

SCHUBERT PUZZLES AND INTEGRABILITY I: INVARIANT TRILINEAR FORMS

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ABSTRACT. The *puzzle rules* for computing Schubert calculus on d -step flag manifolds, proven in [KT03] for 1-step, in [BKPT16] for 2-step, and conjectured in [CV09] for 3-step, lead to vector configurations (one vector for each puzzle edge label) that we recognize as the weights of some minuscule representations. The R -matrices of those representations (which, for 2-step flag manifolds, involve triality of D_4) degenerate to give us puzzle formulae for two previously unsolved Schubert calculus problems: K_T (2-step flag manifolds) and K (3-step flag manifolds). The K (3-step flag manifolds) formula, which involves 151 new puzzle pieces, implies Buch's correction to the first author's 1999 conjecture for H^* (3-step flag manifolds).

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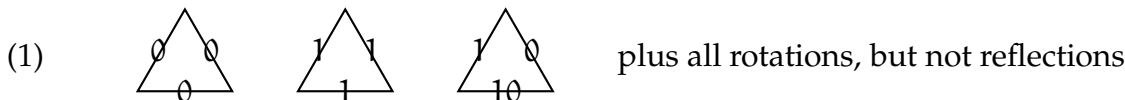
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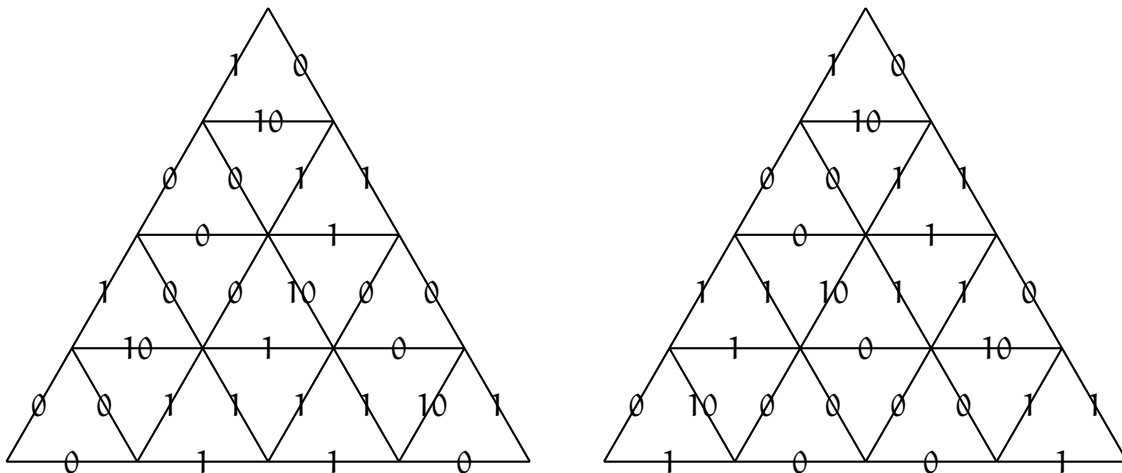
1. INTRODUCTION

In 1912 Weyl asked what the spectrum of a sum of two Hermitian matrices could be, knowing the spectra of the matrices individually. This received two solutions in the 1990s (we recommend the surveys [Ful98, Zel99]): a geometric one due to Klyachko (extending work of Helmke–Rosenthal, Hersch–Zwahlen, and Totaro, among others) based on Schubert calculus of Grassmannians, and a combinatorial one [KTW04] in terms of certain triangular tilings called “puzzles”. We take from this coincidence the oracular statement that *puzzles should be related to Schubert calculus*; hopefully not just for ordinary cohomology of Grassmannians, but also for more exotic cohomology theories of **d-step flag manifolds** $\{V_1 \leq V_2 \leq \dots \leq V_d \leq \mathbb{C}^n : \dim V_i = n_i\}$. Many Schubert calculus problems have been solved with puzzles, some in no other ways; see [KT03, Vak06, BKPT16, KP11, Buc15, PY15, WZJ16a, WZJ16b].

1.1. Puzzles for the $d = 1$ case, of Grassmannians. We will index the Schubert classes $\{S^\lambda\}$ on $P_- \backslash GL_n(\mathbb{C})$ (precise definitions given in §2) by strings λ in $0, 1, \dots, d$, where the number of j s in the string is $n_{j+1} - n_j$ (by convention $n_0 = 0, n_{1+d} = n$). In particular those on the Grassmannian $Gr_k(\mathbb{C}^n)$ of k -planes have¹ content $0^k 1^{n-k}$. We define three **edge labels** $L_1 := \{0, 1, 10\}$, and three **puzzle pieces**:



A **Grassmannian puzzle of size n** is a size n triangle (oriented like Δ) decomposed into n^2 puzzle pieces, whose boundary labels are only 0 or 1, not 10. For example, here are the two puzzles whose NW and NE boundaries are both labeled 0, 1, 0, 1:



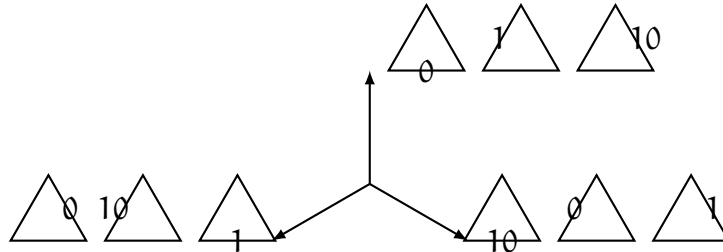
The connection to **Schubert calculus**, the study of the product of Schubert classes $\{S^\lambda\}$, is simple [KTW04]:

$$(2) \quad S^\lambda S^\mu = \sum_{\nu} \# \left(\text{puzzles labeled } \begin{array}{c} \lambda \quad \mu \\ \nu \end{array} \text{ left to right (not clockwise)} \right) S^\nu$$

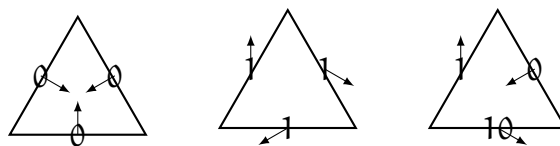
¹One can get away with the dual convention $k \leftrightarrow n - k$ when dealing with only Grassmannians, but this becomes untenable when going to higher d .

In particular, the right-hand side would be nonsensical if λ, μ, ν were strings with different content (numbers of 0s and 1s). Here is one proof that that can't happen:

Lemma 1. *To each edge of a Δ triangle in a puzzle P , we assign a 2-dimensional vector, rotation-equivariantly:*



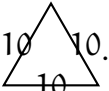
Then every puzzle piece has vector sum $\vec{0}$:



To edges of ∇ triangles, we assign the negatives of these vectors, obtaining a cancelation at each internal edge of P .

Then by Green's theorem the boundary of P has vector sum $\vec{0}$, which forces the content (the numbers of 0s and 1s) on the three sides to be the same.

The original result [KTW04] about ordinary cohomology of Grassmannians has been generalized (for Grassmannians) to equivariant cohomology [KT03, ZJ09a], to K-theory [Vak06], and recently to equivariant K-theory [PY15, WZJ16b], in each case adding some pieces. Our methods in this paper are closest to those of [WZJ16b], which recognizes the three vectors above as the *weights of the standard representation* $V_{\omega_1}^{\Lambda_2}$ of $SL_3(\mathbb{C})$ and makes use of the corresponding R-matrix and the Yang–Baxter equation it satisfies. However, our proof in §3 is cleaner than that of [WZJ16b] even in the $d = 1$ case, through exploitation of the bootstrap equation of the relevant R-matrix.

There is another way to label the edges of a Δ and still get vector sum $\vec{0}$: 

And indeed, if we add this piece *but only in the Δ orientation*² we get a rule for the K-theory product, with respect to the basis of structure sheaves of Schubert varieties [Vak06, theorem 4.6]; whereas if we add this piece *but only in the ∇ orientation* we get a rule for the K-theory product, with respect to the dual basis (the ideal sheaves of boundaries of Schubert varieties) [WZJ16b].

To be precise, equation (2) is modified in this K-theory product, because each K-piece carries a **fugacity**³ of -1 , and the **fugacity of a puzzle** is defined as the product of the fugacities of its pieces. Then we sum the fugacities of the puzzles, rather than simply counting them. Luckily, in this formula all the puzzle fugacities contributing to a given coefficient have the same sign so there is no cancelation.

²If one includes both orientations, the resulting product rule is again commutative associative. We will address this in a future publication [KZJ].

³In prior publications it seemed natural to call this a “weight”, but that might cause confusion with the “weights” we assign to edge labels, and “fugacity” is a standard term in the physics literature for such local contributions.

An important difference between $K(\mathrm{Gr}_k(\mathbb{C}^n))$ and $H^*(\mathrm{Gr}_k(\mathbb{C}^n))$ is that the latter is a graded ring. Equation (2) for H^* then implies that $\ell(\lambda) + \ell(\mu) = \ell(\nu)$, where

$$\ell(\kappa) := \#\{i < j : \kappa_i > \kappa_j\} = \frac{1}{2} \deg S^\kappa$$

is the number of **inversions** of the string κ .

In §2.7 we define the “inversion number” of a path through a puzzle, and in particular, the inversion number of (a clockwise path around) a puzzle piece. If two paths γ, γ' through a puzzle have the same start and end points, then the difference of their inversion numbers is proven in §2.7 to be the sum of the inversion numbers of the pieces in between γ and γ' .

It is straightforward to calculate the inversion numbers of puzzle pieces; the ones in (1) have vanishing inversion number, whereas the K-pieces have inversion number 1. We'll

always draw a piece with its inversion number in the center, e.g. $\begin{array}{c} 10 \textcircled{1} 10 \\ \hline 10 \end{array}$, omitting it only when the inversion number is zero.

1.2. Puzzles for $d = 2$ flag manifolds. In 1999 the first author trusted the oracle and looked for puzzles to compute Schubert calculus of 2-step flag manifolds, where the Schubert classes (hence the boundaries of the puzzles) are strings in 0, 1, 2. Already with puzzles of edge-length 3, one discovers the following eight labels L_2 and puzzle pieces:

$$(3) \quad \begin{array}{c} \triangle \\ \textcircled{0} \quad \textcircled{0} \\ \hline \textcircled{0} \end{array} \quad \begin{array}{c} \triangle \\ \textcircled{1} \quad \textcircled{1} \\ \hline \textcircled{1} \end{array} \quad \begin{array}{c} \triangle \\ \textcircled{2} \quad \textcircled{2} \\ \hline \textcircled{2} \end{array} \quad \begin{array}{c} \triangle \\ \textcircled{1} \quad \textcircled{0} \\ \hline \textcircled{10} \end{array} \quad \begin{array}{c} \triangle \\ \textcircled{2} \quad \textcircled{0} \\ \hline \textcircled{20} \end{array} \quad \begin{array}{c} \triangle \\ \textcircled{2} \quad \textcircled{1} \\ \hline \textcircled{21} \end{array} \quad \begin{array}{c} \triangle \\ \textcircled{21} \quad \textcircled{0} \\ \hline \textcircled{(21)0} \end{array} \quad \begin{array}{c} \triangle \\ \textcircled{2} \quad \textcircled{10} \\ \hline \textcircled{2(10)} \end{array}$$

(again permitting rotations but not reflections). We will call parenthesized expressions X such as these (really encoding planar binary trees with labeled leaves) **multinumbers**, and call a multinumber X **valid** if it appears in a puzzle rule (i.e., on one of the puzzle piece edges here in (3), or later in §1.3). Write $|X|$ for the number of digits in X (i.e. $|X| = 1$ exactly for “single-numbers”). Proving the analogue of equation (2) for $d = 2$, and its generalization to equivariant cohomology, took until 2014 [BKPT16, Buc15].

The $d = 2$ analogue of lemma 1 is an assignment with values in D_4 's weight lattice of row vectors $\{(a_1, a_2, a_3, a_4) : 2a_i \in \mathbb{Z}, a_i - a_j \in \mathbb{Z}\}$. We regard the (counterclockwise) 120° rotation τ as acting on this lattice on the right by

$$\tau = \frac{1}{2} \begin{bmatrix} - & - & - & - \\ + & - & - & + \\ + & + & - & - \\ + & - & + & - \end{bmatrix}$$

and mention that the corresponding automorphism of the Lie algebra D_4 is outer, the triality automorphism.

Lemma 2. *To each of the NW labels L_2 in the puzzle pieces above, we assign a weight in the D_4 weight lattice:*

$$\begin{array}{cccc} 2 \mapsto (+, 0, 0, 0) & 1 \mapsto (0, +, 0, 0) & 0 \mapsto (0, 0, +, 0) & 10 \mapsto (0, 0, 0, +) \\ (21)0 \mapsto (-, 0, 0, 0) & 20 \mapsto (0, -, 0, 0) & 2(10) \mapsto (0, 0, -, 0) & 21 \mapsto (0, 0, 0, -) \end{array}$$

and assign D_4 weights to labels on other sides in a rotation-equivariant way, using the 4×4 matrix above.

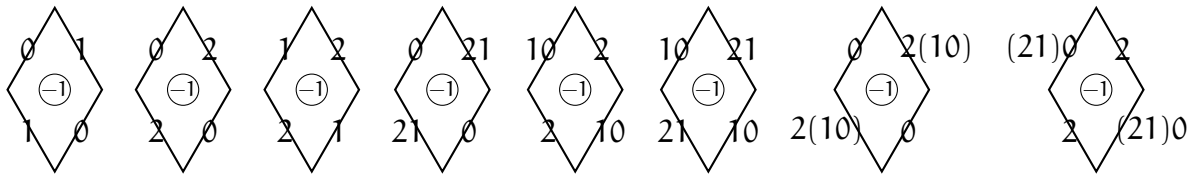
Then every puzzle piece has vector sum $\vec{0}$. By Green's theorem the boundary of the whole puzzle does too, which forces the content (the numbers of 0s, 1s, and 2s) on the three sides to be the same.

These eight vectors are the weights of the standard representation $V_{\omega_1}^{D_4}$ of $\mathfrak{so}(8)$, whose $\mathbb{Z}/3$ rotations are the two spin representations $\tau \cdot V_{\omega_1}^{D_4} \cong V_{\omega_2}^{D_4}$ and $\tau^2 \cdot V_{\omega_1}^{D_4} \cong V_{\omega_3}^{D_4}$.

At this point, the assignments in lemmas 1 and 2 may look quite magical. In §2 we will explain a recipe that produces them without effort, at least up to $d \leq 4$.

With the clue provided by lemma 2, the methods of [WZJ16b] generalize straightforwardly to give our first major theorem of this paper. To state it, we first recall

Theorem 1. [Buc15][KT03, for $d = 1$] *To compute Schubert calculus in T -equivariant cohomology of a 2-step flag manifold, whose coefficients live in $H_T^*(pt) := \mathbb{Z}[y_1, \dots, y_n]$, one needs the following **equivariant rhombi**:*



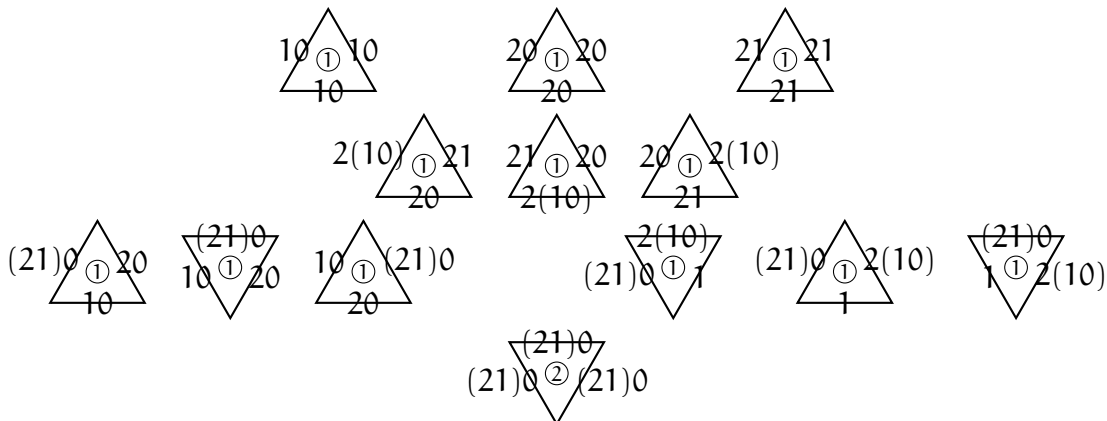
These may not be rotated. The **fugacity of an equivariant rhombus** is $y_i - y_j \in H_T^*(pt)$, where lines drawn Southwest and Southeast from the rhombus hit the bottom edge at positions $i < j$. The **fugacity $\text{fug}(P)$ of a puzzle P** is the product of the fugacities of its rhombi. Then

$$(4) \quad S^\lambda S^\mu = \sum_{\nu} \left(\sum \left\{ \text{fug}(P) : \text{puzzles } P \text{ with boundary } \begin{array}{c} \lambda \quad \mu \\ \nu \end{array} \right\} \right) S^\nu$$

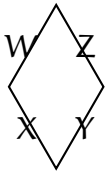
The -1 s in these pieces are to indicate that they, too, spoil lemma 5's inversion count, but in the opposite way from K-theory; more detail will come in §2.7. This formula (4) is positive in the sense of [Gra01].

We come to our first major new result of the paper, the extension of the above to T -equivariant K-theory, for which there was no known positive formula (in the sense of [AGM11]) or even extant conjecture for the structure constants. These live in $K_T(pt) \cong \mathbb{Z}[u_1^\pm, \dots, u_n^\pm]$.

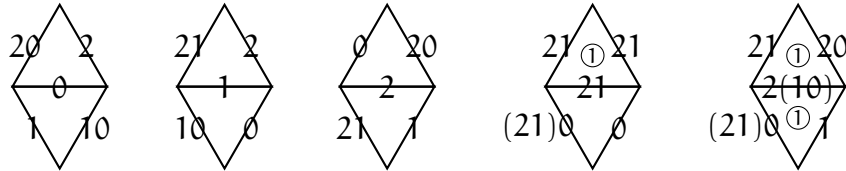
Theorem 2. *Equation (4) generalizes to compute Schubert calculus in equivariant K-theory of a 2-step flag manifold. One needs the equivariant rhombi from theorem 1, whose fugacities are now $1 - u_j/u_i$, and also the following **K-pieces**:*



(no other rotations!) each with fugacity -1 , or $(-1)^2$ for the last one.

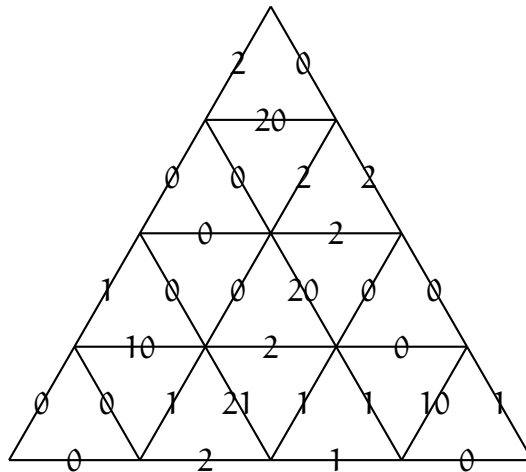
Some vertical rhombi  also acquire an extra monomial fugacity u_j/u_i (which does not

factor as a product over triangles): a rhombus gets a u_j/u_i if $|W| + |Z| > |X| + |Y|$, or (in the case $|W| + |Z| = |X| + |Y|$) if equal to

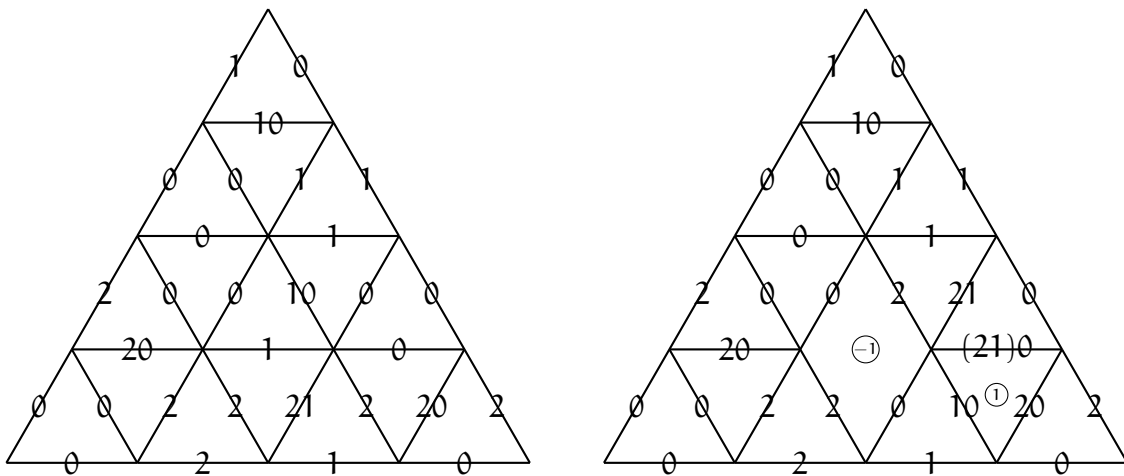


In particular, rotation-invariant rhombi (such as equivariant rhombi) get no u_j/u_i , and among a rhombus and its 180° rotation, assuming they're distinct and valid (i.e., not involving unrotatable K-pieces), exactly one gets a u_j/u_i .

