Karlsruhe
July 13 2017

On Sums of Powers of Arithmetic Progressions, and Generalized Stirling, Eulerian and Bernoulli numbers

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Abstract

For finite sums of non-negative powers of arithmetic progressions the generating functions (ordinary and exponential ones) for given powers are computed. This leads to a two parameter generalization of *Stirling* and *Eulerian* numbers. A direct generalization of Bernoulli numbers and their polynomials follows. On the way to find the *Faulhaber* formula for these sums of powers in terms of generalized *Bernoulli* polynomials one is led to a one parameter generalization of *Bernoulli* numbers and their polynomials. Generalized *Lah* numbers are also considered.

1 Introduction and Summary

A) Generating functions of power sums and powers. Generalized Stirling2 and Eulerian numbers.

Finite sums of non-negative powers of positive integers have been studied by many authors. See *Edwards*, [6], [7] and *Knuth* [12] for some history, and the books on *Johannes Faulhaber* by *Hawlitschek* [11] and *Schneider* [23]. We are interested in finite sums of power of arithmetic progressions (PS for power sums)

$$\boxed{PS(d,a;n,m)} := \sum_{j=0}^{m} (a\,+\,d\,j)^n \text{ with } \; n\,\in\,\mathbb{N}_0,\; m\,\in\,\mathbb{N}_0,\; d\,\in\,\mathbb{N},\; a\,\in\,N_0\,. \tag{1}$$

Note that the lower summation index for j is 0. We put $0^0 := 1$ if $a = 0$ and $n = 0$. (In Maple 13 [19] 0^0 is put to 0). It is sufficient to consider $a = 0$ if $d = 1$, and $a \in RRS(d)$ for $d \geq 2$, where $RRS(d)$ denotes the smallest positive restricted residue system modulo d, *i.e.*, $RRS(d) := \{k \in RS(d)\,|\,\gcd(k,d) = 1\}$ with $RS(d) := \{0,\,1,\,...,\,d-1\}$, the smallest non-negative residue system modulo d.

The aim of the first part of this paper is is to compute the ordinary (*o.g.f.* , symbolized by G) and exponential generating functions (*e.g.f.* , symbolized by E) for given powers n. Such functions are considered in the framework of formal power series, without considering questions of convergence. Proofs will be given in section 2.

The *o.g.f.* (indeterminate x) is

$$\boxed{GPS(d,a;n,x)} := \sum_{m=0}^{\infty} PS(d,a;n,m)\,x^m, \quad n\,\in\,\mathbb{N}_0\,. \tag{2}$$

The *e.g.f.* (indeterminate t) is

$$\boxed{EPS(d,a;n,t)} := \sum_{m=0}^{\infty} PS(d,a;n,m)\,\frac{t^m}{m!}, \quad n\,\in\,\mathbb{N}_0\,. \tag{3}$$

As is known, the *e.g.f.* is obtained from the *o.g.f.* *via* inverse *Laplace* transform as

$$E(t) \,=\, \mathcal{L}^{-1}\left[\frac{1}{p}\,G\left(\frac{1}{p}\right)\right]\,, \tag{4}$$

and *vice versa* by a direct *Laplace* transform to get G from E.

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Of course, application of the binomial theorem immediately leads, after an exchange of the two finite sums, to a formula for $PS(d,a;n,m)$ in terms of the ordinary power sums $PS(n,m) = PS(1,0;n,m)$, *viz*

$$PS(d,a;n,m) = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, d^k\, PS(k,m)\,, \tag{5}$$

and therefore, if we interchange an infinite sum with a finite sum,

$$GPS(d,a;n,x) := \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, d^k\, GPS(k,x)\,, \tag{6}$$

with $GPS(k,x) = GPS(1,0;k,x)$. Similarly,

$$EPS(d,a;n,t) := \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, d^k\, EPS(k,t)\,, \tag{7}$$

with $EPS(k,t) = EPS(1,0;k,t)$. Therefore it is in principle sufficient to compute $GPS(k,x)$ and use an inverse *Laplace* transform to find $EPS(k,t)$. It may however be difficult (or impossible) to give its explicit form.

Instead of $GPS(n,x)$ we prefer to compute the general $GPS(d,a;n,x)$ directly. In this way we find

$$\boxed{GPS(d,a;n,x)} = \sum_{k=0}^{n} S2(d,a;n,k)\, k!\, \frac{x^k}{(1-x)^{k+2}}\,, \tag{8}$$

where the generalized *Stirling* numbers of the second kind (generalized subset numbers) enter *via* the reordering process

$$(a\,\mathbf{1} + d\,\mathbf{E}_x)^n =: \sum_{m=0}^{n} S2(d,a;n,m)\, x^m\, \mathbf{d}_x^{\,m}\,, \tag{9}$$

with the *Euler* operator $\mathbf{E}_x := x\,\mathbf{d}_x$ where $\mathbf{d}_x$ is the differentiation operator, and $\mathbf{1}$ is the identity operator.

