,0,-1}\ +2\ \varphi_{3,0,1}\ = 0$$
$$1\ \varphi_{3,0,-2}\ +5\ \varphi_{3,0,1}\ = 0$$
$$1\ \varphi_{3,0,-3}\ +9\ \varphi_{3,0,1}\ = 0$$
$$1\ \varphi_{3,1,-1}\ -2\ \varphi_{3,1,0}\ = 0$$
$$1\ \varphi_{3,1,-2}\ -3\ \varphi_{3,1,0}\ = 0$$
$$3\ \varphi_{3,1,-3}\ -4\ \varphi_{3,1,-2}\ = 0$$

The last six of the above deserve particular comment. In the middle three above, in the first terms (with coefficient 1) the cubic function f is being averaged over the root families $\mathcal{R}^{(-1)}$, $\mathcal{R}^{(-2)}$, $\mathcal{R}^{(-3)}$, which are the root families of $\int f$, $\iint f$, and $\iiint f$. As noted in an earlier proposition, these results are independent of choice of constants of integration.

The following essentially restate above results with a different perspective. See Tables $\boldsymbol{\alpha.4}$ through $\boldsymbol{\alpha.6}$,

Note: Regarding the alphabetical labels to the right of some relationships, we have used double-letter hybrid labels to indicate that the relationship is linearly dependent upon the denoted pair of preceding relationships, in cases of worthy of attention since the relationship coefficients vector is highly symmetric,

Proposition 11. *In any quartic polynomial the following relations hold:*

(0)	$5\ \varphi_{4,0,1} - 6\,\varphi_{4,0,2} + 1\,\varphi_{4,0,3} = 0$	(b)
(1)	$1\,\varphi_{4,1,0} + 2\,\varphi_{4,1,2} = 0$	(f)
	$1\ \varphi_{4,1,2} - 1\ \varphi_{4,1,3} = 0$	(a')= D(a)
(2)	$1\,\varphi_{4,2,0} - 2\ \varphi_{4,2,1} = 0$	(g)
	$1\,\varphi_{4,2,1} + 1\,\varphi_{4,2,3} = 0\,.$	(e')= D(e)

Proposition 12. *In any quintic polynomial the following relations hold:*

(0)	$1\ \varphi_{5,0,1} - 3\,\varphi_{5,0,3} + 2\,\varphi_{5,0,4} = 0$	(c)
	$2\ \varphi_{5,0,2} - 5\,\varphi_{5,0,2} + 3\,\varphi_{5,0,4} = 0$	(d)
	$1\ \varphi_{5,0,1} - 2\,\varphi_{5,0,2} + 2\,\varphi_{5,0,3} - 1\,\varphi_{5,0,4} = 0$	(cd)
(1)	$5\ \varphi_{5,1,2} - 6\,\varphi_{5,1,3} + 1\,\varphi_{5,1,4} = 0$	(b')= D(b)
(2)	$1\ \varphi_{5,2,0} + 5\,\varphi_{5,2,3} = 0$	(h)
	$1\ \varphi_{5,2,1} + 2\,\varphi_{5,2,3} = 0$	(f')= D(f)
	$1\ \varphi_{5,2,3} - 1\,\varphi_{5,2,4} = 0$	(a")= D(a')= D^2(a)
(3)	$1\ \varphi_{5,3,0} - 3\,\varphi_{5,3,2} = 0$	(i)
	$1\ \varphi_{5,3,1} - 2\,\varphi_{5,3,2} = 0$	(g')= D(g)
	$1\ \varphi_{5,3,2} + 1\,\varphi_{5,3,4} = 0.$	(e")= D(e')= D^2(e)

Also note that item $(0\mathrm{f})$ states that for any quintic f, the average of the function values at the two roots of f''' equals the $2:1$ weighted average of (i) the value of the function at the sole root of $f^{(4)}$ and (ii) the average of the value of the function at the four roots of f'. Item $(0\mathrm{g})$ states that for any quintic f, the average of the function values at the two roots of f''' also equals the $3:2$ weighted average of (i) the value of the function at the sole root of $f^{(4)}$ and (ii) the average of the value of the function at the three roots of f''. We leave the remaining interpretations to the reader.

Proposition 13. *In any sextic polynomial the following relations hold:*

$$
\begin{array}{lll}
(0) & 77\,\varphi_{6,0,1} - 120\,\varphi_{6,0,2} + 60\,\varphi_{6,0,3} - 20\,\varphi_{6,0,4} + 3\,\varphi_{6,0,5} = 0 & \text{(j)} \\
(1) & 1\,\varphi_{6,1,2} - 3\,\varphi_{6,1,4} + 2\,\varphi_{6,1,5} = 0 & \text{(c')}= D\text{(c)} \\
 & 2\,\varphi_{6,1,3} - 5\,\varphi_{6,1,4} + 3\,\varphi_{6,1,5} = 0 & \text{(d')}= D\text{(d)} \\
(2) & 5\,\varphi_{6,2,3} - 6\,\varphi_{6,2,4} + 1\,\varphi_{6,2,5} = 0 & \text{(b'')}= D\text{(b')}= D^2\text{(b)} \\
(3) & 1\,\varphi_{6,3,0} + 9\varphi_{6,3,4} = 0 & \text{(k)} \\
 & 1\,\varphi_{6,3,1} + 5\,\varphi_{6,3,4} = 0 & \text{(h')}= D\text{(h)} \\
 & 1\,\varphi_{6,3,2} + 2\,\varphi_{6,3,4} = 0 & \text{(f'')}= D\text{(f')}= D^2\text{(f)} \\
 & 1\,\varphi_{6,3,4} - 1\,\varphi_{6,3,5} = 0 & \text{(a''')}= D\text{(a'')}= D^2\text{(a')}= D^3\text{(a)} \\
(4) & 3\,\varphi_{6,4,0} - 4\,\varphi_{6,4,1} = 0 & \text{(l)} \\
 & 1\,\varphi_{6,4,1} - 3\,\varphi_{6,4,3} = 0 & \text{(i')}= D\text{(i)} \\
 & 1\,\varphi_{6,4,2} - 2\,\varphi_{6,4,3} = 0 & \text{(g'')}= D\text{(g')}= D^2\text{(g)} \\
 & 1\,\varphi_{6,4,3} + 1\,\varphi_{6,4,5} = 0 & \text{(e''')}= D\text{(e'')}= D^2\text{(e')}= D^3\text{(e)}
\end{array}
$$

Proposition 14. *In any septic polynomial the following relations hold:*

$$
\begin{array}{lll}
(0) & 85\,\varphi_{7,0,1} - 144\,\varphi_{7,0,2} + 90\,\varphi_{7,0,3} - 40\,\varphi_{7,0,4} + 9\,\varphi_{7,0,5} = 0 & \text{(m)} \\
 & 82\,\varphi_{7,0,1} - 135\,\varphi_{7,0,2} + 75\,\varphi_{7,0,3} - 25\,\varphi_{7,0,4} + 3\,\varphi_{7,0,6} = 0 & \text{(n)} \\
 & 1\,\varphi_{7,0,1} - 3\,\varphi_{7,0,2} + 5\,\varphi_{7,0,3} - 5\,\varphi_{7,0,4} + 3\,\varphi_{7,0,5} - 1\,\varphi_{7,0,6} = 0 & \text{(mn)}
\end{array}
$$

We find these relationships distinctively impressive. There are many more identities, not recorded above, that are present when we open up the limitless world of roots $\mathcal{R}^{(-n)}$ of $f^{(-n)} := \underbrace{\iint \cdots \int}_{n\text{copies}} f$, i.e. when we consider the limitless world of mean values $\varphi_{D,\delta,\rho}$ where $\rho < 0$.

Ancillary Computational Results Consider the following observations regarding the coefficients of $\overset{\max}{r} \equiv \overset{D}{r}$ in $\varphi_{D,0,\rho}$. Let $h_D(n)$ be the polynomial in n predicting the coefficient of $\overset{D}{r}$ for the $\boldsymbol{\varphi}$.D family $\left(\varphi_{D,0,\rho}\right)_{\rho<D-1}$. This polynomial $h_D(n)$ is always of the form $h_D(n) = \frac{(-1)^D D}{D!}\rho\; n^{\chi} g_D(n)$ where $g_D(n)$ is a monic irreducible (over $\mathbb{Z}$) polynomial of degree $M := D - (2+\chi)$ with integer coefficients, where $\chi = 1$ when D is odd and $\chi = 0$ when D is even. Let $g_D(n) = \sum_{k=2\ldots D} t_k(D)\, n^{D-(k+\chi)}$. Note that $t_2(D) \equiv 1$ (monic condition); also for odd k, it holds that $t_k(D) = 0$ & $t_{k+1}(D) = 0$ if $D \leq k$. The $t_k(D)$ are of the form

$\frac{1}{Q_k}u_k(D)$ where Q_k is the least common denominator of the terms in $t_k(D)$, such that $u_k(D)$ is a (generally not monic) polynomial in D with integer coefficients, with leading coefficient denoted as $\mathfrak{N}_k$. Curiously, denominators Q_k are the OEIS [**12**] sequence A053657, described in OEIS as "Denominators of integer-valued polynomials on prime numbers (with degree n): 1/a(n) is a generator of the ideal formed by the leading coefficients of integer-valued polynomials on prime numbers with degree less than or equal to n", or equivalently "Also the least common multiple of the orders of all finite subgroups of $GL_n(\mathbb{Q})$ [Minkowski]". Strikingly, the leading coefficients $\mathfrak{N}_k$ of the polynomials $u_k(D)$ are coefficients of Nørlund's polynomials, see OEIS A260326 (which lists only 7 values), described there as "Common denominator of coefficients in Nørlund's polynomial D_2n(x)." [The OEIS citation refers to Nørlund's 1924 book [**8**] (which discusses the higher order Bernoulli & Euler polynomials), Table 6 (p. 460), which only lists 7 values, for *even* indices 0 through 12, but Table 5 (p. 459) includes both even & odd entries as the leading coefficients of the primary components of the numerator polynomials. Our work has produced 198 entries with values eventually exceeding 5.8×10^{40}.]