Example. $c_{0210}^{0102,0201}$ is computed from one puzzle



whose fugacity is u_3/u_2 , whereas the opposite multiplication $c_{0210}^{0201,0102}$ comes from two puzzles



whose fugacities are 1 and $(-1)(1 - u_3/u_2)$, respectively.

Theorem 1 is recovered by writing $u_i = \exp(y_i)$ and expanding at first nontrivial order in the y_i .

Define **dual Schubert classes** $\{S_\lambda\}$ by $\langle S^\lambda S_\mu \rangle = \delta_{\mu, \lambda}$, where $\langle \rangle$ is the K-theoretic analogue of integration, definable by $\langle S^\lambda \rangle = 1 \forall \lambda$. Then theorem 2 also provides a dual puzzle rule

$$S_\lambda S_\mu = \sum_\nu \left(\sum \left\{ \text{fug}(P) : \text{puzzles } P \text{ with boundary } \begin{array}{c} \nu \\ \mu \quad \lambda \end{array} \right\} \right) S_\nu$$

where boundary labels are again read left-to-right.

For future reference, we comment that the tensor square of $V_{\omega_1}^{A_2} \cong \tau \cdot V_{\omega_1}^{A_2}$ is

$$(V_{\omega_1}^{A_2})^{\otimes 2} \cong V_{2\omega_1}^{A_2} \oplus (V_{\omega_1}^{A_2})^*$$

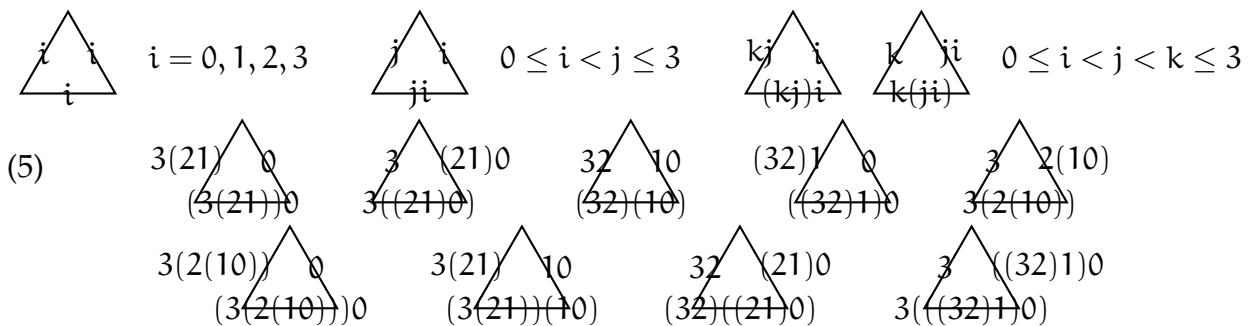
and similarly, the tensor product of two of D_4 's minuscule fundamental irreps is

$$V_{\omega_1}^{D_4} \otimes V_{\omega_2}^{D_4} \cong V_{\omega_1 + \omega_2}^{D_4} \oplus (V_{\omega_3}^{D_4})^* \quad (\text{though note, } (V_{\omega_3}^{D_4})^* \cong V_{\omega_3}^{D_4})$$

We get away with tensoring $V_{\omega_1}^{A_2}$ with "itself" because the action of $\mathbb{Z}/3$ on the A_2 lattice is by an inner automorphism, but with D_4 we must face up to the three different representations associated with the three edge orientations.

Our rules for $K_T(\text{Gr}_k(\mathbb{C}^n))$ and $H_T^*(2\text{-step})$ have a duality symmetry: one may reflect a puzzle left-right while replacing each number i with $d - i$. This $K_T(2\text{-step})$ rule does not, directly, enjoy this symmetry (even though the structure constants do); rather this gives us a second puzzle rule for $K_T(2\text{-step})$.

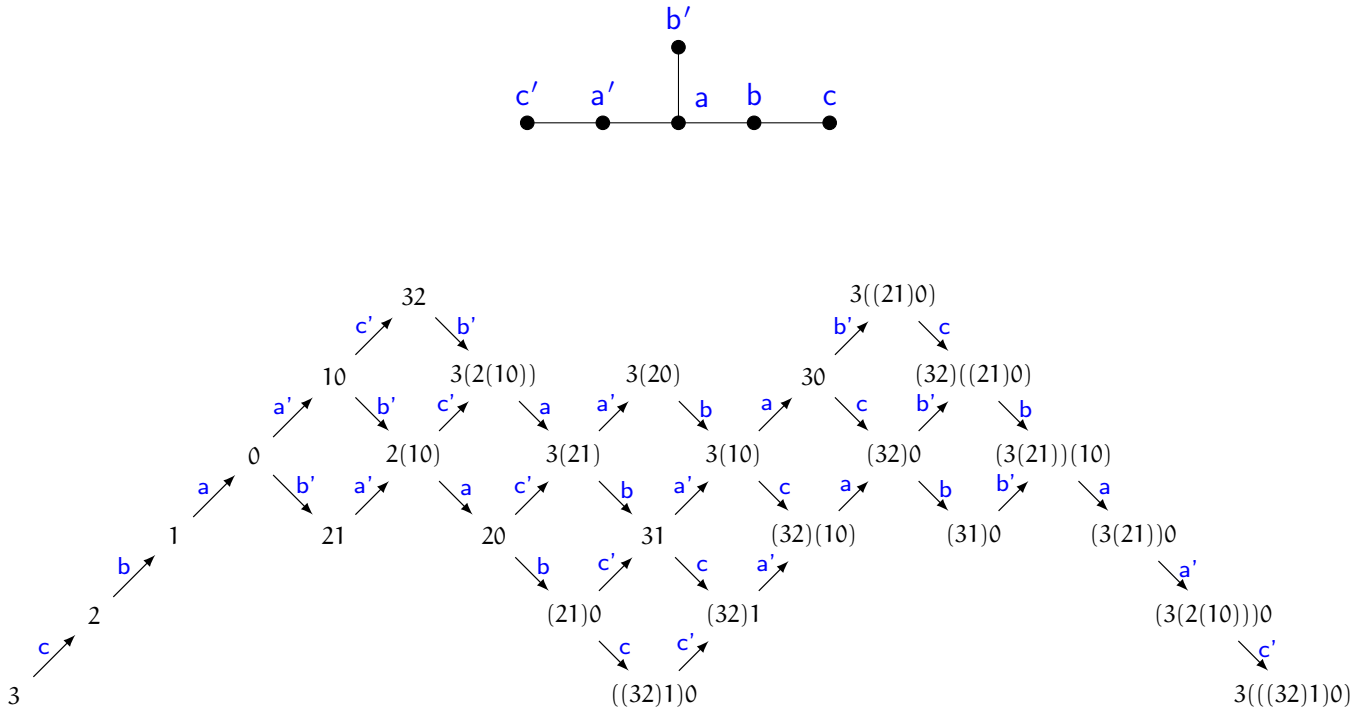
1.3. Puzzles for $d = 3$ flag manifolds. The $d = 3$ labels and pieces turn out to be (up to rotation)



where the bottom four are Buch's correction [CV09] to the first author's incomplete 1999 conjecture.

Lemma 3. *There is an action of $\mathbb{Z}/3$ on E_6 's weight lattice, and a map from the 27 edge labels above to the weights of the standard representation $V_{\omega_1}^{E_6}$ of E_6 (then extended to other edge orientations $\mathbb{Z}/3$ -equivariantly), such that each puzzle piece has vector sum $\vec{0}$. By Green's theorem the boundary of the whole puzzle does too, which forces the content (the numbers of 0s, 1s, 2s, and 3s) on the three sides to be the same.*

Rather than writing out the vectors, we give the correspondence with the crystal of the 27-dimensional representation, whose edges are labeled by the simple roots of E_6 :



In this case the techniques of [WZ]16b) do *not* work right out of the box; during the degeneration there is a divergence, due to the proliferation of equivariant rhombi with negative inversion number. It also seems related to the fact that

$$(V_{\omega_1}^{E_6})^{\otimes 2} \cong V_{2\omega_1}^{E_6} \oplus (V_{\omega_1}^{E_6})^* \oplus V_{\omega_3}^{E_6}$$

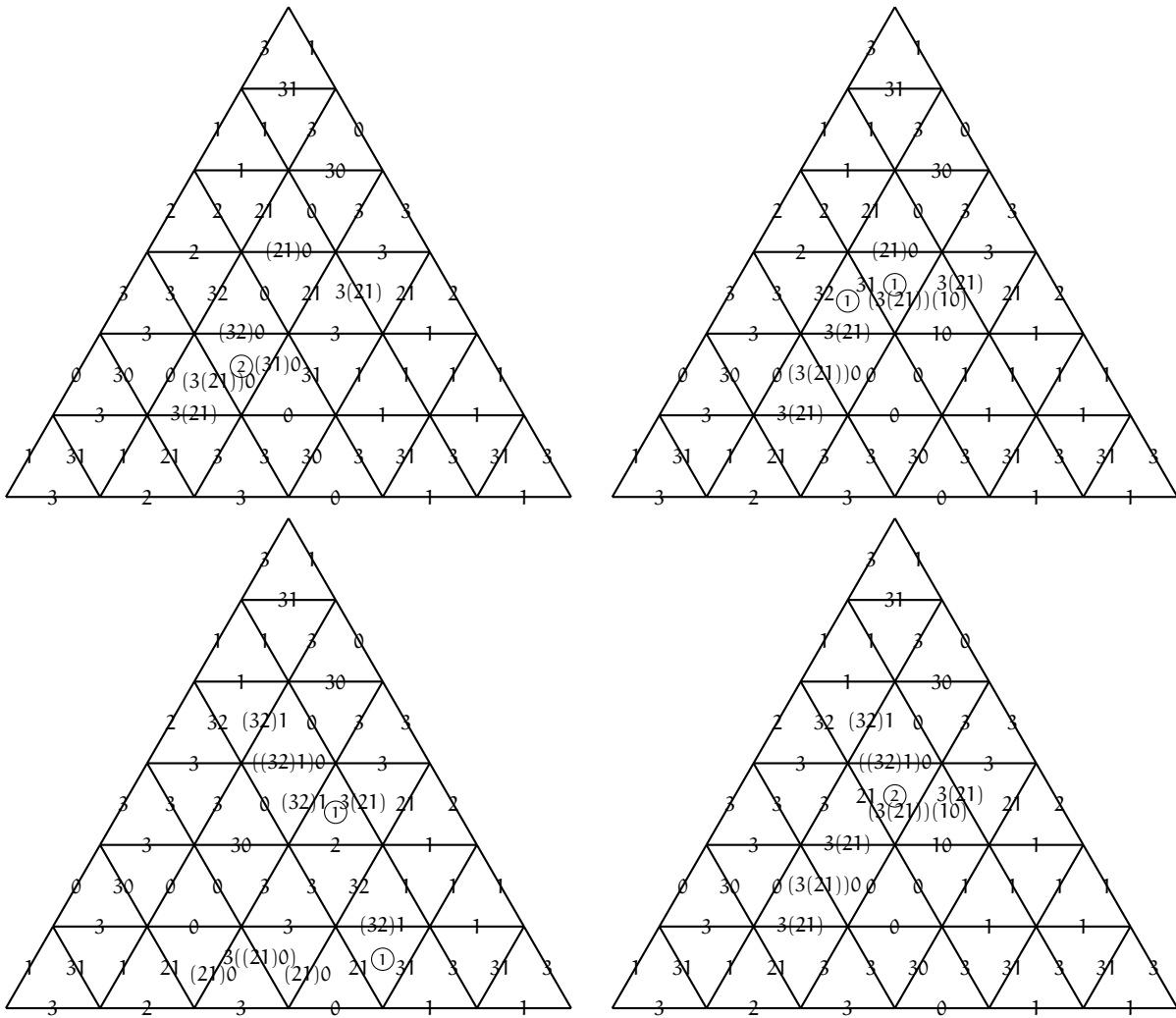
has three summands (the Cartan product, the invariant trilinear form, and another) unlike the $d = 1, 2$ cases that had only two summands.

We can partially rescue the argument, but at the cost of T-equivariance. This gives our other main result:

Theorem 3. *Buch’s correction holds: the puzzle pieces above (and their rotations) correctly compute Schubert calculus in the ordinary cohomology of 3-step flag manifolds.*

One can also obtain formulæ for Schubert calculus in the K-theory of 3-step flag manifolds, with respect to either the standard basis (of structure sheaves) or the dual basis (of ideal sheaves of the boundaries). This requires another 151 pieces, which we list in appendix B.

Example. $c_{323011}^{103213, 103213} = 4$ (in nonequivariant K-theory), as counted by the following puzzles:



The situation is even worse for $d = 4$, in the sense that even nonequivariant triangles with negative inversion number proliferate, leading to more divergences. We shall not formulate any puzzle rule for $d \geq 4$ in the present paper.

In all three cases $d = 1, 2, 3$, as observed above, there exists a trilinear invariant form acting on a tensor product of three minuscule representations of A_2, D_4, E_6 , related by an automorphism of order 3 (whose action on the weight lattice is given by τ). In fact, we shall see that the nonequivariant puzzle rules are nothing but a diagrammatic description of (a degenerate, and in K-theory, q -deformed, version of) this trilinear invariant form.

1.4. Plan of the paper. In §2, after providing basic definitions (§2.1), we lay out the general construction of the vectors \vec{f}_X we assign to multinumbers X (§2.3), from which we derive the A_2, D_4, E_6 representations above (§2.6). We also explain the counting of inversions (§2.7). §3 is concerned with the proofs of the main results: first, a general framework is introduced, leading to our main theorem (§3.4), which is then applied to $d = 2$ equivariant Schubert calculus in §3.5, and to $d = 3$ nonequivariant Schubert calculus in §3.7, thus proving theorems 2 and 3.

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2. THE GENERAL CONSTRUCTION

2.1. Schubert classes. Let $B_{\pm} \leq GL_n(\mathbb{C})$ denote the upper/lower triangular matrices, with intersection T the diagonal matrices, and let $P_- \geq B_-$ be a standard parabolic with Levi factor $\prod_{i=0}^d GL_{p_i}(\mathbb{C})$. Our Schubert cycles $X^w := P_- \backslash \overline{P_- w B_+}$ are right B_+ -orbit closures on $P_- \backslash GL_n(\mathbb{C})$, indexed by $\prod_{i=0}^d \mathcal{S}_{p_i} \backslash \mathcal{S}_n$, which we identify with strings of length n in an ordered alphabet with multiplicities given by the (p_i) . The codimension of the cycle is given by the number of inversions in the string. Schubert cycles define classes in cohomology, T -equivariant cohomology, K -theory, and T -equivariant K -theory.

In each case, these classes form a basis for whatever sort of cohomology group, considered as a module over the same cohomology of a point: $H(\text{pt}) \cong K(\text{pt}) \cong \mathbb{Z}$, $H_T(\text{pt}) \cong \mathbb{Z}[y_1, \dots, y_n]$, $K_T(\text{pt}) \cong \mathbb{Z}[u_1^{\pm}, \dots, u_n^{\pm}]$. Our goal is to compute as in (4) the coefficients, living in this base ring, of the expansion in the basis of the product of two basis elements.

One benefit of working equivariantly is that we can replace equation (4) with its restriction to T -fixed points, as this restriction operation is well-known to be injective. The equation becomes a *list* of equations between elements of the base ring K_T , one for each T -fixed point σ :

$$(6) \quad S^{\lambda}|_{\sigma} S^{\mu}|_{\sigma} = \sum_{\nu} \left(\sum \left\{ \text{fug}(P) : \text{puzzles } P \text{ with boundary } \begin{array}{c} \lambda \quad \mu \\ \nu \end{array} \right\} \right) S^{\nu}|_{\sigma}$$

These restrictions $S^{\pi}|_{\sigma}$ themselves have interpretations as state sums, much like puzzles, and in §3.4 it is this equation (6) we will find most amenable to the framework of quantum integrable systems.

2.2. Functoriality. Consider d -step flags $V_1 \leq V_2 \leq \dots \leq V_d \leq \mathbb{C}^n$, $\dim V_i = n_i$, and *don't* assume strict containments. Then if some $n_j = n_{j+1}$, this “ d -step flag” is really only a d' -step flag for some $d' < d$, and the Schubert classes on this d -step flag manifold are indexed by words that skip the letter j entirely. We hereafter make the inductive assumption that the puzzle rule we seek for d -step flag manifolds should specialize (when $n_j = n_{j+1}$) to the one for $(d-1)$ -step, under the label bijection $\{0, \dots, d\} \setminus \{j\} \cong \{0, \dots, d-1\}$.

In this way, for each $d > 0$ we get a lower bound on the set of expected multinumber labels, derived from each $d' < d$. For example, at $d = 2$ we expect to see 10, 20, 21 (in addition to the numbers 0, 1, 2) derived from $d = 1$, but the 2(10), (21)0 labels are new. At $d = 3$ we expect to see ji for $3 \geq j > i \geq 0$ from $d = 1$, and $k(ji)$, $(kj)i$ for $3 \geq k > j > i \geq 0$ from $d = 2$, but e.g. the (32)(10) label is new.

More generally, by forgetting each of the d subspaces we get maps from the d -step flag manifolds to d different $(d-1)$ -step flag manifolds, and the above concerns the cases where these maps are isomorphisms. Now we consider the richer situation where they need not be.

If $Q_- \geq P_-$, then we have a projection $P_- \backslash GL_n(\mathbb{C}) \twoheadrightarrow Q_- \backslash GL_n(\mathbb{C})$, and pullback maps on these cohomology theories. Under these maps Schubert classes pull back to Schubert

classes (and products to products), taking a string with content (q_i) to the unique string with content (p_i) that refines it and has the same number of inversions. For example,

$$\begin{array}{ccccccc} 2 & + & (3+2) & + & (1+3) & \rightarrow & 2 & + & 5 & + & 4 \\ aa & & fffgg & & jkkk & & aa & & fffff & & jjjj \\ & & f j a k f a k f g k g & \leftarrow & f j a j f a j f f j f & & & & & & \end{array}$$

where the top two rows indicate the coarsening ($Q_- \geq P_-$) and the bottom row gives an example of a refinement. This rule tells us what to do with the boundary labels on a (Q_-) -puzzle to make them into the boundary of a (P_-) -puzzle computing the same coefficient. So we can naturally hope that the refinement rule extends to the interior of the puzzle as well, giving a correspondence on puzzles, not just on their boundaries.

Each T -fixed point on $P_- \backslash GL_n(\mathbb{C})$ is uniquely of the form $P_- \backslash P_- w$ for $w \in \prod_i S_{p_i} \backslash S_n$; for convenience we write this set $W_P \backslash W$. Let λ be an element of $W_P \backslash W$, and λ' its image in $W_Q \backslash W$. Let $\mu \in W_Q \backslash W$, and $\tilde{\mu} \in W_P \backslash W$ its refinement as discussed above. (So $(\tilde{\mu})' = \mu$, but (λ') isn't necessarily λ .) Then considering the functoriality of the composite

$$pt \mapsto P_- \backslash P_- \lambda \mapsto Q_- \backslash Q_- \lambda'$$

gets us a match on restrictions, $S^{\tilde{\mu}}|_{\lambda} = S^{\mu}|_{\lambda'}$. In particular, if $\lambda_1, \lambda_2 \in W_P \backslash W$ lie over the same $\lambda' \in W_Q \backslash W$, then $S^{\tilde{\mu}}|_{\lambda_1} = S^{\tilde{\mu}}|_{\lambda_2}$.

2.3. The vectors \vec{f}_X . Fix d , the number of steps in the flag. At $d = 1, 2$ we learn there are two types of puzzle pieces, up to rotation:

$$(7) \quad \begin{array}{c} \triangle \\ / \quad \backslash \\ i \quad i \\ \backslash \quad / \\ i \end{array} \quad i = 0, \dots, d, \quad \text{and} \quad \begin{array}{c} \triangle \\ / \quad \backslash \\ Y \quad X \\ \backslash \quad / \\ Y \quad X \end{array} \quad \text{for certain pairs of labels } X, Y$$

Let $\Lambda := A_2^*$ denote the 2-dim weight lattice of A_2 , on which τ acts by 120° counterclockwise rotation. Let $\vec{f} \in \Lambda$ denote the first fundamental weight, so $\vec{f}, \tau\vec{f}$ are a basis and $\vec{f} + \tau\vec{f} + \tau^2\vec{f} = \vec{0}$.

