Fibonacci-Lucas SIC-POVMs

Markus Grassl[1, a)] and Andrew J. Scott[b)]
[1]*Max-Planck-Institut für die Physik des Lichts, 91058 Erlangen, Germany*

We present a conjectured family of SIC-POVMs which have an additional symmetry group whose size is growing with the dimension. The symmetry group is related to Fibonacci numbers, while the dimension is related to Lucas numbers. The conjecture is supported by exact solutions for dimensions $d = 4, 8, 19, 48, 124$, as well as numerical solutions for dimensions $d = 323, 844$.

I. INTRODUCTION

The study of equiangular complex lines in quantum information theory started with the thesis of Zauner,[1,2] and independently by Renes et al.,[3] who also coined the term symmetric informationally-complete positive operator valued measure (SIC-POVM). Both conjectured that a SIC-POVM exists for all dimensions and that it could be constructed as the orbit under the Weyl-Heisenberg (WH) group. In addition, Zauner made the remark that one could choose the so-called fiducial vector in a particular eigenspace of an order-three unitary—this is now known as "Zauner's conjecture." In Ref. 4, we provided further support for these conjectures via numerical solutions up to dimension 67, as well as a couple of exact solutions for various dimensions. The second author extended the list of numerical solutions up to dimension 121, confirming Zauner's conjecture.[5] Imposing additional symmetries allowed to find solutions for certain dimensions up to 323. Using the very program from the second author, the list of numerical solutions was completed up to dimension 151 in Ref. 6.

Appleby[7] studied the candidates for additional symmetries of a SIC-POVM that is covariant with respect to the Weyl-Heisenberg group. He made the even stronger Conjecture C that every fiducial vector for such a SIC-POVM was an eigenvector of an order-three unitary that is conjugated to Zauner's matrix. This stronger version of Zauner's conjecture turned out to be false,[8] but there is a lot of support for Conjecture A by Appleby.[7] It states that WH-covariant SIC-POVMs exist in all dimensions and that the fiducial vector is an eigenvector of a canonical order-three unitary. All the solutions up do dimension 67 presented in Ref. 4 possess such an additional order-three symmetry—or even more—without imposing the symmetry.

In Ref. 4, we have also identified putative families of additional symmetries, and those symmetries enabled us to find exact solutions in dimensions as large as 48. The list of putative symmetries was extended in Ref. 5 and helped to find numerical solutions in dimensions $d = 99, 111, 120, 124, 143, 147, 168, 172, 195, 199, 228, 259$, and 323 which, in addition to being WH-covariant, have symmetries of order 6 or 9.

Analyzing the symmetry group of the numerical solutions,[9] it turned out that the solution in dimension 124 did not only have the imposed order-six symmetry, but a symmetry group of order 30. This discovery lead to the conjectured family of symmetries and new solutions presented here.

II. BACKGROUND AND NOTATION

Throughout this article, we mainly use the notation from quantum information.[4] A SIC-POVM in dimension d can be described by a set of d^2 rank-one projectors $\Pi_i = |\psi_i\rangle\langle\psi_i|$ on

a)Electronic mail: markus.grassl@mpl.mpg.de
b)Electronic mail: dr.andrew.scott@gmail.com

$\mathbb{C}^d$ such that

$$\text{tr}(\Pi_i \Pi_j) = \frac{1 + \delta_{ij} d}{1 + d}. \tag{1}$$

The Weyl-Heisenberg group in dimension d is generated by the cyclic shift operator $\hat{X}$ and its Fourier-transformed version $\hat{Z}$, given by

$$\hat{X} = \sum_{i=0}^{d-1} |i+1\rangle\langle i| \qquad \text{and} \qquad \hat{Z} = \sum_{j=0}^{d-1} \omega^j |j\rangle\langle j|, \tag{2}$$

where $\omega = \exp(\frac{2\pi i}{d})$ is a complex primitive d-th root of unity and addition is modulo d. Ignoring phase factors, the elements of the Weyl-Heisenberg group $X^a Z^b$ can be identified with pairs $(a, b) \in \mathbb{Z}_d \times \mathbb{Z}_d$ of integers modulo d. Appleby[7] makes a particular choice for the phases and defines the displacement operators as

$$\hat{D}_{(a,b)} = \tau^{ab} \hat{X}^a \hat{Z}^b, \tag{3}$$

where $\tau = -\exp(\frac{\pi i}{d})$. Then a Weyl-Heisenberg covariant SIC-POVM is given by a fiducial vector $|\psi_{(0,0)}\rangle$ and the vectors

$$|\psi_{(a,b)}\rangle = \hat{D}_{(a,b)}|\psi_{(0,0)}\rangle, \qquad \text{where } (a, b) \in \mathbb{Z}_d^2. \tag{4}$$

In all known cases, additional symmetries are elements of the extended Clifford group.[7] Again ignoring phase factors, the action of a unitary symmetry on the displacement operators (3) can be described by an invertible 2×2 matrix F over $\mathbb{Z}_d$ with determinant $+1$. Likewise, anti-unitary symmetries correspond to matrices with determinant -1. (In order to simplify the presentation here, we do not distinguish between odd and even dimensions; a more rigorous discussion can be found in Ref. 7.) A canonical order-three unitary[7] is an element of the Clifford group for which the corresponding matrix F has trace -1 and order three. The symmetry of Zauner's conjecture corresponds to the matrix

$$F_z = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}. \tag{5}$$

For dimensions d with $d \not\equiv 3 \bmod 9$, every canonical order-three unitary is equivalent to Zauner's matrix. For $d = 9\ell + 3$, $\ell \geq 1$, however, there is additionally the matrix

$$F_a = \begin{pmatrix} 1 & 3 \\ 3\ell & -2 \end{pmatrix} \tag{6}$$

which corresponds to a canonical order-three unitary that is not conjugated to Zauner's matrix. In Refs. 4 and 5, several putative families of additional symmetries for certain dimensions have been identified. Families of unitary symmetries of order 2 and 9 are described by matrices F_b and F_d, respectively, anti-unitary symmetries of order 2 and 6 are given by F_c and F_e, respectively. The latter can be combined[5] to a family of order-six anti-unitary symmetries $F_{e'}$. While the sequence of dimensions for which such additional symmetries might exist is unbounded, the order of the symmetry is at most 9. Here we make the following conjecture.

