The Nu Class of Low-Degree-Truncated Rational Multifunctions. Ib. Integrals of Matérn-correlation functions for all odd-half-integer class parameters

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Abstract

This paper is an extension of Parts I and Ia of a series about Nu-class multifunctions. We provide hand-generated algebraic expressions for integrals of single Matérn-covariance functions, as well as for products of two Matérn-covariance functions, for all odd-half-integer class parameters. These are useful both for IMSPE-optimal design software and for testing universality of Nu-class–multifunction properties, across covariance classes.

Key Words: IMSPE, Matérn process, correlation functions, covariance matrix, universality

1. Introduction

The present paper is Part Ib of the Roman-numeralled series Parts I through V, with new-Latin–lowercase-lettered sub-parts, reporting research into Nu-class multifunctions [1].

We begin with a short history of relevant earlier papers in this series.

Part I, Appendices H and J [1] included lengthy, verbatim, closed-form, symbolic-manipulation-software–generated, algebraic expressions of one-dimensional integrals, named I_5 and I_7, of single Matérn-covariance functions, with respective class parameters $\nu = 3/2$ and $5/2$. In the same paper, Appendices I and K included integrals, named I_6 and I_8, for the respective products of two identical-class Matérn-covariance functions. These last two integrals were left in rough, symbolic-manipulation-software output format. These expressions removed the need for computationally intensive Monte-Carlo approximations that previously had been common practice [3], when the integrals were required, as in the computation of an IMSPE-optimal design based on one of these Matérn-covariance functions.

Part Ia [2] then provided succinct algebraic expressions for these two integrals, more suitable, compared to those in Paper I, for software development involving computation of the IMSPE. However, the expressions were restricted to class parameters $3/2$ and $5/2$, and their correctness depended upon the correctness of the outputs of the original symbolic-manipulation software, thus ruling out rigorous mathematical proofs based on these expressions.

The present Paper Ib provides exclusively–hand-generated algebraic expressions of products of Matérn-covariance functions, for all odd-half-integer class parameters. These expressions agree with those given in Parts I and Ia. The general expressions are algebraic sums succinct enough for use in software development for moderate-size class parameters.

In a subsequent part, the expressions derived here shall be used to test mathematically rigorous universality of Nu-class–multifunction properties, across covariance classes. For example, we shall test for the universal occurrence of quantum phase transitions of designs for computer experiments, across their hyper-parameter spaces [4].

2. Outline

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5. Identities and Elementary Integrals

5.1 Double-factorial identity

$$\frac{(2p)!}{p!} = (2p-1)!!\,2^p,\;\; p \in \mathbb{N}_{\geq 0}, \tag{5.1}$$

where $n!! \equiv \prod_{i=0}^{\lceil n/2 \rceil -1}(n-2i) = n(n-2)(n-4)\cdots$, and $0!! \equiv (-1)!! \equiv 1$.

Demonstration:

$$\frac{(2p)!}{p!} = \frac{(2p)(2p-1)\cdots 1}{p!} = \frac{\overbrace{[(2p)(2p-2)\cdots 2]}^{p \text{ even factors}}\cdot[(2p-1)(2p-3)\cdots 1]}{p!} = \frac{2^p p!\cdot[(2p-1)(2p-3)\cdots 1]}{p!} = 2^p(2p-1)!!.$$

5.2 Variation of Finite Companion Binomial Theorem

$$\sum_{j=0}^{p}\frac{(2p-j)!}{(p-j)!}2^j = 4^p p!\,,\;\; p \in \mathbb{N}_{\geq 0}. \tag{5.2}$$

Demonstration: This follows from the "Variation of Finite Companion Binomial Theorem" tabulated, albeit unproved, in [5] as $\sum_{k=0}^{n}\binom{2n-k}{n}2^k = 2^{2n}$, where $\binom{2n-k}{n} \equiv \frac{(2n-k)!}{n!(n-k)!}$. Changing variables $[n,k]$ to $[p,j]$ and identifying $2^{2p} = 4^p$ gives Eq. 5.2. The Appendix of the present paper provides a detailed graphical demonstration.

5.3 Fubini-principle identity

$$\sum_{j=0}^{p} f(j)\sum_{m=0}^{j} g(m) = \sum_{m=0}^{p} g(m)\sum_{j=m}^{p} f(j) \;\; [6]. \tag{5.3}$$

Example: $\sum_{j=0}^{p}\frac{(2p-j)!}{(p-j)!}2^j\sum_{m=0}^{j}\frac{\sqrt{(2p+1)\theta}^{\,m}(1+a)^m}{m!} = \sum_{m=0}^{p}\frac{\sqrt{(2p+1)\theta}^{\,m}(1+a)^m}{m!}\sum_{j=m}^{p}\frac{(2p-j)!}{(p-j)!}2^j$.

5.4 Symmetry operators

We reintroduce the algebraic symmetry operators $\mathcal{S}_w$ and $\mathcal{T}_{a,b}^{(+)}$ that were defined in Part I, Sub-appendix R.5 [1] and Part Ia, Section 3 [2], respectively:

$\mathcal{S}_w f(w,x) \equiv f(-w,x)$ changes the sign on all algebraic quantities denoted by w, e.g., $\mathcal{S}_w(a+bw+cw^2+dx+ex^2) = a-bw+cw^2+dx+ex^2$, where x has no dependence on w.

$\mathcal{T}_{a;b}^{(+)} \equiv \mathcal{I} + \mathcal{S}_a\mathcal{S}_b$, where $\mathcal{I}$ is the identity operator. Example: $\mathcal{T}_{a;b}\left(1+\frac{a+b}{2}\right) = 2$.

We also define additional algebraic symmetry operators, $\mathcal{T}_a^{(+)}$ and $\mathcal{T}_{a;b}^{(-)}$, as follows:

$$\mathcal{T}_a^{(+)} \equiv \mathcal{I} + \mathcal{S}_a. \quad \mathcal{T}_{x_{i,k};x_{j,k}} \text{ is in Eqs. 9.1 \& 9.} \tag{5.4}$$

$$\mathcal{T}_{a;b}^{(-)} \equiv \mathcal{I} - \mathcal{S}_a \mathcal{S}_b. \tag{5.5}$$

5.5 Elementary integrals

Dwight 567.9 gives almost literally [7]

$$\int x^n e^{bx} dx = e^{bx}\left[\frac{x^n}{b} - \frac{nx^{n-1}}{b^2} + \frac{n(n-1)x^{n-2}}{b^3} - \cdots + (-1)^{n-1}\frac{n!x}{b^n} + (-1)^n \frac{n!}{b^{n+1}}\right], \ \ n \geq 0.$$

Specific examples are the following:

for $[n, b] = [0, -1]$:

$$\int e^{-x} dx = -e^{-x};$$

for $[n, b] = [j \geq 1, -1]$:

$$\begin{aligned}\int x^j e^{-x} dx &= -e^{-x}\left[j! + j!\,x + \cdots + j(j-1)x^{j-2} + jx^{j-1} + x^j\right] \\ &= -e^{-x}\, j!\left[\frac{x^0}{0!} + \frac{x^1}{1!} + \cdots + \frac{x^{j-2}}{(j-2)!} + \frac{x^{j-1}}{(j-1)!} + \frac{x^j}{j!}\right] \\ &= -e^{-x} j! \sum_{m=0}^{j} \frac{x^m}{m!}; \text{ and}\end{aligned} \tag{5.6}$$

for $[n, b] = [j + k - l, -2]$ (These specific values will be used between Eqs. 8.6 and 8.7.)

$$\begin{aligned}\int x^{j+k-l}\, e^{-2x} dx &= e^{-2x}\left[\frac{x^{j+k-l}}{b} - \frac{(j+k-l)x^{j+k-l-1}}{b^2} + \frac{(j+k-l)(j+k-l-1)x^{j+k-l-2}}{b^3} - \cdots \right. \\ &\qquad \left. + (-1)^{j+k-l-1}\frac{(j+k-l)!x}{b^{j+k-l}} + (-1)^{j+k-l}\frac{(j+k-l)!}{b^{j+k-l+1}}\right] \\ &= e^{-2x}(j+k-l)!\left[\frac{x^{j+k-l}}{(j+k-l)!b} - \frac{(j+k-l)x^{j+k-l-1}}{(j+k-l)!b^2} + \frac{(j+k-l)(j+k-l-1)x^{j+k-l-2}}{(j+k-l)!b^3} - \cdots \right. \\ &\qquad \left. + (-1)^{j+k-l-1}\frac{x}{b^{j+k-l}1!} + (-1)^{j+k-l}\frac{1}{b^{j+k-l+1}0!}\right] \\ &= e^{-2x}(j+k-l)!\left[\frac{x^{j+k-l}}{(j+k-l)!b} - \frac{x^{j+k-l-1}}{(j+k-l-1)!b^2} + \frac{x^{j+k-l-2}}{(j+k-l-2)!b^3} - \cdots \right. \\ &\qquad \left. + (-1)^{j+k-l-1}\frac{x^1}{b^{j+k-l}} + (-1)^{j+k-l}\frac{x^0}{b^{j+k-l+1}}\right] \\ &= e^{-2x}(j+k-l)! \sum_{m=0}^{j+k-l} \frac{(-1)^{j+k-l-\ell} x^m}{m! b^{j+k-l-\ell+1}} \\ &= -\frac{(j+k-l)!}{2^{j+k-l+1}} e^{-2x} \sum_{m=0}^{j+k-l} \frac{(2x)^m}{m!}.\end{aligned} \tag{5.7}$$

6. Integrals of Single Matérn-Correlation Functions

Our interest is the integration of odd-half-integer-parameter Matérn-correlation functions defined as functions of a radial distance r as the following, which is taken literally from Eq. 4.16 of Rasmussen and Williams [8]:

$$K_{\nu=p+1/2}(r) = exp\left(\frac{-\sqrt{2\nu}r}{\ell}\right)\frac{\Gamma(p+1)}{\Gamma(2p+1)}\sum_{i=0}^{p}\frac{(p+i)!}{i!(p-i)!}\left(\frac{\sqrt{8\nu}r}{\ell}\right)^{p-i}, \tag{6.1}$$

where $p \in \mathbb{N}_{\geq 0}$, ν is as defined on the left-hand side of Eq. 6.1, and ℓ is a positive parameter.

For the purposes of the present section, we are interested in this correlation as a function of the distance between an arbitrary Cartesian-coordinate location x and the k'th Cartesian coordinate of the i'th design point, i.e., $x_{i,k}$. Throughout most of this section, we use the following definitions:

$$a \equiv x_{i,k} \text{ and} \tag{6.2}$$

$$\theta \equiv \theta_k. \tag{6.3}$$

After using the property of gamma functions of natural numbers $\Gamma(p+1) = p!$ and making connection with the covariance hyper-parameter used in Parts I and Ia of this series of papers [1,2], via $\theta \equiv 1/\ell^2$, Eq. 6.1 becomes

$$K_{p+1/2}(|a-x|) = \frac{p!}{(2p)!}\sum_{i=0}^{p}\frac{(p+i)!}{i!(p-i)!}\left(\sqrt{4(2p+1)\theta|a-x|^2}\right)^{p-i} e^{-\sqrt{(2p+1)\theta|a-x|^2}}.$$

Defining $j \equiv p - i$, reversing the order in the summation, and rearranging slightly, gives

$$K_{p+1/2}(|a-x|) = \frac{p!}{(2p)!}\sum_{j=0}^{p}\frac{(2p-j)!}{(p-j)!j!}2^j\sqrt{(2p+1)\theta|a-x|^2}^{\,j}\, e^{-\sqrt{(2p+1)\theta|a-x|^2}}.$$

The two most frequently discussed such functions in the statistics literature [9,10] are the following:

for $p = 1$: $\quad K_{3/2}(|a-x|) \equiv \left(1 + \sqrt{3\theta|a-x|^2}\right)e^{-\sqrt{3\theta|a-x|^2}}$, and

for $p = 2$: $\quad K_{5/2}(|a-x|) \equiv \left(1 + \sqrt{5\theta|a-x|^2} + \frac{5\theta|a-x|^2}{3}\right)e^{-\sqrt{5\theta|a-x|^2}}$.

The definite integrals of these functions over the range $-1 \leq x \leq 1$ are the subject of interest:

$$I_{p+1/2}(a,\theta) = \frac{p!}{(2p)!}\sum_{j=0}^{p}\frac{(2p-j)!}{(p-j)!j!}2^j\,\frac{1}{2}\left(\begin{array}{c}\int_{-1}^{a}\sqrt{(2p+1)\theta|a-x|^2}^{\,j}\,e^{-\sqrt{(2p+1)\theta|a-x|^2}}dx\\ +\int_{a}^{1}\sqrt{(2p+1)\theta|a-x|^2}^{\,j}\,e^{-\sqrt{(2p+1)\theta|a-x|^2}}\,dx\end{array}\right). \tag{6.4}$$

We introduce the following changes in variables:

for $x < a$: $\tilde{x} \equiv [(2p+1)\theta]^{1/2}\,(a-x)$, $\quad d\tilde{x} = -[(2p+1)\theta]^{1/2}dx$, and

$[(2p+1)\theta|a-X|^2]^{j/2} = [(2p+1)\theta]^{j/2}(a-x)^j = \tilde{x}^j$;

for $x \geq a$: $\tilde{\tilde{x}} \equiv [(2p+1)\theta]^{1/2}\,(x-a)$, $\quad d\tilde{\tilde{x}} = [(2p+1)\theta]^{1/2}dx$, and

$[(2p+1)\theta|a-X|^2]^{j/2} = [(2p+1)\theta]^{j/2}(x-a)^j = \tilde{\tilde{x}}^j$.