Here is a summary of our tabulated results.

Tables $\boldsymbol{\varphi}$.D: Function value means $\varphi_{D,\delta,\rho}$ in terms of the polynomial representation parameters $\overset{i}{\vec{r}}$. These tables extend the Girard-Waring formula.

Tables $\boldsymbol{\alpha}$.D: Linear Relationship Coefficients $\alpha_{D,\delta,\rho}$ such that $\sum_{\rho} \alpha_{D,\delta,\rho}\,\varphi_{D,\delta,\rho} = 0$. Some (D,δ,ρ) triplets admit more than one $\left(\alpha_{D,\delta,\rho}\right)$ family.

Tables $\overline{\textbf{GW}}.\mathfrak{n}$=2 through $\overline{\textbf{GW}}.\mathfrak{n}$=7: Normalized Girard-Waring coefficients $c_{\vec{\kappa}} := \frac{j\,(-1)^j}{n}\,\frac{(-1)^{|\vec{\kappa}|}}{|\vec{\kappa}|}\binom{|\vec{\kappa}|}{\vec{\kappa}} \prod_{i\,\in\,\mathfrak{n}} \binom{n}{i}^{k_i}$ where $j =$ degree and where $\vec{\kappa} \equiv \langle k_1, k_2, \ldots, k_n \rangle$ is an integer partition vector of j, i.e. $\sum_i k_i \cdot i = j$.

Tables $\overline{\textbf{GW}}$.deg=2 through $\overline{\textbf{GW}}$.deg=7: Normalized Girard-Waring coefficients $c_{\vec{\kappa}} := \frac{j\,(-1)^j}{n}\,\frac{(-1)^{|\vec{\kappa}|}}{|\vec{\kappa}|}\binom{|\vec{\kappa}|}{\vec{\kappa}} \prod_{i\,\in\,\mathfrak{n}} \binom{n}{i}^{k_i}$ as above.

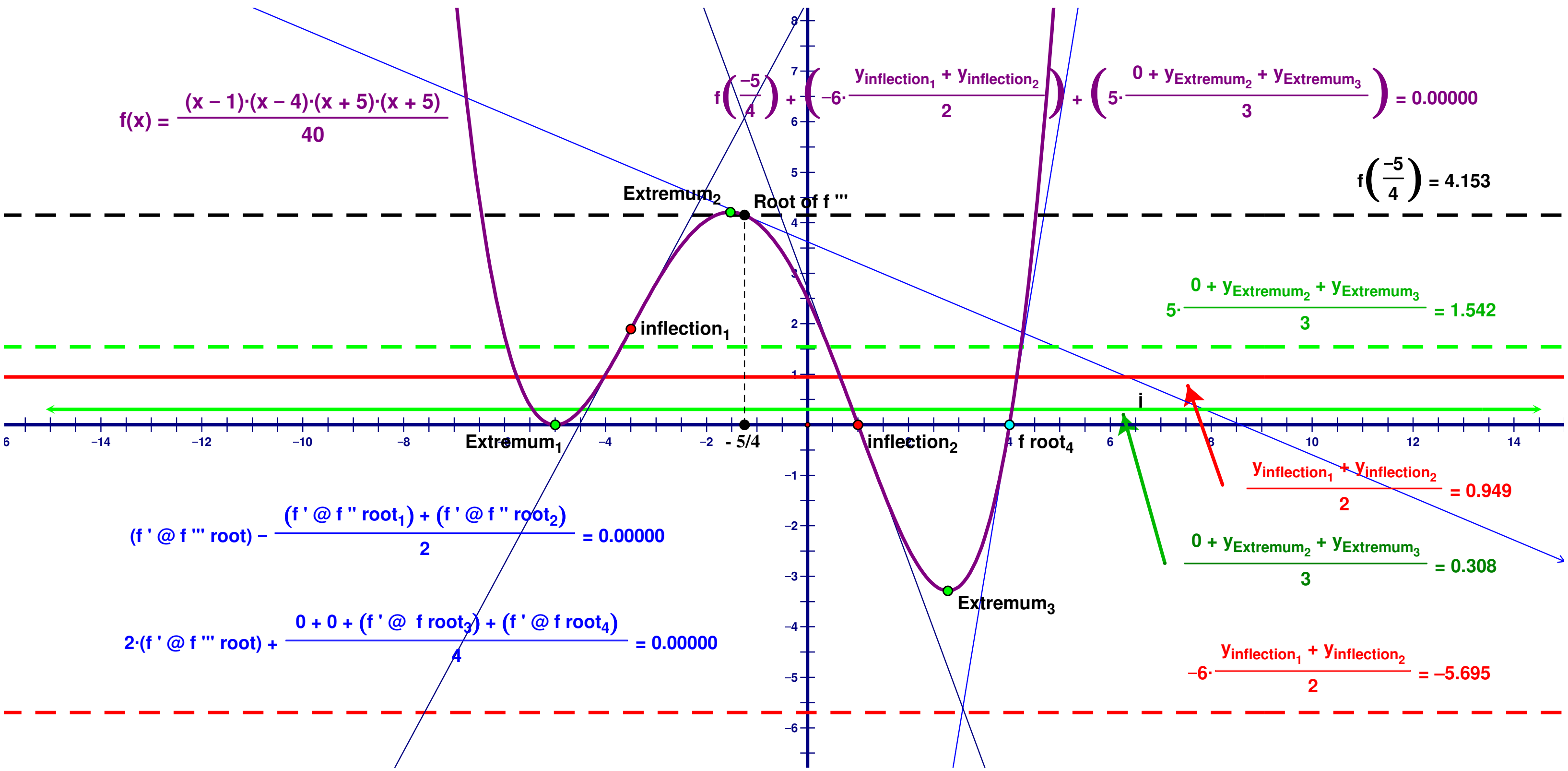

Figure 1. Mean Relationships in all quartics. This example manifests:

(a) $5\overline{f\left(\mathcal{R}'\right)} - 6\,\overline{f\left(\mathcal{R}''\right)} + 1\overline{f\left(\mathcal{R}'''\right)} = 0$ [note: $\mathcal{R}''' = \{-5/4\}$]

(b) $1\overline{f'\left(\mathcal{R}'''\right)} - 1\overline{f'\left(\mathcal{R}''\right)} = 0$

(c) $2\overline{f'\left(\mathcal{R}'''\right)} + 1\overline{f'\left(\mathcal{R}\right)} = 0$

Supportive Results: Function Value Means $\varphi_{D\,\delta\,\rho}$ in Terms of Polynomial Representation Parameters $\overset{i}{\overline{r}}$.

Table φ.2

# roots n	ρ	$\varphi_{D\,\delta\,\rho} = \varphi_{2\,0\,\rho}$	$\sum_{\text{coef}>0}$
1	1	$-\left(\overline{r}^{\,2} - \overline{\overline{r}}\right)$	1
2	0	0	0
3	-1	$\overline{r}^{\,2} - \overline{\overline{r}}$	1
4	-2	$2\left(\overline{r}^{\,2} - \overline{\overline{r}}\right)$	2
5	-3	$3\left(\overline{r}^{\,2} - \overline{\overline{r}}\right)$	3
6	-4	$4\left(\overline{r}^{\,2} - \overline{\overline{r}}\right)$	4
7	-5	$5\left(\overline{r}^{\,2} - \overline{\overline{r}}\right)$	5
8	-6	$6\left(\overline{r}^{\,2} - \overline{\overline{r}}\right)$	6
9	-7	$7\left(\overline{r}^{\,2} - \overline{\overline{r}}\right)$	7
10	-8	$8\left(\overline{r}^{\,2} - \overline{\overline{r}}\right)$	8

Table φ.3

# roots n	ρ	$\varphi_{D\,\delta\,\rho} = \varphi_{3\,0\,\rho}$	$\sum_{\text{coef}>0}$
1	2	$-\left(2\,\overline{r}^{\,3} - 3\,\overline{r}\;\overline{\overline{r}} + \overline{\overline{\overline{r}}}\right)$	3
2	1	$-\left(2\,\overline{r}^{\,3} - 3\,\overline{r}\;\overline{\overline{r}} + \overline{\overline{\overline{r}}}\right)$	3
3	0	0	0
4	-1	$2\left(2\,\overline{r}^{\,3} - 3\,\overline{r}\;\overline{\overline{r}} + \overline{\overline{\overline{r}}}\right)$	6
5	-2	$5\left(2\,\overline{r}^{\,3} - 3\,\overline{r}\;\overline{\overline{r}} + \overline{\overline{\overline{r}}}\right)$	15
6	-3	$9\left(2\,\overline{r}^{\,3} - 3\,\overline{r}\;\overline{\overline{r}} + \overline{\overline{\overline{r}}}\right)$	27
7	-4	$14\left(2\,\overline{r}^{\,3} - 3\,\overline{r}\;\overline{\overline{r}} + \overline{\overline{\overline{r}}}\right)$	42
8	-5	$20\left(2\,\overline{r}^{\,3} - 3\,\overline{r}\;\overline{\overline{r}} + \overline{\overline{\overline{r}}}\right)$	60
9	-6	$27\left(2\,\overline{r}^{\,3} - 3\,\overline{r}\;\overline{\overline{r}} + \overline{\overline{\overline{r}}}\right)$	81
10	-7	$35\left(2\,\overline{r}^{\,3} - 3\,\overline{r}\;\overline{\overline{r}} + \overline{\overline{\overline{r}}}\right)$	105