Now we consider $(\vec{f}_i, \tau\vec{f}_i)_{i=0, \dots, d}$ as a basis of Λ^{1+d} . We associate \vec{f}_i (resp. $\tau\vec{f}_i, \tau^2\vec{f}_i$) to the edge label i on the NW (resp. S, NE) of a \triangle triangle. For YX any multinumber (not necessarily valid) we define

$$\vec{f}_{YX} := -\tau\vec{f}_X - \tau^2\vec{f}_Y$$

with the salutary effect that the vectors on the edges of a puzzle piece $\begin{array}{c} \triangle \\ / \quad \backslash \\ Y \quad X \\ \backslash \quad / \\ Y \quad X \end{array}$ (and its rotations) add to $\vec{0}$. For $d = 1, 2, 3$, these vectors $\{\vec{f}_X : X \text{ a valid multinumber}\}$ are *not* yet quite the ones we used in lemmas 1, 2, 3, since they live in $2(1+d)$ -dimensional space instead of $2d$ -dimensional. They are quite close to the "auras" used in [Buc15] for similar Green's theorem bookkeeping, though he worked in \mathbb{C}^{1+d} and didn't worry about integrality.

2.4. The Gram matrix. In order to make use of the R-matrices of irreps of the quantized affine algebras, our weights (\vec{f}_X) will need to be the weights of a representation. In the simplest possible case, the representation is **minuscule**, meaning it has only extremal weights, which are therefore all of the same norm. So we make the following guess:

There should be a τ -invariant symmetric form on Λ^{1+d} with respect to which,

$$\text{for each valid multinumber } X, \text{ we have } \left| \vec{f}_X \right|^2 = 2.$$

In particular, on each individual Λ the form is determined by the above condition (as we will now recalculate), but the different Λ factors will *not* be orthogonal to one another.

2.4.1. $d = 0$. Here $|\vec{f}_0|^2 = |\tau\vec{f}_0|^2 = 2 = |\tau^2\vec{f}_0|^2 = |-\vec{f}_0 - \tau\vec{f}_0|^2 = |\vec{f}_0|^2 + 2\langle\vec{f}_0, \tau\vec{f}_0\rangle + |\tau\vec{f}_0|^2 = 4 + 2\langle\vec{f}_0, \tau\vec{f}_0\rangle$. Hence the Gram matrix w.r.t. the ordered basis $(\vec{f}_0, \tau\vec{f}_0)$ is the A_2 Cartan matrix $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

When we go to $d > 0$, the 2×2 blocks on the diagonal will all look like this.

2.4.2. $d = 1$. To begin with, the Gram matrix in the basis $\vec{f}_0, \tau\vec{f}_0, \vec{f}_1, \tau\vec{f}_1$ is

$$\begin{bmatrix} 2 & -1 & A & B \\ -1 & 2 & C & D \\ A & C & 2 & -1 \\ B & D & -1 & 2 \end{bmatrix}$$

Since 10 is a valid multinumber,

$$2 = |\vec{f}_{10}|^2 = |-\tau\vec{f}_0 - \tau^2\vec{f}_1|^2 = |\vec{f}_0 + \tau\vec{f}_1|^2 = 2 + 2\langle\vec{f}_0, \tau\vec{f}_1\rangle + 2$$

so as in $d = 0$, we learn that a certain off-diagonal entry (in this case, B) is -1 .

The required τ -invariance gives us

$$\begin{aligned} A &= \langle\vec{f}_0, \vec{f}_1\rangle = \langle\tau\vec{f}_0, \tau\vec{f}_1\rangle = D \\ &= \langle\tau^2\vec{f}_0, \tau^2\vec{f}_1\rangle \\ &= \langle-\vec{f}_0 - \tau\vec{f}_0, -\vec{f}_1 - \tau\vec{f}_1\rangle = \langle\vec{f}_0 + \tau\vec{f}_0, \vec{f}_1 + \tau\vec{f}_1\rangle \\ &= \langle\vec{f}_0, \vec{f}_1\rangle + \langle\vec{f}_0, \tau\vec{f}_1\rangle + \langle\tau\vec{f}_0, \vec{f}_1\rangle + \langle\tau\vec{f}_0, \tau\vec{f}_1\rangle \\ &= A + B + C + D = 2A + C - 1 \end{aligned}$$

so $C = 1 - A$. The off-diagonal block begins to look like the diagonal: to emphasize the similarity we will take $A = 2 - a$, and obtain

$$\begin{bmatrix} 2 & -1 & 2-a & -1 \\ -1 & 2 & a-1 & 2-a \\ 2-a & a-1 & 2 & -1 \\ -1 & 2-a & -1 & 2 \end{bmatrix}$$

When we go to $d > 1$, the 2×2 blocks above the diagonal will each have the same form as this off-diagonal upper block.

2.4.3. $d = 2$. From the analysis above, and the valid multinumbers 10, 20, 21, we already know the Gram matrix looks like

$$\begin{bmatrix} 2 & -1 & 2-a & -1 & 2-b & -1 \\ -1 & 2 & a-1 & 2-a & b-1 & 2-b \\ 2-a & a-1 & 2 & -1 & 2-c & -1 \\ -1 & 2-a & -1 & 2 & c-1 & 2-c \\ 2-b & b-1 & 2-c & c-1 & 2 & -1 \\ -1 & 2-b & -1 & 2-c & -1 & 2 \end{bmatrix}$$

Now we use

$$\vec{f}_{(21)0} = -\tau^2 \vec{f}_{21} - \tau \vec{f}_0 = -\tau^2(-\tau^2 \vec{f}_2 - \tau \vec{f}_1) - \tau \vec{f}_0 = \tau \vec{f}_2 + \vec{f}_1 - \tau \vec{f}_0$$

so that

$$\begin{aligned} |\vec{f}_{(21)0}|^2 &= \left| \tau \vec{f}_2 + \vec{f}_1 - \tau \vec{f}_0 \right|^2 \\ &= |\vec{f}_2|^2 + |\vec{f}_1|^2 + |\vec{f}_0|^2 + 2 \left(\langle \tau \vec{f}_2, \vec{f}_1 \rangle + \langle \tau \vec{f}_2, -\tau \vec{f}_0 \rangle + \langle \vec{f}_1, -\tau \vec{f}_0 \rangle \right) \\ &= 6 + 2(-1 - (2 - b) - (a - 1)) = 2 + 2(b - a) \end{aligned}$$

to learn $a = b$. The validity of 2(10) gives us $b = c$ by a similar computation. In short, the 2×2 blocks above the diagonal are all equal; the only freedom left is in the parameter a .

2.4.4. $d \geq 3$. The multinumbers derived using §2.2's functoriality from $d \leq 2$ are already enough to tie down the matrix, except for this single parameter a in the off-diagonal blocks. For $d = 3$ it remains to verify that the other multinumbers X from §1.3 already have $|\vec{f}_X|^2 = 2$, and indeed they do, putting no condition on a .

2.5. **Signature, and special values of the parameter a .** This matrix G_d is rank 2 at $a = 0$, i.e. its determinant is a multiple of a^{2d} . Mysteriously, this multiple seems to be periodic in d of period 6 for $d > 0$, much as in [BGZ06, example 1.6]. The multiples (which are necessarily squares, since the τ -invariance leads to duplicity in the eigenvalues) are

$$\begin{array}{cccccccc} d & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ a^{-2d} \det G_d & (a-3)^2 & 3(a-2)^2 & (2a-3)^2 & 3(a-1)^2 & a^2 & 3a^2 & \dots \end{array}$$

$$\text{so we define } \alpha_d := \begin{array}{cccccccc} & 3 & 2 & 3/2 & 1 & 0 & 0 & \dots \end{array}$$

We checked this 6-periodicity to $d = 25$, but since we will soon restrict to d at most 4, we didn't pursue it further.

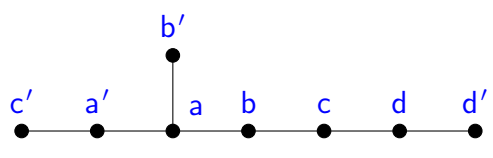
For each $d > 0$ to at least 25 (and probably forever), there is an interval $[0, \alpha_d]$ of values of a outside which G_d is indefinite. For $d = 1, 2, 3, 4$, the matrix is positive definite for a in the open interval $(0, \alpha_d)$. For $d = 5, 6$ there is no open interval, since $\alpha_d = 0$. For $d = 7$ (and $d \not\equiv 5, 6 \pmod{6}$, at least up to 25) there is again an interval, but G_d is not definite in there.

Proposition 1. *For $d = 1, 2, 3$, the only integer vectors in Λ^{1+d} with norm-square 2 (independent of a !) are $\left\{ \pm \tau \bullet \vec{f}_X \text{ for valid multinumbers } X \right\}$, i.e. those associated to a label on a side of a Δ or ∇ .*

We omit the proof as this is a quick computer search, since for $a \in (0, \alpha_d)$, the positive definiteness says we only have to check the integral vectors inside a certain compact ball. For $d = 4$ there are 240 such vectors, arranged in the root system of E_8 , about which more in a moment.

Note that our previous definition of "valid" multinumber X was historical, based on whether X had occurred in a proven or conjectural puzzle rule. Now we can use proposition 1 to give an alternate definition: those X such that $|\vec{f}_X|^2 = 2$.

Here is a simple system satisfying the conditions, and its induced Dynkin diagram.⁶

$$\begin{array}{c}
 \alpha_1 \quad \tau\alpha_1 \quad \alpha_2 \quad \tau\alpha_2 \quad \alpha_3 \quad \tau\alpha_3 \quad \alpha_4 \quad \tau\alpha_4 \\
 \begin{array}{l}
 a \\
 a' \\
 b \\
 b' \\
 c \\
 c' \\
 d \\
 d'
 \end{array}
 \begin{pmatrix}
 1 & & & & & & & \\
 -1 & -1 & & & & & & \\
 0 & 0 & 1 & & & & & \\
 -1 & 0 & -1 & -1 & & & & \\
 0 & 0 & 0 & 0 & 1 & & & \\
 0 & 1 & -1 & 0 & -1 & -1 & & \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
 0 & 1 & -1 & 1 & -2 & 0 & -2 & -1
 \end{pmatrix}
 \end{array}$$


The Dynkin diagram consists of a horizontal line of nodes labeled c', a', a, b, c, d, d' from left to right. A vertical edge connects node a to node b' above it.

Checking the first two conditions is straightforward from the Gram matrices. The third is much more tedious, and for $d = 3$ results in the crystal pictured after lemma 3. (Note that the crystal of a minuscule representation is particularly simple – given two weights $\lambda, \lambda + \alpha$ where α is a simple root, there is a directed edge $\lambda \rightarrow \lambda + \alpha$.) \square

In fact the conditions on the simple system in theorem 4 make the choice of the first seven simple roots unique; alas, there is no choice for the eighth that would make \vec{f}_4 fundamental.

For $d = 1, 2, 3, 4$, if instead of $a = 1$ we set a equal to the α_d from §2.5 (3, 2, $1\frac{1}{2}$, 1 respectively), then the form on Λ^{1+d} becomes positive semidefinite with two null directions, and its quotient is the weight lattice of A_2, D_4, E_6, E_8 respectively. This is how the vectors in lemmas 1-3 arise.

Theorem 4 is *very* suggestive that 4-step Schubert calculus might be approached using edge labels based on the roots of E_8 . However, as briefly indicated in the introduction, the situation is somewhat complicated, and will be discussed in a separate publication [KZJ].

2.7. Inversion numbers of paths. This subsection does not particularly follow the flow from the previous subsections, but its results are of intrinsic interest, besides which we need it for the proofs in the next section.

Fix an antisymmetric form B on Λ^{1+d} , to be specified soon, but whose actual shape is not yet important.

Let P be a puzzle and γ a path through P 's edges. For each edge γ_i in this oriented path, we coorient it by considering it as part of a triangle to its left, and associate a vector $\vec{f}(\gamma_i) = \pm \tau^k \vec{f}_x$ depending on the coorientation and label. Define the **inversion number of γ** as $\frac{1}{2} \sum_{i < j} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j))$. It is something like the area of a surface bounded by γ :

Lemma 4. *If the N -step path γ repeats a vertex, i.e. after step b finds itself in the same location as before step a , then we can break γ into the path $\gamma_{[1,a]} \sqcup \gamma_{[b,N]}$ and the loop $\gamma_{[a,b]}$. In this situation, the inversion number of γ is the sum of the inversion numbers of the path and the loop.*

⁶To check this, let the two matrices above be G and S , and observe that $2 - SGS_T$ is the adjacency matrix for the graph pictured.

Proof.

$$\begin{aligned}
\sum_{i < j} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j)) &= \sum_{i < j < a} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j)) + \sum_{i < a \leq j \leq b} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j)) + \sum_{i < a, b < j} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j)) \\
&+ \sum_{a \leq i < j \leq b} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j)) + \sum_{a \leq i \leq b < j} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j)) + \sum_{b < i < j} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j)) \\
&= \sum_{i < j < a} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j)) + \sum_{i < a} B\left(\vec{f}(\gamma_i), \sum_{a \leq j \leq b} \vec{f}(\gamma_j)\right) + \sum_{i < a, b < j} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j)) \\
&+ \sum_{a \leq i < j \leq b} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j)) + \sum_{b < j} B\left(\sum_{a \leq i \leq b} \vec{f}(\gamma_i), \vec{f}(\gamma_j)\right) + \sum_{b < i < j} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j))
\end{aligned}$$

The Green's theorem property of the vectors $\vec{f}(\gamma_j)$, in the closed loop $\gamma_{[a,b]}$, says that $\sum_{a \leq j \leq b} \vec{f}(\gamma_j) = \vec{0}$. Canceling those two terms,

$$\begin{aligned}
\sum_{i < j} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j)) &= \sum_{i < j < a} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j)) + \sum_{i < a, b < j} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j)) + \sum_{b < i < j} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j)) \\
&+ \sum_{a \leq i < j \leq b} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j))
\end{aligned}$$

giving the stated addition of inversion numbers. \square

(What's "really" going on is that the 2-form B on Λ^{1+d} with constant coefficients is exact, i.e. is of the form $d\alpha$ for some 1-form α with linear coefficients, and the inversion number is the integral over the triangle $0 \leq i \leq j \leq N$ of the pullback of B . We could rewrite the inversion number as the integral of the pullback of α to the boundary of that triangle, but this didn't seem useful.)

In the simplest case of the lemma, $b = a + 1$, i.e. the $(a + 1)$ st step retraces the a th step, and the inversion number of the 2-step closed loop is 0. Consequently we can consider these paths as equivalent modulo insertion or removal of such retracings.

The next simplest application comes when γ_a, γ_{a+1} are two sides of a small Δ or ∇ . We first use the lemma to insert two more steps, jogging back and forth along the third side of the triangle. Now by second use of the lemma, the triangle can be broken off as a closed loop.

We now get specific about B on Λ^{1+d} , defining it on our bases as

$$B(\tau^r \vec{f}_i, \tau^s \vec{f}_j) = \text{sign}(i - j) \begin{cases} 1 & s \equiv r \pmod{3} \\ -1 & s \equiv r + \text{sign}(i - j) \pmod{3} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5. *Fix a partial flag manifold $\{V_1 \leq V_2 \leq \dots \leq V_d \leq \mathbb{C}^n : \dim V_i = n_i\}$, therefore of dimension $D := \sum_{i < j} (n_i - n_{i-1})(n_j - n_{j-1})$.*

Let P be a puzzle with boundaries λ, μ, ν as usual, and γ, γ' paths through P from the SW corner to the SE corner.

- *If γ is the straight path across the bottom, then the inversion number of γ is $\ell(\nu) - \frac{D}{2}$.*
- *If γ' follows the NW then NE sides of P , then the inversion number of γ' is $\ell(\lambda) + \ell(\mu) - \frac{D}{2}$.*

- If one path γ' is always weakly above another path γ (e.g. like the two just mentioned), then the difference in their inversion numbers is the sum of the inversion numbers of the pieces in between them.

Proof. The inversion number of the path along the Southern edge is

$$\frac{1}{2} \sum_{1 \leq i < j \neq n} B(\vec{f}(\gamma_i), \vec{f}(\gamma_j)) = \frac{1}{2} \sum_{1 \leq i < j \neq n} \text{sign}(v_i - v_j) = \ell(v) - \frac{D}{2}$$

Similarly, the inversion number of the path along the NW then NE edge is $\ell(\lambda) + \ell(\mu) - D$ plus the cross-terms from the first half of the path with the second half. The only nonvanishing cross-terms $\frac{1}{2}B(\tau^2 \vec{f}_i, \tau \vec{f}_j)$ come from pairs $i < j$ with i on the NW and j on the NE. There are D such pairs, so the inversion number of the second path is $\ell(\lambda) + \ell(\mu) - \frac{D}{2}$.

The last statement follows directly from lemma 5. \square

3. PROOFS OF THE MAIN THEOREMS

The proofs of theorems 2 and 3 follow the same general philosophy as in [ZJ09a] and [WZJ16b]: they consist in finding an appropriate “quantum integrable system” (that is, in the present context, a set of fugacities collectively satisfying the Yang–Baxter equation), from which can be built both the Schubert basis of $K_T(G/P)$ and the structure constants of that basis. However, we use a much more straightforward approach here based on the bootstrap equation (12)-(13), sidestepping some of the difficulties found in these articles (choice of representative, issues of stability). In appendix C, we briefly sketch an alternative route which is closer to [ZJ09a, WZJ16b]. In the whole of this section, $d \leq 3$.

3.1. Puzzles via the calculus of tensors. We begin with a brief attempt to bring Schubert calculators into the integrable-systems tent, which essentially requires a proper linear algebra understanding of puzzles and their matching rule. Return to equation (4) of theorem 2. The requirement of matching between an edge label on a Δ , and that on a neighboring ∇ , we will reinterpret as a dot product $\langle \vec{v}_x, \vec{v}^y \rangle = \delta_x^y$ between elements of dual bases. To set this up, we need to assign two vector spaces in involution to the South side of Δ and North side of ∇ , with dual bases indexed by L_d (and similarly choose pairs of dual vector spaces for the other two orientations). For the moment, let’s call these vector spaces N, S, NW, SE, NE, SW .

Let us start with nonequivariant, K -theoretic, Schubert calculus. Define a tensor $U \in NW \otimes NE \otimes S$

$$U := \sum \left\{ \text{fug} \left(\begin{array}{c} \triangle \\ \text{X} \quad \text{Z} \\ \text{Y} \end{array} \right) \vec{v}_x \otimes \vec{v}_z \otimes \vec{v}_y : \begin{array}{c} \triangle \\ \text{X} \quad \text{Z} \\ \text{Y} \end{array} \text{ is a valid piece} \right\} \in NW \otimes NE \otimes S$$

and $D \in SE \otimes N \otimes SW$ using ∇ pieces similarly, but using dual basis elements $\vec{v}^x \otimes \vec{v}^y \otimes \vec{v}^z$.

Now consider a diagram \mathcal{D} made of $N_\Delta \Delta$ s and $N_\nabla \nabla$ s, some edges shared, some edges labeled. To such a diagram we assign a tensor $\text{fug}(\mathcal{D})$ as follows: start with $U^{\otimes N_\Delta} \otimes D^{\otimes N_\nabla}$, contract the dual vector spaces at each shared unlabeled edge, and contract with the dual basis vector at each labeled edge. The result is a tensor $\text{fug}(\mathcal{D})$ living in the tensor product of the vector spaces of all unmatched, unlabeled edges of \mathcal{D} . (Observe that if \mathcal{D} is a just a puzzle piece, then this definition gives the usual fugacity ± 1 for the piece, and gives 0 if applied to labeled triangles that aren’t valid puzzle pieces. Or if \mathcal{D} is an unlabeled Δ , then $\text{fug}(\mathcal{D}) = U$.)

The basic case to keep in mind is \mathcal{D} a size n triangle made of n^2 little Δ s and ∇ s, with boundary labeled by λ, μ, ν as usual. Now all internal edges are matched and all external edges are labeled, so the resulting tensor is just a number – exactly the coefficient appearing in the nonequivariant puzzle rule, the $d = 1$ case being equation (2).