Reversing the limits on the second integral in Eq. 6.4 gives

$$I_{p+1/2}(a,\theta) = -\frac{p!}{2\sqrt{(2p+1)\theta}(2p)!}\sum_{j=0}^{p}\frac{(2p-j)!}{(p-j)!j!}2^j\left(\begin{array}{c}\int_{\tilde{x}=\sqrt{(2p+1)\theta}\,(1+a)}^{0}\tilde{x}^j e^{-\tilde{x}}\,d\tilde{x}\\ +\int_{\tilde{\tilde{x}}=\sqrt{(2p+1)\theta}\,(1-a)}^{0}\tilde{\tilde{x}}^j e^{-\tilde{\tilde{x}}}\,d\tilde{\tilde{x}}\end{array}\right).$$

Using the twin operator $\mathcal{T}_a^{(+)}$ of Eq. 5.4 and dropping all tildes gives

$$I_{p+1/2}(a,\theta) = -\frac{p!}{2\sqrt{(2p+1)\theta}(2p)!}\sum_{j=0}^{p}\frac{(2p-j)!}{(p-j)!j!}2^j\mathcal{T}_a^{(+)}\int_{x=\sqrt{(2p+1)\theta}(1+a)}^{0}x^j e^{-x}\,dx. \tag{6.5}$$

Substituting the elementary integral of $x^j e^{-x}$ of Eq. 5.6 into Eq. 6.5, canceling j!'s, and moving left both $\mathcal{T}_a^{(+)}$ and e^{-x} gives

$$I_{p+1/2}(a,\theta) = \frac{p!}{2\sqrt{(2p+1)\theta}(2p)!}\mathcal{T}_a^{(+)} e^{-x} \sum_{j=0}^{p} \frac{(2p-j)!}{(p-j)!} 2^j \sum_{m=0}^{j} \frac{x^m}{m!} \Big|_{x=\sqrt{(2p+1)\theta}(1+a)}^{0}.$$

Noting the upper evaluation value $x = 0$ contributes only via $0^0 = 1$ gives

$$I_{p+1/2}(a,\theta) = \frac{p!}{2\sqrt{(2p+1)\theta}(2p)!}\mathcal{T}_a^{(+)} \begin{bmatrix} \sum_{j=0}^{p} \frac{(2p-j)!}{(p-j)!} 2^j \\ -e^{-\sqrt{(2p+1)\theta}(1+a)} \sum_{j=0}^{p} \frac{(2p-j)!}{(p-j)!} 2^j \sum_{m=0}^{j} \frac{\sqrt{(2p+1)\theta}^{\,m}(1+a)^m}{m!} \end{bmatrix}.$$

Reversing the order of the double sum via the Fubini-principle identity of Eq. 5.3, gives

$$I_{p+1/2}(a,\theta) = \frac{p!}{2\sqrt{(2p+1)\theta}(2p)!}\mathcal{T}_a^{(+)} \begin{bmatrix} \sum_{j=0}^{p} \frac{(2p-j)!}{(p-j)!} 2^j \\ -\sum_{m=0}^{p} \frac{\sqrt{(2p+1)\theta}^{\,m}(1+a)^m}{m!} \sum_{j=m}^{p} \frac{(2p-j)!}{(p-j)!} 2^j \, e^{-\sqrt{(2p+1)\theta}(1+a)} \end{bmatrix}.$$

Expanding the second sum and dropping the explicit argument of $I_{p+1/2}$ gives

$$I_{p+1/2} = \frac{p!}{2\sqrt{(2p+1)\theta}(2p)!}\mathcal{T}_a^{(+)} \left(\begin{array}{c} \sum_{j=0}^{p} \frac{(2p-j)!}{(p-j)!} 2^j \\ -\left\{ \begin{array}{c} \frac{1}{0!}\sum_{j=0}^{p} \frac{(2p-j)!}{(p-j)!} 2^j \sqrt{(2p+1)\theta}^{\,0}(1+a)^0 \\ +\frac{1}{1!}\left[\sum_{j=1}^{p} \frac{(2p-j)!}{(p-j)!} 2^j\right] \sqrt{(2p+1)\theta}^{\,1}(1+a)^1 \\ +\frac{1}{2!}\left[\sum_{j=2}^{p} \frac{(2p-j)!}{(p-j)!} 2^j\right] \sqrt{(2p+1)\theta}^{\,2}(1+a)^2 \\ \vdots \\ +2^p \sqrt{(2p+1)\theta}^{\,p}(1+a)^p \end{array} \right\} e^{-\sqrt{(2p+1)\theta}(1+a)} \end{array} \right).$$

Using the double-factorial identity $\frac{p!}{(2p)!} = \frac{1}{(2p-1)!!2^p}$ from Eq. 5.1, using subscripts on square brackets to denote evaluation at specific values of j, and replacing θ and a with θ_k and $x_{i,k}$, respectively, gives

$$I_{p+1/2} = \frac{1}{2(2p-1)!!\sqrt{(2p+1)\theta_k}}\mathcal{T}_{x_{i,k}}^{(+)} \begin{bmatrix} \frac{1}{2^p}\sum_{j=0}^{p} \frac{(2p-j)!}{(p-j)!} 2^j \\ -\left(\begin{array}{c} \frac{1}{0!2^p}\left[\sum_{j=0}^{p} \frac{(2p-j)!}{(p-j)!} 2^j\right] \sqrt{(2p+1)\theta_k}^{\,0}\left(1+x_{i,k}\right)^0 \\ +\frac{1}{1!2^p}\left\{ \begin{array}{c} \sum_{j=0}^{p} \frac{(2p-j)!}{(p-j)!} 2^j \\ -\left[\frac{(2p-j)!}{(p-j)!} 2^j\right]_{j=0} \end{array} \right\} \sqrt{(2p+1)\theta_k}^{\,1}\left(1+x_{i,k}\right)^1 \\ +\frac{1}{2!2^p}\left\{ \begin{array}{c} \sum_{j=0}^{p} \frac{(2p-j)!}{(p-j)!} 2^j \\ -\left[\frac{(2p-j)!}{(p-j)!} 2^j\right]_{j=0} \\ -\left[\frac{(2p-j)!}{(p-j)!} 2^j\right]_{j=1} \end{array} \right\} \sqrt{(2p+1)\theta_k}^{\,2}\left(1+x_{i,k}\right)^2 \\ \vdots \\ +\sqrt{(2p+1)\theta_k}^{\,p}\left(1+x_{i,k}\right)^p \end{array} \right) e^{-\sqrt{(2p+1)\theta_k}(1+x_{i,k})} \end{bmatrix}.$$

The sums with index j can be removed by Identity 5.2, $\sum_{j=0}^{p} \frac{(2p-j)!}{(p-j)!} 2^j = 4^p p!$. Then, evaluating the ultimate equation for its stated values of j gives

$$I_{p+1/2} = \frac{1}{2(2p-1)!!\sqrt{(2p+1)\theta_k}} \mathcal{T}_{x_{i,k}}^{(+)} \left(\begin{array}{c} \frac{1}{2^p}(4^p p!) \\ - \left\{ \begin{array}{c} \frac{1}{0!2^p}(4^p p!) \quad [(2p+1)\theta_k]^{0/2}(1+x_{i,k})^0 \\ +\frac{1}{1!2^p}\left[\begin{array}{c} 4^p p! \\ -\frac{(2p)!}{(p)!} \end{array} \right] [(2p+1)\theta_k]^{1/2}(1+x_{i,k})^1 \\ +\frac{1}{2!2^p}\left[\begin{array}{c} 4^p p! \\ -\frac{(2p)!}{(p)!} \\ -\frac{(2p-1)!}{(p-1)!}2 \end{array} \right] [(2p+1)\theta_k]^{2/2}(1+x_{i,k})^2 \\ \vdots \\ +[(2p+1)\theta_k]^{p/2}(1+x_{i,k})^p \end{array} \right\} e^{-\sqrt{(2p+1)\theta_k}(1+x_{i,k})} \end{array} \right).$$

Expressing this with the natural-number functions,

$a_0(p) \equiv \frac{1}{2^p} 4^p p! = 2^p p!$ and

$b_j(p) \equiv \frac{1}{2^p} \frac{(2p-j)!}{(p-j)!} 2^j, \;\; 0 \le j \le p-1$, gives

$$I_{p+1/2} = \frac{1}{2(2p-1)!!\sqrt{(2p+1)\theta_k}} \mathcal{T}_{x_{i,k}}^{(+)} \left\{ \begin{array}{c} a_0 \\ - \left[\begin{array}{c} a_0 \sqrt{(2p+1)\theta_k}^{\,0}(1+x_{i,k})^0 \\ +\frac{1}{1!}(a_0-b_0)\sqrt{(2p+1)\theta_k}^{\,1}(1+x_{i,k})^1 \\ +\frac{1}{2!}(a_0-b_0-b_1)\sqrt{(2p+1)\theta_k}^{\,2}(1+x_{i,k})^2 \\ \vdots \\ +\sqrt{(2p+1)\theta_k}^{\,p}(1+x_{i,k})^p \end{array} \right] e^{-\sqrt{(2p+1)\theta_k}(1+x_{i,k})} \end{array} \right\}. \tag{6.6}$$

The natural numbers can be computed simply and recursively from Table 1, immediately below, or picked off directly from Table 3.

N.B.: Identity 5.2, viz., $\sum_{j=0}^{p} \frac{(2p-j)!}{(p-j)!} 2^j = 4^p p!$, was used to simplify the left heading in Table 1 to $2^p p!$

	$a_0(p) \equiv 2^p p!$	$b_j(p) \equiv \frac{1}{2^p}\frac{(2p-j)!}{(p-j)!}2^j$						
		j						
p		*0*	*1*	*2*	*3*	*4*	*5*	*6*
1	2	1	1					
2	8	3	3	2				
3	48	15	15	12	6			
4	384	105	105	90	60	24		
5	3840	945	945	840	630	360	120	
6	46080	10395	10395	9450	7560	5040	2520	720
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
12 *wrapped*	1 961 990 553 600	316 234 143 225	316 234 143 225	302 484 832 650	274 986 211 500	235 702 467 000	188 561 973 600	138 940 401 600
OEIS	A002866	A001147	A001147	A001879	A000457	A001880	A001881	A038121

Table 1. Tabulated values of the natural numbers a_0 and b_j appearing in Eq. 6.6, as functions of p. The putatively identical sequence found in the OEIS [11] is given for each column. Along the right-most diagonal, i.e., for $j = p$, $b_p(p) = p! = p \cdot b_{p-1}(p-1)$. Along the k'th-penultimate-RHS main diagonal, $b_{p-k}(p)! = (p+k) \cdot b_{p-k-1}(p-1)$.

			$\frac{1}{(k+1)!}\left(a_0 - \sum_{j=0}^{k} b_j\right)$					
			k					
p	a_0	a_0	*0*	*1*	*2*	*3*	*4*	*5*
1	2	2	1					
2	8	8	5	1				
3	48	48	33	9	1			
4	384	384	279	87	14	1		
5	3840	3840	2895	975	185	20	1	
6	46080	46080	35685	12645	2640	345	27	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
12 *wrapped*	1 961 990 553 600	1 961 990 553 600	1 645 756 410 375	664 761 133 575	171 172 905 750	31 335 467 625	4 302 906 300	455 259 420
OEIS	A002866	A002866	A129890	A035101	A263384	none	none	none

Table 2. Tabulated values of the sequential natural numbers a_0 (twice), $\frac{1}{1!}(a_0 - b_0)$, $\frac{1}{2!}(a_0 - b_0 - b_1)$, $\cdots$ appearing in Eq. 6.6, as calculated from Table 1 and as functions of p. The putatively corresponding sequence found in the OEIS [11] is given for most columns. The sequences of numbers in the rows do not lead to OEIS entries. For example, the reversed $p = 6$ row, viz., (1, 27, 345, 2640, 12645, 35685, 46080), is not in the OEIS. Details for numbers appearing on the upper-left-to-lower-right diagonals for $k \geq 0$: the rightmost diagonal elements are all unity; while the next-to-rightmost

diagonal elements, viz., (5, 9, 14, 20, 27), follow the sequence $(p+1)(p+2)/2-1$, $p \geq 2$, which is the sequence of triangle numbers minus unity.

7. Four Example Integrals of Single Matérn-Correlation Functions: $I_{3/2}, I_{5/2}, I_{7/2}$, and $I_{9/2}$

We now use Eq. 6.6 in conjunction with Tables 1 and 2 to generate algebraic expressions for integrals $I_{3/2}, I_{5/2}, I_{7/2}$, and $I_{9/2}$.