Conjecture II.1. *For the infinite sequence of dimensions $d_k = \varphi^{2k} + \varphi^{-2k} + 1$, where $\varphi = (1 + \sqrt{5})/2$, $k \geq 1$, there exists a Weyl-Heisenberg covariant SIC-POVM that has an additional anti-unitary symmetry of order $6k$ given by the Fibonacci matrix*

$$F_f = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{7}$$

We term such a set of vectors a Fibonacci-Lucas SIC-POVM.

The conjecture is supported by exact solutions for dimensions $d = 4, 8, 19, 48, 124$ as well as numerical solutions for $d_6 = 323$ and $d_7 = 844$.

III. THE FIBONACCI SYMMETRY

In this section, we study some properties of the additional symmetry given by the Fibonacci matrix. We start with the definition of Fibonacci and Lucas numbers.

Definition III.1 (Fibonacci numbers). Let F_n denote the n-th Fibonacci number given by the linear recurrence relation $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$, with $F_0 = 0$, $F_1 = 1$.

Note that we follow the general convention to denote the n-th Fibonacci number by F_n. At the same time, we use the symbol F to denote the 2×2 matrix over $\mathbb{Z}_d$ corresponding to the symmetry of a WH-covariant SIC-POVM. It should always be clear from the context whether F refers to a Fibonacci number or a matrix.

Definition III.2 (Lucas numbers). Let L_n denote the n-th Lucas number given by the linear recurrence relation $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$, with $L_0 = 2$, $L_1 = 1$.

Instead of the matrix F_f over the integers modulo d, we consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}) \tag{8}$$

with $\det(A) = -1$, over the integers. First note that the matrix A factorises as

$$A = F_z \cdot J = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{9}$$

where J is the matrix corresponding to complex conjugation in the standard basis.

Proposition III.3. *For $n > 0$, the n-th power of A is given by*

$$A^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} \tag{10}$$

Proof. The claim is true for $n = 1$. By induction, we find

$$A^{n+1} = A^n A = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} F_n & F_{n-1} + F_n \\ F_{n+1} & F_n + F_{n+1} \end{pmatrix} = \begin{pmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2.} \end{pmatrix} \tag{11}$$

$\square$

The dimension d_k in Conjecture II.1 is given by $d_k = L_{2k} + 1$ for $k \geq 1$ (see (A2)). The first ten dimensions in this sequence are $d = 4, 8, 19, 48, 124, 323, 844, 2208, 5779, 15128$. We summarise some properties of the sequence of dimensions, which is listed as sequence A065034 in the On-Line Encyclopedia of Integer Sequences.[10]

Proposition III.4.

1. *The sequence d_k obeys the linear recurrence relation*

$$d_{k+3} = 4d_{k+2} - 4d_{k+1} + d_k. \tag{12}$$

2. *Considered modulo 3, the sequence $\tilde{d}_k = d_k \bmod 3$ has period four and is given by $1, 2, 1, 0, 1, 2, 1, 0, \ldots$ (for $k = 1, 2, 3, \ldots$). This implies that d_k is divisible by 3 if and only if $k = 4\ell$. Then $d_{4\ell} \equiv 3 \bmod 9$.*

3. *The square-free part of $(d_k + 1)(d_k - 3)$ equals 5.*

Proof.

1. From the recurrence relation for the Lucas numbers, we obtain $L_{n+2} = 3L_n - L_{n-2}$. Together with the definition $d_k = L_{2k} + 1$, we first compute

$$d_{k+2} = L_{2k+4} + 1 = 3L_{2k+2} - L_{2k} + 1 = 3d_{k+1} - d_k - 1. \tag{13}$$

Then, using (13) twice, we get

$$\begin{aligned} d_{k+3} = 3d_{k+2} - d_{k+1} - 1 &= 4d_{k+2} - d_{k+2} - d_{k+1} - 1 \\ &= 4d_{k+2} - (3d_{k+1} - d_k - 1) - d_{k+1} - 1 \\ &= 4d_{k+2} - 4d_{k+1} + d_k. \end{aligned} \tag{14}$$

2. Consider the generating function $D(z) = \sum_{k \geq 0} d_k z^k$. Using standard techniques[11] one can show that

$$\begin{aligned} D(z) = \frac{-4z^2 + 8z - 3}{z^3 - 4z^2 + 4z - 1} &\equiv \frac{-z^2 - z}{z^3 - z^2 + z - 1} \pmod 3 \\ &= \frac{z^3 + 2z^2 + z}{1 - z^4} = (z^3 + 2z^2 + z)(1 + z^4 + z^8 + \ldots). \end{aligned} \tag{15}$$

This shows that $\tilde{d}_k$ has period 4, and that $d_k \equiv 0 \bmod 3$ if and only if $k = 4\ell$. For $k = 4\ell$, we compute

$$d_{4\ell} = L_{8\ell} + 1 = L_{4\ell}^2 - 1 = 5F_{4\ell}^2 + 3, \tag{16}$$

where we have used (A7) and (A5). From (A3) it follows that $F_{4l} \equiv 0 \bmod F_4$. Together with $F_4 = 3$, this implies that $d_{4\ell} \equiv 3 \bmod 9$.