Because we use the same labels L_d for our basis and dual basis, if E is a shared unlabeled edge of \mathcal{D} , then

$$(8) \quad \text{fug}(\mathcal{D}) = \sum_{X \in L_d} \{ \text{fug}(\mathcal{D}') : \mathcal{D}' \text{ is } \mathcal{D} \text{ with edge } E \text{ labeled } X \}$$

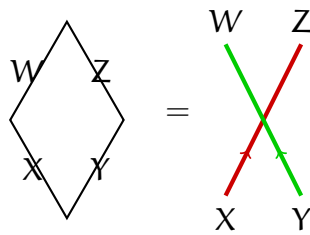
To involve equivariant pieces, we also define a tensor \check{R} living in $NW \otimes NE \otimes SE \otimes SW \cong \text{End}(NW \otimes NE)$, constructed as a sum

$$\check{R} := \sum \left\{ \text{fug} \left(\begin{array}{c} \text{W} \quad \text{Z} \\ \diagdown \quad \diagup \\ \text{X} \quad \text{Y} \end{array} \right) \vec{v}^W \otimes \vec{v}^Z \otimes \vec{v}^Y \otimes \vec{v}^X : \begin{array}{c} \text{W} \quad \text{Z} \\ \diagdown \quad \diagup \\ \text{X} \quad \text{Y} \end{array} \text{ can be filled in} \right\}$$

where the fugacities depend on the location of the rhombus (ultimately, they will depend on the equivariant parameters).

Finally, each vector space associated to edges will be endowed with the action of a *Hopf algebra*, in such a way that U, D and \check{R} are invariant under that action (i.e., they live in the trivial representation); in particular, U and D are the trilinear invariant forms advertised in the title. A subtlety here is that the lack of involutivity of the antipode means that the notion of dual representation is not uniquely defined. This will force us to coorient all edges, as will be discussed in the next section.

3.2. The required properties: quantum integrability. It is convenient to introduce the “dual” graphical representation which is more traditional in quantum integrable systems: we denote a rhombus with its four edges labeled by L_d



The colors of lines are a reminder of the direction of the corresponding edge of the rhombus. This information may seem redundant since lines themselves have a given direction; however it is sometimes useful to deform the lines (in a way that is harder to realize with rhombi), and then their direction may vary (but not their color). We will say more about this after proposition 2.

We then attach a parameter (usually called the spectral parameter) to each line, in such a way that the fugacity of the rhombus / crossing depends on the ratio⁷ of these two

⁷We work in the multiplicative group, or equivalently with trigonometric solutions of the Yang-Baxter equation, since our goal is to compute in K-theory. The corresponding additive group / rational solutions compute Schubert calculus in ordinary cohomology.

parameters:

$$(9) \quad \begin{array}{c} W \quad Z \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ X \quad Y \end{array} = \check{R}(u)_{XY}^{WZ} \in \mathbb{Q}(u_1, u_2, \dots) \quad u = u_1/u_2$$

The fugacity lives in some appropriate ring of rational functions in u_1, u_2 and possibly other indeterminates.

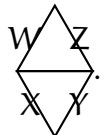
The graphical convention (as explained in §3.1) is that a drawn rhombus / crossing represents its own fugacity, and that when rhombi are glued together, the labels of the internal edges are summed over as in (8), or in the dual language, when lines emerging from crossings are reconnected, the labels of these lines should match and are summed over.

We impose an important condition on the fugacity of (9): that there is a special value α/β of the ratio of spectral parameters such that the fugacity “factorizes”:

Property 1.

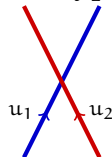
$$(10) \quad \begin{array}{c} W \quad Z \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ X \quad Y \end{array} = \begin{array}{c} W \quad Z \\ \beta u \quad \alpha u \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ \alpha u \quad \beta u \\ X \quad Y \end{array}$$

where the blue line, or equivalently the diagonal of the rhombus, carries a label in L_d .

Equivalently, in the original graphical language, the r.h.s. can be pictured as .

This is the way that the “invariant trilinear form” appears in this integrable setting. The assignment of the parameter u to the intermediate blue line is at the moment purely a definition, but it will be useful below. The constants α and β could be set to 1 by a redefinition of the spectral parameters, but as we shall see this would not match with usual quantum group conventions.

We need more general types of crossings, where two lines with arbitrary colors can

cross each other, e.g., . ⁸ In the argument from proposition 2, it will become clear

that the same-color crossings are closely related to the $(150^\circ, 30^\circ)$ rhombi in [Pur08] and more distantly related to the gashes in [KT03, BKPT16].

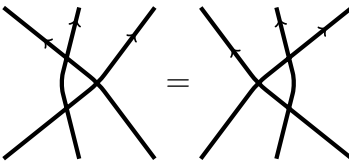
We also associate to each labeled line a weight: the weight of a green (resp. blue, red) line labeled X is \vec{f}_X (resp. $-\tau \vec{f}_X, \tau^2 \vec{f}_X$). The minus sign for blue lines comes from their opposite orientation at the trivalent vertices.

⁸Many of these R-matrices were defined in the study of the cohomology of $d = 1$ in [ZJ09a]; in particular, crossings of identical colors were called R^* there.

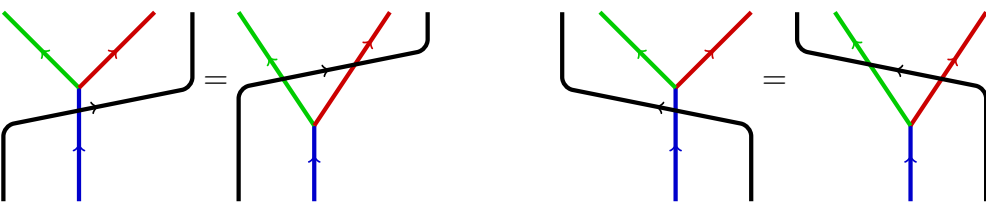
We then require the following conditions:

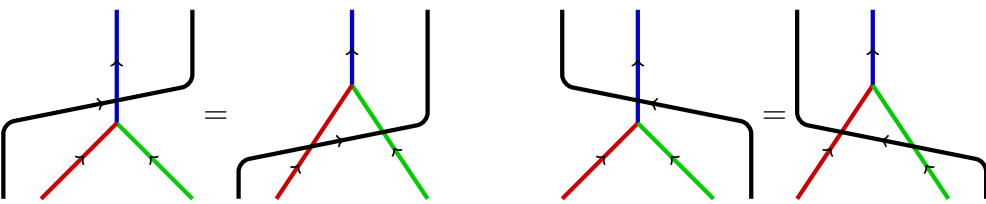
Property 2. *In the pictures below, black lines can have arbitrary (independent) colors, and all lines can have arbitrary spectral parameters (as long as they match between l.h.s. and r.h.s.).*

- *The fugacity of any vertex is nonzero only if weight is conserved, i.e., if the sum of incoming weights is equal to the sum of outgoing weights.*
- *Yang–Baxter equation:*

(11) 

- *Bootstrap equations:*

(12) 

(13) 

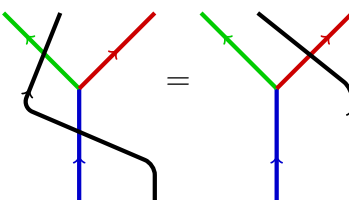
- *Unitarity equation:*

(14) 

- *Value at equal spectral parameters: (here, the lines must have the same color for the equality to make sense)*

(15) 

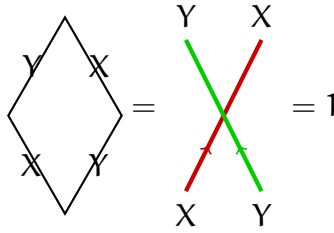
Morally, properties (12)–(14) mean that one can freely slide lines across vertices/intersections of other lines. For example, combining say the second equation of (12) with (14), we can also write

(16) 

which is an equality that will be used in what follows.

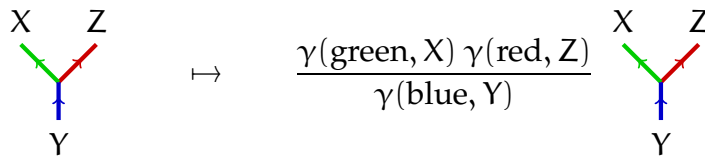
We also impose the following normalization condition on the fugacities:

Property 3. For any $X, Y \in L_d$ such that $\langle \vec{f}_X, \tau \vec{f}_Y \rangle = -1$,



(independent of u).

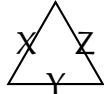
There is a remaining gauge freedom on the fugacities, namely one can multiply any vertex by the product of $\prod_{\text{edges}} \gamma(\text{color, label})^{\pm 1}$ where the γ s are arbitrary parameters and the sign of the exponent depends on the orientation of the edge; e.g.,



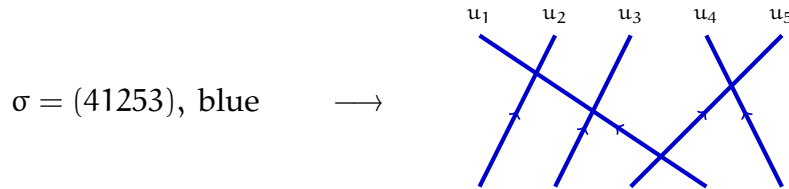
These rescalings preserve properties 1–3. We now get rid of this freedom by imposing

$$(17) \quad \begin{array}{c} X \\ \diagdown \quad \diagup \\ Y \quad X \\ \diagup \quad \diagdown \\ Y \end{array} = \begin{array}{c} i \\ \diagdown \quad \diagup \\ i \quad i \\ \diagup \quad \diagdown \\ i \end{array} = 1 \quad \text{likewise for all rotations } \Delta \text{ and } \nabla$$

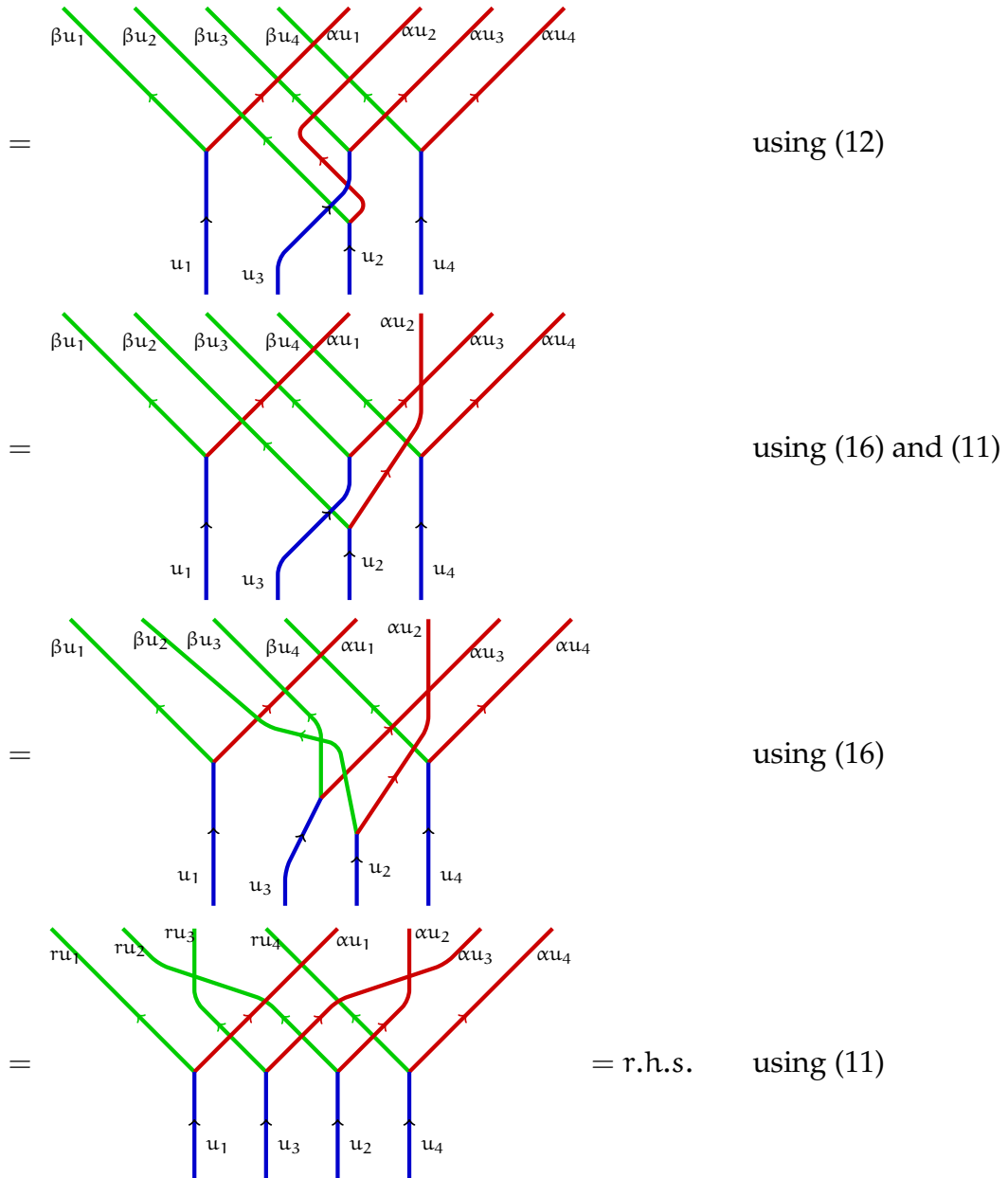
where YX varies over valid multinumbers and i varies over $0, \dots, d$. Indeed, we can use the Δ triangles alone (say) to fix the $\gamma(\text{color, } X)$ inductively in $|X|$ (number of digits of X), and then the ∇ triangle condition follows from property 3 (noting that the weight

conservation from property 2 for the triangle  implies $\langle \vec{f}_X, \tau \vec{f}_Y \rangle = -1$). The only remaining gauge freedom in the choice of basis vectors comes in choosing a weight vector for each weight $\vec{f}_i, \tau \vec{f}_i, i = 0, \dots, d$. These $2(d + 1)$ remaining scale parameters can be blamed on our $2(d + 1)$ -torus whose weight lattice is Λ^{1+d} .

Finally, one more graphical notation is needed. Given a positive integer n , a permutation $\sigma \in \mathcal{S}_n$, and a color $\in \{\text{red, green, blue}\}$, we consider a wiring diagram for σ made of n lines in that color, with parameters u_1, \dots, u_n , e.g.



This results in an endomorphism of $N^{\otimes n}$ (in the blue case; or of $NE^{\otimes n}, NW^{\otimes n}$ in the red and green cases), which, thanks to the Yang–Baxter equation (11) and the unitarity equation (14), is independent of the choice of wiring diagram for the permutation. Here the convention is that the numbering of the spectral parameters is the increasing one at the top, and $u_{\sigma^{-1}(1)}, u_{\sigma^{-1}(2)}, \dots, u_{\sigma^{-1}(n)}$ at the bottom (consistent with the fact that our fixed points are indexed by left cosets in $W_p \setminus W$). In the dual picture, we'll denote the endomorphism by a rectangle labeled σ , the orientation of the rectangle determining as usual the color of the corresponding lines.

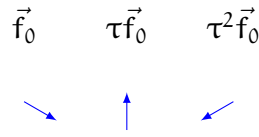


□

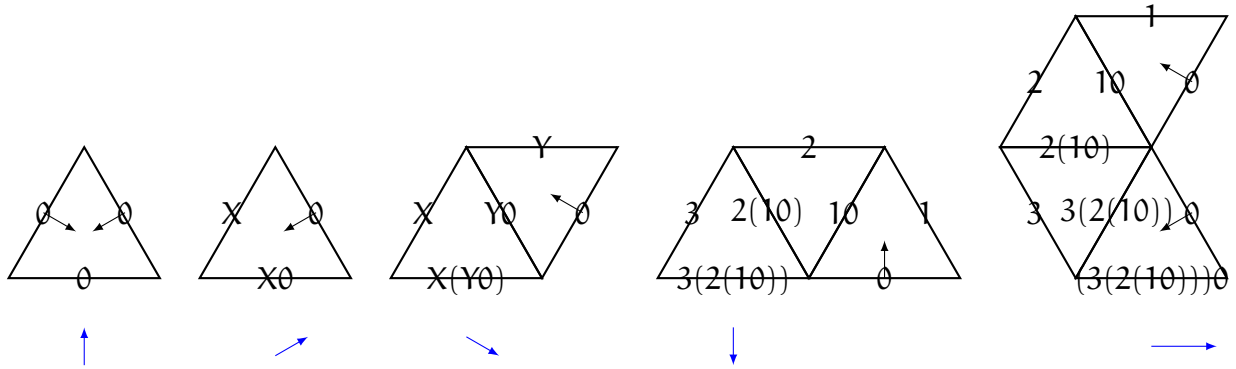
It is interesting to note that if we allowed puzzles to have arbitrary boundary labels (multinumerals, not just single numbers) then the multiplication they define would not be associative. We need some results about the importance of this single-number sector, based on the following technical lemma.

Lemma 6. *Let X be a valid multinumber, and $\vec{f}_X \in \Lambda^{1+d}$ its associated weight. Then the projection of $\tau \vec{f}_X$ to the 0th factor of Λ points weakly to the right, and strictly to the right unless X is 0, is $3(2(10))$, or lacks 0 entirely.*

To put this statement in context, we draw the vectors $\vec{f}_0, \tau \vec{f}_0, \tau^2 \vec{f}_0 \in \Lambda$:



Proof. Multinumbers come in six types: 0, X0, X(Y0), 3(2(10)), (3(2(10)))0, and those not involving 0. We draw the first five here in their minimal puzzle environments, the better to calculate the contributions (shown here in blue) of their $\tau\vec{f}$ s to the 0th factor of Λ . (The multinumbers not involving 0 contribute nothing in the 0th factor.)



□

Proposition 3. *Let P be a size n triangle made of puzzle pieces, where the Northwest and Northeast sides have only single-number labels $0, 1, 2, 3$.*

If P has the same content on NW and NE, then the South side is also single-number labels (with that same content). The $120^\circ, 240^\circ$ rotations of the statement also hold. Conversely, if the contents on the (all single-number) NW and NE sides differ, then the S side cannot be all single-number.

Proof. Let $X(j), j = 1, \dots, n$ be the multinumbers across the South side. The Green’s theorem argument and the assumptions about the NW, NE sides tell us that $\sum_{j=1}^n \vec{f}_{X(j)} \in \text{span}(\vec{f}_0, \vec{f}_1, \vec{f}_2, \vec{f}_3)$, i.e. not using the $\tau\vec{f}_0, \tau\vec{f}_1, \tau\vec{f}_2, \tau\vec{f}_3$ basis elements.

This motivates an equivalent, if more general-looking, proposition: if $X(j), j = 1, \dots, n$ is a list of multinumbers such that $\sum_{j=1}^n \vec{f}_{X(j)}$ lies in $\text{span}(\vec{f}_0, \vec{f}_1, \vec{f}_2, \vec{f}_3)$, then the only $X(j)$ s involving 0 actually have $X(j) = 0$. Once we excise them from the list, subtract 1 from every number in every remaining $X(j)$ (again obtaining valid multinumbers, by the functoriality property of multinumbers from §2.2), and use induction on d , we find that each multinumber $X(j)$ is a single-number.

We consider now the contribution “ $\pi_0(\vec{f}_{X(j)})$ ” of each $\vec{f}_{X(j)}$ to the 0th copy of Λ in Λ^{1+d} .

A $d = 0$ puzzle has only 0-labels, so trivially achieves the condition we seek, namely that the only multinumbers $X(j)$ involving 0s are exactly 0s.

In a $d = 1$ puzzle, only the labels 0, 1, 10 occur. Since we know the sum $\sum_{j=1}^n \pi_0(\vec{f}_{X(j)})$ must be parallel to \vec{f}_0 (drawn here as vertical), and by lemma 6 these vectors $\pi_0(\tau\vec{f}_0), \pi_0(\tau\vec{f}_1)$ point weakly right while $\pi_0(\vec{f}_{10})$ points strictly right, there can be no $\pi_0(\vec{f}_{10})$ s. For $d = 1$ that rules out the 10 label, leaving only the 0, 1 labels.

In a $d \leq 2$ puzzle, the only labels including 0 are 0, X0, 2(10), and the same rightward-pointing argument applies. Again we learn the only $\{X(j)\}$ including 0 are equal to 0.

In a $d = 3$ puzzle, this rightward-pointing argument shows only that any label involving 0 must be 0 or $3(2(10))$. But the same argument applied dually to study 3s in labels shows that any label involving 3 must be 3 or $((32)1)0$; in particular, not $3(2(10))$. Consequently, neither the $3(2(10))$ nor $((32)1)0$ labels can appear. So once again, any label involving 0 must be 0, and we can apply the inductive argument from above.