Example: p=1: $I_{\nu=p+1/2} \equiv I_{3/2} \equiv R_{0,i}^{(p=1)}$

Using Eq. 6.5, using a_0 and b_0 from the $p=1$ row of Table 2, and replacing a and θ with $x_{i,k}$ and θ_k, respectively, via Eqs. 6.2 and 6.3, gives

$$I_{3/2}(a,\theta) = \frac{1}{2\sqrt{3\theta_k}} \mathcal{T}_{x_{i,k}}^{(+)} \left\{ \begin{array}{c} a_0 \\ -\left[\begin{array}{l} a_0 \\ +(a_0-b_0)\sqrt{3\theta_k}(1+x_{i,k}) \end{array} \right] e^{-\sqrt{3\theta_k}(1+x_{i,k})} \end{array} \right\}, \quad 1 \leq i \leq \mathrm{n}.$$

$$= \frac{1}{2\sqrt{3\theta_k}} \mathcal{T}_{x_{i,k}}^{(+)} \left\{ \begin{array}{c} 2 \\ -\left[\begin{array}{l} 2 \\ +\sqrt{3\theta_k}(1+x_{i,k}) \end{array} \right] e^{-\sqrt{3\theta_k}(1+x_{i,k})} \end{array} \right\}. \tag{7.1}$$

This agrees, as it should, with the expression given in Part I, Table 4.2 of this series of papers, viz.,

$$R_{0,i}^{(p=1)} = \frac{1}{2\sqrt{3\theta_k}} \left\{ \begin{array}{r} 2\left[\begin{array}{l} 1-e^{-\sqrt{3\theta_k}\,(1+x_{i,k})} \\ +1-e^{-\sqrt{3\theta_k}\,(1-x_{i,k})} \end{array} \right] \\ -\sqrt{3\theta_k}\left[\begin{array}{l} (1+x_{i,k})e^{-\sqrt{3\theta_k}\,(1+x_{i,k})} \\ +(1-x_{i,k})e^{-\sqrt{3\theta_k}\,(1-x_{i,k})} \end{array} \right] \end{array} \right\}.$$

Example: p=2: $I_{\nu=p+1/2} \equiv I_{5/2} \equiv R_{0,i}^{(p=2)}$

In similar fashion to the example, immediately above, but using a_0, b_0, and b_1 from the $p=2$ row of Table 2 gives

$$I_{5/2}(x_{i,k},\theta_k) = \frac{1}{6\sqrt{5\theta_k}} \mathcal{T}_{x_{i,k}}^{(+)} \left\{ \begin{array}{c} a_0 \\ -\left[\begin{array}{l} a_0 \\ + \quad (a_0-b_0)\ \sqrt{5\theta_k}\ (1+x_{i,k}) \\ +\frac{1}{2}(a_0-b_0-b_1)\ \ 5\theta_k\ \ (1+x_{i,k})^2 \end{array} \right] e^{-\sqrt{5\theta_k}(1+x_{i,k})} \end{array} \right\}$$

$$= \frac{1}{6\sqrt{5\theta_k}} \mathcal{T}_{x_{i,k}}^{(+)} \left\{ \begin{array}{c} 8 \\ -\left[\begin{array}{l} 8 \\ +5\sqrt{5\theta_k}\ (1+x_{i,k}) \\ +1\cdot\ 5\theta_k\ \ (1+x_{i,k})^2 \end{array} \right] e^{-\sqrt{5\theta_k}(1+x_{i,k})} \end{array} \right\}. \tag{7.2}$$

This agrees with the expression given in Part I, Table 4.2 of this series of papers [1], viz.,

$$R_{0,i}^{(p=2)} = \frac{1}{6\sqrt{5\theta_k}}\begin{Bmatrix} 8\begin{bmatrix} 1 - e^{-\sqrt{5\theta_k}\,(1+x_{i,k})} \\ +1 - e^{-\sqrt{5\theta_k}\,(1-x_{i,k})} \end{bmatrix} \\ -5\sqrt{5\theta_k}\begin{bmatrix} (1+x_{i,k})e^{-\sqrt{5\theta_k}\,(1+x_{i,k})} \\ +(1-x_{i,k})e^{-\sqrt{5\theta_k}\,(1-x_{i,k})} \end{bmatrix} \\ -5\theta_k\begin{bmatrix} (1+x_{i,k})^2 e^{-\sqrt{5\theta_k}\,(1+x_{i,k})} \\ +(1-x_{i,k})^2 e^{-\sqrt{5\theta_k}\,(1-x_{i,k})} \end{bmatrix} \end{Bmatrix}.$$

Additional examples of integrals, generated from Eq. 4.5 and Table 2, are given below:

$$R_{0,i}^{(p=3)} = I_{7/2}(x_{i,k},\theta_k) = \frac{1}{30\sqrt{7\theta_k}}\mathcal{T}_{x_{i,k}}^{(+)}\begin{Bmatrix} 48 \\ -\begin{bmatrix} 48 \\ +33\sqrt{7\theta_k}\;(1+x_{i,k}) \\ +\;9\sqrt{7\theta_k}^{\,2}\;(1+x_{i,k})^2 \\ +\;1\sqrt{7\theta_k}^{\,3}\;(1+x_{i,k})^3 \end{bmatrix} e^{-\sqrt{7\theta_k}(1+x_{i,k})} \end{Bmatrix}. \quad (7.3)$$

$$R_{0,i}^{(p=4)} = I_{9/2}(a,\theta) = \frac{1}{210\sqrt{9\theta_k}}\mathcal{T}_{x_{i,k}}^{(+)}\begin{Bmatrix} 384 \\ -\begin{bmatrix} 384 \\ +279\sqrt{9\theta_k}\;(1+x_{i,k}) \\ +\;87\sqrt{9\theta_k}^{\,2}\;(1+x_{i,k})^2 \\ +\;14\sqrt{9\theta_k}^{\,3}\;(1+x_{i,k})^3 \\ +\;\;1\sqrt{9\theta_k}^{\,4}\;(1+x_{i,k})^4 \end{bmatrix} e^{-\sqrt{9\theta}(1+x_{i,k})} \end{Bmatrix}. \quad (7.4)$$

8. Integrals of Products of Two Matérn-Correlation Functions

Sec. 6 provided algebraic expressions for integrals of single Matérn-correlation functions. In the present section we derive an expression for integrals of the products of the following two Matérn-correlation functions, as a function of the distance between an arbitrary Cartesian-coordinate location x and the k'th Cartesian coordinates of the i'th and j'th design points, i.e., $x_{i,k}$ and $x_{j,k}$, respectively:

$$K_{p+1/2}(|a-x|) = \frac{p!}{(2p)!}\sum_{j=0}^{p}\frac{(2p-j)!}{(p-j)!\,j!}2^j\sqrt{(2p+1)\theta|x_{i,k}-x|^2}^{\,j}\; e^{-\sqrt{(2p+1)\theta|x_{i,k}-x|^2}} \text{ and}$$

$$K_{p+1/2}(|b-x|) = \frac{p!}{(2p)!}\sum_{k=0}^{p}\frac{(2p-k)!}{(p-k)!\,k!}2^k\sqrt{(2p+1)\theta|x_{j,k}-x|^2}^{\,k}\; e^{-\sqrt{(2p+1)\theta|x_{j,k}-x|^2}}.$$

Throughout most of this section, we substitute the letter a for $x_{i,k}$ and the letter b for $x_{j,k}$, i.e.,

$a \equiv x_{i,k}$, which repeats Eq. 6.2, and

$b \equiv x_{j,k}$. (8.1)

The definite integrals of the products of these functions over the range $-1 \le x \le 1$ and for $-1 \le a \le b \le 1$ are the subject of interest, viz.,

$$J_{p+1/2}(|a-x|,|b-x|) \equiv \frac{1}{2}\int_{-1}^{1}\left[K_{p+1/2}(|a-x|)\right]\left[K_{p+1/2}(|b-x|)\right]dx$$

$$= \left[\frac{p!}{(2p)!}\right]^2 \sum_{j,k=0}^{p} \frac{(2p-j)!(2p-k)!2^{j+k}}{(p-j)!j!(p-k)!k!}$$

$$\cdot \frac{1}{2}\left\{\begin{array}{l} \int_{x=-1}^{a}\left[\sqrt{(2p+1)\theta}(a-x)\right]^j\left[\sqrt{(2p+1)\theta}(b-x)\right]^k e^{-\sqrt{(2p+1)\theta}[(a-x)+(b-x)]}dx \\ + \int_{x=a}^{b}\left[\sqrt{(2p+1)\theta}(x-a)\right]^j\left[\sqrt{(2p+1)\theta}(b-x)\right]^k e^{-\sqrt{(2p+1)\theta}[(x-a)+(b-x)]}\,dx \\ + \int_{x=b}^{1}\left[\sqrt{(2p+1)\theta}(x-a)\right]^j\left[\sqrt{(2p+1)\theta}(x-b)\right]^k e^{-\sqrt{(2p+1)\theta}[(x-a)+(x-b)]}\,dx \end{array}\right\}.$$

As in Sec. 6, we proceed by changing variables.

For $x < a \le b$:

$$\tilde{x}_a \equiv \sqrt{(2p+1)\theta}(a-x), \quad \tilde{x}_b \equiv \sqrt{(2p+1)\theta}(b-x), \quad dx = \frac{-d\tilde{x}_a}{\sqrt{(2p+1)\theta}}, \tag{8.2}$$

$$\left[\sqrt{(2p+1)\theta}(a-x)\right]^j\left[\sqrt{(2p+1)\theta}(b-x)\right]^k = \tilde{x}_a^j\tilde{x}_b^k .$$

Solving the equations in Line 8.2 for $\tilde{x}_b$ in terms of $\tilde{x}_a$ gives

$$\tilde{x}_b = \sqrt{(2p+1)\theta}\left(b-a+\frac{\tilde{x}_a}{\sqrt{(2p+1)\theta}}\right) = \tilde{x}_a + \sqrt{(2p+1)\theta}(b-a) \text{ and}$$

$$\sqrt{(2p+1)\theta}[(a-x)+(b-x)] = \tilde{x}_a + \tilde{x}_b = 2\tilde{x}_a + \sqrt{(2p+1)\theta}(b-a). \tag{8.3}$$

For $a \le x \le b$, and in similar fashion:

$$\tilde{\tilde{x}}_a \equiv \sqrt{(2p+1)\theta}(x-a)\,, \quad \tilde{\tilde{x}}_b = \sqrt{(2p+1)\theta}(b-x)\,, \quad dx = \frac{d\tilde{\tilde{x}}_a}{\sqrt{(2p+1)\theta}},$$

$$\left[\sqrt{(2p+1)\theta}(x-a)\right]^j\left[\sqrt{(2p+1)\theta}(b-x)\right]^k = \tilde{\tilde{x}}_a^j\tilde{\tilde{x}}_b^k ,$$

$$\tilde{\tilde{x}}_b = \sqrt{(2p+1)\theta}\left(b-a-\frac{\tilde{\tilde{x}}_a}{\sqrt{(2p+1)\theta}}\right) = -\tilde{\tilde{x}}_a + \sqrt{(2p+1)\theta}(b-a) \text{ and}$$

$$\sqrt{(2p+1)\theta}[(x-a)+(b-x)] = \tilde{\tilde{x}}_a + \tilde{\tilde{x}}_b = \sqrt{(2p+1)\theta}(b-a). \tag{8.4}$$

For $a \le b < x$, and in similar fashion:

$$\tilde{\tilde{\tilde{x}}}_a \equiv \sqrt{(2p+1)\theta}(x-a)\,, \quad \tilde{\tilde{\tilde{x}}}_b \equiv \sqrt{(2p+1)\theta}(x-b)\,, \quad dx = \frac{d\tilde{\tilde{\tilde{x}}}_a}{\sqrt{(2p+1)\theta}},$$

$$\left[\sqrt{(2p+1)\theta}(x-a)\right]^j\left[\sqrt{(2p+1)\theta}(x-b)\right]^k = \tilde{\tilde{\tilde{x}}}_a^j\tilde{\tilde{\tilde{x}}}_b^k ,$$

$$\tilde{\tilde{\tilde{x}}}_b = \sqrt{(2p+1)\theta}\left(\frac{\tilde{\tilde{\tilde{x}}}_a}{\sqrt{(2p+1)\theta}}+a-b\right) = \tilde{\tilde{\tilde{x}}}_a - \sqrt{(2p+1)\theta}(b-a), \text{ and}$$

$$\sqrt{(2p+1)\theta}[(x-a)+(x-b)] = \tilde{\tilde{\tilde{x}}}_a + \tilde{\tilde{\tilde{x}}}_b = 2\tilde{\tilde{\tilde{x}}}_a - \sqrt{(2p+1)\theta}(b-a). \tag{8.5}$$

$$J_{p+1/2} = \frac{1}{\sqrt{(2p+1)\theta}}\left[\frac{p!}{(2p)!}\right]^2 \sum_{j,k=0}^{p} \frac{(2p-j)!(2p-k)!2^{j+k}}{2(p-j)!j!(p-k)!k!}\left(\begin{array}{ll} -\int_{\tilde{x}_a=\sqrt{(2p+1)\theta}(1+a)}^{0} & \tilde{x}_a^j\tilde{x}_b^k e^{-(\tilde{x}_a+\tilde{x}_b)}d\tilde{x}_a \\ +\int_{\tilde{\tilde{x}}_a=0}^{\sqrt{(2p+1)\theta}(b-a)} & \tilde{\tilde{x}}_a^j\tilde{\tilde{x}}_b^k e^{-(\tilde{\tilde{x}}_a+\tilde{\tilde{x}}_b)}\,d\tilde{\tilde{x}}_a \\ +\int_{\tilde{\tilde{\tilde{x}}}_a=\sqrt{(2p+1)\theta}(b-a)}^{\sqrt{(2p+1)\theta}(1-a)} & \tilde{\tilde{\tilde{x}}}_a^j\tilde{\tilde{\tilde{x}}}_b^k e^{-\left(\tilde{\tilde{\tilde{x}}}_a+\tilde{\tilde{\tilde{x}}}_b\right)}\,d\tilde{\tilde{\tilde{x}}}_a \end{array}\right).$$