3. We have

$$(d_k + 1)(d_k - 3) = (d_k - 1)^2 - 4 = L_{2k}^2 - 4 = 5F_{2k}^2, \tag{17}$$

where the last equality follows from (A5).

$\square$

The last statement that the squarefree part of $(d_k + 1)(d_k - 3)$ equals $D = 5$ has to be seen in the context of the conjecture[12] that the number field of smallest degree containing the fiducial projector in the standard basis is a ray class field over $\mathbb{Q}(\sqrt{D})$. It turned out that the conjecture is true for all our exact solutions.

Next we show that the matrix F_f determining the additional symmetry for dimension d_k has the desired properties.

Proposition III.5. *When considered as an element $\tilde{A} \in \mathrm{GL}(2, \mathbb{Z}_d)$ with $d = d_k = L_{2k} + 1$, the matrix $F_f = \tilde{A}$ has the following properties:*

1. *F_f^{2k} has trace -1.*

2. *For k even, F_f^{3k} is a scalar multiple of identity.*

3. *F_f has order $6k$.*

4. *For $k \equiv 0 \bmod 4$, F_f^{2k} is conjugated to F_a; otherwise, it is conjugated to Zauner's matrix F_z.*

Proof. From (10) it follows that the entries of the matrix A^{2k} over $\mathbb{Z}$ are strongly monotonic increasing in k. We have

$$A^{2k} = \begin{pmatrix} F_{2k-1} & F_{2k} \\ F_{2k} & F_{2k+1} \end{pmatrix}. \tag{18}$$

From (A4) it follows that $d_k = L_{2k} + 1 > F_{2k+1} > F_{2k}$, and hence the order of $\tilde{A} \in$ $\mathrm{GL}(2, \mathbb{Z}_{d_k})$ is strictly larger than $2k$. The determinant is $\det(A^{2k}) = 1$, and

$$\mathrm{tr}(A^{2k}) = F_{2k-1} + F_{2k+1} = L_{2k} \equiv -1 \bmod d_k, \tag{19}$$

where we have again used (A4). It follows that the characteristic polynomial of F_f^{2k} is $x^2 + x + 1$, and F_f^{2k} corresponds to a canonical order-three unitary.[7] As F_f^{2k} has order three, the order of F_f is a divisor of $6k$. We already know that the order is strictly larger than $2k$, and therefore we are left with the possibilities $3k$ and $6k$.

When k is odd, the determinant of A^{3k} equals -1, and therefore the order of F_f is $6k$. Now assume that $k = 2\ell$ is even. Then

$$A^{3k} = A^{6\ell} = \begin{pmatrix} F_{6\ell-1} & F_{6\ell} \\ F_{6\ell} & F_{6\ell+1} \end{pmatrix} = \begin{pmatrix} F_{6\ell-1} & F_{6\ell} \\ F_{6\ell} & F_{6\ell-1} + F_{6\ell} \end{pmatrix}. \tag{20}$$

From (A6), we have $F_{6\ell} \equiv 0 \bmod (L_{4\ell} + 1)$, and hence

$$\tilde{A}^{3k} = \begin{pmatrix} F_{6\ell-1} & 0 \\ 0 & F_{6\ell-1} \end{pmatrix}. \tag{21}$$

From (A8), we have $F_{6\ell-1} + L_{2\ell} = F_{2\ell-1}(L_{4\ell}+1)$, which implies $F_{6\ell-1} \equiv -L_{2\ell} \bmod (L_{4\ell}+1)$. In particular, we have $F_{6\ell-1} \not\equiv 1 \bmod (L_{4\ell}+1)$, i.e., $\tilde{A}^{3k}$ is different from identity, and hence the order of F_f is $6k$ in this case as well. Note that the matrix (21) is related to the matrix F_b in Ref. 4. From (A7) it follows that the dimension $d_{2\ell} = L_{4\ell} + 1$ is $L_{2\ell}^2 - 1$, i.e., a square minus one.

In order to prove the final part, first note that the dimension d_k is co-prime to three when k is not a multiple of four, and for those dimensions any matrix of order three with trace -1 is conjugated to Zauner's matrix. For $k = 4\ell$, we have

$$A^{2k} = A^{8\ell} = \begin{pmatrix} F_{8\ell-1} & F_{8\ell} \\ F_{8\ell} & F_{8\ell+1} \end{pmatrix} = \begin{pmatrix} F_{8\ell-1} & F_{8\ell} \\ F_{8\ell} & F_{8\ell-1} + F_{8\ell} \end{pmatrix}. \tag{22}$$

Again using (A3), $F_{8\ell} \equiv 0 \bmod F_8$, and with $F_8 = 21$, it follows that $A^{2k} = A^{8l} \bmod 3$ is proportional to identity. Furthermore, as $\tilde{A}^{2k}$ has order three, it follows that $A^{2k} = I \bmod 3$. Therefore, $\tilde{A}^{2k}$ is not conjugated to Zauner's matrix, but to F_a. $\square$

In summary, the matrix F_f is a candidate for an anti-unitary symmetry of order $6k$ in dimension $d_k = L_{2k} + 1$. In the following we show that exact fiducial vectors with this additional symmetry exist for the first five dimensions up to $d_5 = 124$. We have numerical solutions for the next two dimensions $d_6 = 323$ and $d_7 = 844$ as well.