Table φ.4

# roots n	ρ	$\varphi_{D\,\delta\,\rho} = \varphi_{4\,0\,\rho}$	$\sum_{\text{coef}>0}$
1	3	$-3\,\overline{r}^{\,4} + 6\,\overline{r}^{\,2}\,\overline{\overline{r}} - 4\,\overline{r}\,\overline{\overline{\overline{r}}} - 0\,\overline{\overline{r}}^{\,2} + 1\,\overline{\overline{\overline{\overline{r}}}}$	7
2	2	$-8\,\overline{r}^{\,4} + 16\,\overline{r}^{\,2}\,\overline{\overline{r}} - 4\,\overline{r}\,\overline{\overline{\overline{r}}} - 5\,\overline{\overline{r}}^{\,2} + 1\,\overline{\overline{\overline{\overline{r}}}}$	17
3	1	$-9\,\overline{r}^{\,4} + 18\,\overline{r}^{\,2}\,\overline{\overline{r}} - 4\,\overline{r}\,\overline{\overline{\overline{r}}} - 6\,\overline{\overline{r}}^{\,2} + 1\,\overline{\overline{\overline{\overline{r}}}}$	19
4	0	0	0
5	-1	$25\,\overline{r}^{\,4} - 50\,\overline{r}^{\,2}\,\overline{\overline{r}} + 12\,\overline{r}\,\overline{\overline{\overline{r}}} + 16\,\overline{\overline{r}}^{\,2} - 3\,\overline{\overline{\overline{\overline{r}}}}$	53
6	-2	$72\,\overline{r}^{\,4} - 144\,\overline{r}^{\,2}\,\overline{\overline{r}} + 36\,\overline{r}\,\overline{\overline{\overline{r}}} + 45\,\overline{\overline{r}}^{\,2} - 9\,\overline{\overline{\overline{\overline{r}}}}$	153
7	-3	$147\,\overline{r}^{\,4} - 294\,\overline{r}^{\,2}\,\overline{\overline{r}} + 76\,\overline{r}\,\overline{\overline{\overline{r}}} + 90\,\overline{\overline{r}}^{\,2} - 19\,\overline{\overline{\overline{\overline{r}}}}$	313
8	-4	$256\,\overline{r}^{\,4} - 512\,\overline{r}^{\,2}\,\overline{\overline{r}} + 136\,\overline{r}\,\overline{\overline{\overline{r}}} + 154\,\overline{\overline{r}}^{\,2} - 34\,\overline{\overline{\overline{\overline{r}}}}$	546
9	-5	$405\,\overline{r}^{\,4} - 810\,\overline{r}^{\,2}\,\overline{\overline{r}} + 220\,\overline{r}\,\overline{\overline{\overline{r}}} + 240\,\overline{\overline{r}}^{\,2} - 55\,\overline{\overline{\overline{\overline{r}}}}$	865
10	-6	$600\,\overline{r}^{\,4} - 1200\,\overline{r}^{\,2}\,\overline{\overline{r}} + 332\,\overline{r}\,\overline{\overline{\overline{r}}} + 351\,\overline{\overline{r}}^{\,2} - 83\,\overline{\overline{\overline{\overline{r}}}}$	1283

Tables α.4

ρ	$\alpha_{4,0,\rho}$						
3	615						
2		615					
1			205				
0							
–1				123			
–2					205		
–3						205	
–4							615
–5	–351	64	49	–55	–192	–267	–848
–6	240	–35	–30	32	105	130	310

ρ	$\alpha_{4,0,\rho}$																				
3	**1**	1		9			1				67					155					
2	**–6**		2		9			9				134					310				
1	**5**	8	1			9			15				67					31			
0																					
–1		3	1	–45		9				45				67					155		
–2				16	1	–2	–2	1	8	–34					134					310	
–3							1			9	–77	16	25	41	–144						155
–4											45	–5	–12	17	45	–120	35	9	–80	–315	–195
–5																77	–16	–5	41	144	67
–6																					

Tables φ.5

# roots n	ρ	$\varphi_{D\,\delta\,\rho} = \varphi_{5\,0\,\rho}$	$\sum_{\text{coef}>0}$
1	4	$-4\,\bar{r}^{\,5} + 10\,\bar{r}^{\,3}\bar{\bar{r}} - 10\,\bar{r}^{\,2}\bar{\bar{\bar{r}}} - 0\,\bar{r}\,\bar{\bar{r}}^{\,2} + 5\,\bar{r}\,\bar{\bar{\bar{\bar{r}}}} + 0\,\bar{\bar{r}}\,\bar{\bar{\bar{r}}} - 1\,\bar{\bar{\bar{\bar{\bar{r}}}}}$	15
2	3	$-24\,\bar{r}^{\,5} + 60\,\bar{r}^{\,3}\bar{\bar{r}} - 20\,\bar{r}^{\,2}\bar{\bar{\bar{r}}} - 30\,\bar{r}\,\bar{\bar{r}}^{\,2} + 5\,\bar{r}\,\bar{\bar{\bar{\bar{r}}}} + 10\,\bar{\bar{r}}\,\bar{\bar{\bar{r}}} - 1\,\bar{\bar{\bar{\bar{\bar{r}}}}}$	75
3	2	$-54\,\bar{r}^{\,5} + 135\,\bar{r}^{\,3}\bar{\bar{r}} - 35\,\bar{r}^{\,2}\bar{\bar{\bar{r}}} - 75\,\bar{r}\,\bar{\bar{r}}^{\,2} + 5\,\bar{r}\,\bar{\bar{\bar{\bar{r}}}} + 25\,\bar{\bar{r}}\,\bar{\bar{\bar{r}}} - 1\,\bar{\bar{\bar{\bar{\bar{r}}}}}$	165
4	1	$-64\,\bar{r}^{\,5} + 160\,\bar{r}^{\,3}\bar{\bar{r}} - 40\,\bar{r}^{\,2}\bar{\bar{\bar{r}}} - 90\,\bar{r}\,\bar{\bar{r}}^{\,2} + 5\,\bar{r}\,\bar{\bar{\bar{\bar{r}}}} + 30\,\bar{\bar{r}}\,\bar{\bar{\bar{r}}} - 1\,\bar{\bar{\bar{\bar{\bar{r}}}}}$	195
5	0	0	0
6	-1	$216\,\bar{r}^{\,5} - 540\,\bar{r}^{\,3}\bar{\bar{r}} + 140\,\bar{r}^{\,2}\bar{\bar{\bar{r}}} + 300\,\bar{r}\,\bar{\bar{r}}^{\,2} - 20\,\bar{r}\,\bar{\bar{\bar{\bar{r}}}} - 100\,\bar{\bar{r}}\,\bar{\bar{\bar{r}}} + 4\,\bar{\bar{\bar{\bar{\bar{r}}}}}$	660
7	-2	$686\,\bar{r}^{\,5} - 1715\,\bar{r}^{\,3}\bar{\bar{r}} + 455\,\bar{r}^{\,2}\bar{\bar{\bar{r}}} + 945\,\bar{r}\,\bar{\bar{r}}^{\,2} - 70\,\bar{r}\,\bar{\bar{\bar{\bar{r}}}} - 315\,\bar{\bar{r}}\,\bar{\bar{\bar{r}}} + 14\,\bar{\bar{\bar{\bar{\bar{r}}}}}$	2100
8	-3	$1536\,\bar{r}^{\,5} - 3840\,\bar{r}^{\,3}\bar{\bar{r}} + 1040\,\bar{r}^{\,2}\bar{\bar{\bar{r}}} + 2100\,\bar{r}\,\bar{\bar{r}}^{\,2} - 170\,\bar{r}\,\bar{\bar{\bar{\bar{r}}}} - 700\,\bar{\bar{r}}\,\bar{\bar{\bar{r}}} + 34\,\bar{\bar{\bar{\bar{\bar{r}}}}}$	4710
9	-4	$2916\,\bar{r}^{\,5} - 7290\,\bar{r}^{\,3}\bar{\bar{r}} + 2010\,\bar{r}^{\,2}\bar{\bar{\bar{r}}} + 3960\,\bar{r}\,\bar{\bar{r}}^{\,2} - 345\,\bar{r}\,\bar{\bar{\bar{\bar{r}}}} - 1320\,\bar{\bar{r}}\,\bar{\bar{\bar{r}}} + 69\,\bar{\bar{\bar{\bar{\bar{r}}}}}$	8955
10	-5	$5000\,\bar{r}^{\,5} - 12500\,\bar{r}^{\,3}\bar{\bar{r}} + 3500\,\bar{r}^{\,2}\bar{\bar{\bar{r}}} + 6750\,\bar{r}\,\bar{\bar{r}}^{\,2} - 625\,\bar{r}\,\bar{\bar{\bar{\bar{r}}}} - 2250\,\bar{\bar{r}}\,\bar{\bar{\bar{r}}} + 125\,\bar{\bar{\bar{\bar{\bar{r}}}}}$	15375