Then, using Eqs. 8.3, 8.4, and 8.5 to express $\tilde{x}_b$ (respectively, $\tilde{\tilde{x}}_b$ or $\tilde{\tilde{\tilde{x}}}_b$) in terms of $\tilde{x}_a$ (resp., $\tilde{\tilde{x}}_a$ or $\tilde{\tilde{\tilde{x}}}_a$), dropping all tildes and subscripts on the x's, and making slight rearrangements, gives

$$J_{p+1/2} = \frac{1}{\sqrt{(2p+1)\theta}}\left[\frac{p!}{(2p)!}\right]^2 \sum_{j,k=0}^{p} \frac{(2p-j)!(2p-k)!2^{j+k}}{2(p-j)!j!(p-k)!k!}$$

$$\cdot\left\{\begin{array}{ll} -e^{-\sqrt{(2p+1)\theta}(b-a)}\int_{\sqrt{(2p+1)\theta}(1+a)}^{0} & x^j\left[x+\sqrt{(2p+1)\theta}(b-a)\right]^k e^{-2x}dx \\ +\ e^{-\sqrt{(2p+1)\theta}(b-a)}(-1)^k\int_{0}^{\sqrt{(2p+1)\theta}(b-a)} & x^j\left[x-\sqrt{(2p+1)\theta}(b-a)\right]^k dx \\ +e^{\sqrt{(2p+1)\theta}(b-a)}\int_{\sqrt{(2p+1)\theta}(b-a)}^{\sqrt{(2p+1)\theta}(1-a)} & x^j\left[x-\sqrt{(2p+1)\theta}(b-a)\right]^k e^{-2x}dx \end{array}\right\}.$$

Applying the binomial theorem, $(x \pm y)^n = \sum_{i=0}^{n}\frac{n!}{i!(n-i)!}(\pm 1)^i x^{n-i}y^i$, and factoring out $e^{-\sqrt{(2p+1)\theta}(b-a)}$ gives

$$J_{p+1/2} = \frac{1}{\sqrt{(2p+1)\theta}}\left[\frac{p!}{(2p)!}\right]^2 \sum_{j,k=0}^{p} \frac{(2p-j)!(2p-k)!2^{j+k}}{2(p-j)!j!(p-k)!k!}e^{-\sqrt{(2p+1)\theta}(b-a)}$$

$$\cdot\left\{\begin{array}{ll} -\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}\int_{\sqrt{(2p+1)\theta}(1+a)}^{0} & x^{j+k-l}\left[\sqrt{(2p+1)\theta}(b-a)\right]^l e^{-2x}dx \\ +(-1)^{k+l}\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}\int_{0}^{\sqrt{(2p+1)\theta}(b-a)} & x^{j+k-l}\left[\sqrt{(2p+1)\theta}(b-a)\right]^l dx \\ +(-1)^l e^{2\sqrt{(2p+1)\theta}(b-a)}\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}\int_{\sqrt{(2p+1)\theta}(b-a)}^{\sqrt{(2p+1)\theta}(1-a)} & x^{j+k-l}\left[\sqrt{(2p+1)\theta}(b-a)\right]^l e^{-2x}dx \end{array}\right\}.$$

Moving the summations and $\frac{k!}{l!(k-l)!}$ left, canceling the $k!$ in the numerator and denominator, and factoring $\left[\sqrt{(2p+1)\theta}(b-a)\right]^l$ gives

$$J_{p+1/2} = \frac{1}{\sqrt{(2p+1)\theta}}\left[\frac{p!}{(2p)!}\right]^2 \sum_{j,k=0}^{p}\sum_{l=0}^{k} \frac{(2p-j)!(2p-k)!2^{j+k}}{2(p-j)!j!(p-k)!l!(k-l)!}e^{-\sqrt{(2p+1)\theta}(b-a)}\left[\sqrt{(2p+1)\theta}(b-a)\right]^l$$

$$\cdot\left[\begin{array}{r} -\int_{\sqrt{(2p+1)\theta}(1+a)}^{0} x^{j+k-l}e^{-2x}dx \\ +(-1)^{k+l}\int_{0}^{\sqrt{(2p+1)\theta}(b-a)} x^{j+k-l}\ dx \\ +(-1)^l e^{2\sqrt{(2p+1)\theta}(b-a)}\int_{\sqrt{(2p+1)\theta}(b-a)}^{\sqrt{(2p+1)\theta}(1-a)} x^{j+k-l}e^{-2x}dx \end{array}\right]. \quad (8.6)$$

Substituting the integral of $x^{j+k-l}\,e^{-2x}$, Eq. 5.7, in the first and last rows of the large parentheses of Eq. 8.6, as well as the trivial integral $\int x^{j+k-l}dx = \frac{x^{j+k-l+1}}{j+k-l+1}$ in the second row of the large parentheses of Eq. 8.6, and some rearranging gives

$$J_{p+1/2} = \frac{1}{\sqrt{(2p+1)\theta}}\left[\frac{p!}{(2p)!}\right]^2 \sum_{j,k=0}^{p}\sum_{l=0}^{k} \frac{(2p-j)!(2p-k)!2^{j+k}}{2(p-j)!j!(p-k)!l!(k-l)!}\left[\sqrt{(2p+1)\theta}(b-a)\right]^l e^{-\sqrt{(2p+1)\theta}(b-a)}$$

$$\cdot\left[\begin{array}{r} \frac{(j+k-l)!}{2^{j+k-l+1}} e^{-2x}\sum_{m=0}^{j+k-l}\frac{(2x)^m}{m!}\Big|_{\sqrt{(2p+1)\theta}(1+a)}^{0} \\ +(-1)^{k+l}\frac{x^{j+k-l+1}}{(j+k-l+1)}\Big|_{x=0}^{\sqrt{(2p+1)\theta}(b-a)} \\ -(-1)^l \frac{(j+k-l)!}{2^{j+k-l+1}} e^{2\sqrt{(2p+1)\theta}(b-a)} e^{-2x}\sum_{m=0}^{j+k-l}\frac{(2x)^m}{m!}\Big|_{\sqrt{(2p+1)\theta}(b-a)}^{\sqrt{(2p+1)\theta}(1-a)} \end{array}\right]. \qquad (8.7)$$

Evaluation of the integrals gives the following, where the identity $0^0 = 1$ has been used in the evaluation of the upper limit of the first integral in Eq. 8.7, the double-factorial identity of Eq. 5.1 has been used, and $\frac{(j+k-l)!}{2^{j+k-l+1}}$ has been factored out. Each row is given a different color, as this will prove useful in what follows.

$$J_{p+1/2} = \frac{1}{\sqrt{(2p+1)\theta}4^p[(2p-1)!!]^2}\sum_{j,k=0}^{p}\sum_{l=0}^{k}\frac{(2p-j)!(2p-k)!2^{j+k}}{2(p-j)!j!(p-k)!l!(k-l)!}\left[\sqrt{(2p+1)\theta}(b-a)\right]^l e^{-\sqrt{(2p+1)\theta}(b-a)}$$

$$\cdot\frac{(j+k-l)!}{2^{j+k-l+1}}\left\{\begin{array}{c} 1 \\ -\sum_{m=0}^{j+k-l}\frac{1}{m!}\left[2\sqrt{(2p+1)\theta}(1+a)\right]^m e^{-2\sqrt{(2p+1)\theta}(1+a)} \\ +(-1)^{k+l}\frac{1}{(j+k-l+1)!}\left[2\sqrt{(2p+1)\theta}(b-a)\right]^{j+k-l+1} \\ -(-1)^l\sum_{m=0}^{j+k-l}\frac{1}{m!}\left[2\sqrt{(2p+1)\theta}(1-a)\right]^m e^{-2\sqrt{(2p+1)\theta}(1-a)}e^{2\sqrt{(2p+1)\theta}(b-a)} \\ +(-1)^l\sum_{m=0}^{j+k-l}\frac{1}{m!}\left[2\sqrt{(2p+1)\theta}(b-a)\right]^m e^{-2\sqrt{(2p+1)\theta}(b-a)}e^{2\sqrt{(2p+1)\theta}(b-a)} \end{array}\right\}. \qquad (8.8)$$

Using the identities

$$e^{-\sqrt{(2p+1)\theta}(b-a)}e^{-2\sqrt{(2p+1)\theta}(1+a)} = e^{-2\sqrt{(2p+1)\theta}\left(1+\frac{a+b}{2}\right)} \text{ and}$$

$$e^{\sqrt{(2p+1)\theta}(b-a)}e^{-2\sqrt{(2p+1)\theta}(1-a)} = e^{-2\sqrt{(2p+1)\theta}\left(1-\frac{a+b}{2}\right)},$$

collecting terms with common exponents, moving the factor $\frac{(j+k-l)!}{2^{j+k-l+1}}$ left, distributing $\left[\sqrt{(2p+1)\theta}(b-a)\right]^l$, using the elementary identity $\frac{2^{j+k}}{2^{j+k-l+1}} = \frac{2^l}{2}$, and maintaining the colors of Eq. 8.8 to readily demonstrate the rearrangement gives

$$J_{p+1/2} = \frac{1}{\sqrt{(2p+1)\theta}}\left[\frac{p!}{(2p)!}\right]^2\sum_{j,k=0}^{p}\sum_{l=0}^{k}\frac{(2p-j)!(2p-k)!2^l(j+k-l)!}{4(p-j)!j!(p-k)!l!(k-l)!}\left[\sqrt{(2p+1)\theta}(b-a)\right]^l$$

$$\cdot\left(\begin{array}{c} 1 \\ \left\{\begin{array}{c} +(-1)^{k+l}\frac{1}{(j+k-l+1)!}\left[2\sqrt{(2p+1)\theta}(b-a)\right]^{j+k-l+1} \\ +(-1)^l\sum_{m=0}^{j+k-l}\frac{1}{m!}\left[2\sqrt{(2p+1)\theta}(b-a)\right]^m \end{array}\right\} e^{-2\sqrt{(2p+1)\theta}\left(\frac{b-a}{2}\right)} \\ -\left\{\sum_{\ell=0}^{j+k-l}\frac{1}{m!}\left[2\sqrt{(2p+1)\theta}(1+a)\right]^m\right\} e^{-2\sqrt{(2p+1)\theta}\left(1+\frac{a+b}{2}\right)} \\ +\left\{(-1)^l\sum_{m=0}^{j+k-l}\frac{1}{\ell!}\left[2\sqrt{(2p+1)\theta}(1-a)\right]^{\ell}\right\} e^{-2\sqrt{(2p+1)\theta}\left(1-\frac{a+b}{2}\right)} \end{array}\right). \qquad (8.9)$$

Consolidating the 2^l and the $\left[\sqrt{(2p+1)\theta}(b-a)\right]^l$, and using the assumption $b \geq a$, gives

$$J_{p+1/2} = \frac{1}{\sqrt{(2p+1)\theta}} \left[\frac{p!}{(2p)!}\right]^2 \sum_{j,k=0}^{p} \sum_{l=0}^{k} \frac{(2p-j)!(2p-k)!(j+k-l)!}{4(p-j)!j!(p-k)!l!(k-l)!} \left[2\sqrt{(2p+1)\theta}\,|b-a|\right]^l$$

$$\cdot \left(\begin{array}{l} \left\{ \begin{array}{c} 1 \\ +(-1)^{k+l} \frac{1}{(j+k-l+1)!} \left[2\sqrt{(2p+1)\theta}|b-a|\right]^{j+k-l+1} \\ +(-1)^l \sum_{m=0}^{j+k-l} \frac{1}{m!} \left[2\sqrt{(2p+1)\theta}|b-a|\right]^m \end{array} \right\} e^{-2\sqrt{(2p+1)\theta}\left|\frac{b-a}{2}\right|} \\ -\left\{ \sum_{m=0}^{j+k-l} \frac{1}{m!} \left[2\sqrt{(2p+1)\theta}(1+a)\right]^m \right\} e^{-2\sqrt{(2p+1)\theta}\left(1+\frac{a+b}{2}\right)} \\ +\left\{ (-1)^l \sum_{m=0}^{j+k-l} \frac{1}{m!} \left[2\sqrt{(2p+1)\theta}(1-a)\right]^m \right\} e^{-2\sqrt{(2p+1)\theta}\left(1-\frac{a+b}{2}\right)} \end{array} \right), \quad b \geq a. \tag{8.10}$$

Introducing the definitions

$$A_{\pm} \equiv 2\sqrt{(2p+1)\theta}\,(1 \pm a) \text{ and} \tag{8.11}$$

$$B \equiv 2\sqrt{(2p+1)\theta}\,|b-a| \tag{8.12}$$

gives the following alternative equation that carries through the sums over m:

$$J_{p+1/2} = \frac{1}{\sqrt{(2p+1)\theta}} \left[\frac{p!}{(2p)!}\right]^2 \sum_{j,k=0}^{p} \sum_{l=0}^{k} \frac{(2p-j)!(2p-k)!(j+k-l)!}{4(p-j)!j!(p-k)!l!(k-l)!}$$

$$\cdot \left(\begin{array}{r} \left\{ B^l + (-1)^l B^l \left[1 + B + \frac{B^2}{2} + \cdots + \frac{B^{j+k-l}}{(j+k-l)!}\right] + \frac{(-1)^{k+l} B^{j+k+1}}{(j+k-l+1)!} \right\} e^{-2\sqrt{(2p+1)\theta}\left|\frac{b-a}{2}\right|} \\ -B^l \left\{1 + A_+ + \frac{A_+^2}{2} + \cdots + \frac{A_+^{j+k-l}}{(j+k-l)!}\right\} e^{-2\sqrt{(2p+1)\theta}\left(1+\frac{a+b}{2}\right)} \\ +(-1)^l B^l \left\{1 + A_- + \frac{A_-^2}{2} + \cdots + \frac{A_-^{j+k-l}}{(j+k-l)!}\right\} e^{-2\sqrt{(2p+1)\theta}\left(1-\frac{a+b}{2}\right)} \end{array} \right), \quad b \geq a. \tag{8.13}$$