IV. EXACT SOLUTIONS

We can factorise F_f as

$$F_f = JF_f' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \tag{23}$$

and obtain the corresponding unitary for the latter $\mathrm{SL}(2, \mathbb{Z}_d)$ matrix as[7]

$$\hat{U}_{F'_f} = \frac{1}{\sqrt{d}} \sum_{r,s=0}^{d-1} \tau^{-r^2-2rs} |r\rangle\langle s|, \tag{24}$$

where $\tau = -\exp(\pi i/d)$. Fiducial vectors $|\psi\rangle$ with the F_f symmetry are con-eigenvectors of this unitary, i.e.

$$\hat{J}\hat{U}_{F'_f}|\psi\rangle = e^{i\phi}|\psi\rangle \tag{25}$$

for some irrelevant phase ϕ, where $\hat{J}$ denotes the anti-linear operator corresponding to complex conjugation in the standard basis.

A rigorous description of the extended Clifford group in even dimensions in fact requires 2×2 matrices over $\mathbb{Z}_{2d}$ rather than $\mathbb{Z}_d$, a complication that has been avoided up until now. To relate the following solutions to previous work, however, we now quote matrices over $\mathbb{Z}_{2d}$ as appropriate and follow Appleby's correspondence precisely (Theorem 2 of Ref. 7). The quoted symmetry group of 2×2 matrices over $\mathbb{Z}_{2d}$ then doubly covers the corresponding group of extended Clifford operators (modulo its center).

A. Dimension $d = 4$

In Ref. 4, the numerical solution labeled $4a$ has a symmetry group of order 6 generated by

$$F_c F_z = \begin{pmatrix} 1 & 2 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 7 & 3 \end{pmatrix}. \tag{26}$$

The 7-th power of this matrix is conjugated to F_f by

$$G = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}, \tag{27}$$

as

$$G(F_c F_z)^7 G^{-1} = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 7 & 3 \end{pmatrix}^7 \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = F_f. \tag{28}$$

A non-normalised exact fiducial vector with the symmetry F_f is given by

$$|\psi^{4a}\rangle = \begin{pmatrix} \left((\sqrt{10}+\sqrt{2})\sqrt{1+\sqrt{5}}+4\sqrt{2}-4\right) - \left((-\sqrt{10}-\sqrt{2}+2\sqrt{5}+2)\sqrt{1+\sqrt{5}}+4\right)I \\ 8\sqrt{2}-8 \\ 8I \\ \left(-(\sqrt{10}+\sqrt{2})\sqrt{1+\sqrt{5}}+4\sqrt{2}-4\right) - \left((\sqrt{10}+\sqrt{2}-2\sqrt{5}-2)\sqrt{1+\sqrt{5}}+4\right)I \end{pmatrix}, \tag{29}$$

where $I^2 = -1$. This is the numerical solution $4a$ translated by the unitary corresponding to G. The number field containing a fiducial projector is $\mathbb{E}^{4a} = \mathbb{Q}(\sqrt{5}, \sqrt{2}, \sqrt{1+\sqrt{5}}, \sqrt{-1})$, which is an Abelian extension of $\mathbb{Q}(\sqrt{5})$.

B. Dimension $d = 8$

In Ref. 4, the numerical solution labeled $8b$ has a symmetry group of order 12 generated by

$$F = \begin{pmatrix} 6 & 11 \\ 5 & 1 \end{pmatrix}. \tag{30}$$

The 11-th power of this matrix is conjugated to F_f by

$$G = \begin{pmatrix} 5 & 5 \\ 4 & 1 \end{pmatrix}, \tag{31}$$

as

$$GF^{11}G^{-1} = \begin{pmatrix} 5 & 5 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 6 & 11 \\ 5 & 1 \end{pmatrix}^{11} \begin{pmatrix} 5 & 5 \\ 4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & 5 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 7 & 3 \\ 13 & 10 \end{pmatrix} \begin{pmatrix} 1 & 11 \\ 12 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = F_f. \tag{32}$$

A non-normalised exact fiducial vector with the symmetry F_f is given by

$$|\psi^{8b}\rangle = \begin{pmatrix} -4\sqrt{5} + 12 \\ ((2\sqrt{2} - 2)s_1 s_2 + (-\sqrt{10} + 3\sqrt{2} + 2\sqrt{5} - 6)s_1 - 4s_2)I + 2s_1 s_2 + (\sqrt{10} - 3\sqrt{2})s_1 - 2\sqrt{10} + 6\sqrt{2} \\ -4\sqrt{2}s_2 I \\ ((2\sqrt{2} - 2)s_1 s_2 + (\sqrt{10} - 3\sqrt{2} - 2\sqrt{5} + 6)s_1 + 4s_2)I + 2s_1 s_2 + (-\sqrt{10} + 3\sqrt{2})s_1 - 2\sqrt{10} + 6\sqrt{2} \\ 4\sqrt{5} - 12 \\ ((-2\sqrt{2} + 2)s_1 s_2 + (\sqrt{10} - 3\sqrt{2} - 2\sqrt{5} + 6)s_1 - 4s_2)I - 2s_1 s_2 + (-\sqrt{10} + 3\sqrt{2})s_1 - 2\sqrt{10} + 6\sqrt{2} \\ 4\sqrt{2}s_2 I \\ ((-2\sqrt{2} + 2)s_1 s_2 + (-\sqrt{10} + 3\sqrt{2} + 2\sqrt{5} - 6)s_1 + 4s_2)I - 2s_1 s_2 + (\sqrt{10} - 3\sqrt{2})s_1 - 2\sqrt{10} + 6\sqrt{2} \end{pmatrix}, \tag{33}$$

where $s_1 = \sqrt{2 + \sqrt{2}}$ and $s_2 = \sqrt{\sqrt{5} - 1}$. This is the numerical solution $8b$ translated by the unitary corresponding to G. The number field containing the fiducial projector is an Abelian extension of $\mathbb{Q}(\sqrt{5})$ given by

$$\mathbb{E}^{8b} = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{\sqrt{5} - 1}, \sqrt{2 + \sqrt{2}}, \sqrt{-1}). \tag{34}$$