For the converse, pick i such that there is a different number of i labels on the NW and NE sides. Then the projection to the i th \wedge factor is not vertical, and cannot be canceled by single-number \vec{f}_x s from the South side, since each \vec{f}_i is vertical and each $\vec{f}_{j \neq i}$ projects to $\vec{0}$ in the i th factor. \square

Proposition 4. (1) *Single-color crossings preserve single-numbers, i.e.,*

$$(18) \quad \text{if } \begin{array}{c} k \quad l \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ i \quad j \end{array} \neq 0, \text{ then } i, j \in \{0, \dots, d\} \Leftrightarrow k, l \in \{0, \dots, d\}$$

(where the lines have arbitrary, but identical, colors).

(2) For $i \in \{0, \dots, d\}$,

$$(19) \quad \begin{array}{c} j \quad k \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ i \quad i \end{array} \neq 0 \text{ or } \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ j \quad k \end{array} \neq 0 \Rightarrow i = j = k$$

where again the lines have identical colors.

(3) If ω is a weakly increasing single-number string, then

$$(20) \quad \begin{array}{c} \lambda \quad \mu \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \omega \end{array} \neq 0 \Rightarrow \lambda = \mu = \omega$$

for all single-number strings λ and μ , i.e., having ω at the bottom forces it on the other two sides. Moreover, the puzzle is unique, and along each diagonal (NW/SE or NE/SW) the labels on the non-horizontal edges are constant.

Similarly, if ω is a weakly decreasing single-number string, then

$$(21) \quad \begin{array}{c} \omega \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \mu \quad \lambda \end{array} \neq 0 \Rightarrow \lambda = \mu = \omega$$

for all single-number strings λ and μ .

The first two of these will derive from the same source:

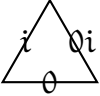
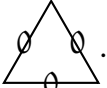
Lemma 7. Let V be a T -representation with weights P and weight basis (\vec{v}_p) , and F a face of the convex hull of P . Let $R \in \text{End}(V \otimes V)$ be T -equivariant. If $\vec{v}_p, \vec{v}_q, \vec{v}_r, \vec{v}_s$ are basis vectors with weights $p, q \in F$, and $R_{\vec{v}_p \vec{v}_s}^{\vec{v}_r \vec{v}_q} \neq 0$ or $R_{\vec{v}_p \vec{v}_q}^{\vec{v}_r \vec{v}_s} \neq 0$, then $r, s \in F$ as well.

Proof. Since F is a face of the convex hull, it is defined by the tightness of some inequality $\langle \vec{a}, p \rangle \geq c$ for some $\vec{a} \in \mathfrak{t}$ generating some one-parameter subgroup $A \leq T$. Equivalently, all the A -weights in V are $\geq c$, and those on F span the c weight space.

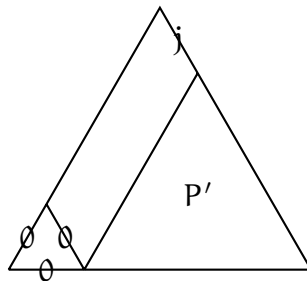
Then the only basis vectors $\vec{v}_m \otimes \vec{v}_n$ of A -weight $2c$ are those with $m, n \in F$, and R must preserve this A -weight space of $V \otimes V$, giving the claim. \square

Proof of proposition 4. We'll apply this lemma when V is a representation $V_\lambda^{X_{2d}}$ of $X_{2d} = A_2, D_4, E_6$, and $P = W_{X_{2d}} \cdot \lambda$; our λ will be such that its projection to the A_d weight lattice is the fundamental weight ω_1 . Hence the A_d -subrepresentation containing the high weight vector has the standard basis indexed by $\{0, \dots, d\}$. To get part (1), we take F to be the face containing exactly the vertices $\{0, \dots, d\}$, and use theorem 4(2). Shrinking F to its a th vertex (then applying the lemma) gets us part (2).

For part (3), first observe that the Green's theorem argument shows that the content on the three sides must be the same. For convenience we assume that the lowest number occurring is 0, since if it is $i > 0$ we can simply subtract i from all numbers appearing and work instead with that equally valid puzzle.

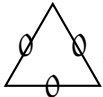
By the weak increase across the South side, the leftmost label is smallest, hence 0. There are no  pieces for $i > 0$ (since $|\vec{f}_{0i}|^2 = 4 - 2a \neq 2$), so the leftmost triangle on the South side must be a .

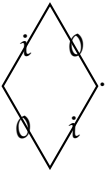
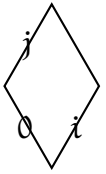
Let j be the leftmost label on the Northeast side, and let P' be the size $n - 1$ triangle made by removing the pieces touching the Northwest side. (We don't know yet that the newly exposed Northwest labels are single numbers, i.e. that P' is itself a puzzle.)



Assume for contradiction that $j > 0$. Rotate P 120° counterclockwise for ease of comparison to proposition 3. The NW side of the rotated P' has one more 0 than the NE side, so the vector sum over the NW, NE boundary labels (in the projection to the 0th factor Λ of Λ^{1+d}) points strictly rightward. Now we add in the vectors from the South boundary labels of P' (as determined in lemma 6), each of which points weakly rightward, and we reach a contradiction with the Green's theorem argument.

Now that we've established $j = 0$, the content of the NE and S side of (unrotated) P' is the same, and we can apply proposition 3 to P' . We learn that its Northwest side is all single numbers. Of course its South side is weakly increasing, so by induction on n the present proposition applies, i.e. P' is uniquely determined by its South side, has fugacity 1, and its NW side matches its South side (both read left to right).

It remains to determine the pieces in the strip $P \setminus P'$, which we do one rhombus at a time from SW to NE, having already determined the bottom . We will show inductively

that each rhombus is of the form . A priori, the rhombus bears  where i, j

are single numbers, so the weight on the NE side is $\vec{f}_{(i0)j}$ (whether or not that's a valid multinumber). Its norm-square is

$$\begin{aligned} |\vec{f}_{(i0)j}|^2 &= |\vec{f}_{i0}|^2 + |\vec{f}_j|^2 + 2\langle \vec{f}_{i0}, \tau \vec{f}_j \rangle \\ &= 4 + 2\langle -\tau \vec{f}_i - \tau^2 \vec{f}_0, \tau \vec{f}_j \rangle = 4 - 2\langle \tau \vec{f}_i, \tau \vec{f}_j \rangle - 2\langle \vec{f}_j, \tau \vec{f}_0 \rangle = 6 - 2 \begin{cases} 2 & \text{if } i = j \\ 2 - \alpha & \text{if } i \neq j \end{cases} \end{aligned}$$

which is 2 iff $i = j$, in which case (by Green's theorem) the NE label is 0, continuing the induction. Hence the Northwest labels on P' are copied across these rhombi to the Northwest side of P , after the initial 0, and the fugacities are 1 as claimed. \square

Given a permutation $\sigma \in \mathcal{S}_n$ and a single-number string λ of length n , define

$$(22) \quad \mathbf{S}_{N|\sigma}^\lambda := \begin{array}{|c|} \hline \lambda \\ \hline \sigma \\ \hline \omega \\ \hline \end{array} \quad \mathbf{S}_{NW|\sigma}^\lambda := \begin{array}{|c|} \hline \lambda \\ \hline \delta \\ \hline \omega \\ \hline \end{array} \quad \mathbf{S}_{NE|\sigma}^\lambda := \begin{array}{|c|} \hline \omega \\ \hline \alpha \\ \hline \lambda \\ \hline \end{array}$$

where here ω is the unique weakly increasing string with the same content as λ .

3.4. Main theorems. This leads us to the heart of the paper:

Theorem 5. *The following equality holds:*

$$\sum_{\nu} \begin{array}{|c|} \hline \lambda \\ \hline \nu \\ \hline \mu \\ \hline \omega \\ \hline \end{array} \mathbf{S}_{N|\sigma}^\nu = \mathbf{S}_{NW|\sigma}^\lambda \mathbf{S}_{NE|\sigma}^\mu$$

where the summation is over single-number strings with the same content as λ and μ .

Proof. Start from proposition 2 and set the NW side to λ , the NE side to μ and the bottom side to ω , the unique weakly increasing string with the same content as λ and μ .

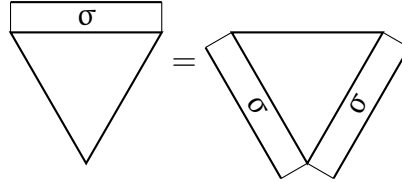
$$\sum_{\nu} \begin{array}{|c|} \hline \lambda \\ \hline \nu \\ \hline \mu \\ \hline \omega \\ \hline \end{array} = \begin{array}{|c|} \hline \lambda \\ \hline \sigma \\ \hline \omega \\ \hline \end{array} = \begin{array}{|c|} \hline \lambda \\ \hline \delta \\ \hline \omega \\ \hline \end{array} \begin{array}{|c|} \hline \alpha \\ \hline \mu \\ \hline \omega \\ \hline \end{array} = \sum_{\tilde{\lambda}, \tilde{\mu}} \begin{array}{|c|} \hline \lambda \\ \hline \delta \\ \hline \tilde{\lambda} \\ \hline \omega \\ \hline \end{array} \begin{array}{|c|} \hline \tilde{\mu} \\ \hline \alpha \\ \hline \mu \\ \hline \omega \\ \hline \end{array} = \begin{array}{|c|} \hline \lambda \\ \hline \delta \\ \hline \omega \\ \hline \end{array} \begin{array}{|c|} \hline \omega \\ \hline \alpha \\ \hline \mu \\ \hline \end{array}$$

The first and third equations are essentially applications of (8). By the first part of proposition 4, the internal labels $\nu, \tilde{\lambda}, \tilde{\mu}$ are single-number strings (or more precisely, all other contributions to the sum vanish).

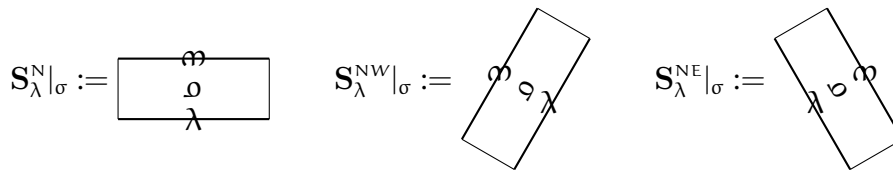
The third part of proposition 4 tells us that $\tilde{\lambda} = \tilde{\mu} = \omega$, and that the triangle of the r.h.s. is made of triangles and rhombi which are all of fugacity 1 according to property 3 and gauge fixing condition (17). Once rid of this triangle, we recognize on both sides the definitions (22), hence the result. \square

One can redo the whole reasoning after “reversal of all the arrows”. More precisely, we note that property 1 defines two types of trivalent vertices, one where two edges are outgoing, as was used so far, and one where two edges are incoming, which we shall use now. In property 2, equations (11), (14) and (15) are invariant by reversal of arrows, whereas eq. (12) turns into eq. (13).

In the puzzle picture, it is customary to rotate the picture by 180° so that lines are still oriented upwards; we then have an obvious analogue of proposition 2, namely



We furthermore define



where λ is read from left to right, but m denotes ω read right to left (equivalently, it is the weakly decreasing string with the same content as λ), and similarly o denotes the diagram of σ rotated 180° .

Using the exact same reasoning as above (noting in particular the 180° symmetry of proposition 4), we have the “dual” statement:

Theorem 6. *The following equality holds:*

$$\sum_{\nu} \left(\text{trivalent vertex with top edge } \nu, \text{ left edge } \mu, \text{ right edge } \lambda \right) S_{\nu}^N|_{\sigma} = S_{\lambda}^{NW}|_{\sigma} S_{\mu}^{NE}|_{\sigma}$$

(where all labels are read as usual from left to right).

This is to be compared with the primed theorems of [WZJ16b]. With not much more work, we could also obtain analogues of the double primed theorems of [WZJ16b], but we shall refrain from doing so here.

In the rest of this section, the strategy is to define, using the representation theory of quantized affine algebras, systems of fugacities which satisfy the properties stated in §3.2. The fugacities of rhombi obtained this way will be more general than needed for the purposes of this paper, and we shall take a limit ($q \rightarrow 0$ where q is a parameter in these fugacities) to recover the puzzle rule that we are trying to prove. (We will interpret geometrically this more general solution in [KZJ].)

3.5. $d = 2$ and triality in D_4 .

3.5.1. *R-matrices.* To each line (or color of line) we now attach a representation of the quantized affine algebra $U_q(\mathfrak{d}_4^{(1)})$. There are three colors of lines, which correspond to the three fundamental representations of $U_q(\mathfrak{d}_4)$ that are related by triality. We assign to green (resp. blue, red) lines the representation V_{ω_1} (resp. $V_{\omega_2}, V_{\omega_3}$) with weights given by lemma 2 (resp. with τ, τ^2 times these weights). It is known [CP94] that all three representations can then be extended to evaluation representations $V_{\omega_i}(u)$ of $U_q(\mathfrak{d}_4^{(1)})$, where u is

the spectral (evaluation) parameter. Note that these $U_q(\mathfrak{d}_4)$ representations are self-dual, so that their $U_q(\mathfrak{d}_4^{(1)})$ counterparts only differ from their dual by a shift of the spectral parameter.

Next we define R-matrices⁹ (which correspond to crossings of lines of various colors): $\check{R}_{i,j}$ will be an $U_q(\mathfrak{d}_4^{(1)})$ -intertwiner from $V_{\omega_i}(u_1) \otimes V_{\omega_j}(u_2)$ to $V_{\omega_j}(u_2) \otimes V_{\omega_i}(u_1)$. Let us for example consider representations $V_{\omega_3}(u_1)$ and $V_{\omega_1}(u_2)$, i.e., red and green lines respectively. Then the corresponding R-matrix can be written as

$$(23) \quad \check{R}_{3,1} = \frac{1}{u_2 - q^4 u_1} ((u_1 - q^4 u_2) P_{\omega_2} + (u_2 - q^4 u_1) P_{\omega_3 + \omega_1})$$

where q is another indeterminate. Here P_{ω_2} and $P_{\omega_3 + \omega_1}$ are operators of rank 8 and 56 respectively, implementing the two channels of decomposition $V_{\omega_3} \otimes V_{\omega_1} \cong V_{\omega_2} \oplus V_{\omega_3 + \omega_1}$ for the horizontal subalgebra $U_q(\mathfrak{d}_4)$ (here horizontal means that its action is independent of the spectral parameter). Note that we do not require $\check{R}_{3,1}$ to have the normalization of the universal R-matrix, only that it commute with the $U_q(\mathfrak{d}_4^{(1)})$ action; its normalization is chosen for convenience.

In terms of puzzles, recall that $\check{R}_{3,1}$ parametrizes the fugacities of rhombi, cf Eq. (9):

Next, we observe that the R-matrix (23) satisfies

$$(24) \quad \check{R}_{3,1}|_{u_1=q^{-2}u, u_2=q^2u} = -\frac{(1+q^4)(1+q^2)}{q^2} P_{\omega_2} = UD$$

where $D : V_{\omega_3}(q^{-2}u) \otimes V_{\omega_1}(q^2u) \rightarrow V_{\omega_2}(u)$ and $U : V_{\omega_2}(u) \rightarrow V_{\omega_1}(q^2u) \otimes V_{\omega_3}(q^{-2}u)$ commute with the $U_q(\mathfrak{d}_4^{(1)})$ action. This is the analogue of property 1, with $\alpha = q^{-2}$, $\beta = q^2$; that is, we can write

$\check{R}_{3,1}$, as well as the matrices U and D , are given in appendix A. In particular we can check explicitly that property 3 is satisfied, as well as (17).

The definitions of $\check{R}_{1,2}$, $\check{R}_{2,3}$ are similar to that of $\check{R}_{3,1}$, except we choose the normalization differently:

$$(25) \quad \check{R}_{i,i+1} = \frac{u_1 - q^4 u_2}{u_2 - q^4 u_1} P_{\omega_{i-1}} + P_{\omega_i + \omega_{i+1}}, \quad i = 1, 2$$

⁹We use here so-called \check{R} -matrices, which are related to usual R-matrices by permutation of the factors of the tensor product.

Next, we consider $\check{R}_{i+1,i}$, $i = 1, 2, 3$ (with indices understood mod 3); to ensure that (14) is automatically satisfied for distinct colors of lines, we define them as

$$\check{R}_{i+1,i}(u_1, u_2) = (\check{R}_{i,i+1}(u_2, u_1))^{-1}$$

where the notation indicates that we've permuted the roles of the spectral parameters u_1 and u_2 on the r.h.s.

Finally, we must define $\check{R}_{i,i}$:

$$(26) \quad \check{R}_{i,i} = \frac{(u_1 - q^6 u_2)(u_1 - q^2 u_2)}{(u_2 - q^6 u_1)(u_2 - q^2 u_1)} P_0 + \frac{u_1 - q^2 u_2}{u_2 - q^2 u_1} P_{\omega_4} + P_{2\omega_i}$$

where the P_* are projectors onto the various irreducible subrepresentations of $V_{\omega_i} \otimes V_{\omega_i}$ as $U_q(\mathfrak{d}_4)$ -modules. Note that $\check{R}_{i,i}$ satisfies $\check{R}_{i,i}(u_1, u_2)\check{R}_{i,i}(u_2, u_1) = 1$, which is nothing but (14) for identical colors of lines; in addition $\check{R}_{i,i}|_{u_1=u_2} = 1$, which is (15).

These R-matrices, with the prescribed choice of bases of V_{ω_i} ($i = 1, 2, 3$) specify the crossings of lines of all possible colors.

Looking at each equation of property 2, we note that l.h.s. and r.h.s. commute with the $U_q(\mathfrak{d}_4^{(1)})$ action, so by Schur's lemma they must be proportional. In fact, the Yang–Baxter equation (11) is homogeneous (i.e., independent of the normalization of R-matrices) and is known to be satisfied by the universal R-matrix. As explained above, our normalization also ensures that unitarity equation (14), as well as (15), are satisfied. There remain only the bootstrap equations (12)–(13). We leave it to the reader to check that with the choice of normalization of (23), (25) and (26), these are indeed satisfied as well.

3.5.2. *The limit $q \rightarrow 0$.* Since all the properties of §3.2 are satisfied, we can take the equality of theorem 5 and carefully compute the leading order as $q \rightarrow 0$.

We first discuss the “rectangles” in their three orientations, that is the $S^\lambda|_\sigma$. According to part (2) of proposition 4, we can restrict $\check{R}_{i,i}$ to the single-number sector, i.e., in the given basis, to a 9×9 matrix. One checks explicitly that these three matrices are *identical*, and equal to

$$(27) \quad \check{R}_A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{(1-q^2)u_2}{u_2-q^2u_1} & 0 & \frac{q(u_1-u_2)}{q^2u_1-u_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-q^2)u_2}{u_2-q^2u_1} & 0 & 0 & 0 & \frac{q(u_1-u_2)}{q^2u_1-u_2} & 0 & 0 \\ 0 & \frac{q(u_1-u_2)}{q^2u_1-u_2} & 0 & \frac{(1-q^2)u_1}{u_2-q^2u_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-q^2)u_2}{u_2-q^2u_1} & 0 & \frac{q(u_1-u_2)}{q^2u_1-u_2} & 0 \\ 0 & 0 & \frac{q(u_1-u_2)}{q^2u_1-u_2} & 0 & 0 & 0 & \frac{(1-q^2)u_1}{u_2-q^2u_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{q(u_1-u_2)}{q^2u_1-u_2} & 0 & \frac{(1-q^2)u_1}{u_2-q^2u_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Theorem 5 can then be rewritten

$$(28) \quad \sum_{\nu} \begin{array}{c} \lambda \\ \triangle \\ \nu \end{array} \begin{array}{c} \mu \\ \triangle \\ \nu \end{array} S^\nu|_\sigma = S^\lambda|_\sigma S^\mu|_\sigma$$

where the subscript of the Ss is irrelevant.

Define the **nilHecke** R-matrix to be

$$(29) \quad \check{R}_{\text{nil}ij}^{kl} := \begin{cases} 1 & (k, l) = (i, j), i \leq j \\ u_1/u_2 & (k, l) = (i, j), i > j \\ 1 - u_1/u_2 & (k, l) = (j, i), i < j \\ 0 & \text{otherwise} \end{cases} \quad i, j, k, l \in \{0, \dots, d\}$$

By direct inspection, we have:

Lemma 8. *As $q \rightarrow 0$ at fixed u_1, u_2 , one has*

$$\check{R}_{Aij}^{kl} = q^{\frac{1}{2}(-B(\vec{f}_i, \vec{f}_j) + B(\vec{f}_k, \vec{f}_l))} (\check{R}_{\text{nil}ij}^{kl} + O(q^2))$$

for all $i, j, k, l \in \{0, \dots, d\}$, noting that $B(\vec{f}_i, \vec{f}_j) = \text{sign}(i - j)$.