Finally, identifying $J_{p+1/2}$, a, and b of Eq. 8.13 with $R_{i,j}^{(p)}$, $x_{i,k}$, and $x_{j,k}$, respectively, provides the following two alternative expressions that can replace the "machine-readable symbolic expressions," i.e., the "MRSE's," in Part I, Table 4.3 [1]:

$$R_{i,j}^{(p)} = \frac{1}{\sqrt{(2p+1)\theta}} \left[\frac{p!}{(2p)!}\right]^2 \sum_{j,k=0}^{p} \sum_{l=0}^{k} \frac{(2p-j)!(2p-k)!(j+k-l)!}{4(p-j)!j!(p-k)!l!(k-l)!} \left[2\sqrt{(2p+1)\theta}\,|b-a|\right]^l$$

$$\cdot \left(\begin{array}{l} \left\{ \begin{array}{c} 1 \\ +(-1)^{k+l} \frac{1}{(j+k-l+1)!} \left[2\sqrt{(2p+1)\theta}|b-a|\right]^{j+k-l+1} \\ +(-1)^l \sum_{m=0}^{j+k-l} \frac{1}{m!} \left[2\sqrt{(2p+1)\theta}|b-a|\right]^m \end{array} \right\} e^{-2\sqrt{(2p+1)\theta}\left|\frac{x_{j,k}-x_{i,k}}{2}\right|} \\ -\left\{ \sum_{m=0}^{j+k-l} \frac{1}{m!} \left[2\sqrt{(2p+1)\theta}(1+a)\right]^m \right\} e^{-2\sqrt{(2p+1)\theta}\left(1+\frac{x_{j,k}+x_{i,k}}{2}\right)} \\ +\left\{ (-1)^l \sum_{m=0}^{j+k-l} \frac{1}{m!} \left[2\sqrt{(2p+1)\theta}(1-a)\right]^m \right\} e^{-2\sqrt{(2p+1)\theta}\left(1-\frac{x_{j,k}+x_{i,k}}{2}\right)} \end{array} \right), \quad x_{j,k} \geq x_{i,k}, \tag{8.14}$$

or

$$R_{i,j}^{(p)} = \frac{1}{\sqrt{(2p+1)\theta}}\left[\frac{p!}{(2p)!}\right]^2 \sum_{j,k=0}^{p}\sum_{l=0}^{k}\frac{(2p-j)!(2p-k)!(j+k-l)!}{4(p-j)!j!(p-k)!l!(k-l)!}$$

$$\cdot\left(\begin{array}{r}\left\{B^l + (-1)^l B^l\left[1 + B + \frac{B^2}{2} + \cdots + \frac{B^{j+k-l}}{(j+k-l)!}\right] + \frac{(-1)^{k+l}B^{j+k+1}}{(j+k-l+1)!}\right\} e^{-2\sqrt{(2p+1)\theta}\left|\frac{x_{j,k}-x_{i,k}}{2}\right|} \\ -B^l\left\{1 + A_+ + \frac{A_+^2}{2} + \cdots + \frac{A_+^{j+k-l}}{(j+k-l)!}\right\} e^{-2\sqrt{(2p+1)\theta}\left(1+\frac{x_{j,k}+x_{i,k}}{2}\right)} \\ +(-1)^l B^l\left\{1 + A_- + \frac{A_-^2}{2} + \cdots + \frac{A_-^{j+k-l}}{(j+k-l)!}\right\} e^{-2\sqrt{(2p+1)\theta}\left(1-\frac{x_{j,k}+x_{i,k}}{2}\right)}\end{array}\right), \quad x_{j,k} \geq x_{i,k}. \tag{8.15}$$

9. Two Example Integrals of Products of Two Matérn-Correlation Functions: $J_{3/2}$ and $J_{5/2}$

Each example of this section uses Eq. 8.15.

Example: p=1: $J_{\nu=p+1/2} \equiv J_{3/2} \equiv R_{i,j}^{(p=1)}$, $1 \leq i,j \leq$ n.

$$J_{3/2} = \frac{1}{16\sqrt{3\theta}}\sum_{j,k=0}^{1}\sum_{l=0}^{k}(2-j)!\,(2-k)!\,(j+k-l)!$$

$$\cdot\left(\begin{array}{r}\left\{B^l + (-1)^l B^l\left[1 + B + \frac{B^2}{2} + \cdots + \frac{B^{j+k-l}}{(j+k-l)!}\right] + \frac{(-1)^{k+l}B^{j+k+1}}{(j+k-l+1)!}\right\} e^{-2\sqrt{3\theta}\left(\frac{b-a}{2}\right)} \\ -B^l\left\{1 + A_+ + \frac{A_+^2}{2} + \cdots + \frac{A_+^{j+k-l}}{(j+k-l)!}\right\} e^{-2\sqrt{3\theta}\left(1+\frac{a+b}{2}\right)} \\ +(-1)^l B^l\left\{1 + A_- + \frac{A_-^2}{2} + \cdots + \frac{A_-^{j+k-l}}{(j+k-l)!}\right\} e^{-2\sqrt{3\theta}\left(1-\frac{a+b}{2}\right)}\end{array}\right).$$

Terms in the last equation's large parentheses are evaluated individually for each $[j,k,l]$ set, to give the sub-terms of $J_{3/2}$, each with an integer 16 in its denominator, as follows:

$$[0,0,0]: \frac{4}{16\sqrt{3\theta}}\left[\begin{array}{l}(2+B)e^{-2\sqrt{3\theta}\left(\frac{b-a}{2}\right)} \\ -(1)e^{-2\sqrt{3\theta}\left(1+\frac{a+b}{2}\right)} \\ +(1)e^{-2\sqrt{3\theta}\left(1-\frac{a+b}{2}\right)}\end{array}\right].$$

$$[0,1,0]: \frac{2}{16\sqrt{3\theta}}\left[\begin{array}{l}\left(2+B-\frac{1}{2}B^2\right)e^{-2\sqrt{3\theta}\left(\frac{b-a}{2}\right)} \\ -(1+A_+)e^{-2\sqrt{3\theta}\left(1+\frac{a+b}{2}\right)} \\ +(1+A_-)e^{-2\sqrt{3\theta}\left(1-\frac{a+b}{2}\right)}\end{array}\right].$$

$$[0,1,1]: \frac{2}{16\sqrt{3\theta}}\left[\begin{array}{l}(B^2)e^{-2\sqrt{3\theta}\left(\frac{b-a}{2}\right)} \\ -B(1)e^{-2\sqrt{3\theta}\left(1+\frac{a+b}{2}\right)} \\ +B(-1)e^{-2\sqrt{3\theta}\left(1-\frac{a+b}{2}\right)}\end{array}\right].$$

$$[1,0,0]: \frac{2}{16\sqrt{3\theta}}\left[\begin{array}{l}\left(2+B+\frac{1}{2}B^2\right)e^{-2\sqrt{3\theta}\left(\frac{b-a}{2}\right)} \\ -(1+A_+)e^{-2\sqrt{3\theta}\left(1+\frac{a+b}{2}\right)} \\ +(1+A_-)e^{-2\sqrt{3\theta}\left(1-\frac{a+b}{2}\right)}\end{array}\right].$$

$$[1,1,0]: \frac{2}{16\sqrt{3\theta}}\left[\begin{array}{l}\left(2+B+\frac{1}{2}B^2-\frac{1}{6}B^3\right)e^{-2\sqrt{3\theta}\left(\frac{b-a}{2}\right)} \\ -\left(1+A_++\frac{1}{2}A_+^2\right)e^{-2\sqrt{3\theta}\left(1+\frac{a+b}{2}\right)} \\ +\left(1+A_-+\frac{1}{2}A_-^2\right)e^{-2\sqrt{3\theta}\left(1-\frac{a+b}{2}\right)}\end{array}\right].$$

$$[1,1,1]: \frac{1}{16\sqrt{3\theta}}\left[\begin{array}{l}\left(-B^2+\frac{1}{2}B^3\right)e^{-2\sqrt{3\theta}\left(\frac{b-a}{2}\right)} \\ -B(1+A_+)e^{-2\sqrt{3\theta}\left(1+\frac{a+b}{2}\right)} \\ +B(-1-A_-)e^{-2\sqrt{3\theta}\left(1-\frac{a+b}{2}\right)}\end{array}\right].$$

Summing these six terms gives

$$J_{3/2} = \frac{1}{16\sqrt{3\theta}} \left\{ \begin{array}{c} \left[20 + 10B + 2B^2 + \frac{1}{6}B^3\right] e^{-2\sqrt{3\theta}\left(\frac{b-a}{2}\right)} \\ -[(10 + 6A_+ + A_+^2) + B(3 + A_+)]e^{-2\sqrt{3\theta}\left(1+\frac{a+b}{2}\right)} \\ +[(10 + 6A_- + A_-^2) - B(3 + A_-)]e^{-2\sqrt{3\theta}\left(1-\frac{a+b}{2}\right)} \end{array} \right\}.$$

Then, changing the denominator factor to 24 for comparison with previous results in Parts I and Ia of this series of papers, gives

$$J_{3/2} = \frac{1}{24\sqrt{3\theta}} \left\{ \begin{array}{c} \left[30 + 15B + 3B^2 + \frac{1}{4}B^3\right] e^{-2\sqrt{3\theta}\left(\frac{b-a}{2}\right)} \\ -\left[\left(15 + 9A_+ + \frac{3}{2}A_+^2\right) + B\left(\frac{9}{2} + \frac{3}{2}A_+\right)\right] e^{-2\sqrt{3\theta}\left(1+\frac{a+b}{2}\right)} \\ +\left[\left(15 + 9A_- + \frac{3}{2}A_-^2\right) - B\left(\frac{9}{2} + \frac{3}{2}A_-\right)\right] e^{-2\sqrt{3\theta}\left(1-\frac{a+b}{2}\right)} \end{array} \right\}.$$

Substituting back in the definitions of $A_\pm$ and B in Eqs. 8.11 and 8.12 gives

$$J_{3/2} = \frac{1}{24\sqrt{3\theta}} \left(\begin{array}{c} \left\{ \begin{array}{l} 30 \\ +30\sqrt{3\theta}\ (b-a) \\ +12 \cdot 3\theta\ (b-a)^2 \\ +\ 2\sqrt{3\theta}^3 (b-a)^3 \end{array} \right\} e^{-2\sqrt{3\theta}\left(\frac{b-a}{2}\right)} \\ -\left\{ \left[\begin{array}{l} 15 \\ +18\sqrt{3\theta}(1+a) \\ +\ \ 6 \cdot 3\theta(1+a)^2 \end{array} \right] + \sqrt{3\theta}(b-a) \left[\begin{array}{l} 9 \\ +6\sqrt{3\theta}(1+a) \end{array} \right] \right\} e^{-2\sqrt{3\theta}\left(1+\frac{a+b}{2}\right)} \\ +\left\{ \left[\begin{array}{l} 15 \\ +18\sqrt{3\theta}(1-a) \\ +\ \ 6 \cdot 3\theta(1-a)^2 \end{array} \right] - \sqrt{3\theta}(b-a) \left(\begin{array}{l} 9 \\ +6\sqrt{3\theta}(1-a) \end{array} \right) \right\} e^{-2\sqrt{3\theta}\left(1-\frac{a+b}{2}\right)} \end{array} \right)$$

$$= \frac{1}{24\sqrt{3\theta}} \left\{ \begin{array}{c} \left[\begin{array}{l} 30 \\ +30\sqrt{3\theta}(b-a) \\ +12 \cdot 3\theta(b-a)^2 \\ +2\sqrt{3\theta}^3 (b-a)^3 \end{array} \right] e^{-2\sqrt{3\theta}\left(\frac{b-a}{2}\right)} \\ -\left[\begin{array}{l} 15 \\ +\ \ 9\sqrt{3\theta}(2+a+b) \\ +\ \ 6 \cdot 3\theta(1+a+b+ab) \end{array} \right] e^{-2\sqrt{3\theta}\left(1+\frac{a+b}{2}\right)} \\ +\left[\begin{array}{l} 15 \\ +\ 9\sqrt{3\theta}(2-a-b) \\ +\ 6 \cdot 3\theta(1-a-b+ab) \end{array} \right] e^{-2\sqrt{3\theta}\left(1-\frac{a+b}{2}\right)} \end{array} \right\}.$$

Factoring some more gives

$$J_{3/2}=\frac{1}{24\sqrt{3\theta_k}}\left\{\begin{matrix} 2\begin{bmatrix} 15 \\ +15\left|x_{i,k}-x_{j,k}\right| \ \sqrt{3\theta_k} \\ +\ 6\left|x_{i,k}-x_{j,k}\right|^2 \sqrt{3\theta_k}^{\,2} \\ + \ \left|x_{i,k}-x_{j,k}\right|^3 \sqrt{3\theta_k}^{\,3} \end{bmatrix} e^{-2\sqrt{3\theta_k}\left|\frac{x_{i,k}-x_{j,k}}{2}\right|} \\ -3\mathcal{T}^{(-)}_{x_{i,k};x_{j,k}} \begin{bmatrix} 5 \\ +3\begin{pmatrix} 2 \\ +x_{i,k}+x_{j,k} \end{pmatrix} \ \sqrt{3\theta_k} \\ +2\begin{pmatrix} 1 \\ +x_{i,k}+x_{j,k} \\ +x_{i,k}x_j,k \end{pmatrix} \sqrt{3\theta_k}^{\,2} \end{bmatrix} e^{-2\sqrt{3\theta_k}\left(1+\frac{x_{i,k}+x_{j,k}}{2}\right)} \end{matrix}\right\}. \tag{9.1}$$