The exact solution $8b$ in Ref. 4 has a symmetry group generated by

$$F' = \begin{pmatrix} 1 & 5 \\ 13 & 0 \end{pmatrix}, \tag{35}$$

which is again conjugated to F_f, i.e., $HF'H^{-1} = F_f$ for the choice

$$H = \begin{pmatrix} 1 & 4 \\ 5 & 5 \end{pmatrix}. \tag{36}$$

The unitary corresponding to H translates the exact solution in Ref. 4 to that given above.

C. Dimension $d = 19$

In Ref. 4, the numerical solution labeled $19e$ has a symmetry group of order 18 generated by

$$F = \begin{pmatrix} 3 & 12 \\ 7 & 15 \end{pmatrix}. \tag{37}$$

The 17-th power of this matrix is conjugated to F_f by

$$G = \begin{pmatrix} 11 & 10 \\ 0 & 7 \end{pmatrix}, \tag{38}$$

as

$$GF^{17}G^{-1} = \begin{pmatrix} 11 & 10 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} 3 & 12 \\ 7 & 15 \end{pmatrix}^{17} \begin{pmatrix} 11 & 10 \\ 0 & 7 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 11 & 10 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} 4 & 12 \\ 7 & 16 \end{pmatrix} \begin{pmatrix} 7 & 9 \\ 0 & 11 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = F_f. \tag{39}$$

A non-normalised exact fiducial vector with the symmetry F_f can be found online.[13] This is the numerical solution $19e$ translated by the unitary corresponding to G. The exact solution $19e$ in Ref. 4 has a symmetry group generated by the diagonal matrix $F' = \mathrm{diag}(15, 5)$, which yields a unitary symmetry that is just a permutation. Hence, the corresponding fiducial vector has a very compact representation. For

$$H = \begin{pmatrix} 8 & 5 \\ 6 & 6 \end{pmatrix} \tag{40}$$

we have $HF'H^{-1} = F_f$, i.e., the symmetry is conjugated to F_f as well. The unitary corresponding to H also translates the exact solution in Ref. 4 to the alternative given online.[13]

D. Dimension $d = 48$

The numerical solution in Ref. 4 given for orbit $48g$ has a symmetry group of order 24 generated by

$$F = \begin{pmatrix} 4 & 37 \\ 25 & 63 \end{pmatrix}. \tag{41}$$

The 41-st power of this matrix is conjugated to F_f by

$$G = \begin{pmatrix} 10 & 47 \\ 21 & 22 \end{pmatrix}, \tag{42}$$

as

$$
\begin{aligned}
GF^{41}G^{-1} &= \begin{pmatrix} 10 & 47 \\ 21 & 22 \end{pmatrix} \begin{pmatrix} 4 & 37 \\ 25 & 63 \end{pmatrix}^{41} \begin{pmatrix} 10 & 47 \\ 21 & 22 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 10 & 47 \\ 21 & 22 \end{pmatrix} \begin{pmatrix} 61 & 25 \\ 61 & 36 \end{pmatrix} \begin{pmatrix} 22 & 49 \\ 75 & 10 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = F_f.
\end{aligned} \tag{43}
$$

A non-normalised exact fiducial vector with the symmetry F_f can be found online.[13] This is the numerical solution $48g$ translated by the unitary corresponding to G. The exact solution $48g$ in Ref. 4 also has the symmetry group generated by F once it is displaced by $\hat{D}_{(41,15)}$. The unitary corresponding to

$$H = \begin{pmatrix} 5 & 16 \\ 33 & 29 \end{pmatrix} \tag{44}$$

then translates this exact solution to the alternative given online.[13]

E. Dimension $d = 124$

The numerical solution of Ref. 5 labeled $124a$, which was obtained by imposing the symmetries F_z and F_b of combined order 6, turned out to have a symmetry of order 30 generated by

$$F = \begin{pmatrix} 58 & 133 \\ 115 & 191 \end{pmatrix}. \tag{45}$$

This symmetry is conjugated to F_f by

$$G = \begin{pmatrix} 100 & 15 \\ 85 & 45 \end{pmatrix}, \tag{46}$$

as

$$GFG^{-1} = \begin{pmatrix} 100 & 15 \\ 85 & 45 \end{pmatrix} \begin{pmatrix} 58 & 133 \\ 115 & 191 \end{pmatrix} \begin{pmatrix} 100 & 15 \\ 85 & 45 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 100 & 15 \\ 85 & 45 \end{pmatrix} \begin{pmatrix} 58 & 133 \\ 115 & 191 \end{pmatrix} \begin{pmatrix} 45 & 233 \\ 163 & 100 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = F_f. \tag{47}$$