Tables $\alpha.5$

ρ	$\alpha_{5,0,\rho}$																				
4	**–1**	**1**				2				28					26						171
3	**2**		**1**		1				28					182						38	
2	**–2**	**–6**	**–4**	4				4					182						228		
1	**1**	**5**	**3**		3	10	28					182						342			
0																					
–1				1	1	3	21	1	–35	–63	91						57				
–2							–4		12	20	–60	64	30	–72	–20	684					
–3											14	–21	–7	35	9	–651	–27	75	27	–7	–66
–4																182	10	–32	–10	4	35
–5																					

ρ	$\alpha_{5,0,\rho}$						
4							325
3						975	
2					1950		
1				13			
0							
–1			975				
–2		650					
–3	975						
–4	–1100	–525	–350	2	175	–300	–75
–5	342	217	162	–1	–81	63	44

Table φ.6

# roots n	ρ	$\varphi_{D\,\delta\,\rho} = \varphi_{6\,0\,\rho}$	$\sum_{\text{coef}>0}$
1	5	$-5\,\overline{r}^{6}+15\,\overline{r}^{4}\overline{\overline{r}}-20\,\overline{r}^{3}\overline{\overline{\overline{r}}}+0\,\overline{r}^{2}\,\overline{\overline{r}}^{2}+15\,\overline{r}^{2}\overline{\overline{\overline{\overline{r}}}}+0\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{r}}}-6\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{r}}}}}+0\,\overline{\overline{r}}^{3}+0\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}}+0\,\overline{\overline{\overline{r}}}^{2}+1\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}$	31
2	4	$-64\,\overline{r}^{6}+192\,\overline{r}^{4}\overline{\overline{r}}-80\,\overline{r}^{3}\overline{\overline{\overline{r}}}-132\,\overline{r}^{2}\,\overline{\overline{r}}^{2}+30\,\overline{r}^{2}\overline{\overline{\overline{\overline{r}}}}+60\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{r}}}-6\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{r}}}}}+14\,\overline{\overline{r}}^{3}-15\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}}+0\,\overline{\overline{\overline{r}}}^{2}+1\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}$	297
3	3	$-243\,\overline{r}^{6}+729\,\overline{r}^{4}\overline{\overline{r}}-216\,\overline{r}^{3}\overline{\overline{\overline{r}}}-567\,\overline{r}^{2}\,\overline{\overline{r}}^{2}+45\,\overline{r}^{2}\overline{\overline{\overline{\overline{r}}}}+234\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{r}}}-6\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{r}}}}}+72\,\overline{\overline{r}}^{3}-30\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}}-19\,\overline{\overline{\overline{r}}}^{2}+1\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}$	1081
4	2	$-512\,\overline{r}^{6}+1536\,\overline{r}^{4}\overline{\overline{r}}-416\,\overline{r}^{3}\overline{\overline{\overline{r}}}-1224\,\overline{r}^{2}\,\overline{\overline{r}}^{2}+66\,\overline{r}^{2}\overline{\overline{\overline{\overline{r}}}}+492\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{r}}}-6\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{r}}}}}+162\,\overline{\overline{r}}^{3}-51\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}}-48\,\overline{\overline{\overline{r}}}^{2}+1\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}$	2257
5	1	$-625\,\overline{r}^{6}+1875\,\overline{r}^{4}\overline{\overline{r}}-500\,\overline{r}^{3}\overline{\overline{\overline{r}}}-1500\,\overline{r}^{2}\,\overline{\overline{r}}^{2}+75\,\overline{r}^{2}\overline{\overline{\overline{\overline{r}}}}+600\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{r}}}-6\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{r}}}}}+200\,\overline{\overline{r}}^{3}-60\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}}-60\,\overline{\overline{\overline{r}}}^{2}+1\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}$	2751
6	0	0	0
7	-1	$2401\,\overline{r}^{6}-7203\,\overline{r}^{4}\overline{\overline{r}}+1960\,\overline{r}^{3}\overline{\overline{\overline{r}}}+5733\,\overline{r}^{2}\,\overline{\overline{r}}^{2}-315\,\overline{r}^{2}\overline{\overline{\overline{\overline{r}}}}-2310\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{r}}}+30\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{r}}}}}-756\,\overline{\overline{r}}^{3}+240\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}}+225\,\overline{\overline{\overline{r}}}^{2}-5\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}$	10589
8	-2	$8192\,\overline{r}^{6}-24576\,\overline{r}^{4}\overline{\overline{r}}+6784\,\overline{r}^{3}\overline{\overline{\overline{r}}}+19488\,\overline{r}^{2}\,\overline{\overline{r}}^{2}-1140\,\overline{r}^{2}\overline{\overline{\overline{\overline{r}}}}-7896\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{r}}}+120\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{r}}}}}-2548\,\overline{\overline{r}}^{3}+840\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}}+756\,\overline{\overline{\overline{r}}}^{2}-20\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}$	36180
9	-3	$19683\,\overline{r}^{6}-59049\,\overline{r}^{4}\overline{\overline{r}}+16524\,\overline{r}^{3}\overline{\overline{\overline{r}}}+46656\,\overline{r}^{2}\,\overline{\overline{r}}^{2}-2889\,\overline{r}^{2}\overline{\overline{\overline{\overline{r}}}}-19008\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{r}}}+330\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{r}}}}}-6048\,\overline{\overline{r}}^{3}+2064\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}}+1792\,\overline{\overline{\overline{r}}}^{2}-55\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}$	87049
10	-4	$40000\,\overline{r}^{6}-120000\,\overline{r}^{4}\overline{\overline{r}}+34000\,\overline{r}^{3}\overline{\overline{\overline{r}}}+94500\,\overline{r}^{2}\,\overline{\overline{r}}^{2}-6150\,\overline{r}^{2}\overline{\overline{\overline{\overline{r}}}}-38700\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{r}}}+750\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{r}}}}}-12150\,\overline{\overline{r}}^{3}+4275\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}}+3600\,\overline{\overline{\overline{r}}}^{2}-125\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}$	177125

Table α.6

ρ	$\alpha_{6,0,\rho}$														
5	**3**	1		1			3				125				
4	**–20**		2		1			4				125			
3	**60**	–30	–15			3			2				125		
2	**–120**	160	60	–140	–45	–30				12				125	
1	**77**	–81	–32	–161	–36	–8	1372	651	83	53					125
0						–15									
–1		–10	3	–140	–36	2	1680	756	90	60	67200	–25200	–6825	–700	300
–2				20	5		–640	–280	–32	–20	64800	23800	6300	675	–200
–3							105	45	5	3	–29925	–10800	–2800	–300	75
–4											5488	1953	498	53	–12