Corollary 1. *As at the end of §2.2, let $\sigma \in \mathcal{S}_n$ be a permutation and σ' be its image in $\prod_i \mathcal{S}_{p_i} \setminus \mathcal{S}_n$, giving a T-fixed point on $P_- \setminus \text{GL}_n(\mathbb{C})$. Then*

$$(30) \quad S^\lambda|_\sigma = q^{-\ell(\lambda)} (S^\lambda|_{\sigma'} + O(q^2))$$

where $S^\lambda|_{\sigma'}$ is the restriction to the σ' fixed point of the λ Schubert class.

Proof. $S^\lambda|_\sigma$ is defined using a reduced wiring diagram Q for σ , and extracting a matrix coefficient from a product $\prod_Q \check{R}$ of \check{R} -matrices. Each term in that product corresponds to a selection, at each crossing in Q , of either no crossing $(k, l) = (i, j)$ or a transposition $(k, l) = (j, i)$; in the latter case we can only cross two wires so as to create an inversion (since $(\check{R}_{\text{nil}})_{ij}^{jk} = 0$ if the crossing would destroy an inversion, namely, if $i > j$).

As such, the (λ, σ) matrix entry is a sum over *reduced* subwords of Q with product λ . Note that this is not the usual formula [FK94, theorem 2.3] that gives an alternating sum over subwords with nilHecke product λ ;¹⁰ rather, it is the same indexing set as in the formula [KM04, theorem 4.4].

It remains to match up the summands in $S^\lambda|_\sigma$ with those in [KM04, theorem 4.4]. The “absorbable reflections” of that theorem 4.4 are exactly the near-misses of wires that have already crossed once and choose to not cross a second time, contributing factors u_1/u_2 to the product as they should. \square

Next we proceed similarly with the equivariant puzzle itself. Denote by $|a|$ the cardinality of a multinumber a (number of 0, 1, 2 it contains).

¹⁰The usual formula is based on Möbius inversion on a subword complex, and its coefficients ± 1 derive from that complex being homeomorphic to a ball. The formula used here is subtler, in being based on the subword complex being shellable; see [KM04]. A similar subtlety will be discussed in the proof of proposition 5.

Lemma 9. *At fixed u_1, u_2 ,*

$$\check{R}_{3,1}(q^{-2}u_1, q^2u_2)_{XY}^{WZ} = (-q)^{\frac{1}{2}(B(\vec{f}_X, \vec{f}_Y) - B(\vec{f}_W, \vec{f}_Z))} \left(O(q^2) + \left\{ \begin{array}{l} 1 \text{ or } u_2/u_1 \quad \exists T : \begin{array}{c} \triangle \\ \text{X} \quad \text{Y} \\ \text{T} \end{array} \text{ and } \begin{array}{c} \triangle \\ \text{W} \quad \text{Z} \\ \text{T} \end{array} \text{ admissible,} \\ \text{the factor of } u_2/u_1 \text{ occurring in the cases specified in Theorem 2} \\ u_2/u_1 - 1 \quad \begin{array}{c} \diamond \\ \text{W} \quad \text{Z} \\ \text{X} \quad \text{Y} \end{array} \text{ is an equivariant piece from Theorem 1} \\ 0 \quad \text{otherwise} \end{array} \right) \right)$$

where an admissible triangle is one which is either of the form of (3) (up to rotation), or a K-piece of Theorem 2.

Similarly,

$$u_Y^{XZ} = (-q)^{-\frac{1}{2}B(\vec{f}_X, \vec{f}_Z)} \left(\left\{ \begin{array}{l} 1 \quad \begin{array}{c} \triangle \\ \text{X} \quad \text{Z} \\ \text{Y} \end{array} \text{ admissible} \\ 0 \quad \text{otherwise} \end{array} \right\} + O(q^2) \right)$$

Proof. This can be checked explicitly on the entries of the R-matrix and of U , which are listed in appendix A, noting that $\frac{1}{2}(B(\vec{f}_X, \vec{f}_Y) - B(\vec{f}_W, \vec{f}_Z))$ (resp. $-\frac{1}{2}B(\vec{f}_X, \vec{f}_Z)$) is the inversion number of the rhombus (resp. triangle), given on the pictures inside a circle at the center of it. \square

Summing the exponents $\frac{1}{2}Bs$, the whole puzzle has a leading order in q which is (at most) $q^{\sum\{\text{inversion numbers}\}}$, where the sum is over all triangles and rhombi of the puzzle. According to lemma 5, this sum computes the variation of inversion numbers $\ell(\nu) - \ell(\lambda) - \ell(\mu)$. The expression of lemma 9 also coincides with the fugacities from theorem 2, including the $(-1)^{\sum\{\text{inversion numbers}\}}$ of K-pieces. We conclude that the contribution of a puzzle is $q^{\ell(\nu) - \ell(\lambda) - \ell(\mu)}$ times its fugacity as in theorem 2 plus smaller terms as $q \rightarrow 0$. Combining with corollary 1, we find that both l.h.s. and r.h.s. of (28) are of order (at most) $q^{-\ell(\lambda) - \ell(\mu)}$, and that the coefficient of that order of the equation is exactly the product rule for Schubert classes as stated in theorem 2.

As a consequence of theorem 6, we also obtain for free the puzzle rule for *dual* Schubert classes in equivariant K-theory of the 2-step flag manifold, thus concluding the proof of theorem 2.

3.6. Interlude: double Grothendieck polynomials. As promised, we now justify (under certain assumptions) the notation $S^\lambda|_\sigma$ that was used above. In this subsection we are back to $d \in \{1, 2, 3\}$.

We assume a further normalization condition:

Property 4.

$$(31) \quad \begin{array}{c} i \quad i \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ i \quad i \end{array} = 1, \quad i \in \{0, \dots, d\}$$

where lines have arbitrary, but identical, colors.

Given a single-number string λ of length n , we now define S_N^λ as follows: (on the picture $n = 4$)

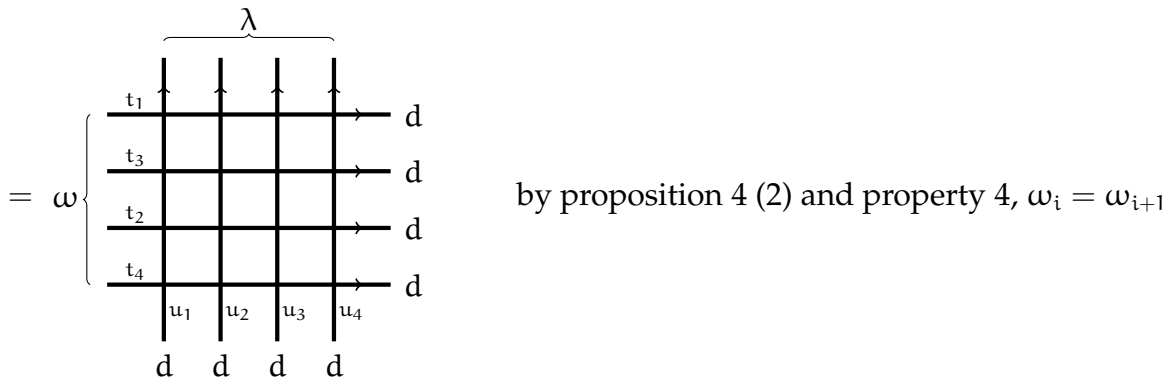
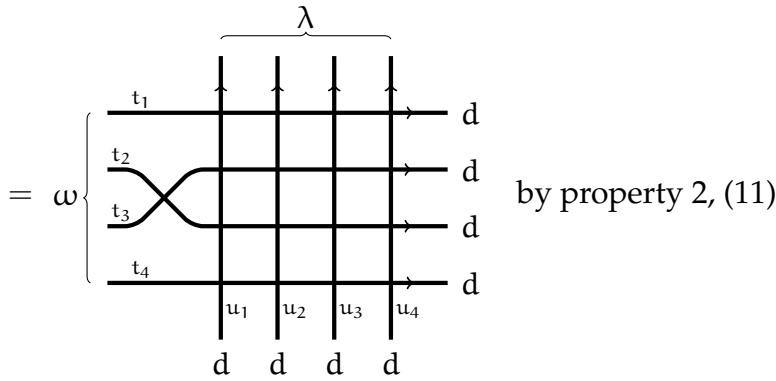
$$(32) \quad S_N^\lambda := \omega \left\{ \begin{array}{c} \overbrace{\quad \quad \quad}^{\lambda} \\ \begin{array}{cccc} \begin{array}{c} \uparrow \\ t_1 \end{array} & \begin{array}{c} \uparrow \\ t_2 \end{array} & \begin{array}{c} \uparrow \\ t_3 \end{array} & \begin{array}{c} \uparrow \\ t_4 \end{array} \\ \hline \begin{array}{cccc} \rightarrow & \rightarrow & \rightarrow & \rightarrow \\ d & d & d & d \end{array} \\ \hline \begin{array}{cccc} \begin{array}{c} \downarrow \\ u_1 \end{array} & \begin{array}{c} \downarrow \\ u_2 \end{array} & \begin{array}{c} \downarrow \\ u_3 \end{array} & \begin{array}{c} \downarrow \\ u_4 \end{array} \\ \hline \begin{array}{cccc} d & d & d & d \end{array} \end{array} \right.$$

where the string ω (the unique weakly increasing string with the same content as λ) is read from top to bottom. S_{NW}^λ and S_{NE}^λ are defined identically, except with green and red lines, respectively. All three live in $\mathbb{Q}(t_1, \dots, t_n, u_1, \dots, u_n, \dots)$. More precisely, denoting them collectively by S^λ , and writing $\omega = 0^{p_0} 1^{p_1} \dots d^{p_d}$, with $p_i = n_{i+1} - n_i$ (and $n_0 = 0$, $n_{d+1} = n$), we have

Lemma 10. S^λ is symmetric in $t_{1+n_i}, \dots, t_{n_{i+1}}$ for $i = 0, \dots, d$.

Proof. We show invariance by elementary transposition $i, i + 1$ where $\omega_i = \omega_{i+1}$: ($i = 2$ on the picture; all lines are of the same color)

$$S^\lambda = \omega \left\{ \begin{array}{c} \overbrace{\quad \quad \quad}^{\lambda} \\ \begin{array}{cccc} \begin{array}{c} \uparrow \\ t_1 \end{array} & \begin{array}{c} \uparrow \\ t_2 \end{array} & \begin{array}{c} \uparrow \\ t_3 \end{array} & \begin{array}{c} \uparrow \\ t_4 \end{array} \\ \hline \begin{array}{cccc} \rightarrow & \rightarrow & \rightarrow & \rightarrow \\ d & d & d & d \end{array} \\ \hline \begin{array}{cccc} \begin{array}{c} \downarrow \\ u_1 \end{array} & \begin{array}{c} \downarrow \\ u_2 \end{array} & \begin{array}{c} \downarrow \\ u_3 \end{array} & \begin{array}{c} \downarrow \\ u_4 \end{array} \\ \hline \begin{array}{cccc} d & d & d & d \end{array} \end{array} \right. \quad \text{by proposition 4 (2) and property 4}$$

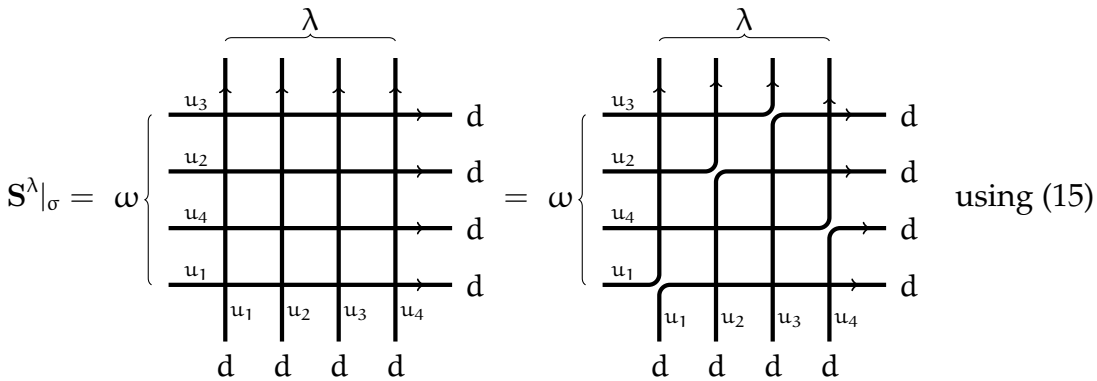


□

The justification of the notation $S^\lambda|_\sigma$ is then

Lemma 11. For each permutation $\sigma \in \mathcal{S}_n$, we have $S^\lambda|_\sigma = S^\lambda|_{t_i = u_{\sigma(i)} \ \forall i \in \{1, \dots, n\}}$.

Proof. The proof is again an obvious check of connectivity of lines. Let us work it out on an example, such as $\sigma = 3241$. Then






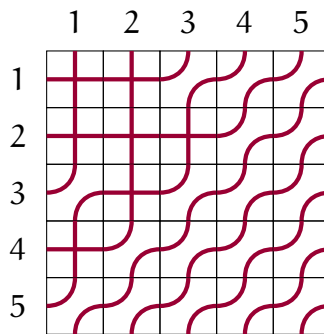
which is the dual picture from the one in the statement of the proposition.



□

The observant reader may have recognized in the definition of S^λ a pipe dream formula in disguise. More precisely, we have the following statement:

Proposition 5. *If the R-matrix defining crossings in the definition (32) is given by (29), i.e., of nilHecke type, then S^λ coincides with the **double Grothendieck polynomial** associated to λ .*

Proof. A **pipe dream** is a filling of a $n \times n$ square with “plaquettes”  and , where the Southeast triangle is entirely made of , e.g.,



We say that a pipe dream has connectivity $\pi \in S_n$ if the i th left mid edge (numbered from top to bottom) is connected to top mid edges numbered (from left to right) π_i , in the following sense. One follows the line formed by the plaquettes, with one exception: if one crosses another given line multiple times in the process, then all  after the first one are ignored (for connectivity purposes they’re equivalent to a ). Following [KM04], we call such crossings **absorbable**. In the example above, $\pi = 35142$ (paying attention to the absorbable crossing at row 3, column 2).

Given a single-number string λ , define π to be the (unique) permutation such that $\lambda_{\pi_i} = \omega_i$, and $\pi_i > \pi_{i+1}$ implies $\omega_i < \omega_{i+1}$, $i = 1, \dots, n-1$, as in §2.2. The **double Grothendieck polynomial** associated to λ , or more traditionally to π , is given by

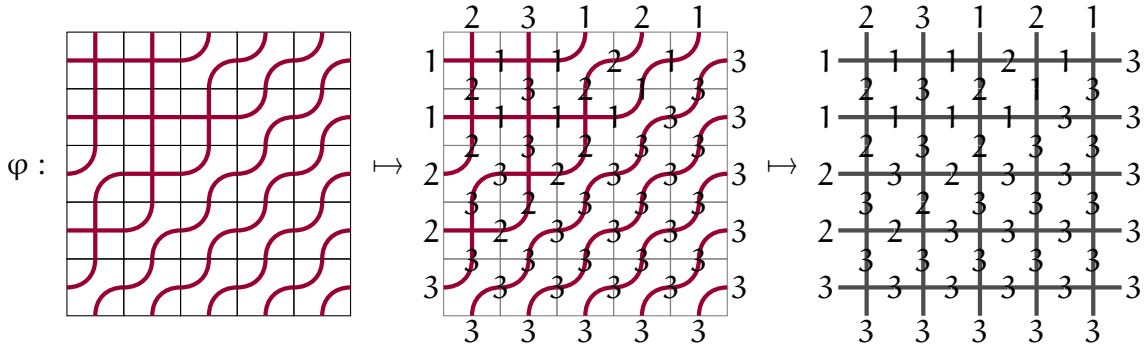
$$(33) \quad S^\lambda := \sum_{\substack{\text{pipe dreams with} \\ \text{connectivity } \pi}} (-1)^{\#\{\text{absorbable crossings}\}} \prod_{\substack{\text{at row } i, \\ \text{column } j}} (1 - t_i/u_j)$$

(see e.g. [FK94, theorem 2.3]).

We want to show that if the R-matrix defining the crossings of the picture of (32) is given by (29), then $S^\lambda = S^\lambda$. We proceed by defining a map φ from the pipe dreams of (33) to *configurations* of (32), that is fillings of the edges of the grid with labels in $\{0, \dots, d\}$ such that the associated fugacity is nonzero.

Given a pipe dream, we number the lines starting on the left side (resp. bottom side) of the grid according to ω (resp. “ d ”), thus matching the left and bottom boundary labels of (32). We then continue labeling the lines as they propagate inside the grid, in the same sense as the connectivity of a pipe dream; that is, the labels follow the lines except across absorbable crossings, in which case the labels move Northeast as if the crossing was absent. Finally, we remove all the original lines and replace them with a square grid.

On the same example, assuming $d = 3, \lambda = 23121$,



It is easy to check that the result is a valid configuration of (32), and that the top labels reproduce λ .

Inversely, consider the preimage under φ of a configuration of (32). The labels around

each vertex $i \begin{matrix} k \\ \text{---} \\ j \end{matrix} l$ can be of three types, according to (29):

- $i = k, j = l, i \leq j$: the latter condition means that the two lines arriving from the bottom and left sides have not crossed yet, and they are not either at this vertex; symbolically,

$$\phi^{-1} \left(i \begin{matrix} k \\ \text{---} \\ j \end{matrix} l \right) = \begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix} \quad i = k, \quad j = l, \quad i \leq j$$

- Similarly, if $i = l, j = k, i < j$, the two lines that have not crossed yet do cross at the vertex:



$$\phi^{-1} \left(i \begin{matrix} k \\ \text{---} \\ j \end{matrix} l \right) = \begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix} \quad i = l, \quad j = k, \quad i < j$$

- The nontrivial case is $i = k, j = l, i > j$, in which case the two lines have crossed somewhere Southwest of the vertex. In this case, the corresponding plaquette can either be a , or an absorbable

$$\phi^{-1} \left(i \begin{matrix} k \\ \text{---} \\ j \end{matrix} l \right) = \left\{ \begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix}, \begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix} \right\} \quad i = k, \quad j = l, \quad i > j$$

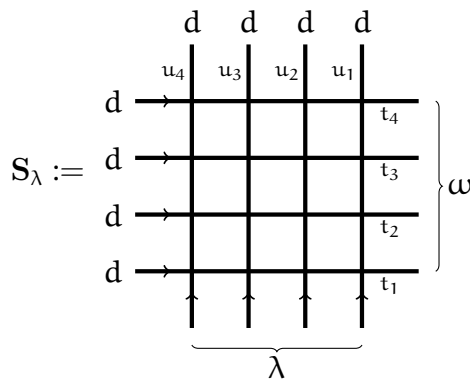
Since identically labeled lines cannot cross each other, lines go straight from the South side to the East side, and similarly the requirement that the top labels form λ leads to a connectivity from West to North sides which is given by π . Therefore any preimage under φ is a pipe dream configuration with the correct connectivity.

Furthermore, since the correspondence is purely local, we can compare fugacities one vertex/plaquette at a time. In the first two cases, one directly finds fugacities 1 and 1 –

t_i/u_j , respectively, in both (29) and (33). In the third case, we find t_i/u_j in (29), whereas  (resp. ) contributes 1 (resp. $(-1) \times (1 - t_i/u_j)$, since the crossing is absorbable) to the sum of (33), which also matches. \square

Geometrically, this proposition is saying that if the R-matrix is of the form of (29), then S^λ is the expression of a Schubert class (in K_T) in terms of Chern roots. Lemma 11 then tells us how to obtain from this expression its restriction at each fixed point. The proof amounts to providing a direct connection between pipe dream-type formulæ for the former, and AJS/Billey/Graham–Willems-type formulæ for the latter.

Similar “dual” statements can be made by defining



where ω is read from bottom to top. Geometrically, if the R-matrix is of the form of (29), S_λ is the expression of a dual Schubert class (in K_T) in terms of first Chern classes (or really Bott classes, being in K-theory).

Remark. The connection between the nilHecke algebra and Grothendieck polynomials was first observed in [FK93]. There, a different representation is used, which is more appropriate for the *full* flag variety. The idea to introduce the “vertex” representation we use here is mentioned (at $d = 1$: the so-called “5-vertex model”) in [ZJ09b, §5.2.5].

3.7. $d = 3$ and E_6 . The reasoning is extremely similar to the case of $d = 2$, but for technical reasons, we conclude in a slightly different manner; in particular we use the results of the previous subsection.