In order to find an exact fiducial vector with the anti-unitary symmetry F_f, we use the factorization (9) of F_f into Zauner's matrix F_z and complex conjugation. We identify $\mathbb{C}^{124}$ with the real space $\mathbb{R}^{248}$ and find a 10-dimensional eigenspace of the corresponding $\mathbb{R}$-linear transformation, i.e., we have to solve for 10 real variables. As $124 = 2^2 \times 31$, we can apply a global change of basis in order to obtain more sparse vectors.[14] It turned out that it sufficient to consider only those vectors of the SIC-POVM that are obtained as the orbit of the fiducial vector with respect to cyclic shift operation $\hat{X}$, i.e., the equations

$$\langle \psi_{(0,0)} | \psi_{(0,0)} \rangle = 1, \tag{48}$$

$$\text{and} \qquad |\langle \psi_{(0,0)} | \hat{X}^a | \psi_{(0,0)} \rangle|^2 = \frac{1}{125} \qquad \text{for } a = 1, \dots, 123. \tag{49}$$

Moreover, the equations for the exponents a and $-a$ in (49) are identical, so that we have only 63 equations in total. Computing a Gröbner basis with the modular approach for polynomial equations over number fields implemented in Magma[15] V2.22-6 on a system with an Intel Xeon X5680 processor with 3.33 GHz clock speed took about 27 CPU hours and 33 GB of RAM. It has been conjectured that the minimal field containing a fiducial projector is the ray class field $\mathbb{E}$ of $\mathbb{Q}(\sqrt{5})$ with conductor $d' = 2d = 248$ and ramification at both infinite places.[12] Using Magma,[15] we computed a slightly optimised representation of $\mathbb{E}$ as

$$\mathbb{E}^{124a} = \mathbb{Q}(\sqrt{5}, \sqrt{2}, \sqrt{31}, \sqrt{-1}, \sqrt{(\sqrt{5}-1)/2}, \sqrt{6+\sqrt{5}}, s_1, s_2, s_3), \tag{50}$$

where s_1 is a root of the polynomial $f_1(x) = x^3 + x^2 - 10x - 8$, s_2 is a root of the polynomial $f_2(x) = x^5 - x^4 - 12x^3 + 21x^2 + x - 5$, and s_3 is a root of the polynomial $f_3(x) = x^3 + x^2 - 41x + 85$. These three polynomials have only real roots, i.e., the only complex number among the generators of $\mathbb{E}^{124a}$ is the imaginary unit $\sqrt{-1}$. The generators of the Galois group of $\mathbb{E}^{124a}/\mathbb{Q}$ are given as follows (only the non-trivial action on the generators is listed):

$$g_1: \qquad \sqrt{5} \mapsto -\sqrt{5}; \qquad \sqrt{(\sqrt{5}-1)/2} \mapsto \frac{(\sqrt{5}-1)\sqrt{-1}}{2}\sqrt{(\sqrt{5}-1)/2};$$

$$\sqrt{6+\sqrt{5}} \mapsto \frac{(\sqrt{5}-6)\sqrt{31}}{31}\sqrt{6+\sqrt{5}}$$

$$g_2: \qquad \sqrt{2} \mapsto -\sqrt{2}$$

$$g_3: \qquad \sqrt{31} \mapsto -\sqrt{31}$$

$$g_4: \qquad \sqrt{-1} \mapsto -\sqrt{-1}$$

$$g_5: \sqrt{(\sqrt{5}-1)/2} \mapsto -\sqrt{(\sqrt{5}-1)/2}$$

$$g_6: \qquad \sqrt{6+\sqrt{5}} \mapsto -\sqrt{6+\sqrt{5}}$$

$$g_7: \qquad s_1 \mapsto (s_1^2 - s_1 - 8)/2$$

$$g_8: \qquad s_2 \mapsto \left(2s_2^4 - s_2^3 - 22s_2^2 + 31s_2\right)/5$$

$$g_9: \qquad s_3 \mapsto \left(4\sqrt{5}s_3^2 + (17\sqrt{5}-5)s_3 - 105\sqrt{5} - 5\right)/10 \tag{51}$$

The Galois group has order 2880 and is isomorphic to $C_{30} \times \left((C_6 \times C_2 \times C_2 \times C_2) \rtimes C_2\right)$. It turns out that a fiducial vector can indeed be found in the ray class field. The most complex

step in the computation, however, was to find a representation of the fiducial vector in $\mathbb{E}^{124a}$. For this, we had to factorise several polynomials over $\mathbb{E}^{124a}$. A non-normalised exact fiducial vector can be found online.[13] This is the numerical solution $124a$ of Ref. 5 translated by the unitary corresponding to G above.

V. NUMERICAL SOLUTIONS

Our numerical search for solutions followed Refs. 4 and 5 by first translating the Welch bound on a set of unit vectors in $\mathbb{C}^d$ to the case of a SIC-POVM: for any $|\phi\rangle \in \mathbb{C}^d$,

$$\frac{1}{d^3} \sum_{a,b,a',b'} |\langle\phi|\hat{D}^{\dagger}_{(a',b')}\hat{D}_{(a,b)}|\phi\rangle|^4 = \sum_{j,k} \left| \sum_l \langle\phi|j+l\rangle\langle l|\phi\rangle\langle\phi|k+l\rangle\langle j+k+l|\phi\rangle \right|^2 \geq \frac{2}{d+1} ,$$
(52)

with equality if and only if $|\phi\rangle$ is a fiducial vector for a WH covariant SIC-POVM. The condition for equality follows[3] from the 2-design property of a SIC-POVM. We may now search for fiducial vectors by simply minimising the LHS of the inequality in Eq. (52), parametrised as a function of the real and imaginary parts of the d complex numbers $\langle j|\phi\rangle$, until the bound on the RHS is met.