This agrees with the result for this integral in Paper Ia, Sec. 4 of this series of papers [2], viz.,

$$R^{(1)}_{i,j}=\frac{1}{24\sqrt{3\theta_k}}\left\{\begin{matrix} 2\begin{bmatrix} 15 \\ +15\left|x_{i,k}-x_{j,k}\right| \ \sqrt{3\theta_k} \\ + \ 6\left|x_{i,k}-x_{j,k}\right|^2 \sqrt{3\theta_k}^{\,2} \\ + \ \left|x_{i,k}-x_{j,k}\right|^3 \sqrt{3\theta_k}^{\,3} \end{bmatrix} e^{-2\sqrt{3\theta_k}\left|\frac{x_{i,k}-x_{j,k}}{2}\right|} \\ -3\mathcal{T}^{(-)}_{x_{i,k};x_{j,k}} \begin{bmatrix} 5 \\ +3\begin{pmatrix} 2 \\ +x_{i,k}+x_{j,k} \end{pmatrix} \sqrt{3\theta_k}^{\,1} \\ +2\begin{pmatrix} 1 \\ +x_{i,k}+x_{j,k} \\ +x_{i,k}x_{j,k} \end{pmatrix} \sqrt{3\theta_k}^{\,2} \end{bmatrix} e^{-2\sqrt{3\theta_k}\left(1+\frac{x_{i,k}+x_{j,k}}{2}\right)} \end{matrix}\right\}.$$

Example: p=2: $J_{\nu=p+1/2}\equiv J_{5/2}\equiv R^{(p=2)}_{i,j},\ \ 1\le i,j\le n.$

$$J_{5/2}=\frac{1}{576\sqrt{5\theta}}\sum_{j,k=0}^{2}\sum_{l=0}^{k}\frac{(4-j)!(4-k)!(j+k-l)!}{(2-j)!j!(2-k)!l!(k-l)!}$$

$$\cdot\left(\begin{matrix} \left\{B^l+(-1)^l B^l\left[1+B+\frac{B^2}{2}+\cdots+\frac{B^{j+k-l}}{(j+k-l)!}\right]+\frac{(-1)^{k+l}B^{j+k+1}}{(j+k-l+1)!}\right\}e^{-2\sqrt{5\theta}\left(\frac{b-a}{2}\right)} \\ -B^l\left\{1+A_+ +\frac{A_+^2}{2}+\cdots+\frac{A_+^{j+k-l}}{(j+k-l)!}\right\}e^{-2\sqrt{5\theta}\left(1+\frac{a+b}{2}\right)} \\ +(-1)^l B^l\left\{1+A_- +\frac{A_-^2}{2}+\cdots+\frac{A_-^{j+k-l}}{(j+k-l)!}\right\}e^{-2\sqrt{5\theta}\left(1-\frac{a+b}{2}\right)} \end{matrix}\right).$$

Terms in the last equation's large parentheses are evaluated individually for each $[j,k,l]$ set, to give the following sub-terms of $J_{5/2}$, each with an integer 576 in its denominator, with the help of the definitions

$E_\delta\equiv e^{-2\sqrt{5\theta}\left(\frac{b-a}{2}\right)}$ and (9.2)

$E_\pm\equiv e^{-2\sqrt{5\theta}\left(1\pm\frac{a+b}{2}\right)}$: (9.3)

$$[0,0,0]:\ \frac{144}{576\sqrt{5\theta}}\begin{bmatrix}(2+B)E_\delta\\ -(1)E_+\\ +(1)E_-\end{bmatrix}\qquad [0,1,0]:\ \frac{72}{576\sqrt{5\theta}}\begin{bmatrix}\left(2+B-\frac{1}{2}B^2\right)E_\delta\\ -(1+A_+)E_+\\ +(1+A_-)E_-\end{bmatrix}$$

$$[0,1,1]:\ \frac{72}{576\sqrt{5\theta}}\begin{bmatrix}(B^2)E_\delta\\ -B(1)E_+\\ +B(-1)E_-\end{bmatrix}\qquad [1,0,0]:\ \frac{72}{576\sqrt{5\theta}}\begin{bmatrix}\left(2+B+\frac{1}{2}B^2\right)E_\delta\\ -(1+A_+)E_+\\ +(1+A_-)E_-\end{bmatrix}$$

$$[1,1,0]:\ \frac{72}{576\sqrt{5\theta}}\begin{bmatrix}\left(2+B+\frac{1}{2}B^2-\frac{1}{6}B^3\right)E_\delta\\ -\left(1+A_++\frac{1}{2}A_+^2\right)E_+\\ +\left(1+A_-+\frac{1}{2}A_-^2\right)E_-\end{bmatrix}\qquad [1,1,1]:\ \frac{36}{576\sqrt{5\theta}}\begin{bmatrix}\left(-B^2+\frac{1}{2}B^3\right)E_\delta\\ -B(1+A_+)E_+\\ +B(-1-A_-)E_-\end{bmatrix}$$

$$[0,2,0]:\ \frac{24}{576\sqrt{5\theta}}\begin{bmatrix}\left(2+B+\frac{1}{2}B^2+\frac{1}{6}B^3\right)E_\delta\\ -\left(1+A_++\frac{1}{2}A_+^2\right)E_+\\ +\left(1+A_-+\frac{1}{2}A_-^2\right)E_-\end{bmatrix}\qquad [0,2,1]:\ \frac{24}{576\sqrt{5\theta}}\begin{bmatrix}\left(-B^2-\frac{1}{2}B^3\right)E_\delta\\ -B(1+A_+)E_+\\ +B(-1-A_-)E_-\end{bmatrix}$$

$$[0,2,2]:\ \frac{12}{576\sqrt{5\theta}}\begin{bmatrix}(2B^2+B^3)E_\delta\\ -B^2(1)E_+\\ +B^2(1)E_-\end{bmatrix}\qquad [1,2,0]:\ \frac{36}{576\sqrt{5\theta}}\begin{bmatrix}\left(2+B+\frac{1}{2}B^2+\frac{1}{6}B^3+\frac{1}{24}B^4\right)E_\delta\\ -\left(1+A_++\frac{1}{2}A_+^2+\frac{1}{6}A_+^3\right)E_+\\ +\left(1+A_-+\frac{1}{2}A_-^2+\frac{1}{6}A_-^3\right)E_-\end{bmatrix}$$

$$[1,2,1]:\ \frac{24}{576\sqrt{5\theta}}\begin{bmatrix}\left(-B^2-\frac{1}{2}B^3-\frac{1}{6}B^4\right)E_\delta\\ -B\left(1+A_++\frac{1}{2}A_+^2\right)E_+\\ +B\left(-1-A_--\frac{1}{2}A_-^2\right)E_-\end{bmatrix}\qquad [1,2,2]:\ \frac{6}{576\sqrt{5\theta}}\begin{bmatrix}\left(2B^2+B^3+\frac{1}{2}B^4\right)E_\delta\\ -B^2(1+A_+)E_+\\ +B^2(1+A_-)E_-\end{bmatrix}$$

$$[2,0,0]:\ \frac{24}{576\sqrt{5\theta}}\begin{bmatrix}\left(2+B+\frac{1}{2}B^2+\frac{1}{6}B^3\right)E_\delta\\ -\left(1+A_++\frac{1}{2}A_+^2\right)E_+\\ +\left(1+A_-+\frac{1}{2}A_-^2\right)E_-\end{bmatrix}\qquad [2,1,0]:\ \frac{36}{576\sqrt{5\theta}}\begin{bmatrix}\left(2+B+\frac{1}{2}B^2+\frac{1}{6}B^3-\frac{1}{24}B^4\right)E_\delta\\ -\left(1+A_++\frac{1}{2}A_+^2+\frac{1}{6}A_+^3\right)E_+\\ +\left(1+A_-+\frac{1}{2}A_-^2+\frac{1}{6}A_-^3\right)E_-\end{bmatrix}$$

$$[2,1,1]:\ \frac{12}{576\sqrt{5\theta}}\begin{bmatrix}\left(-B^2-\frac{1}{2}B^3+\frac{1}{6}B^4\right)E_\delta\\ -B\left(1+A_++\frac{1}{2}A_+^2\right)E_+\\ +B\left(-1-A_--\frac{1}{2}A_-^2\right)E_-\end{bmatrix}\qquad [2,2,0]:\ \frac{24}{576\sqrt{5\theta}}\begin{bmatrix}\left(2+B+\frac{B^2}{2}+\frac{B^3}{6}+\frac{B^4}{24}+\frac{B^5}{120}\right)E_\delta\\ -\left(1+A_++\frac{A_+^2}{2}+\frac{A_+^3}{6}+\frac{A_+^4}{24}\right)E_+\\ +\left(1+A_-+\frac{A_-^2}{2}+\frac{A_-^3}{6}+\frac{A_-^4}{24}\right)E_-\end{bmatrix}$$

$$[2,2,1]:\ \frac{12}{576\sqrt{5\theta}}\begin{bmatrix}\left(-B^2-\frac{1}{2}B^3-\frac{1}{6}B^4-\frac{1}{24}B^5\right)E_\delta\\ -B\left(1+A_++\frac{1}{2}A_+^2+\frac{1}{6}A_+^3\right)E_+\\ +B\left(-1-A_--\frac{1}{2}A_-^2-\frac{1}{6}A_-^3\right)E_-\end{bmatrix}\qquad [2,2,2]:\ \frac{2}{576\sqrt{5\theta}}\begin{bmatrix}\left(2B^2+B^3+\frac{1}{2}B^4+\frac{1}{6}B^5\right)E_\delta\\ -B^2\left(1+A_++\frac{1}{2}A_+^2\right)E_+\\ +B^2\left(1+A_-+\frac{1}{2}A_-^2\right)E_-\end{bmatrix}.$$

Summing these eighteen terms gives

$$J_{5/2}=\frac{1}{576\sqrt{5\theta}}\left(\begin{array}{l}\left\{\begin{array}{l}(288+144+144+144+48+72+48+72+48)\\+B\ \ (144+72+72+72+24+36+24+36+24)\\+B^2\begin{pmatrix}-36+72+36+36-36+12-24+24\\+18-24+12+12+18-12+12-12+4\end{pmatrix}\\+B^3(-12+18+4-12+12+6-12+6+4+6-6+4-6+2)\\+B^4\left(\frac{3}{2}-4+3-\frac{3}{2}+2+1-2+1\right)\\+B^5\left(\frac{1}{5}-\frac{1}{2}+\frac{1}{3}\right)\end{array}\right\}E_\delta\\ -\left\{\begin{array}{l}\left[\begin{array}{l}(144+72+72+72+24+36+24+36+24)\\+(72+72+72+24+36+24+36+24)A_+\\+(36+12+18+12+18+12)A_+^2\\+(6+6+4)A_+^3+A_+^4\end{array}\right]\\+B\left[\begin{array}{l}(72+36+24+24+12+12)\\+(36+24+24+12+12)A_+\\+(12+6+6)A_+^2+2A_+^3\end{array}\right]\\+B^2[(12+6+2)+(6+2)A_++A_+^2]\end{array}\right\}E_+\\ +\left\{\begin{array}{l}\left[\begin{array}{l}(144+72+72+72+24+36+24+36+24)\\+(72+72+72+24+36+24+36+24)A_-\\+(36+12+18+12+18+12)A_-^2\\+(6+6+4)A_-^3+A_-^4\end{array}\right]\\-B\left[\begin{array}{l}(72+36+24+24+12+12)\\+(36+24+24+24+12+12)A_-\\+(12+6+6)A_-^2+2A_-^3\end{array}\right]\\+B^2[(12+6+2)+(6+2)A_-+A_-^2]\end{array}\right\}E_-\end{array}\right)$$

$$=\frac{1}{576\sqrt{5\theta}}\left\{\begin{array}{l}\left[1008+504B+112B^2+14B^3+B^4+\frac{1}{30}B^5\right]E_\delta\\-\left[\begin{array}{l}(504+360A_++108A_+^2+16A_+^3+A_+^4)\\+B(180+108A_++24A_+^2+2A_+^3)+B^2(20+8A_++A_+^2)\end{array}\right]E_+\\+\left[\begin{array}{l}(504+360A_++108A_+^2+16A_+^3+A_+^4)\\-B(180+132A_++24A_+^2+2A_+^3)+B^2(24+8A_++A_+^2)\end{array}\right]E_-\end{array}\right\}.$$

Multiplying both numerator and denominator by $\frac{15}{8}$ gives a denominator of 1080 for comparison with previous results in Parts I and Ia of this series of papers:

$$J_{5/2}=\frac{1}{1080\sqrt{5\theta}}\left\{\begin{array}{l}\left[1890+945B+210B^2+\frac{105}{4}B^3+\frac{15}{8}B^4+\frac{1}{16}B^5\right]E_\delta\\-\left[\begin{array}{l}\left(945+675A_++\frac{405}{2}A_+^2+30A_+^3+\frac{15}{8}A_+^4\right)\\+B\left(\frac{675}{2}+\frac{405}{2}A_++45A_+^2+\frac{15}{4}A_+^3\right)+B^2\left(\frac{75}{2}+15A_++\frac{15}{8}A_+^2\right)\end{array}\right]E_+\\+\left[\begin{array}{l}\left(945+675A_-+\frac{405}{2}A_-^2+30A_-^3+\frac{15}{8}A_-^4\right)\\-B\left(\frac{675}{2}+\frac{495}{2}A_-+45A_-^2+\frac{15}{4}A_-^3\right)+B^2\left(45+15A_-+\frac{15}{8}A_-^2\right)\end{array}\right]E_-\end{array}\right\}.$$