3.7.1. *R-matrices.* In order to avoid repeats, we only mention the setup insofar as it differs from the one at $d = 2$. To each line we now attach a representation of the quantized affine algebra $U_q(\mathfrak{e}_6^{(1)})$: to red and green lines we attach the evaluation representation (also called affinization [CP94]) of the V_{ω_1} of lemma 3, whereas to blue lines we attach the evaluation representation based on the dual $V_{\omega_6} \cong V_{\omega_1}^*$.

Next we define R-matrices. Explicitly, in terms of the projectors $P_{\omega_6}, P_{2\omega_1}, P_{\omega_3}$ that intertwine the action of the horizontal subalgebra $U_q(\mathfrak{e}_6)$, we have

$$(34) \quad \check{R}_{3,1} = \frac{1}{q^2(u_2 - q^6 u_1)(u_2 - u_1)} \left((u_1 - q^8 u_2)(u_1 - q^2 u_2) P_{\omega_6} + (u_2 - q^8 u_1)(u_1 - q^2 u_2) P_{\omega_3} + (u_2 - q^8 u_1)(u_2 - q^2 u_1) P_{2\omega_1} \right)$$

All other R-matrices involving red and green lines are proportional, since they're based on the same representations. However, we choose their normalizations differently:

$$(35) \quad \check{R}_{i,i} = \frac{(u_1 - q^8 u_2)(u_1 - q^2 u_2)}{(u_2 - q^8 u_1)(u_2 - q^2 u_1)} P_{\omega_6} + \frac{u_1 - q^2 u_2}{u_2 - q^2 u_1} P_{\omega_3} + P_{2\omega_1} \quad i = 1, 3$$

$\check{R}_{2,2}$ is defined identically up to the outer automorphism of E_6 that switches ω_1 and ω_6 .

It is simplest to define $\check{R}_{2,i}$, $i = 1, 3$, using the ‘‘crossing symmetry’’; namely,

$$\check{R}_{2,i}(u_1, u_2) = \mathcal{P}_{\omega_6, \omega_1}(\mathcal{P}_{\omega_1, \omega_1} \check{R}_{3,1}(u_2, u_1))^{T_2} \quad i = 1, 3$$

where $\mathcal{P}_{\omega_i, \omega_j}$ is the operator from $V_{\omega_i} \otimes V_{\omega_j}$ to $V_{\omega_j} \otimes V_{\omega_i}$ that switches the factors of the tensor product, and T_2 means partial transpose of the second factor of the tensor product. As usual, we define all remaining R-matrices such that

$$\check{R}_{i,j}(u_1, u_2) = \check{R}_{j,i}(u_2, u_1)^{-1}$$

is satisfied for all $i, j = 1, 2, 3$.

As in (24) at $d = 2$, we get the factorization

$$\check{R}_{3,1}|_{u_1=q^{-16}u, u_2=q^{-8}u} = \frac{(1 - q^{16})(1 - q^{10})}{q^8(1 - q^8)(1 - q^2)} P_{\omega_6} = UD$$

where $D : V_{\omega_1}(q^{-16}u) \otimes V_{\omega_1}(q^{-8}u) \rightarrow V_{\omega_6}(u)$ and $U : V_{\omega_6}(u) \rightarrow V_{\omega_1}(q^{-8}u) \otimes V_{\omega_3}(q^{-16}u)$ commute with the $U_q(\mathfrak{e}_6^{(1)})$ action. This is the analogue of property 1, with $\alpha = q^{-16}$, $\beta = q^{-8}$.

It is a tedious but easy exercise to check that all properties 1–4 are satisfied in this setup. We also impose the gauge fixing condition (17).

3.7.2. *The limit $q \rightarrow 0$.* We start once again from theorem 5, and rewrite it using lemma 11:

$$\sum_{\nu} \begin{array}{c} \lambda \\ \triangle \\ \mu \\ \nu \end{array} S_N^{\nu} = S_{NW}^{\lambda} S_{NE}^{\mu} \quad \text{in } \mathbb{Q}(t_1, \dots, t_n, u_1, \dots, u_n, q)^{\prod_{i=0}^d S_{p_i}} / \mathcal{I}$$

where $\prod_{i=0}^d S_{p_i}$ acts by permutations of the t_i , cf lemma 10, and \mathcal{I} is the ideal of rational functions that vanish when the t s are specialized to a permutation of the u s. \mathcal{I} is generated by the $P(t_1, \dots, t_n) - P(u_1, \dots, u_n)$, where P runs over symmetric polynomials.

We set all u s to one: we obtain

$$\sum_{\nu} \begin{array}{c} \lambda \\ \triangle \\ \mu \\ \nu \end{array} |_{u_i=1} S_N^{\nu}|_{u_i=1} = S_{NW}|_{u_i=1} S_{NE}|_{u_i=1} \quad \text{in } \mathbb{Q}(t_1, \dots, t_n, q)^{\prod_{i=0}^d S_{p_i}} / \mathcal{I}_1$$

where \mathcal{I}_1 is the ideal generated by $P(t_1, \dots, t_n) - P(1, \dots, 1)$, $P \in \mathbb{Q}[t_1, \dots, t_n]^{S_n}$.

We are now (and only now) ready to take the limit $q \rightarrow 0$ as before:

- According to proposition 4 part 1, we can restrict $\check{R}_{i,i}$ to the single-number sector, check explicitly that these three matrices are identical for $i = 1, 2, 3$, and then check that lemma 8 holds without any change at $d = 3$.
- According to proposition 5, this implies that

$$S^{\lambda} = q^{-\ell(\lambda)}(S^{\lambda} + O(q^2))$$

where S^λ is the double Grothendieck polynomial associated to λ . In particular, $S^\lambda|_{u_i=1} = q^{-\ell(\lambda)}(S^\lambda|_{u_i=1} + O(q^2))$, where $S^\lambda|_{u_i=1}$ is the (ordinary) Grothendieck polynomial associated to λ .

- Finally, we examine the (nonequivariant) puzzle, i.e., we take the $q \rightarrow 0$ limit of $\check{R}_{3,1}(q^{-16}u, q^8u) = UD$. We find:

Lemma 12. *As $q \rightarrow 0$,*

$$U_Y^{XZ} = (-q)^{-\frac{1}{2}B(\vec{f}_X, \vec{f}_Z)} \left(\begin{cases} 1 & \begin{array}{c} X \\ \triangle \\ Y \end{array} \text{ admissible} \\ 0 & \text{otherwise} \end{cases} + O(q^2) \right)$$

$$D_{ZX}^Y = (-q)^{\frac{1}{2}B(\vec{f}_Z, \vec{f}_X)} \left(\begin{cases} 1 & \begin{array}{c} Y \\ \triangle \\ X \end{array} \text{ admissible} \\ 0 & \text{otherwise} \end{cases} + O(q^2) \right)$$

where an admissible triangle is one which is either of the form of (5) (up to rotation), or a K-piece as listed in appendix B.

The proof is a (computer-assisted) expansion at first nontrivial order in q of the 27^3 entries of U and D . The Mathematica code is available upon request.

Finally,

$$\sum_{\nu} \begin{array}{c} \lambda \\ \triangle \\ \mu \\ \nu \end{array} S^\nu|_{u_i=1} = S^\lambda|_{u_i=1} S^\mu|_{u_i=1} \quad \text{in } \mathbb{Z}[t_1^\pm, \dots, t_n^\pm]^{\prod_{i=0}^d S_{p_i}} / \mathcal{I}'$$

where we have changed the ambient ring because the $S^\lambda|_{u_i=1}$ are polynomials in the t_i (in fact, we only care that they're Laurent polynomials), and $\mathcal{I}' = \mathcal{I}_1 \cap \mathbb{Z}[t_1^\pm, \dots, t_n^\pm]^{\prod_{i=0}^d S_{p_i}}$.

The triangle now stands for the summation over *nonequivariant* puzzles, i.e., $\begin{array}{c} \lambda \\ \triangle \\ \mu \\ \nu \end{array} =$

$(-1)^{\ell(\nu) - \ell(\lambda) - \ell(\mu)} \#\{\text{such puzzles}\}$. The quotient ring $\mathbb{Z}[t_1^\pm, \dots, t_n^\pm]^{\prod_{i=0}^d S_{p_i}} / \mathcal{I}'$ is nothing but the K-theory ring of the d -step flag variety $\{0 = V_0 \leq V_1 \leq \dots \leq V_d \leq V_{d+1} = \mathbb{C}^n\}$ with $\dim V_i = n_i$, $p_i = n_{i+1} - n_i$, $i = 0, \dots, d$, where $d = 3$ here, and the t s play the role of Chern roots. The $S^\lambda|_{u_i=1}$ are then the nonequivariant K-theoretic Schubert classes.

As explained at the end of §3.6, K-theoretic dual Schubert classes can be obtained by 180° degree rotation of the pictures for Schubert classes, and application of theorem 6 leads to a puzzle formula for these dual classes:

$$\sum_{\nu} \begin{array}{c} \nu \\ \triangle \\ \mu \\ \lambda \end{array} S_\nu|_{u_i=1} = S_\lambda|_{u_i=1} S_\mu|_{u_i=1}$$

where once again $\begin{array}{c} \nu \\ \triangle \\ \mu \\ \lambda \end{array} = (-1)^{\ell(\nu) - \ell(\lambda) - \ell(\mu)} \#\{\text{such puzzles}\}$.

This concludes the proof of theorem 3.

APPENDIX A. THE $d = 2$ R-MATRIX

These are the nonzero entries of $\check{R}_{3,1}(q^{-2}u_1, q^2u_2)$ as defined in (23) and used in lemma 9:

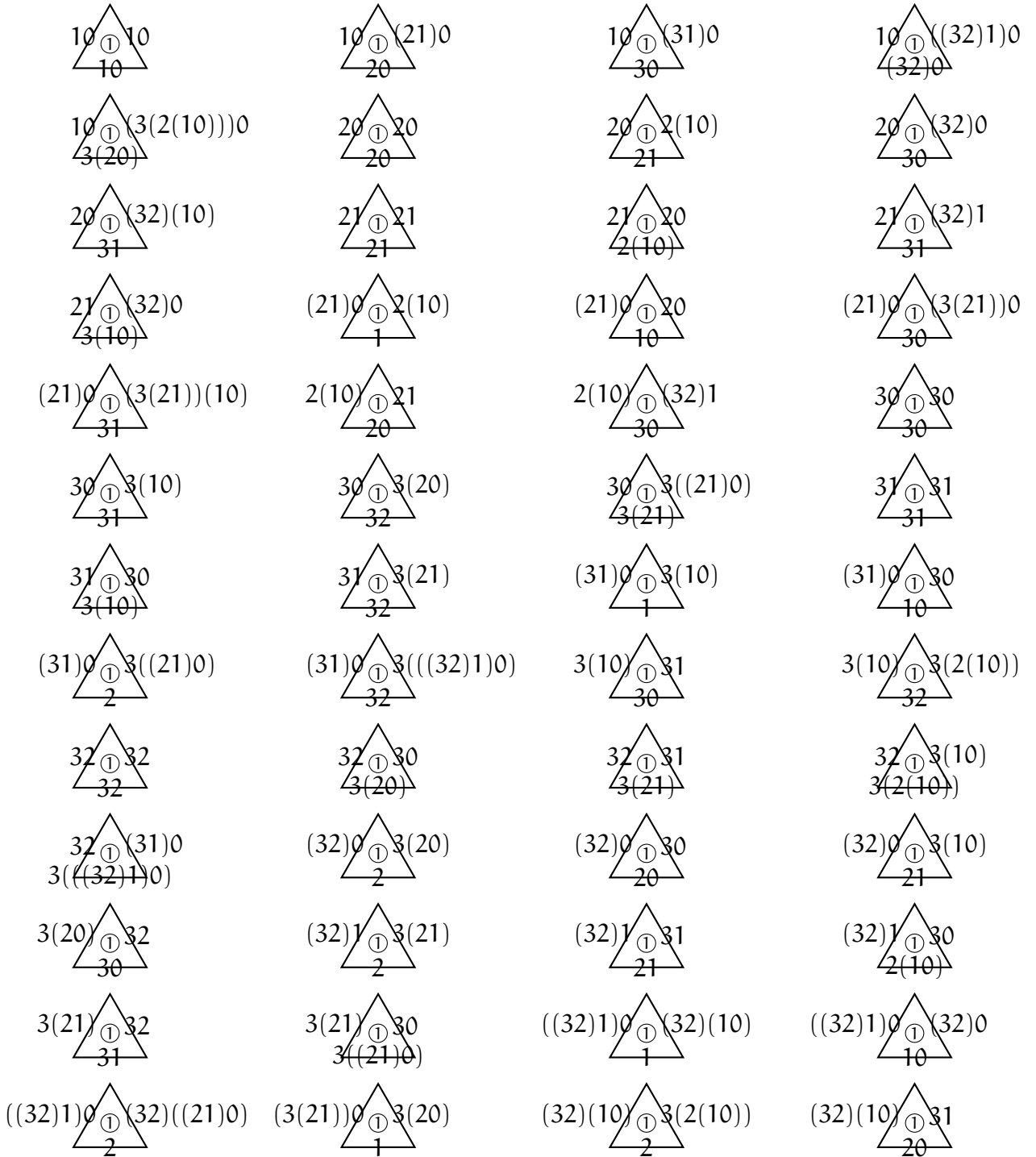
- $\begin{array}{c} 0 & 0 \\ \textcircled{0} \\ 0 & 0 \end{array} = \begin{array}{c} 0 & 10 \\ \textcircled{0} \\ 10 & 0 \end{array} = \begin{array}{c} 0 & 20 \\ \textcircled{0} \\ 20 & 0 \end{array} = \begin{array}{c} 0 & (21)0 \\ \textcircled{0} \\ (21)0 & 0 \end{array} = \begin{array}{c} 1 & 0 \\ \textcircled{0} \\ 0 & 1 \end{array} = \begin{array}{c} 1 & 1 \\ \textcircled{0} \\ 1 & 1 \end{array} = \begin{array}{c} 1 & 21 \\ \textcircled{0} \\ 21 & 1 \end{array} =$
- $\begin{array}{c} 1 & (21)0 \\ \textcircled{2} \\ (21)0 & 1 \end{array} = \begin{array}{c} 2 & 0 \\ \textcircled{0} \\ 0 & 2 \end{array} = \begin{array}{c} 2 & 1 \\ \textcircled{0} \\ 1 & 2 \end{array} = \begin{array}{c} 2 & 2 \\ \textcircled{0} \\ 2 & 2 \end{array} = \begin{array}{c} 2 & 10 \\ \textcircled{0} \\ 10 & 2 \end{array} = \begin{array}{c} 10 & 1 \\ \textcircled{0} \\ 1 & 10 \end{array} = \begin{array}{c} 10 & 10 \\ \textcircled{2} \\ 10 & 10 \end{array} =$
- $\begin{array}{c} 10 & (21)0 \\ \textcircled{2} \\ (21)0 & 10 \end{array} = \begin{array}{c} 10 & 2(10) \\ \textcircled{0} \\ 2(10) & 10 \end{array} = \begin{array}{c} 20 & 2 \\ \textcircled{0} \\ 2 & 20 \end{array} = \begin{array}{c} 20 & 10 \\ \textcircled{2} \\ 10 & 20 \end{array} = \begin{array}{c} 20 & 20 \\ \textcircled{2} \\ 20 & 20 \end{array} = \begin{array}{c} 20 & 2(10) \\ \textcircled{2} \\ 2(10) & 20 \end{array} =$
- $\begin{array}{c} 21 & 0 \\ \textcircled{0} \\ 0 & 21 \end{array} = \begin{array}{c} 21 & 2 \\ \textcircled{0} \\ 2 & 21 \end{array} = \begin{array}{c} 21 & 20 \\ \textcircled{2} \\ 20 & 21 \end{array} = \begin{array}{c} 21 & 21 \\ \textcircled{2} \\ 21 & 21 \end{array} = \begin{array}{c} (21)0 & 20 \\ \textcircled{2} \\ 20 & (21)0 \end{array} = \begin{array}{c} (21)0 & 21 \\ \textcircled{0} \\ 21 & (21)0 \end{array} = \begin{array}{c} (21)0 & (21)0 \\ \textcircled{4} \\ (21)0 & (21)0 \end{array} =$
- $\begin{array}{c} (21)0 & 2(10) \\ \textcircled{2} \\ 2(10) & (21)0 \end{array} = \begin{array}{c} 2(10) & 1 \\ \textcircled{2} \\ 1 & 2(10) \end{array} = \begin{array}{c} 2(10) & 2 \\ \textcircled{0} \\ 2 & 2(10) \end{array} = \begin{array}{c} 2(10) & 21 \\ \textcircled{2} \\ 21 & 2(10) \end{array} = \begin{array}{c} 2(10) & 2(10) \\ \textcircled{4} \\ 2(10) & 2(10) \end{array} = 1$
- $\begin{array}{c} 21 & 0 \\ \textcircled{2} \\ (21)0 & (21)0 \end{array} = \frac{(1-q^2)u_1}{q^2(u_1-q^2u_2)}$
- $\begin{array}{c} 2 & 10 \\ \textcircled{1} \\ (21)0 & 1 \end{array} = \begin{array}{c} 10 & 10 \\ \textcircled{1} \\ 2 & 2(10) \end{array} = \begin{array}{c} 20 & 10 \\ \textcircled{3} \\ (21)0 & (21)0 \end{array} = \begin{array}{c} 21 & 0 \\ \textcircled{1} \\ 1 & 2(10) \end{array} = \begin{array}{c} 21 & 0 \\ \textcircled{1} \\ 10 & 20 \end{array} = \begin{array}{c} 21 & 20 \\ \textcircled{3} \\ 2(10) & 2(10) \end{array} =$
- $\begin{array}{c} 2(10) & 1 \\ \textcircled{3} \\ (21)0 & (21)0 \end{array} = \frac{(1-q^2)u_1}{q(u_1-q^2u_2)}$
- $\begin{array}{c} 0 & 0 \\ \textcircled{0} \\ 1 & 10 \end{array} = \begin{array}{c} 0 & 0 \\ \textcircled{0} \\ 2 & 20 \end{array} = \begin{array}{c} 0 & 0 \\ \textcircled{0} \\ 21 & (21)0 \end{array} = \begin{array}{c} 0 & 10 \\ \textcircled{0} \\ 2 & 21 \end{array} = \begin{array}{c} 0 & 20 \\ \textcircled{0} \\ 2(10) & 10 \end{array} = \begin{array}{c} 1 & 0 \\ \textcircled{0} \\ 2 & 2(10) \end{array} = \begin{array}{c} 1 & 1 \\ \textcircled{0} \\ 2 & 21 \end{array} =$
- $\begin{array}{c} 1 & 1 \\ \textcircled{0} \\ 10 & 0 \end{array} = \begin{array}{c} 1 & 21 \\ \textcircled{0} \\ 20 & 0 \end{array} = \begin{array}{c} 1 & 21 \\ \textcircled{0} \\ 2(10) & 10 \end{array} = \begin{array}{c} 2 & 1 \\ \textcircled{0} \\ (21)0 & 0 \end{array} = \begin{array}{c} 2 & 2 \\ \textcircled{0} \\ 20 & 0 \end{array} = \begin{array}{c} 2 & 2 \\ \textcircled{0} \\ 21 & 1 \end{array} = \begin{array}{c} 2 & 2 \\ \textcircled{0} \\ 2(10) & 10 \end{array} =$