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Table φ.7

n	ρ	$\varphi_{D\,\delta\,\rho} = \varphi_{7\,0\,\rho}$	$\sum_{\text{coef}>0}$
1	6	$-6\,\overline{r}^{\,7} + 21\,\overline{r}^{\,5}\overline{\overline{r}} - 35\,\overline{r}^{\,4}\overline{\overline{\overline{r}}} + 0\,\overline{r}^{\,3}\,\overline{\overline{r}}^{\,2} + 35\,\overline{r}^{\,3}\overline{\overline{\overline{\overline{r}}}} + 0\,\overline{r}^{\,2}\overline{\overline{r}}\,\overline{\overline{\overline{r}}} - 21\,\overline{r}^{\,2}\,\overline{\overline{\overline{\overline{\overline{r}}}}} +$ $0\,\overline{r}\,\overline{\overline{r}}^{\,3} + 0\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}} + 0\,\overline{r}\,\overline{\overline{\overline{r}}}^{\,2} + 7\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}} + 0\,\overline{\overline{r}}^{\,2}\overline{\overline{\overline{r}}} - 0\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{\overline{r}}}}} + 0\,\overline{\overline{\overline{r}}}\,\overline{\overline{\overline{\overline{r}}}} - 1\,\overline{\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}}$	63
2	5	$-160\,\overline{r}^{\,7} + 560\,\overline{r}^{\,5}\overline{\overline{r}} - 280\,\overline{r}^{\,4}\overline{\overline{\overline{r}}} - 490\,\overline{r}^{\,3}\,\overline{\overline{r}}^{\,2} + 140\,\overline{r}^{\,3}\overline{\overline{\overline{\overline{r}}}} + 280\,\overline{r}^{\,2}\overline{\overline{r}}\,\overline{\overline{\overline{r}}} - 42\,\overline{r}^{\,2}\,\overline{\overline{\overline{\overline{\overline{r}}}}} +$ $105\,\overline{r}\,\overline{\overline{r}}^{\,3} - 105\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}} + 0\,\overline{r}\,\overline{\overline{\overline{r}}}^{\,2} + 7\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}} - 35\,\overline{\overline{r}}^{\,2}\overline{\overline{\overline{r}}} + 21\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{\overline{r}}}}} + 0\,\overline{\overline{\overline{r}}}\,\overline{\overline{\overline{\overline{r}}}} - 1\,\overline{\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}}$	1113
3	4	$-972\,\overline{r}^{\,7} + 3402\,\overline{r}^{\,5}\overline{\overline{r}} - 1134\,\overline{r}^{\,4}\overline{\overline{\overline{r}}} - 3402\,\overline{r}^{\,3}\,\overline{\overline{r}}^{\,2} + 315\,\overline{r}^{\,3}\overline{\overline{\overline{\overline{r}}}} + 1638\,\overline{r}^{\,2}\overline{\overline{r}}\,\overline{\overline{\overline{r}}} - 63\,\overline{r}^{\,2}\,\overline{\overline{\overline{\overline{\overline{r}}}}} +$ $882\,\overline{r}\,\overline{\overline{r}}^{\,3} - 315\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}} - 140\,\overline{r}\,\overline{\overline{\overline{r}}}^{\,2} + 7\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}} - 294\,\overline{\overline{r}}^{\,2}\overline{\overline{\overline{r}}} + 42\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{\overline{r}}}}} + 35\,\overline{\overline{\overline{r}}}\,\overline{\overline{\overline{\overline{r}}}} - 1\,\overline{\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}}$	6321
4	3	$-3072\,\overline{r}^{\,7} + 10752\,\overline{r}^{\,5}\overline{\overline{r}} - 3136\,\overline{r}^{\,4}\overline{\overline{\overline{r}}} - 11088\,\overline{r}^{\,3}\,\overline{\overline{r}}^{\,2} + 616\,\overline{r}^{\,3}\overline{\overline{\overline{\overline{r}}}} + 5040\,\overline{r}^{\,2}\overline{\overline{r}}\,\overline{\overline{\overline{r}}} - 84\,\overline{r}^{\,2}\,\overline{\overline{\overline{\overline{\overline{r}}}}} +$ $3024\,\overline{r}\,\overline{\overline{r}}^{\,3} - 714\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}} - 532\,\overline{r}\,\overline{\overline{\overline{r}}}^{\,2} + 7\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}} - 1008\,\overline{\overline{r}}^{\,2}\overline{\overline{\overline{r}}} + 63\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{\overline{r}}}}} + 133\,\overline{\overline{\overline{r}}}\,\overline{\overline{\overline{\overline{r}}}} - 1\,\overline{\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}}$	19635
5	2	$-6250\,\overline{r}^{\,7} + 21875\,\overline{r}^{\,5}\overline{\overline{r}} - 6125\,\overline{r}^{\,4}\overline{\overline{\overline{r}}} - 22750\,\overline{r}^{\,3}\,\overline{\overline{r}}^{\,2} + 1050\,\overline{r}^{\,3}\overline{\overline{\overline{\overline{r}}}} + 10150\,\overline{r}^{\,2}\overline{\overline{r}}\,\overline{\overline{\overline{r}}} - 112\,\overline{r}^{\,2}\,\overline{\overline{\overline{\overline{\overline{r}}}}} +$ $6300\,\overline{r}\,\overline{\overline{r}}^{\,3} - 1295\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}} - 1120\,\overline{r}\,\overline{\overline{\overline{r}}}^{\,2} + 7\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}} - 2100\,\overline{\overline{r}}^{\,2}\overline{\overline{\overline{r}}} + 91\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{\overline{r}}}}} + 280\,\overline{\overline{\overline{r}}}\,\overline{\overline{\overline{\overline{r}}}} - 1\,\overline{\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}}$	39753
6	1	$-7776\,\overline{r}^{\,7} + 27216\,\overline{r}^{\,5}\overline{\overline{r}} - 7560\,\overline{r}^{\,4}\overline{\overline{\overline{r}}} - 28350\,\overline{r}^{\,3}\,\overline{\overline{r}}^{\,2} + 1260\,\overline{r}^{\,3}\overline{\overline{\overline{\overline{r}}}} + 12600\,\overline{r}^{\,2}\overline{\overline{r}}\,\overline{\overline{\overline{r}}} - 126\,\overline{r}^{\,2}\,\overline{\overline{\overline{\overline{\overline{r}}}}} +$ $7875\,\overline{r}\,\overline{\overline{r}}^{\,3} - 1575\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}} - 1400\,\overline{r}\,\overline{\overline{\overline{r}}}^{\,2} + 7\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}} - 2625\,\overline{\overline{r}}^{\,2}\overline{\overline{\overline{r}}} + 105\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{\overline{r}}}}} + 350\,\overline{\overline{\overline{r}}}\,\overline{\overline{\overline{\overline{r}}}} - 1\,\overline{\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}}$	49413
7	0	0	0
8	-1	$32768\,\overline{r}^{\,7} - 114688\,\overline{r}^{\,5}\overline{\overline{r}} + 32256\,\overline{r}^{\,4}\overline{\overline{\overline{r}}} + 11968\,\overline{r}^{\,3}\,\overline{\overline{r}}^{\,2} - 5600\,\overline{r}^{\,3}\overline{\overline{\overline{\overline{r}}}} - 53312\,\overline{r}^{\,2}\overline{\overline{r}}\,\overline{\overline{\overline{r}}} + 616\,\overline{r}^{\,2}\,\overline{\overline{\overline{\overline{\overline{r}}}}} -$ $32928\,\overline{r}\,\overline{\overline{r}}^{\,3} + 6860\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}} + 5880\,\overline{r}\,\overline{\overline{\overline{r}}}^{\,2} - 42\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}} + 10976\,\overline{\overline{r}}^{\,2}\overline{\overline{\overline{r}}} - 490\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{\overline{r}}}}} - 1470\,\overline{\overline{\overline{r}}}\,\overline{\overline{\overline{\overline{r}}}} + 6\,\overline{\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}}$	208530
9	-2	$118098\,\overline{r}^{\,7} - 413343\,\overline{r}^{\,5}\overline{\overline{r}} + 117369\,\overline{r}^{\,4}\overline{\overline{\overline{r}}} + 428652\,\overline{r}^{\,3}\,\overline{\overline{r}}^{\,2} - 20979\,\overline{r}^{\,3}\overline{\overline{\overline{\overline{r}}}} - 192780\,\overline{r}^{\,2}\overline{\overline{r}}\,\overline{\overline{\overline{r}}} + 2457\,\overline{r}^{\,2}\,\overline{\overline{\overline{\overline{\overline{r}}}}} -$ $117936\,\overline{r}\,\overline{\overline{r}}^{\,3} + 25326\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}} + 21168\,\overline{r}\,\overline{\overline{\overline{r}}}^{\,2} - 189\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}} + 39312\,\overline{\overline{r}}^{\,2}\overline{\overline{\overline{r}}} - 1890\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{\overline{r}}}}} - 5292\,\overline{\overline{\overline{r}}}\,\overline{\overline{\overline{\overline{r}}}} + 27\,\overline{\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}}}$	752409
10	-3	$300000\,\overline{r}^{\,7} - 1050000\,\overline{r}^{\,5}\overline{\overline{r}} + 301000\,\overline{r}^{\,4}\overline{\overline{\overline{r}}} + 1086750\,\overline{r}^{\,3}\,\overline{\overline{r}}^{\,2} - 55300\,\overline{r}^{\,3}\overline{\overline{\overline{\overline{r}}}} - 491400\,\overline{r}^{\,2}\overline{\overline{r}}\,\overline{\overline{\overline{r}}} + 6846\,\overline{r}^{\,2}\,\overline{\overline{\overline{\overline{\overline{r}}}}} -$ $297675\,\overline{r}\,\overline{\overline{r}}^{\,3} + 65385\,\overline{r}\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{r}}}} + 53760\,\overline{r}\,\overline{\overline{\overline{r}}}^{\,2} - 581\,\overline{r}\,\overline{\overline{\overline{\overline{\overline{\overline{r}}}}}} + 99225\,\overline{\overline{r}}^{\,2}\overline{\overline{\overline{r}}} - 5103\,\overline{\overline{r}}\,\overline{\overline{\overline{\overline{\overline{r}}}}} - 13440\,\overline{\overline{\overline{r}}}\,\overline{\overline{\overline{\overline{r}}}} + 83\,\overline{\overline{\overline{r}}}\overline{\overline{\overline{\overline{r}}}}$	1913499

Sub-Supportive Normalized Girard-Waring Results

To present our results, let $\mathcal{S} := \{s_1, s_2\} = \mathcal{R}_f^{(D-2)}$ where D is the degree of f; let $\mathcal{T} := \{t_1, t_2, t_3\} = \mathcal{R}_f^{(D-3)}$; let $\mathcal{U} := \mathcal{R}_f^{(D-4)}$ etc. We have the following results.

N.B.: The coefficients in Table $\overline{\text{GW}}.\mathfrak{n}$=2 are exactly the coefficients of Tchebyshev polynomials of the first kind. Thus these tables generalize Tchebyshev polynomials.