Substituting back in the definitions of $A_\pm$ and B of Eqs. 8.11 and 8.12, gives

$$J_{5/2} = \frac{1}{1080\sqrt{5\theta}} \left(\begin{array}{l} 2\left\{ \begin{array}{c} 945 + 945(b-a)\sqrt{5\theta} + 420(b-a)^2\sqrt{5\theta}^2 \\ +105(b-a)^3\sqrt{5\theta}^3 + 15(b-a)^4\sqrt{5\theta}^4 + (b-a)^5\sqrt{5\theta}^5 \end{array} \right\} E_\delta \\ -\left\{ \begin{array}{l} \left[\begin{array}{c} 945 + 1350(1+a)\sqrt{5\theta} + 810(1+a)^2\sqrt{5\theta}^2 \\ +240(1+a)^3\sqrt{5\theta}^3 + 30(1+a)^4\sqrt{5\theta}^4 \end{array} \right] \\ +(b-a)\sqrt{5\theta}\left[\begin{array}{c} 675 + 810(1+a)\sqrt{5\theta} \\ +360(1+a)^2\sqrt{5\theta}^2 + 60(1+a)^3\sqrt{5\theta}^3 \end{array} \right] \\ +(b-a)^2\sqrt{5\theta}^2\left[150 + 120(1+a)\sqrt{5\theta} + 30(1+a)^2\sqrt{5\theta}^2\right] \end{array} \right\} E_+ \\ +\left\{ \begin{array}{l} \left[\begin{array}{c} 945 + 1350(1-a)\sqrt{5\theta} + 810(1-a)^2\sqrt{5\theta}^2 \\ +240(1-a)^3\sqrt{5\theta}^3 + 30(1-a)^4\sqrt{5\theta}^4 \end{array} \right] \\ -(b-a)\sqrt{5\theta}\left[\begin{array}{c} 675 + 810(1-a)\sqrt{5\theta} \\ +360(1-a)^2\sqrt{5\theta}^2 + 60(1-a)^3\sqrt{5\theta}^3 \end{array} \right] \\ +(b-a)^2\sqrt{5\theta}^2\left[150 + 120(1-a)\sqrt{5\theta} + 30(1-a)^2\sqrt{5\theta}^2\right] \end{array} \right\} E_- \end{array} \right).$$

Additional factoring gives

$$J_{5/2} = \frac{1}{1080\sqrt{5\theta}} \left(\begin{array}{l} 2\left\{ \begin{array}{c} 945 + 945(b-a)\sqrt{5\theta} + 420(b-a)^2\sqrt{5\theta}^2 \\ +105(b-a)^3\sqrt{5\theta}^3 + 15(b-a)^4\sqrt{5\theta}^4 + (b-a)^5\sqrt{5\theta}^5 \end{array} \right\} E_\delta \\ -15\left\{ \begin{array}{l} \left[\begin{array}{c} 63 + 90(1+a)\sqrt{5\theta} + 54(1+a)^2\sqrt{5\theta}^2 \\ +16(1+a)^3\sqrt{5\theta}^3 + 2(1+a)^4\sqrt{5\theta}^4 \end{array} \right] \\ +(b-a)\sqrt{5\theta}\left[\begin{array}{c} 45 + 54(1+a)\sqrt{5\theta} + 24(1+a)^2\sqrt{5\theta}^2 \\ +4(1+a)^3\sqrt{5\theta}^3 \end{array} \right] \\ +(b-a)^2\sqrt{5\theta}^2\left[10 + 8(1+a)\sqrt{5\theta} + 2(1+a)^2\sqrt{5\theta}^2\right] \end{array} \right\} E_+ \\ +15\left\{ \begin{array}{l} \left[\begin{array}{c} 63 + 90(1-a)\sqrt{5\theta} + 54(1-a)^2\sqrt{5\theta}^2 \\ +16(1-a)^3\sqrt{5\theta}^3 + 2(1-a)^4\sqrt{5\theta}^4 \end{array} \right] \\ -(b-a)\sqrt{5\theta}\left[\begin{array}{c} 45 + 54(1-a)\sqrt{5\theta} + 24(1-a)^2\sqrt{5\theta}^2 \\ +4(1-a)^3\sqrt{5\theta}^3 \end{array} \right] \\ +(b-a)^2\sqrt{5\theta}^2\left[10 + 8(1-a)\sqrt{5\theta} + 2(1-a)^2\sqrt{5\theta}^2\right] \end{array} \right\} E_- \end{array} \right).$$

Collecting common powers of $\sqrt{5\theta}$ gives

$$J_{5/2} = \frac{1}{1080\sqrt{5\theta}} \left(\begin{array}{l} 2\left\{ \begin{array}{l} 945 + 945(b-a)\sqrt{5\theta} + 420(b-a)^2\sqrt{5\theta}^2 \\ +105(b-a)^3\sqrt{5\theta}^3 + 15(b-a)^4\sqrt{5\theta}^4 + (b-a)^5\sqrt{5\theta}^5 \end{array} \right\} E_\delta \\ -15\left\{ \begin{array}{l} 63 \\ + [90(1+a) + 45(b-a)]\sqrt{5\theta} \\ + [54(1+a)^2 + 54(b-a)(1+a) + 10(b-a)^2]\sqrt{5\theta}^2 \\ + \begin{bmatrix} 16(1+a)^3 + 24(b-a)(1+a)^2 \\ +8(b-a)^2(1+a) \end{bmatrix} \sqrt{5\theta}^3 \\ + [\ 2(1+a)^4 + 4(b-a)(1+a)^3 + 2(b-a)^2(1+a)^2]\sqrt{5\theta}^4 \end{array} \right\} E_+ \\ +15\left\{ \begin{array}{l} 63 \\ + [90(1-a) - 45(b-a)]\sqrt{5\theta} \\ + [54(1-a)^2 - 54(b-a)(1-a) + 10(b-a)^2]\sqrt{5\theta}^2 \\ + \begin{bmatrix} 16(1-a)^3 - 24(b-a)(1-a)^2 \\ +8(b-a)^2(1+a) \end{bmatrix} \sqrt{5\theta}^3 \\ + [\ 2(1-a)^4 - 4(b-a)(1-a)^3 + 2(b-a)^2(1-a)^2]\sqrt{5\theta}^4 \end{array} \right\} E_- \end{array} \right).$$

Factoring the square brackets further gives

$$J_{5/2} = \frac{1}{1080\sqrt{5\theta}} \left(\begin{array}{l} 2\left\{ \begin{array}{l} 945 + 945(b-a)\sqrt{5\theta} + 420(b-a)^2(5\theta) \\ +105(b-a)^3\sqrt{5\theta}^3 + 15(b-a)^4\sqrt{5\theta}^4 + (b-a)^5\sqrt{5\theta}^5 \end{array} \right\} E_\delta \\ -15\left\{ \begin{array}{l} 63 \\ +45[2(1+a) + (b-a)]\sqrt{5\theta} \\ +\ 2[27(1+a)^2 + 27(b-a)(1+a) + 5(b-a)^2]\sqrt{5\theta}^2 \\ +\ 8(1+a)[2(1+a)^2 + 3(b-a)(1+a) + (b-a)^2]\sqrt{5\theta}^3 \\ +\ 2(1+a)^2[(1+a)^2 + 2(b-a)(1+a) + (b-a)^2]\sqrt{5\theta}^4 \end{array} \right\} E_+ \\ +15\left\{ \begin{array}{l} 63 \\ +45[2(1-a) - (b-a)]\sqrt{5\theta} \\ +\ 2[27(1-a)^2 - 27(b-a)(1-a) + 5(b-a)^2]\sqrt{5\theta}^2 \\ +\ 8(1-a)[2(1-a)^2 - 3(b-a)(1-a) + (b-a)^2]\sqrt{5\theta}^3 \\ +\ 2(1-a)^2[(1-a)^2 - 2(b-a)(1-a) + (b-a)^2]\sqrt{5\theta}^4 \end{array} \right\} E_- \end{array} \right). \qquad (9.4)$$

---------- *Algebraic expansions for the penultimate and ultimate curly brackets of Eq. 9.4* ----------

$2(1 \pm a) \pm (b-a) = 2 \pm a \pm b.$

$$\begin{aligned} &27(1 \pm a)^2 \pm 27(b-a)(1 \pm a) + 5(b-a)^2 \\ &\quad = 27 \pm 54a + 27a^2 \pm 27b + 27ab \mp 27a \mp 27a^2 + 5b^2 - 10ab + 5a^2 \\ &\quad = 27 \pm 27a \pm 27b + 5a^2 + 5b^2 + 17ab. \end{aligned}$$

$$\begin{aligned} &(1 \pm a)[2(1 \pm a)^2 \pm 3(b-a)(1 \pm a) + (b-a)^2] \\ &\quad = (1 \pm a)(2 \pm 4a \pm 2a^2 \pm 3b + 3ab \mp 3a - 3a^2 + b^2 - 2ab + a^2) \\ &\quad = (1 \pm a)(2 \pm a \pm 3b + b^2 + ab) \\ &\quad = 2 \pm a \pm 3b + b^2 + ab \pm 2a + a^2 + 3ab \pm ab^2 \pm a^2 b \\ &\quad = 2 \pm 3a \pm 3b + b^2 + 4ab + a^2 \pm a^2 b \pm ab^2. \end{aligned}$$

$(1 \pm a)^2[(1 \pm a)^2 \pm 2(b-a)(1 \pm a) + (b-a)^2]$

$$= (1 \pm a)^2[1 \pm 2a + a^2 \pm 2b \pm 2ab \mp 2a - 2a^2 + b^2 - 2ab + a^2]$$
$$= (1 \pm a)^2[1 \pm 2b + b^2]$$
$$= 1 \pm 2b + b^2 \pm 2a + 4ab \pm 2ab^2 + a^2 \pm 2a^2 b + a^2 b^2$$
$$= 1 \pm 2a \pm 2b + a^2 + 4ab + b^2 \pm 2a^2 b \pm 2ab^2 + a^2 b^2.$$

---------- *End* ----------

Using the expansions in the above digression and substituting back in the expressions for E_δ and $E_\pm$ in Eqs. 9.2 and 9.3 gives

$$J_{5/2} = \frac{1}{1080\sqrt{5\theta}} \left\{ \begin{matrix} 2\begin{bmatrix} 945 & \\ +945(b-a) & \sqrt{5\theta} \\ +420(b-a)^2 & \sqrt{5\theta}^2 \\ +105(b-a)^3 & \sqrt{5\theta}^3 \\ +\ 15(b-a)^4 & \sqrt{5\theta}^4 \\ +\quad (b-a)^5 & \sqrt{5\theta}^5 \end{bmatrix} e^{-2\sqrt{5\theta}\left|\frac{x_i - x_j}{2}\right|} \\ -15\begin{bmatrix} 63 & \\ +45\begin{pmatrix} 2 \\ +a+b \end{pmatrix} & \sqrt{5\theta} \\ +2\begin{pmatrix} 27 \\ +27a+27b \\ +5a^2+5b^2+17ab \end{pmatrix} & \sqrt{5\theta}^2 \\ +\ 8\begin{pmatrix} 2 \\ +3a+3bv \\ +b^2+4ab+a^2 \\ +a^2b+ab^2 \end{pmatrix} & \sqrt{5\theta}^3 \\ +\ 2\begin{pmatrix} 1 \\ +2a+2b \\ +a^2+4ab+b^2 \\ +2a^2b+2ab^2 \\ +a^2b^2 \end{pmatrix} & \sqrt{5\theta}^4 \end{bmatrix} e^{-2\sqrt{5\theta}\left(1+\frac{x_i+x_j}{2}\right)} \\ +15\begin{bmatrix} 63 & \\ +45\begin{pmatrix} 2 \\ -a-b \end{pmatrix} & \sqrt{5\theta} \\ +\ 2\begin{pmatrix} 27 \\ -27a-27b \\ +5a^2+5b^2+17ab \end{pmatrix} & \sqrt{5\theta}^2 \\ +\ 8\begin{pmatrix} 2 \\ -3a-3b \\ +b^2+4ab+a^2 \\ -a^2b-ab^2 \end{pmatrix} & \sqrt{5\theta}^3 \\ +\ 2\begin{pmatrix} 1 \\ -2a-2b \\ +a^2+4ab+b^2 \\ -2a^2b-2ab^2 \\ +a^2b^2 \end{pmatrix} & \sqrt{5\theta}^4 \end{bmatrix} e^{-2\sqrt{5\theta}\left(1-\frac{x_i+x_j}{2}\right)} \end{matrix} \right\}. \qquad (9.5)$$

This agrees, as it should, with the result for this integral in Papers Ia, Sec. 5 of this series of papers [2], viz.,

$$
R^{(2)}_{i,j} = \frac{1}{1080\sqrt{5\theta_k}} \left\{
\begin{array}{l}
2\begin{bmatrix} 945 \\ +945\left|x_{i,k}-x_{j,k}\right| \sqrt{5\theta_k} \\ +420\left|x_{i,k}-x_{j,k}\right|^2 \sqrt{5\theta_k}^2 \\ +105\left|x_{i,k}-x_{j,k}\right|^3 \sqrt{5\theta_k}^3 \\ +\ 15\left|x_{i,k}-x_{j,k}\right|^4 \sqrt{5\theta_k}^4 \\ +\ \left|x_{i,k}-x_{j,k}\right|^5 \sqrt{5\theta_k}^5 \end{bmatrix} e^{-2\sqrt{5\theta_k}\left|\frac{x_{i,k}-x_{j,k}}{2}\right|} \\
-15\mathcal{T}^{(-)}_{x_{i,k};x_{j,k}} \begin{bmatrix} 63 \\ +45\begin{pmatrix} 2 \\ +x_{i,k}+x_{j,k} \end{pmatrix} \sqrt{5\theta_k} \\ +\ 2\begin{pmatrix} 27 \\ +27x_{i,k}+27x_{j,k} \\ +\ 5x_{i,k}^2+17x_{i,k}x_{j,k}+5x_{j,k}^2 \end{pmatrix} \sqrt{5\theta_k}^2 \\ +\ 8\begin{pmatrix} 2 \\ +3x_{i,k}+3x_{j,k} \\ +x_{i,k}^2+4x_{i,k}x_{j,k}+x_{j,k}^2 \\ +x_{i,k}^2x_{j,k}+x_{i,k}x_{j,k}^2 \end{pmatrix} \sqrt{5\theta_k}^3 \\ +\ 2\begin{pmatrix} 1 \\ +2x_{i,k}+2x_{j,k} \\ +x_{i,k}^2+4x_{i,k}x_{j,k}+x_{j,k}^2 \\ +2x_{i,k}^2x_{j,k}+2x_{i,k}x_{j,k}^2 \\ +x_{i,k}^2x_{j,k}^2 \end{pmatrix} \sqrt{5\theta_k}^4 \end{bmatrix} e^{-2\sqrt{5\theta_k}\left(1+\frac{x_{i,k}+x_{j,k}}{2}\right)}
\end{array}
\right\}.
$$

10. Bessel Numbers of the First Kind

We conjecture, and leave the proof to the interested researcher, that the integer coefficients in the top curly brackets of Eqs. 8.14 and 8.15 are signless Bessel numbers of the first kind [12,13]. Some of the relevant Bessel numbers of the first kind are tabulated, immediately below.