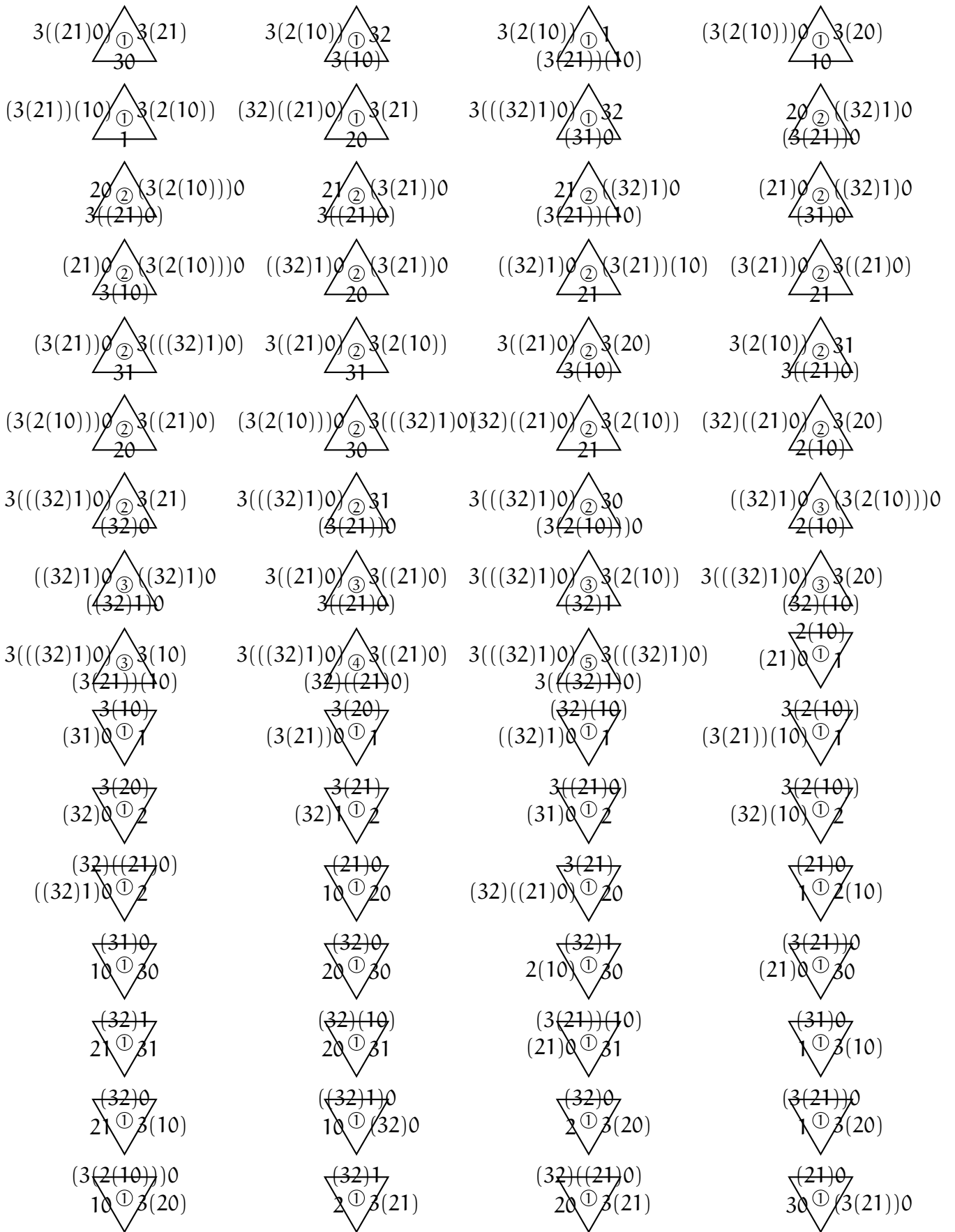
$$\begin{aligned}
 & \begin{array}{c} 2 \quad 10 \\ \textcircled{2} \\ 2(10) \quad 2(10) \end{array} = \begin{array}{c} 10 \quad 1 \\ \textcircled{0} \\ 2 \quad 20 \end{array} = \begin{array}{c} 10 \quad 1 \\ \textcircled{0} \\ 21 \quad (21)0 \end{array} = \begin{array}{c} 10 \quad 10 \\ \textcircled{2} \\ 20 \quad (21)0 \end{array} = \begin{array}{c} 10 \quad (21)0 \\ \textcircled{2} \\ 20 \quad 20 \end{array} = \begin{array}{c} 10 \quad (21)0 \\ \textcircled{2} \\ 21 \quad 2(10) \end{array} = \\
 & \begin{array}{c} 20 \quad 2 \\ \textcircled{0} \\ 21 \quad (21)0 \end{array} = \begin{array}{c} 21 \quad 21 \\ \textcircled{2} \\ 2(10) \quad 20 \end{array} = \begin{array}{c} 2(10) \quad 1 \\ \textcircled{2} \\ 10 \quad 20 \end{array} = \begin{array}{c} 2(10) \quad 21 \\ \textcircled{2} \\ 20 \quad 20 \end{array} = \frac{(1-q^2)u_1}{u_1-q^2u_2} \\
 \bullet & \begin{array}{c} 0 \quad 10 \\ \textcircled{1} \\ 2(10) \quad (21)0 \end{array} = \begin{array}{c} 0 \quad (21)0 \\ \textcircled{1} \\ 21 \quad 21 \end{array} = \begin{array}{c} 0 \quad (21)0 \\ \textcircled{1} \\ 2(10) \quad 20 \end{array} = \begin{array}{c} 1 \quad 0 \\ \textcircled{1} \\ 10 \quad 10 \end{array} = \begin{array}{c} 1 \quad 0 \\ \textcircled{1} \\ 20 \quad (21)0 \end{array} = \begin{array}{c} 1 \quad 1 \\ \textcircled{1} \\ 2(10) \quad (21)0 \end{array} = \\
 & \begin{array}{c} 1 \quad (21)0 \\ \textcircled{3} \\ 2(10) \quad 2(10) \end{array} = \begin{array}{c} 2 \quad 0 \\ \textcircled{1} \\ 20 \quad 20 \end{array} = \begin{array}{c} 2 \quad 0 \\ \textcircled{1} \\ 21 \quad 2(10) \end{array} = \begin{array}{c} 2 \quad 0 \\ \textcircled{1} \\ (21)0 \quad 10 \end{array} = \begin{array}{c} 2 \quad 1 \\ \textcircled{1} \\ 21 \quad 21 \end{array} = \begin{array}{c} 2 \quad 1 \\ \textcircled{1} \\ 2(10) \quad 20 \end{array} = \\
 & \begin{array}{c} 2 \quad 10 \\ \textcircled{1} \\ 20 \quad 21 \end{array} = \begin{array}{c} 21 \quad 2 \\ \textcircled{1} \\ 2(10) \quad (21)0 \end{array} = \begin{array}{c} 2(10) \quad 2 \\ \textcircled{1} \\ 20 \quad (21)0 \end{array} = \frac{q(1-q^2)u_1}{u_1-q^2u_2} \\
 \bullet & \begin{array}{c} 1 \quad (21)0 \\ \textcircled{2} \\ 20 \quad 21 \end{array} = \frac{q^2(1-q^2)u_1}{u_1-q^2u_2} \\
 \bullet & \begin{array}{c} 21 \quad 20 \\ \textcircled{2} \\ (21)0 \quad 1 \end{array} = \frac{(1-q^2)u_2}{q^2(u_1-q^2u_2)} \\
 \bullet & \begin{array}{c} 10 \quad 10 \\ \textcircled{1} \\ 0 \quad 1 \end{array} = \begin{array}{c} 10 \quad (21)0 \\ \textcircled{1} \\ 0 \quad 2 \end{array} = \begin{array}{c} 20 \quad 20 \\ \textcircled{1} \\ 0 \quad 2 \end{array} = \begin{array}{c} 20 \quad 2(10) \\ \textcircled{1} \\ 1 \quad 2 \end{array} = \begin{array}{c} 20 \quad 2(10) \\ \textcircled{1} \\ (21)0 \quad 0 \end{array} = \begin{array}{c} 21 \quad 20 \\ \textcircled{1} \\ 10 \quad 2 \end{array} = \begin{array}{c} 21 \quad 21 \\ \textcircled{1} \\ 1 \quad 2 \end{array} = \\
 & \begin{array}{c} 21 \quad 21 \\ \textcircled{1} \\ (21)0 \quad 0 \end{array} = \begin{array}{c} (21)0 \quad 20 \\ \textcircled{1} \\ 0 \quad 1 \end{array} = \begin{array}{c} (21)0 \quad 20 \\ \textcircled{1} \\ 2 \quad 2(10) \end{array} = \begin{array}{c} (21)0 \quad 2(10) \\ \textcircled{1} \\ 1 \quad 1 \end{array} = \begin{array}{c} (21)0 \quad 2(10) \\ \textcircled{1} \\ 2 \quad 21 \end{array} = \begin{array}{c} (21)0 \quad 2(10) \\ \textcircled{1} \\ 10 \quad 0 \end{array} = \\
 & \begin{array}{c} 2(10) \quad 21 \\ \textcircled{1} \\ 0 \quad 2 \end{array} = \begin{array}{c} 2(10) \quad 2(10) \\ \textcircled{3} \\ (21)0 \quad 1 \end{array} = \frac{(1-q^2)u_2}{q(u_1-q^2u_2)}
 \end{aligned}$$

$$\begin{aligned}
 & \bullet \begin{array}{c} 0 & 10 \\ \textcircled{0} \\ 1 & 1 \end{array} = \begin{array}{c} 0 & 20 \\ \textcircled{0} \\ 2 & 2 \end{array} = \begin{array}{c} 0 & 20 \\ \textcircled{0} \\ 21 & 1 \end{array} = \begin{array}{c} 0 & (21)0 \\ \textcircled{0} \\ 1 & 2 \end{array} = \begin{array}{c} 1 & 21 \\ \textcircled{0} \\ 2 & 2 \end{array} = \begin{array}{c} 10 & 1 \\ \textcircled{0} \\ 0 & 0 \end{array} = \begin{array}{c} 10 & 2(10) \\ \textcircled{0} \\ 2 & 2 \end{array} = \\
 & \begin{array}{c} 10 & 2(10) \\ \textcircled{0} \\ 20 & 0 \end{array} = \begin{array}{c} 10 & 2(10) \\ \textcircled{0} \\ 21 & 1 \end{array} = \begin{array}{c} 20 & 2 \\ \textcircled{0} \\ 0 & 0 \end{array} = \begin{array}{c} 20 & 2 \\ \textcircled{0} \\ 1 & 10 \end{array} = \begin{array}{c} 20 & 10 \\ \textcircled{2} \\ 1 & 2(10) \end{array} = \begin{array}{c} 20 & 20 \\ \textcircled{2} \\ 21 & 2(10) \end{array} = \begin{array}{c} 20 & 20 \\ \textcircled{2} \\ (21)0 & 10 \end{array} = \\
 & \begin{array}{c} 20 & 2(10) \\ \textcircled{2} \\ 21 & 21 \end{array} = \begin{array}{c} 21 & 2 \\ \textcircled{0} \\ 1 & 1 \end{array} = \begin{array}{c} 21 & 2 \\ \textcircled{0} \\ 10 & 0 \end{array} = \begin{array}{c} (21)0 & 20 \\ \textcircled{2} \\ 10 & 10 \end{array} = \begin{array}{c} (21)0 & 21 \\ \textcircled{0} \\ 0 & 0 \end{array} = \begin{array}{c} (21)0 & 21 \\ \textcircled{0} \\ 1 & 10 \end{array} = \begin{array}{c} (21)0 & 21 \\ \textcircled{0} \\ 2 & 20 \end{array} = \\
 & \begin{array}{c} 2(10) & 2 \\ \textcircled{0} \\ 0 & 1 \end{array} = \begin{array}{c} 2(10) & 21 \\ \textcircled{2} \\ (21)0 & 10 \end{array} = \begin{array}{c} 2(10) & 2(10) \\ \textcircled{2} \\ 10 & 2 \end{array} = \frac{(1-q^2)u_2}{u_1-q^2u_2} \\
 & \bullet \begin{array}{c} 1 & (21)0 \\ \textcircled{1} \\ 10 & 2 \end{array} = \begin{array}{c} 20 & 10 \\ \textcircled{1} \\ 0 & 21 \end{array} = \begin{array}{c} (21)0 & (21)0 \\ \textcircled{3} \\ 1 & 2(10) \end{array} = \begin{array}{c} (21)0 & (21)0 \\ \textcircled{3} \\ 10 & 20 \end{array} = \begin{array}{c} 2(10) & 1 \\ \textcircled{1} \\ 0 & 21 \end{array} = \begin{array}{c} 2(10) & 2 \\ \textcircled{1} \\ 10 & 10 \end{array} = \\
 & \begin{array}{c} 2(10) & 2(10) \\ \textcircled{3} \\ 20 & 21 \end{array} = \frac{q(1-q^2)u_2}{u_1-q^2u_2} \\
 & \bullet \begin{array}{c} (21)0 & (21)0 \\ \textcircled{2} \\ 0 & 21 \end{array} = \frac{q^2(1-q^2)u_2}{u_1-q^2u_2} \\
 & \bullet \begin{array}{c} 0 & 1 \\ \textcircled{-1} \\ 1 & 0 \end{array} = \begin{array}{c} 0 & 2 \\ \textcircled{-1} \\ 2 & 0 \end{array} = \begin{array}{c} 0 & 21 \\ \textcircled{-1} \\ 21 & 0 \end{array} = \begin{array}{c} 0 & 2(10) \\ \textcircled{-1} \\ 2(10) & 0 \end{array} = \begin{array}{c} 1 & 2 \\ \textcircled{-1} \\ 2 & 1 \end{array} = \begin{array}{c} 1 & 10 \\ \textcircled{1} \\ 10 & 1 \end{array} = \begin{array}{c} 1 & 20 \\ \textcircled{1} \\ 20 & 1 \end{array} = \\
 & \begin{array}{c} 1 & 2(10) \\ \textcircled{1} \\ 2(10) & 1 \end{array} = \begin{array}{c} 2 & 20 \\ \textcircled{1} \\ 20 & 2 \end{array} = \begin{array}{c} 2 & 21 \\ \textcircled{1} \\ 21 & 2 \end{array} = \begin{array}{c} 2 & (21)0 \\ \textcircled{1} \\ (21)0 & 2 \end{array} = \begin{array}{c} 2 & 2(10) \\ \textcircled{1} \\ 2(10) & 2 \end{array} = \begin{array}{c} 10 & 0 \\ \textcircled{1} \\ 0 & 10 \end{array} = \\
 & \begin{array}{c} 10 & 2 \\ \textcircled{-1} \\ 2 & 10 \end{array} = \begin{array}{c} 10 & 20 \\ \textcircled{1} \\ 20 & 10 \end{array} = \begin{array}{c} 10 & 21 \\ \textcircled{-1} \\ 21 & 10 \end{array} = \begin{array}{c} 20 & 0 \\ \textcircled{1} \\ 0 & 20 \end{array} = \begin{array}{c} 20 & 1 \\ \textcircled{1} \\ 1 & 20 \end{array} = \begin{array}{c} 20 & 21 \\ \textcircled{1} \\ 21 & 20 \end{array} = \begin{array}{c} 20 & (21)0 \\ \textcircled{3} \\ (21)0 & 20 \end{array} =
 \end{aligned}$$

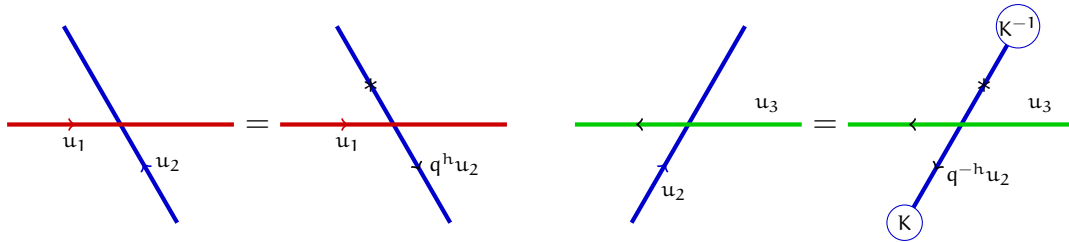
$$\begin{aligned}
 \bullet \quad & \begin{array}{c} 2(10) \\ \triangle \\ 20 \textcircled{1} 21 \\ \hline 10 \end{array} = \begin{array}{c} 20 \\ \triangle \\ 20 \textcircled{1} 20 \\ \hline 10 \end{array} = \begin{array}{c} 20 \\ \triangle \\ 21 \textcircled{1} 2(10) \\ \hline 10 \end{array} = \begin{array}{c} 20 \\ \triangle \\ (21)0 \textcircled{1} 10 \\ \hline 10 \end{array} = \begin{array}{c} 21 \\ \triangle \\ 21 \textcircled{1} 21 \\ \hline 10 \end{array} = \begin{array}{c} 21 \\ \triangle \\ 2(10) \textcircled{1} 20 \\ \hline 10 \end{array} = \begin{array}{c} 1 \\ \triangle \\ 2(10) \textcircled{1} (21)0 \\ \hline 10 \end{array} = \\
 & \begin{array}{c} 10 \\ \triangle \\ 10 \textcircled{1} 10 \\ \hline 10 \end{array} = \begin{array}{c} 10 \\ \triangle \\ 20 \textcircled{1} (21)0 \\ \hline 10 \end{array} = -q
 \end{aligned}$$

APPENDIX B. THE $d = 3$ K-TRIANGLES





We now assume that to each crossing is associated an R-matrix coming from an un-twisted quantized affine algebra $U_q(\mathfrak{g}^{(1)})$. Then one can use the *crossing symmetry* to restore the cyclic symmetry of the picture:



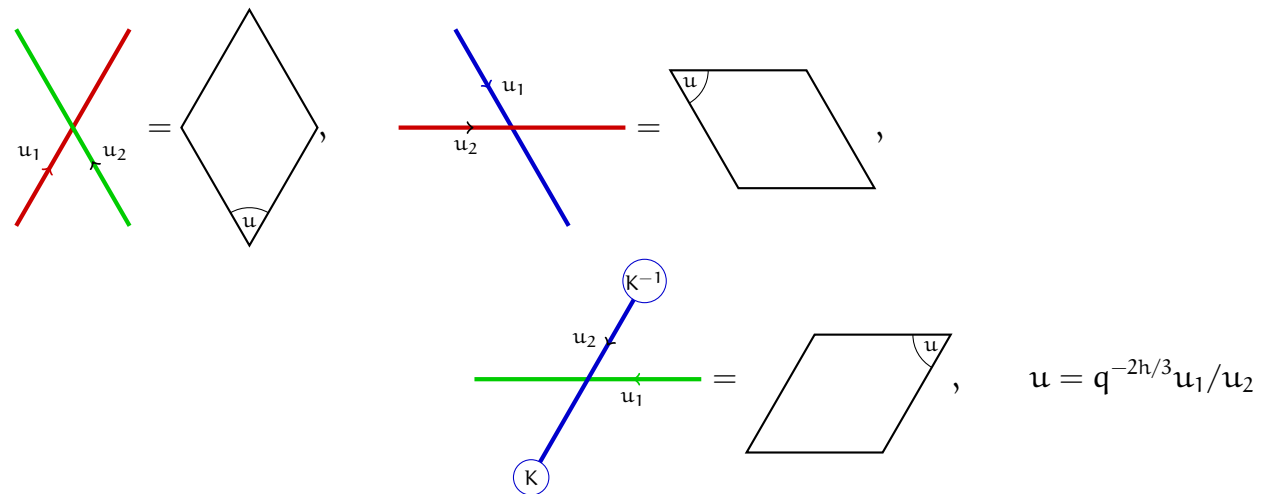
where $*$ denotes the *dual* representation, the circled $K^{\pm 1}$ denotes multiplication by the Cartan element $K = q^\rho$, ρ half the sum of positive roots of \mathfrak{g} , and h is the dual Coxeter number of \mathfrak{g} . Note the conventional choice of dual (rather than predual) representation, hence the apparent discrepancy between the two equalities.

We can thus rewrite the Yang–Baxter equation (36):

(37)

where something slightly strange happens in that the parameter of the blue line “jumps” from $q^{\pm h}u_2$ to $q^{\mp h}u_2$ as it goes across the central circle.

Observe that for $\mathfrak{g} = \mathfrak{a}_2, \mathfrak{d}_4, \mathfrak{e}_6, \mathfrak{e}_8$, the dual Coxeter number $h = 3, 6, 12, 30$ is a multiple of 3. Furthermore, for the representations used (implicitly) at $d = 1$ in [WZJ16b], at $d = 2$ in §3.5, at $d = 3$ in §3.7, the ratio of spectral parameters $q^{2h/3}$ is precisely the value at which the R-matrices corresponding to two types of lines turn into projectors onto the third type of line. It is therefore very tempting to redefine



where the rhombi are not allowed to be rotated. Note that by conjugation, one could share K equally between the three orientations of rhombi, thus restoring $\mathbb{Z}/3$ -symmetry. Up to

such possible conjugations, the first rhombus will eventually play the role of equivariant rhombus, where u is the ratio of equivariant parameters.

Setting $u = q^{h/3}u_2/u_1$, $v = q^{h/3}u_3/u_2$, $w = q^{-2h/3}u_1/u_3$ (with $uvw = 1$), we find that Eq. (37) becomes the “cyclic” Yang–Baxter equation:

$$(38) \quad \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \quad \forall u, v, w : uvw = 1$$

Compare with [ZJ09a, proposition 1] (which corresponds to the rational limit where the spectral parameters are expanded to first order around 1) and [WZJ16b, proposition 3], both related to the $d = 1$ case in the limit $q \rightarrow 0$. We could then proceed as in these papers to derive our puzzle rules for general $d \leq 3$, although this is quite involved technically.

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