Table $\overline{\text{GW}}.\mathfrak{n}$=2

Sum of Positive Coefficients	Data Family $\mathcal{F}_2 \equiv \mathcal{S} = \{s_1, s_2\}$
1 :	$\overline{s^1} = \bar{s}^{\,1}$
2 :	$\overline{s^2} = 2\,\bar{s}^{\,2} - 1\,\bar{\bar{s}}$
4 :	$\overline{s^3} = 4\,\bar{s}^{\,3} - 3\,\bar{s}\,\bar{\bar{s}}$
9 :	$\overline{s^4} = 8\,\bar{s}^{\,4} - 8\,\bar{s}^{\,2}\,\bar{\bar{s}} + 1\,\bar{\bar{s}}^{\,2}$
21 :	$\overline{s^5} = 16\,\bar{s}^{\,5} - 20\,\bar{s}^{\,3}\,\bar{\bar{s}} + 5\,\bar{s}\,\bar{\bar{s}}^{\,2}$
50 :	$\overline{s^6} = 32\,\bar{s}^{\,6} - 48\,\bar{s}^{\,4}\,\bar{\bar{s}} + 18\,\bar{s}^{\,2}\,\bar{\bar{s}}^{\,2} - \bar{\bar{s}}^{\,3}$
120 :	$\overline{s^7} = 64\,\bar{s}^{\,7} - 112\,\bar{s}^{\,5}\,\bar{\bar{s}} + 56\,\bar{s}^{\,3}\,\bar{\bar{s}}^{\,2} - 7\,\bar{s}\,\bar{\bar{s}}^{\,3}$
289 :	$\overline{s^8} = 128\,\bar{s}^{\,8} - 256\,\bar{s}^{\,6}\,\bar{\bar{s}} + 160\,\bar{s}^{\,4}\,\bar{\bar{s}}^{\,2} - 32\,\bar{s}^{\,2}\,\bar{\bar{s}}^{\,3} + 1\,\bar{\bar{s}}^{\,4}$

Table $\overline{\text{GW}}.\mathfrak{n}$=3

Sum of Positive Coefficients	Data Family $\mathcal{F}_3 \equiv \mathcal{T} = \{t_1, t_2, t_3\}$
1 :	$\overline{t^1} = \bar{t}^{\,1}$
3 :	$\overline{t^2} = 3\,\bar{t}^{\,2} - 2\,\bar{\bar{t}}$
10 :	$\overline{t^3} = 9\,\bar{t}^{\,3} - 9\,\bar{t}\,\bar{\bar{t}} + \bar{\bar{\bar{t}}}$
37 :	$\overline{t^4} = 27\,\bar{t}^{\,4} - 36\,\bar{t}^{\,2}\,\bar{\bar{t}} + 4\,\bar{t}\,\bar{\bar{\bar{t}}} + 6\,\bar{\bar{t}}^{\,2}$
141 :	$\overline{t^5} = 81\,\bar{t}^{\,5} - 135\,\bar{t}^{\,3}\,\bar{\bar{t}} + 15\,\bar{t}^{\,2}\,\bar{\bar{\bar{t}}} + 45\,\bar{t}\,\bar{\bar{t}}^{\,2} - 5\,\bar{\bar{t}}\,\bar{\bar{\bar{t}}}$
541 :	$\overline{t^6} = 243\,\bar{t}^{\,6} - 486\,\bar{t}^{\,4}\,\bar{\bar{t}} + 54\,\bar{t}^{\,3}\,\bar{\bar{\bar{t}}} + 243\,\bar{t}^{\,2}\,\bar{\bar{t}}^{\,2} - 36\,\bar{t}\,\bar{\bar{t}}\,\bar{\bar{\bar{t}}} - 18\,\bar{\bar{t}}^{\,3} + 1\,\bar{\bar{\bar{t}}}^{\,2}$
2080 :	$\overline{t^7} = 729\,\bar{t}^{\,7} - 1701\,\bar{t}^{\,5}\,\bar{\bar{t}} + 189\,\bar{t}^{\,4}\,\bar{\bar{\bar{t}}} + 1134\,\bar{t}^{\,3}\,\bar{\bar{t}}^{\,2} - 189\,\bar{t}^{\,2}\,\bar{\bar{t}}\,\bar{\bar{\bar{t}}} - 189\,\bar{t}\,\bar{\bar{t}}^{\,3} + 7\,\bar{t}\,\bar{\bar{\bar{t}}}^{\,2} + 21\,\bar{\bar{t}}^{\,2}\,\bar{\bar{\bar{t}}}$

Table $\overline{\text{GW}}$.n=4

Sum of Positive Coefficients	Data Family $\mathcal{F}_4 \equiv \mathcal{U} = \{u_1,\, u_2,\, u_3,\, u_4\}$
1 :	$\overline{u^1} = \overline{u}^{\,1}$
4 :	$\overline{u^2} = 4\,\overline{u}^{\,2} - 3\,\overline{\overline{u}}$
19 :	$\overline{u^3} = 16\,\overline{u}^{\,3} - 18\,\overline{u}\,\overline{\overline{u}} + 3\,\overline{\overline{\overline{u}}}$
98 :	$\overline{u^4} = 64\,\overline{u}^{\,4} - 96\,\overline{u}^{\,2}\,\overline{\overline{u}} + 16\,\overline{u}\,\overline{\overline{\overline{u}}} + 18\,\overline{\overline{u}}^{\,2} - \overline{\overline{\overline{\overline{u}}}}$
516 :	$\overline{u^5} = 256\,\overline{u}^{\,5} - 480\,\overline{u}^{\,3}\,\overline{\overline{u}} + 80\,\overline{u}^{\,2}\,\overline{\overline{\overline{u}}} + 180\,\overline{u}\,\overline{\overline{u}}^{\,2} - 30\,\overline{\overline{u}}\,\overline{\overline{\overline{u}}} - 5\,\overline{u}\,\overline{\overline{\overline{\overline{u}}}}$
2725 :	$\overline{u^6} = 1024\,\overline{u}^{\,6} - 2304\,\overline{u}^{\,4}\,\overline{\overline{u}} + 384\,\overline{u}^{\,3}\,\overline{\overline{\overline{u}}} + 1296\,\overline{u}^{\,2}\,\overline{\overline{u}}^{\,2} - 288\,\overline{u}\,\overline{\overline{u}}\,\overline{\overline{\overline{u}}} - 108\,\overline{\overline{u}}^{\,3} - 24\,\overline{u}^{\,2}\,\overline{\overline{\overline{\overline{u}}}} + 9\,\overline{\overline{u}}\,\overline{\overline{\overline{\overline{u}}}} + 12\,\overline{\overline{\overline{u}}}^{\,2}$
14400 :	$\overline{u^7} = 4096\,\overline{u}^{\,7} - 10752\,\overline{u}^{\,5}\,\overline{\overline{u}} + 1792\,\overline{u}^{\,4}\,\overline{\overline{\overline{u}}} + 112 \cdot 72\,\overline{u}^{\,3}\,\overline{\overline{u}}^{\,2} - 2016\,\overline{u}^{\,2}\,\overline{\overline{u}}\,\overline{\overline{\overline{u}}} - 1512\,\overline{u}\,\overline{\overline{u}}^{\,3} + 112\,\overline{u}\,\overline{\overline{\overline{u}}}^{\,2} + 252\,\overline{\overline{u}}^{\,2}\,\overline{\overline{\overline{u}}} - 112\,\overline{u}^{\,3}\,\overline{\overline{\overline{\overline{u}}}} + 84\,\overline{u}\,\overline{\overline{u}}\,\overline{\overline{\overline{\overline{u}}}} - 7\,\overline{\overline{\overline{u}}}\,\overline{\overline{\overline{\overline{u}}}}$

Table $\overline{\text{GW}}$.n=5

Sum of Positive Coefficients	Data Family $\mathcal{F}_5 \equiv \mathcal{V} = \{v_1,\, v_2,\, v_3,\, v_4,\, v_5\}$
1 :	$\overline{v^1} = \overline{v}^{\,1}$
5 :	$\overline{v^2} = 5\,\overline{v}^{\,2} - 4\,\overline{\overline{v}}$
31 :	$\overline{v^3} = 25\,\overline{v}^{\,3} - 30\,\overline{v}\,\overline{\overline{v}} + 6\,\overline{\overline{\overline{v}}}$
205 :	$\overline{v^4} = 125\,\overline{v}^{\,4} - 200\,\overline{v}^{\,2}\,\overline{\overline{v}} + 40\,\overline{v}\,\overline{\overline{\overline{v}}} + 40\,\overline{\overline{v}}^{\,2} - 4\,\overline{\overline{\overline{\overline{v}}}}$
1376 :	$\overline{v^5} = 625\,\overline{v}^{\,5} - 1250\,\overline{v}^{\,3}\,\overline{\overline{v}} + 250\,\overline{v}^{\,2}\,\overline{\overline{\overline{v}}} + 500\,\overline{v}\,\overline{\overline{v}}^{\,2} - 100\,\overline{\overline{v}}\,\overline{\overline{\overline{v}}} - 25\,\overline{v}\,\overline{\overline{\overline{\overline{v}}}} + 1\,\overline{\overline{\overline{\overline{\overline{v}}}}}$
9251 :	$\overline{v^6} = 3125\,\overline{v}^{\,6} - 7500\,\overline{v}^{\,4}\,\overline{\overline{v}} + 1500\,\overline{v}^{\,3}\,\overline{\overline{\overline{v}}} + 4500\,\overline{v}^{\,2}\,\overline{\overline{v}}^{\,2} - 1200\,\overline{v}\,\overline{\overline{v}}\,\overline{\overline{\overline{v}}} - 400\,\overline{\overline{v}}^{\,3} - 150\,\overline{v}^{\,2}\,\overline{\overline{\overline{\overline{v}}}} + 60\,\overline{\overline{v}}\,\overline{\overline{\overline{\overline{v}}}} + 60\,\overline{\overline{\overline{v}}}^{\,2} + 6\,\overline{v}\,\overline{\overline{\overline{\overline{\overline{v}}}}}66$
62210 :	$\overline{v^7} = 15625\,\overline{v}^{\,7} - 43750\,\overline{v}^{\,5}\,\overline{\overline{v}} + 8750\,\overline{v}^{\,4}\,\overline{\overline{\overline{v}}} + 35000\,\overline{v}^{\,3}\,\overline{\overline{v}}^{\,2} - 10,500\,\overline{v}^{\,2}\,\overline{\overline{v}}\,\overline{\overline{\overline{v}}} - 7000\,\overline{v}\,\overline{\overline{v}}^{\,3} + 700\,\overline{v}\,\overline{\overline{\overline{v}}}^{\,2} + 1400\,\overline{\overline{v}}^{\,2}\,\overline{\overline{\overline{v}}} - 875\,\overline{v}^{\,3}\,\overline{\overline{\overline{\overline{v}}}} + 700\,\overline{v}\,\overline{\overline{v}}\,\overline{\overline{\overline{\overline{v}}}} - 70\,\overline{\overline{\overline{v}}}\,\overline{\overline{\overline{\overline{v}}}} + 35\,\overline{v}^{\,2}\,\overline{\overline{\overline{\overline{\overline{v}}}}} - 14\,\overline{\overline{v}}\,\overline{\overline{\overline{\overline{\overline{v}}}}}$