To search for Fibonacci-Lucas SIC-POVMs we applied the method of Ref. 5 for general anti-unitary symmetries. Suppose that $\hat{J}\hat{U}|\psi\rangle = \lambda|\psi\rangle$ for some unitary $\hat{U}$ with $(\hat{J}\hat{U})^{2n} = (\hat{J}\hat{U}\hat{J}\hat{U})^n = (\overline{U}\hat{U})^n = I$, where $\overline{U}$ denotes complex conjugation of matrix components in the standard basis. We must have $|\lambda| = 1$ and may in fact assume $\lambda = 1$ for con-eigenvalues. Now given that the projector onto the eigenspace of the unitary $\overline{U}\hat{U}$ with eigenvalue 1 is

$$\hat{Q} = \frac{1}{n} \sum_{j=0}^{n-1} \left(\overline{U}\hat{U}\right)^j$$
(53)

it is easy to check that the unnormalised $|\psi'\rangle = \overline{U\hat{Q}|\phi\rangle} + \hat{Q}|\phi\rangle$ solves our con-eigenvalue problem. We can therefore replace $|\phi\rangle$ with $|\psi\rangle = |\psi'\rangle/\sqrt{\langle\psi'|\psi'\rangle}$ in eq. (52) to search the set of con-eigenvectors of an anti-unitary symmetry. In particular, we take $\hat{U} = \hat{U}_{F'_f}$ to search for fiducial vectors with the Fibonacci symmetry.

In practice the numerical search was performed by repeating a local search from different initial trial vectors until a solution is found. The local search used a C++ implementation of L-BFGS and only a few trials were required to find a Fibonacci-Lucas fiducial vector in each dimension up to 844. Nonetheless, the search in dimension 2208 proved just beyond our reach.

By repeating the search for randomly chosen trial vectors under the unitarily invariant Haar measure on $\mathbb{C}^d$, the entire search space can be exhausted to identify all unique extended Clifford orbits generated by Fibonacci-Lucas fiducial vectors. This was done for dimensions up to 124, where only a single Fibonacci-Lucas extended Clifford orbit was found in each case. In all dimensions calculated (all but 844) the symmetry group of the orbit was found to be no larger than that generated by F_f. For dimension 844, we used the general approach[9] based on triple-products

$$t_{0ij} = \langle\psi_0|\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|\psi_0\rangle = \text{tr}(|\psi_0\rangle\langle\psi_0| \cdot |\psi_i\rangle\langle\psi_i| \cdot |\psi_j\rangle\langle\psi_j|) = \text{tr}(\Pi_0\Pi_i\Pi_j)$$
(54)

to verify that the additional unitary symmetry within the Clifford group has order 21. This implies that again the symmetry group is exactly the one generated by F_f.

All numerical solutions are included as supplementary files in the article source and also made available online.[13] These can be taken to be exact up to the precision quoted, which is 150 digits in dimensions up to 323 and 75 digits in dimension 844. Solutions in dimensions up to 124 match the exact solutions.

VI. CONCLUSIONS

Identifying a putative family of SIC-POVMs for which the size of the additional symmetry grows with the dimension allowed us to find both exact and numerical solutions for the largest dimensions so far. For both dimension 323 and 844, it doesn't seem to be completely out of reach to obtain exact solutions. The number of real parameters is moderate in both cases, we have to solve for at most 19 and 42 real variables, respectively. Furthermore, both dimensions allow a more sparse representation of the symmetry. On the other hand, the degree of the corresponding ray class fields over the rationals is already pretty large, 10 368 and 100 800, respectively. When considered as extensions of the corresponding cyclotomic fields, however, the degrees are only 36 and 120, respectively. There is also some chance to convert the numerical solutions into exact ones.[16]

Similar to the Fibonacci matrix investigated here, there are other candidates for matrices that give rise to putative families of additional symmetries for Weyl-Heisenberg covariant SIC-POVMs. We leave this to future work.[17]

Despite the simplifications due to the additional symmetries, there are clearly limits up to which dimension solutions can be found by direct computation. So we hope that the new families of putative symmetries, in combination with all the structure of SIC-POVMs that has been brought to light so far, will eventually give way to a construction of an infinite family of SIC-POVMs.

ACKNOWLEDGMENTS

[1] G. Zauner, *Grundzüge einer nichtkommutativen Designtheorie*, Ph.D. thesis, Universität Wien (1999).

[2] G. Zauner, "Quantum designs: Foundations of a non-commutative design theory," International Journal of Quantum Information **9**, 445–508 (2011).

[3] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, "Symmetric informationally complete quantum measurements," Journal of Mathematical Physics **45**, 2171 (2004).

[4] A. J. Scott and M. Grassl, "Symmetric informationally complete positive-operator-valued measures: A new computer study," Journal of Mathematical Physics **51**, 042203 (2010).

[5] A. J. Scott, "SICs: extending the list of solutions," arXiv:1703.03993 [quant-ph] (2017).

[6] C. A. Fuchs, M. C. Hoang, and B. C. Stacey, "The SIC question: History and state of play," arXiv:1703.07901 [quant-ph] (2017).

[7] D. M. Appleby, "Symmetric informationally complete-positive operator valued measures and the extended Clifford group," Journal of Mathematical Physics **46**, 052107 (2005).

[8] M. Grassl, "Tomography of quantum states in small dimensions," Electronic Notes in Discrete Mathematics **20**, 151–164 (2005), Workshop on Discrete Tomography and its Applications New York, USA, June 2005.

[9] M. Grassl and S. Waldron, "Computing projective symmetries of frames," (2017), in preparation.

[10] N. J. A. Sloane (editor), "The on-line encyclopedia of integer sequences," published electronically at https://oeis.org (2010).

[11] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd ed. (Addison-Wesley, Boston, MA, USA, 1994).

[12] M. Appleby, S. Flammia, G. McConnell, and J. Yard, "Generating ray class fields of real quadratic fields via complex equiangular lines," arXiv:1604.06098 [math.NT] (2016).