ν	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
3/2	**1**	**6**	**15**	**15**						
2	1	10	45	105	105					
5/2	**1**	**15**	**105**	**420**	**945**	**945**				
3	1	21	210	1260	4725	10395	10395			
7/2	1	28	378	3150	17325	62370	135135	135135		
4	1	36	630	6930	51975	270270	945945	2027025	2027025	
9/2	1	45	990	13860	135135	945945	4729725	16216200	34459425	34459425

Table 4. Bessel numbers of the first kind, from the OEIS [11], are tabulated for $3/2 \le \nu \le 9/2$. The entries in the row with heading $\nu = 3/2$, viz., $[1,6,15,15]$, are the integer coefficients in the top square bracket of Eq. 9.1; while the entries in the row with heading $\nu = 5/2$, viz., [1,15,105,420,945,945], are the integer coefficients in the top square bracket of Eq. 9.5.

11. Summary and Concluding Comments

Eq. 6.6 (respectively, Eqs. 8.14 and 8.15) provides hand-generated algebraic expressions of integrals of single (resp., products of two) Matérn-covariance functions, for all odd-half-integer class parameters. Some derived coefficients in these integrals are conjectured to be Bessel numbers of the first kind.

12. Research Reproducibility

We support the recommendations of ICERM's Workshop on Reproducibility in Computational and Experimental Mathematics Workshop [14]. All data and figure-generation files used in this research are available to responsible parties from the first author at selden_crary (at) yahoo (dot) com.

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Appendix: A Proof of the Variation of Finite Companion Binomial Theorem

We now provide a proof of the variation of finite companion binomial theorem of Sec. 5.2, in an experimental, highly graphical form. The first author takes full responsibility for any abuse of notation.

Statement of the theorem

To show: $\sum_{j=0}^{p} \frac{(2p-j)!}{(p-j)!} 2^j = 4^p p!\,,\ \ p \in \mathbb{N}_{\geq 0}$.

Preliminaries

Rows 0 through 11 of the unmodified Pascal’s triangle (hereafter $\overset{\mathcal{P}}{\triangle}$) are shown below, along with the numbering scheme for rows, diagonals, and anti-diagonals:

```
Row 0 -->            1
Row 1 -->          1   1
Row 2 -->        1   2   1
               1   3   3   1
             1   4   6   4   1
           1   5  10  10   5   1
         1   6  15  20  15   6   1
       1   7  21  35  35  21   7   1
     1   8  28  56  70  56  28   8   1
   1   9  36  84 126 126  84  36   9   1
 1  10  45 120 210 252 210 120  45  10   1
1  11  55 165 330 462 462 330 165  55  11   1
```

Anti-diag. 0 -->
Anti-diag. 1 -->
Anti-diag. 2 -->
...

...
<-- Diagonal 2
<-- Diagonal 1
<-- Diagonal 0

Demonstration

Starting from the binomial coefficients of the Binomial theorem, $\binom{a}{b} \equiv \frac{a!}{b!(a-b)!}$, $a \geq b$,

$$\sum_{j=0}^{p} \frac{(2p-j)!}{(p-j)!} 2^j = p! \sum_{j=0}^{p} 2^j \binom{2p-j}{p} = p! \left[\binom{2p}{p} + 2 \binom{2p-1}{p} + \sum_{j=2}^{p} 2^j \binom{2p-j}{p} \right] \tag{1}$$

$$= p! \left[\frac{2p(2p-1)!}{p!p!} + 2 \frac{(2p-1)!}{(p-1)!p!} + \sum_{j=2}^{p} 2^j \binom{2p-j}{p} \right]$$

$$= p! \left[\frac{2p(2p-1)!}{p!p!} + \frac{2p(2p-1)!}{p!p!} + \sum_{j=2}^{p} 2^j \binom{2p-j}{p} \right]$$

$$= p! \left[\frac{4(2p-1)!}{(p-1)!p!} + \sum_{j=2}^{p} 2^j \binom{2p-j}{p} \right] = p! \left[4 \binom{2p-1}{p} + \sum_{j=2}^{p} 2^j \binom{2p-j}{p} \right]$$

$$= 4 \cdot p! \left[\binom{2p-1}{p} + \sum_{j=2}^{p} 2^{j-2} \binom{2p-j}{p} \right],$$

so it remains to show $\binom{2p-1}{p} + \sum_{j=2}^{p} 2^{j-2} \binom{2p-j}{p} = 4^{p-1}$. (2)

The p binomial coefficients of Eq. 2 lie along Anti-diagonal p in Rows p through $2p-1$ of $\triangle\mathcal{P}$, with the modification that the Row p coefficient is multiplied by 2^{p-2}, and the relevant coefficients in successive rows are multiplied by $2^{p-3}, 2^{p-4}, \cdots, 2, 2^0$, and unity (again). To assist the student, we carry forward the needed demonstration by writing out the relevant rows of $\triangle\mathcal{P}$ for the $p = 6$ case, with red used for the (modified) coefficients in Eq. 1. We seek to show the sum of the red cells in the triangular tableau is 4^{p-1}, i.e. that it is equal to the sum of the unmodified binomial coefficients of Row $(2p-2)$. That is, we seek to show the sum of red integers in $\triangle\mathcal{P}$ segments "a", below, equals the sum of green integers in $\triangle\mathcal{P}$ segments "b", below.

```
(a)          1    6   15   20   15    6   16
          1    7   21   35   35   21   56    1
       1    8   28   56   70   56  112    8    1
    1    9   36   84  126  126  168   36    9    1
 1   10   45  120  210  252  210  120   45   10    1
1   11   55  165  330  462  462  330  165   55   11    1

(b)          1    6   15   20   15    6    1
          1    7   21   35   35   21    7    1
       1    8   28   56   70   56   28    8    1
    1    9   36   84  126  126   84   36    9    1
 1   10   45  120  210  252  210  120   45   10    1
1   11   55  165  330  462  462  330  165   55   11    1
```

Graphic above. For the case $p = 6$, Rows 6 through 11 of $\triangle_{\mathcal{P}}$, are relevant and are shown in each of (a) and (b), above. In (a), the black integers are $\triangle_{\mathcal{P}}$ entries, while the red integers are $\triangle_{\mathcal{P}}$ entries multiplied by $2^4, 2^3, \cdots, 2^0$, or unity; for Rows $6, 7, \cdots, 11$; respectively. In (b), the integers are $\triangle_{\mathcal{P}}$,with green used for the Row 10 entries. For the case at hand, the identity sought in this paper is the equality of the sum of the colored entries in each of (a) and (b) to $4^{p-1} = 4^5 = 1024$.

It will be useful to show the modifications of the entries of $\triangle_{\mathcal{P}}$ explicitly, as in the following:

1 6 15 20 15 6 **$2^4 \cdot 1$**

1 7 21 35 35 21 **$2^3 \cdot 7$** 1

1 8 28 56 70 56 **$2^2 \cdot 28$** 8 1

1 9 36 84 126 126 **$2^1 \cdot 84$** 36 9 1

1 10 45 120 210 252 **$2^0 \cdot 210$** 120 45 10 1

1 11 55 165 330 462 **$1 \cdot 462$** 330 165 55 11 1.

The coefficient of interest in Row $2p - 1$ (the bold 462 in the truncated triangle, above) is the latter of the two equal central coefficients in that row, viz. $\binom{2p-1}{p-1}$ and $\binom{2p-1}{p}$. By the defining recursion of $\triangle_{\mathcal{P}}$, the former is equal to $\binom{2p-2}{p-2} + \binom{2p-2}{p-1}$, so we can, in our sum of sometimes modified coefficients, replace the coefficient $\binom{2p-1}{p-1}$ with two others, viz. $\binom{2p-2}{p-2}$ and $\binom{2p-2}{p-1}$. Then, removing the last row, as it will not be needed subsequently, the colored, truncated $\triangle_{\mathcal{P}}$ becomes

1 6 15 20 15 6 **$2^4 \cdot 1$**

1 7 21 35 35 21 **$2^3 \cdot 7$** 1

1 8 28 56 70 56 **$2^2 \cdot 28$** 8 1

1 9 36 84 126 126 **$2^1 \cdot 84$** 36 9 1

1 10 45 120 **210** **252** **210** 120 45 10 1

The values of the upper $p - 2$ modified coefficients along the p'th anti-diagonal can be split evenly over their own cell in $\triangle_{\mathcal{P}}$ and the mirror-image location, across a vertical bisector of $\triangle_{\mathcal{P}}$, giving, using an obvious definition of $Cell\binom{i}{j}$ as the contents of the Row i and Anti-diagonal j bisector of $\triangle_{\mathcal{P}}$:

$$
\begin{aligned}
Cell\binom{p}{p} &= Cell\binom{p}{0} &&= \tfrac{1}{2}2^{p-2}\binom{p}{p} &&= 2^{p-3},\\
Cell\binom{p+1}{p} &= Cell\binom{p+1}{1} &&= \tfrac{1}{2}2^{p-3}\binom{p+1}{p} &&= 2^{p-4}\binom{p+1}{p},\\
&\vdots\\
Cell\binom{2p-3}{p} &= Cell\binom{p-1}{p-3} &&= \tfrac{1}{2}2^{1}\binom{2p-3}{p} &&= 2^{0}\ \binom{2p-3}{p}.
\end{aligned}
$$

This leads to the graphic,

$\mathbf{2^3\cdot 1}$ 6 15 20 15 6 $\mathbf{2^3\cdot 1}$
1 $\mathbf{2^2\cdot 7}$ 21 35 35 21 $\mathbf{2^2\cdot 7}$ 1
1 8 $\mathbf{2^1\cdot 28}$ 56 70 56 $\mathbf{2^1\cdot 28}$ 8 1
1 9 36 $\mathbf{2^0\cdot 84}$ 126 126 $\mathbf{2^0\cdot 84}$ 36 9 1
1 10 45 120 **210** **252** **210** 120 45 10 1.

At this stage, the p'th row consists of the single entry in the uppermost left location and an equal entry in the uppermost right location, with the latter being on Anti-diagonal p and having value $2^{p-3}\binom{p}{p}$. Successive entries, descending along the anti-diagonal, are $2^{p-4}\binom{p+1}{p}, 2^{p-5}\binom{p+2}{p}, \cdots, 2^{0}\binom{2p-3}{p}$, and $\binom{2p-2}{p}$.

Now familiar with the steps that can be applied to the graphic, including the defining recursion of $\overset{\mathcal{P}}{\triangle}$, the student can follow the steps below that lead to the desired result:

1 6 15 20 15 6 1
$\mathbf{2^3\cdot 1}$ $\mathbf{2^2\cdot 7}$ 21 35 35 21 $\mathbf{2^2\cdot 7}$ $\mathbf{2^3\cdot 1}$
1 8 $\mathbf{2^1\cdot 28}$ 56 70 56 $\mathbf{2^1\cdot 28}$ 8 1
1 9 36 $\mathbf{2^0\cdot 84}$ 126 126 $\mathbf{2^0\cdot 84}$ 36 9 1
1 10 45 120 **210** **252** **210** 120 45 10 1,

1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 $\mathbf{2^1\cdot 28}$ 56 70 56 $\mathbf{2^1\cdot 28}$ $\mathbf{2^2\cdot 8}$ $\mathbf{2^2\cdot 1}$
1 9 36 $\mathbf{2^0\cdot 84}$ 126 126 $\mathbf{2^0\cdot 84}$ 36 9 1
1 10 45 120 **210** **252** **210** 120 45 10 1,

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

$2^1 \cdot \mathbf{1}$ $2^1 \cdot \mathbf{9}$ 36 $2^0 \cdot \mathbf{84}$ 126 126 $2^0 \cdot \mathbf{84}$ $2^1 \cdot \mathbf{36}$ $2^1 \cdot \mathbf{9}$ $2^1 \cdot \mathbf{1}$

1 10 45 120 **210** **252** **210** 120 45 10 1,

and finally,

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 **10** **45** **120** **210** **252** **210** **120** **45** **10** **1**,

which was to be shown.

The astute student will see that this graphical procedure carries through, in general.