Table $\overline{\mathbf{GW}}$.n=6

Sum of Positive Coefficients	Data Family $\mathcal{F}_6 \equiv \mathcal{W} = \{w_1,\, w_2,\, w_3,\, w_4,\, w_5,\, w_6\}$
1 :	$\overline{w^1} \;=\; \overline{w}^{\,1}$
6 :	$\overline{w^2} \;=\; 6\;\overline{w}^{\,2} - 5\,\overline{\overline{w}}$
46 :	$\overline{w^3} \;=\; 36\;\overline{w}^{\,3} - 45\,\overline{w}\;\overline{\overline{w}} + 10\,\overline{\overline{\overline{w}}}$
371 :	$\overline{w^4} \;=\; 216\,\overline{w}^{\,4} - 360\,\overline{w}^{2}\,\overline{\overline{w}} + 80\,\overline{w}\;\overline{\overline{\overline{w}}} + 75\,\overline{\overline{w}}^{\,2} - 10\,\overline{\overline{\overline{\overline{w}}}}$
3026 :	$\overline{w^5} \;=\; 1296\,\overline{w}^{\,5} - 2700\,\overline{w}^{\,3}\,\overline{\overline{w}} + 600\,\overline{w}^{\,2}\,\overline{\overline{\overline{w}}} + 1125\,\overline{w}\;\overline{\overline{w}}^{\,2} - 250\,\overline{\overline{w}}\;\overline{\overline{\overline{w}}} - 75\,\overline{w}\;\overline{\overline{\overline{\overline{w}}}} + 5\,\overline{\overline{\overline{\overline{\overline{w}}}}}$
24707 :	$\overline{w^6} \;=\; 7776\,\overline{w}^{\,6} - 19440\,\overline{w}^{\,4}\,\overline{\overline{w}} + 4320\,\overline{w}^{3}\;\overline{\overline{\overline{w}}} + 12150\;\overline{w}^{2}\;\overline{\overline{w}}^{\,2} - 3600\,\overline{w}\;\overline{\overline{w}}\;\overline{\overline{\overline{w}}} - 1125\,\overline{\overline{w}}^{\,3} - 540\,\overline{w}^{\,2}\;\overline{\overline{\overline{\overline{w}}}} + 225\,\overline{\overline{w}}\;\overline{\overline{\overline{\overline{w}}}} + 200\,\overline{\overline{\overline{w}}}^{\,2} + 36\,\overline{w}\,\overline{\overline{\overline{\overline{\overline{w}}}}} - 1\,\overline{\overline{\overline{\overline{\overline{\overline{w}}}}}}$
201748 :	$\overline{w^7} \;=\; 46656\,\overline{w}^{\,7} - 136080\,\overline{w}^{\,5}\;\overline{\overline{w}} + 30240\,\overline{w}^{\,4}\,\overline{\overline{\overline{w}}} + 113400\,\overline{w}^{\,3}\,\overline{\overline{w}}^{\,2} - 37,800\,\overline{w}^{\,2}\;\overline{\overline{w}}\;\overline{\overline{\overline{w}}} - 23625\,\overline{w}\;\overline{\overline{w}}^{\,3} + 2800\,\overline{w}\;\overline{\overline{\overline{w}}}^{\,2} + 5250\,\overline{\overline{w}}^{\,2}\;\overline{\overline{\overline{w}}} - 3780\,\overline{w}^{\,3}\,\overline{\overline{\overline{\overline{w}}}} + 3150\,\overline{w}\,\overline{\overline{w}}\,\overline{\overline{\overline{\overline{w}}}} - 350\,\overline{\overline{\overline{w}}}\;\overline{\overline{\overline{\overline{w}}}} + 252\,\overline{w}^{\,2}\,\overline{\overline{\overline{\overline{\overline{w}}}}} - 105\,\overline{\overline{w}}\;\overline{\overline{\overline{\overline{\overline{w}}}}} - 7\,\overline{w}\;\overline{\overline{\overline{\overline{\overline{\overline{w}}}}}}$

Collated, instead, along powers, the same information is:

Table $\overline{\mathbf{GW}}$.deg=2

$\lvert\mathcal{F}_n\rvert$	Sum of Positive Coefficients	Quadratic
2	2	$\overline{s^2} \;=\; 2\;\overline{s}^{\,2} - 1\,\overline{\overline{s}}$
3	3	$\overline{t^2} \;=\; 3\;\overline{t}^{\,2} - 2\,\overline{\overline{t}}$
4	4	$\overline{u^2} \;=\; 4\;\overline{u}^{\,2} - 3\,\overline{\overline{u}}$
5	5	$\overline{v^2} \;=\; 5\;\overline{v}^{\,2} - 4\,\overline{\overline{v}}$
6	6	$\overline{w^2} \;=\; 6\;\overline{w}^{\,2} - 5\,\overline{\overline{w}}$

Table $\overline{\text{GW}}$.deg=3

$\lvert\mathcal{F}_n\rvert$	Sum of $(+)$ Coefficients	Cubic
2	4	$\overline{s^3} = 4\,\overline{s}^{\,3} - 3\,\overline{s}\,\overline{\overline{s}}$
3	10	$\overline{t^3} = 9\,\overline{t}^{\,3} - 9\,\overline{t}\,\overline{\overline{t}} + \overline{\overline{\overline{t}}}$
4	19	$\overline{u^3} = 16\,\overline{u}^{\,3} - 18\,\overline{u}\,\overline{\overline{u}} + 3\,\overline{\overline{\overline{u}}}$
5	31	$\overline{v^3} = 25\,\overline{v}^{\,3} - 30\,\overline{v}\,\overline{\overline{v}} + 6\,\overline{\overline{\overline{v}}}$
6	46	$\overline{w^3} = 36\,\overline{w}^{\,3} - 45\,\overline{w}\,\overline{\overline{w}} + 10\,\overline{\overline{\overline{w}}}$

Table $\overline{\text{GW}}$.deg=4

$\lvert\mathcal{F}_n\rvert$	Sum of $(+)$ Coefficients	Quartic
2	9	$\overline{s^4} = 8\,\overline{s}^{\,4} - 8\,\overline{s}^{\,2}\,\overline{\overline{s}} + 1\,\overline{\overline{s}}^{\,2}$
3	37	$\overline{t^4} = 27\,\overline{t}^{\,4} - 36\,\overline{t}^{\,2}\,\overline{\overline{t}} + 4\,\overline{t}\,\overline{\overline{\overline{t}}} + 6\,\overline{\overline{t}}^{\,2}$
4	98	$\overline{u^4} = 64\,\overline{u}^{\,4} - 96\,\overline{u}^{\,2}\,\overline{\overline{u}} + 16\,\overline{u}\,\overline{\overline{\overline{u}}} + 18\,\overline{\overline{u}}^{\,2} - \overline{\overline{\overline{\overline{u}}}}$
5	205	$\overline{v^4} = 125\,\overline{v}^{\,4} - 200\,\overline{v}^{\,2}\,\overline{\overline{v}} + 40\,\overline{v}\,\overline{\overline{\overline{v}}} + 40\,\overline{\overline{v}}^{\,2} - 4\,\overline{\overline{\overline{\overline{v}}}}$
6	371	$\overline{w^4} = 216\,\overline{w}^{\,4} - 360\,\overline{w}^{\,2}\,\overline{\overline{w}} + 80\,\overline{w}\,\overline{\overline{\overline{w}}} + 75\,\overline{\overline{w}}^{\,2} - 10\,\overline{\overline{\overline{\overline{w}}}}$

Table $\overline{\text{GW}}$.deg=5

$\lvert\mathcal{F}_n\rvert$	Sum of $(+)$ Coefficients	Quintic
2	21	$\overline{s^5} = 16\,\overline{s}^{\,5} - 20\,\overline{s}^{\,3}\,\overline{\overline{s}} + 5\,\overline{s}\,\overline{\overline{s}}^{\,2}$
3	141	$\overline{t^5} = 81\,\overline{t}^{\,5} - 135\,\overline{t}^{\,3}\,\overline{\overline{t}} + 15\,\overline{t}^{\,2}\,\overline{\overline{\overline{t}}} + 45\,\overline{t}\,\overline{\overline{t}}^{\,2} - 5\,\overline{\overline{t}}\,\overline{\overline{\overline{t}}}$
4	516	$\overline{u^5} = 256\,\overline{u}^{\,5} - 480\,\overline{u}^{\,3}\,\overline{\overline{u}} + 80\,\overline{u}^{\,2}\,\overline{\overline{\overline{u}}} + 180\,\overline{u}\,\overline{\overline{u}}^{\,2} - 30\,\overline{\overline{u}}\,\overline{\overline{\overline{u}}} - 5\,\overline{u}\,\overline{\overline{\overline{\overline{u}}}}$
5	1376	$\overline{v^5} = 625\,\overline{v}^{\,5} - 1250\,\overline{v}^{\,3}\,\overline{\overline{v}} + 250\,\overline{v}^{\,2}\,\overline{\overline{\overline{v}}} + 500\,\overline{v}\,\overline{\overline{v}}^{\,2} - 100\,\overline{\overline{v}}\,\overline{\overline{\overline{v}}} - 25\,\overline{v}\,\overline{\overline{\overline{\overline{v}}}} + 1\,\overline{\overline{\overline{\overline{\overline{v}}}}}$
6	3026	$\overline{w^5} = 1296\,\overline{w}^{\,5} - 2700\,\overline{w}^{\,3}\,\overline{\overline{w}} + 600\,\overline{w}^{\,2}\,\overline{\overline{\overline{w}}} + 1125\,\overline{w}\,\overline{\overline{w}}^{\,2} - 250\,\overline{\overline{w}}\,\overline{\overline{\overline{w}}} - 75\,\overline{w}\,\overline{\overline{\overline{\overline{w}}}} + 5\,\overline{\overline{\overline{\overline{\overline{w}}}}}$