[13] http://sicpovm.markus-grassl.de, (2017).

[14] D. M. Appleby, I. Bengtsson, S. Brierley, Å. Ericsson, M. Grassl, and J.-Å. Larsson, "Systems of imprimitivity for the Clifford group," Quantum Information & Computation **14**, 339–360 (2014), arXiv:1210.1055 [quant-ph].

[15] W. Bosma, J. J. Cannon, and C. Playoust, "The Magma Algebra System I: The User Language," Journal of Symbolic Computation **24**, 235–265 (1997).

[16] M. Appleby, T.-Y. Chien, S. Flammia, and S. Waldron, "Constructing exact symmetric informationally complete measurements from numerical solutions," arXiv:1703.05981 [quant-ph] (2017).

[17] M. Grassl, A. J. Scott, and U. Seyfarth, "Symmetries of Weyl-Heisenberg SIC-POVMs," (2017), in preparation.

[18]L. Euler, "Observationes analyticæ," Novi commentarii academiæ scientiarum Petropolitanæ, 124–143 (1765), reprinted in his Opera Omnia, series 1, volume 15, 50–69.

Appendix A: Relations for Fibonacci and Lucas Numbers

In the following we will derive a couple of relations for Fibonacci and Lucas numbers.

Proposition A.1. *Let $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden ratio which is the positive root of the equation $x^2 = x + 1$. Then we have the following well-known closed formulas for the Fibonacci[18] and Lucas numbers (see, e. g., Exercise 28 in Ref. 11):*

$$F_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} \tag{A1}$$

$$and \quad L_n = \varphi^n + (1-\varphi)^n = \varphi^n + (-\varphi)^{-n}. \tag{A2}$$

Proposition A.2. *For $k, n \geq 1$, the following identities hold:*

$$i) \qquad\qquad F_n \mid F_{kn} \tag{A3}$$
$$ii) \qquad\qquad L_n = F_{n-1} + F_{n+1} \tag{A4}$$
$$iii) \qquad\qquad L_{2n}^2 = 5F_{2n}^2 + 4 \tag{A5}$$
$$iv) \qquad\qquad F_{6n} = F_{2n}(L_{4n} + 1) \tag{A6}$$
$$v) \qquad L_{4n} + 1 = (L_{2n} + 1)(L_{2n} - 1) = L_{2n}^2 - 1 \tag{A7}$$
$$vi) \qquad F_{6n-1} + L_{2n} = F_{2n-1}(L_{4n} + 1) \tag{A8}$$

Proof. i): By induction it can be shown that F_{kn} is divisible by F_n (see, e. g., Eq. (6.111) in Ref. 11).

ii): We prove the identity by induction. It clearly holds for $n = 1$. Using the defining relations, we compute

$$\begin{aligned} L_{n+1} = L_n + L_{n-1} &= (F_{n-1} + F_{n+1}) + (F_{n-2} + F_n) \\ &= F_{n-1} + F_{n-2} + F_{n+1} + F_n \quad = F_n + F_{n+2}. \end{aligned} \tag{A9}$$

iii): Using the closed formulas (A1) and (A2), we compute

$$\begin{aligned} 5F_{2n}^2 + 4 = (\varphi^{2n} - (-\varphi)^{-2n})^2 + 4 &= \varphi^{4n} - 2 + (-\varphi)^{-4n} + 4 \\ &= \varphi^{4n} + 2 + (-\varphi)^{-4n} \\ &= (\varphi^{2n} + (-\varphi)^{-2n})^2 \quad = L_{2n}^2. \end{aligned} \tag{A10}$$

iv): Similar as before, we compute

$$\begin{aligned} F_{2n}(L_{4n} + 1) &= \frac{1}{\sqrt{5}}(\varphi^{2n} - (-\varphi)^{-2n})(\varphi^{4n} + (-\varphi)^{-4n} + 1) \\ &= \frac{1}{\sqrt{5}}(\varphi^{6n} - (-\varphi)^{-6n}) \qquad\qquad = F_{6n}. \end{aligned} \tag{A11}$$

v): Again, direct computations shows

$$\begin{aligned} (L_{2n} + 1)(L_{2n} - 1) &= (\varphi^{2n} + (-\varphi)^{-2n} + 1)(\varphi^{2n} + (-\varphi)^{-2n} - 1) \\ &= \varphi^{4n} + (-\varphi)^{-4n} + 1 \qquad\qquad = L_{4n} + 1. \end{aligned} \tag{A12}$$

$vi)$: In order to prove the claim, consider

$$
\begin{aligned}
F_{2n-1}(L_{4n}+1) &= \frac{1}{\sqrt{5}}(\varphi^{2n-1} - (-\varphi)^{-2n+1})(\varphi^{4n} + (-\varphi)^{-4n} + 1) \\
&= \frac{1}{\sqrt{5}}(\varphi^{6n-1} + \varphi^{2n+1} + \varphi^{2n-1} + \varphi^{-2n+1} + \varphi^{-2n-1} + \varphi^{-6n+1}) \\
&= \frac{\varphi^{6n-1} + \varphi^{-6n+1}}{\sqrt{5}} + \frac{(\varphi^{2n} + \varphi^{-2n})(\varphi + \varphi^{-1})}{\sqrt{5}} \\
&= \frac{\varphi^{6n-1} - (-\varphi)^{-6n+1}}{\sqrt{5}} + \varphi^{2n} + (-\varphi)^{-2n} \\
&= F_{6n-1} + L_{2n} \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\text{(A13)}
\end{aligned}
$$

$\square$