Table $\overline{\text{GW}}$.deg=6

$\lvert\mathcal{F}_n\rvert$	Sum of $(+)$ Coefficients	Sextic
2	50	$\overline{s^6} = 32\,\overline{s}^{\,6} - 48\,\overline{s}^{\,4}\,\overline{\overline{s}} + 18\,\overline{s}^{\,2}\,\overline{\overline{s}}^{\,2} - \overline{\overline{s}}^{\,3}$
3	541	$\overline{t^6} = 243\,\overline{t}^{\,6} - 486\,\overline{t}^{\,4}\,\overline{\overline{t}} + 54\,\overline{t}^{\,3}\,\overline{\overline{\overline{t}}} -$ $243\,\overline{t}^{\,2}\,\overline{\overline{t}}^{\,2} - 36\,\overline{t}\,\overline{\overline{t}}\,\overline{\overline{\overline{t}}} - 18\,\overline{\overline{t}}^{\,3}$
4	2725	$\overline{u^6} = 1024\,\overline{u}^{\,6} - 2304\,\overline{u}^{\,4}\,\overline{\overline{u}} + 384\,\overline{u}^{\,3}\,\overline{\overline{\overline{u}}} +$ $1296\,\overline{u}^{\,2}\,\overline{\overline{u}}^{\,2} - 288\,\overline{u}\,\overline{\overline{u}}\,\overline{\overline{\overline{u}}} -$ $108\,\overline{\overline{u}}^{\,3} - 24\,\overline{u}^{\,2}\,\overline{\overline{\overline{\overline{u}}}} + 9\,\overline{\overline{u}}\,\overline{\overline{\overline{\overline{u}}}} + 12\,\overline{\overline{\overline{u}}}^{\,2}$
5	9251	$\overline{v^6} = 3125\,\overline{v}^{\,6} - 7500\,\overline{v}^{\,4}\,\overline{\overline{v}} + 1500\,\overline{v}^{\,3}\,\overline{\overline{\overline{v}}} +$ $4500\,\overline{v}^{\,2}\,\overline{\overline{v}}^{\,2} - 1200\,\overline{v}\,\overline{\overline{v}}\,\overline{\overline{\overline{v}}} - 400\,\overline{\overline{v}}^{\,3} -$ $150\,\overline{v}^{\,2}\,\overline{\overline{\overline{\overline{v}}}} + 60\,\overline{\overline{v}}\,\overline{\overline{\overline{\overline{v}}}} + 60\,\overline{\overline{\overline{v}}}^{\,2} + 6\,\overline{v}\,\overline{\overline{\overline{\overline{\overline{v}}}}}$
6	24707	$\overline{w^6} = 7776\,\overline{w}^{\,6} - 19440\,\overline{w}^{\,4}\,\overline{\overline{w}} + 4320\,\overline{w}^{\,3}\,\overline{\overline{\overline{w}}} +$ $12150\,\overline{w}^{\,2}\,\overline{\overline{w}}^{\,2} - 3600\,\overline{w}\,\overline{\overline{w}}\,\overline{\overline{\overline{w}}} - 1125\,\overline{\overline{w}}^{\,3} -$ $540\,\overline{w}^{\,2}\,\overline{\overline{\overline{\overline{w}}}} + 225\,\overline{\overline{w}}\,\overline{\overline{\overline{\overline{w}}}} + 200\,\overline{\overline{\overline{w}}}^{\,2} + 36\,\overline{w}\,\overline{\overline{\overline{\overline{\overline{w}}}}} - 1\,\overline{\overline{\overline{\overline{\overline{\overline{w}}}}}}$

Table $\overline{\text{GW}}$.deg=7

$\lvert\mathcal{F}_n\rvert$	Sum of $(+)$ Coefficients	Septic
2	120	$\overline{s^7} = 64\,\overline{s}^{\,7} - 112\,\overline{s}^{\,5}\,\overline{\overline{s}} + 56\,\overline{s}^{\,3}\,\overline{\overline{s}}^{\,2} - 7\,\overline{s}\,\overline{\overline{s}}^{\,3}$
3	2080	$\overline{t^7} = 7294\,\overline{t}^{\,7} - 1701\,\overline{t}^{\,5}\,\overline{\overline{t}} + 189\,\overline{t}^{\,4}\,\overline{\overline{\overline{t}}} + 1134\,\overline{t}^{\,3}\,\overline{\overline{t}}^{\,2} -$ $189\,\overline{t}^{\,2}\,\overline{\overline{t}}\,\overline{\overline{\overline{t}}} - 189\,\overline{t}\,\overline{\overline{t}}^{\,3} + 7\,\overline{t}\,\overline{\overline{\overline{t}}}^{\,2} + 21\,\overline{\overline{t}}^{\,2}\,\overline{\overline{\overline{t}}}$
4	14400	$\overline{u^7} = 4096\,\overline{u}^{\,7} - 10752\,\overline{u}^{\,5}\,\overline{\overline{u}} + 1792\,\overline{u}^{\,4}\,\overline{\overline{\overline{u}}} +$ $112 \cdot 72\,\overline{u}^{\,3}\,\overline{\overline{u}}^{\,2} - 2016\,\overline{u}^{\,2}\,\overline{\overline{u}}\,\overline{\overline{\overline{u}}} - 1512\,\overline{u}\,\overline{\overline{u}}^{\,3} +$ $112\,\overline{u}\,\overline{\overline{\overline{u}}}^{\,2} + 252\,\overline{\overline{u}}^{\,2}\,\overline{\overline{\overline{u}}} - 112\,\overline{u}^{\,3}\,\overline{\overline{\overline{\overline{u}}}} + 84\,\overline{u}\,\overline{\overline{u}}\,\overline{\overline{\overline{\overline{u}}}} - 7\,\overline{\overline{\overline{u}}}\,\overline{\overline{\overline{\overline{u}}}}$
5	62210	$\overline{v^7} = 15625\,\overline{v}^{\,7} - 43750\,\overline{v}^{\,5}\,\overline{\overline{v}} + 8750\,\overline{v}^{\,4}\,\overline{\overline{\overline{v}}} + 35000\,\overline{v}^{\,3}\,\overline{\overline{v}}^{\,2} -$ $10,500\,\overline{v}^{\,2}\,\overline{\overline{v}}\,\overline{\overline{\overline{v}}} - 7000\,\overline{v}\,\overline{\overline{v}}^{\,3} + 700\,\overline{v}\,\overline{\overline{\overline{v}}}^{\,2} + 1400\,\overline{\overline{v}}^{\,2}\,\overline{\overline{\overline{v}}} -$ $875\,\overline{v}^{\,3}\,\overline{\overline{\overline{\overline{v}}}} + 700\,\overline{v}\,\overline{\overline{v}}\,\overline{\overline{\overline{\overline{v}}}} - 70\,\overline{\overline{\overline{v}}}\,\overline{\overline{\overline{\overline{v}}}} + 35\,\overline{v}^{\,2}\,\overline{\overline{\overline{\overline{\overline{v}}}}} - 14\,\overline{\overline{v}}\,\overline{\overline{\overline{\overline{\overline{v}}}}}$
6	201748	$\overline{w^7} = 46656\,\overline{w}^{\,7} - 136080\,\overline{w}^{\,5}\,\overline{\overline{w}} + 30240\,\overline{w}^{\,4}\,\overline{\overline{\overline{w}}} +$ $113400\,\overline{w}^{\,3}\,\overline{\overline{w}}^{\,2} - 37,800\,\overline{w}^{\,2}\,\overline{\overline{w}}\,\overline{\overline{\overline{w}}} - 23625\,\overline{w}\,\overline{\overline{w}}^{\,3} +$ $2800\,\overline{w}\,\overline{\overline{\overline{w}}}^{\,2} + 5250\,\overline{\overline{w}}^{\,2}\,\overline{\overline{\overline{w}}} - 3780\,\overline{w}^{\,3}\,\overline{\overline{\overline{\overline{w}}}} +$ $3150\,\overline{w}\,\overline{\overline{w}}\,\overline{\overline{\overline{\overline{w}}}} - 350\,\overline{\overline{\overline{w}}}\,\overline{\overline{\overline{\overline{w}}}} + 252\,\overline{w}^{\,2}\,\overline{\overline{\overline{\overline{\overline{w}}}}} - 105\,\overline{\overline{w}}\,\overline{\overline{\overline{\overline{\overline{w}}}}} - 7\,\overline{w}\,\overline{\overline{\overline{\overline{\overline{\overline{w}}}}}}$

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