This definition leads to the three term recurrence relation

$$S2(d,a;n,m) = d\,S2(d,a;n-1,m-1) + (a + d\,m)\,S2(d,a;n-1,m), \quad \text{for } n \geq 1\,,\ m = 0,\,1,\,...,\,n\,, \tag{10}$$

with $S2(d,a;n,-1) = 0$, $S2(d,a;n,m) = 0$ for $n < m$ and $S2(d,a;0,0) = 1$.

This recurrence is obeyed by

$$S2(d,a;n,m) = \frac{1}{m!} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} (a + d\,k)^n\,. \tag{11}$$

These generalized *Stirling* numbers build a lower triangular infinite dimensional matrix, named $\mathbf{S2}[d,a]$ which turns out to be an exponential convolution array like the ordinary *Stirling* **S2** matrix, *i.e.*, a *Sheffer* matrix, denoted by

$$\mathbf{S2}[d,a] = (e^{a\,x},\, e^{d\,x} - 1)\,. \tag{12}$$

For *Sheffer* matrices see the W. Lang link in *OEIS* [21], A006232 called "Sheffer a- and z-sequences", the second part, where also references are given. (Henceforth A-numbers will be given without quoting *OEIS* each time.)

A three parameter generalization of Stirling numbers of the second kind has been proposed in *Bala* [1] as $S_{(a,b,c)}$. The present generalization is $S2(d,a;n,m) = d^m\, S_{(d,0,a)}$. There *Sheffer* arrays are called exponential *Riordan* arrays.

A one parameter generalization is given in *Luschny* [17] called *Stirling-Frobenius* subset numbers, with the scaled version called there [SF-SS] with parameter m corresponding to $\mathbf{S2}[m,m-1]$. The [SF-S] triangle family coincides with *Bala*'s $S_{(m,0,m-1)}$.

The *Sheffer* structure (exponential convolution polynomials) means that the *e.g.f.* of the sequence of column m is

$$ES2Col(d,a;t,m) = e^{a\,t}\, \frac{(e^{d\,t} - 1)^m}{m!}\,, \quad m \in \mathbb{N}_0\,. \tag{13}$$

This corresponds to the *o.g.f.*

$$GS2Col(d,a;x,m) \;=\; \frac{(d\,x)^m}{\prod_{j=0}^{m}(1\;-\;(a+d\,j)\,x)}\,,\quad m\,\in\,\mathbb{N}_0\,. \tag{14}$$

This means that the column scaled *Sheffer* triangle $\widehat{S2}[d,a] \;=\; \left(e^{a\,x},\,\frac{1}{d}\,(e^{d\,x}\;-\;1)\right)$ with elements $\widehat{S2}(d,a;n,m) \;=\; S2(d,a;n,m)\,\frac{1}{d^m}$ are

$$\widehat{S2}(d,a;n,m) \;=\; h_{n-m}^{(m+1)}[d,a], \tag{15}$$

where $h_k^{(m+1)}[d,a]$ are the complete homogeneous symmetric functions of degree k of the $m+1$ symbols $a_j \;=\; a+d\,j$, $j \;=\; 0,\,1,\,...,\,m$, and $h_0^{(m+1)} \;=\; 1$. If $[d,a] \;=\; [1,0]$ the symbol $a_0 \;=\; 0$ can be omitted, and only the m symbols $a_j \;=\; a+\;d\,j$ for $j \;=\; 1,\,2\,...,\,m$ are active. For symmetric functions see *e.g.*, [14], p. 53, and p. 54, eq (46).
The transition matrix property of the $\mathbf{S2}[1,0] \;=\mathbf{S2}$ (see [9] p. 262, eq. (6.10)) generalizes to

$$x^n \;=\; \sum_{m=0}^{n}\widehat{S2}(d,a;n,m)\,fallfac(d,a;x,m)\,,\quad n\,\in\,\mathbb{N}_0\,, \tag{16}$$

where the generalized falling factorial is (see also Bala [1] where this falling factorial appears in eq. (15) as special $[t;d,0,c]_n$ in the signed *Stirling*1 context. See the present part C for the unsigned case)

$$fallfac(d,a;x,m) \;:=\; \prod_{j=0}^{m-1}(x\;-\;(a+j\,d)\quad\text{with}\quad fallfac(d,a;x,0) \;:=\; 1\,. \tag{17}$$

This can also be written in terms of the usual falling factorials $x^{\underline{n}} \;:=\; \prod_{j=0}^{n-1}(x-j)$, for $n\,\in\,\mathbb{N}$ and $x^{\underline{0}} \;:=\; 1$ as $fallfac(d,a;x,m) \;=\; d^m\left(\frac{x-a}{d}\right)^{\underline{m}}$.
Using the binomial theorem in eq. (11) and interchanging the sums shows that $\mathbf{S2}[d,a]$ (when the matrix elements are not specified we use this notation) can be written in terms of the usual *Stirling*2 numbers $\mathbf{S2} \;=\; \mathbf{S2}[1,0]$ as,

$$S2(d,a;n,m) \;=\; \sum_{k=0}^{n}\binom{n}{k}\,a^{n-k}\,d^k\,S2(k,m)\;. \tag{18}$$

For the inverse of this relation see *Lemma 10*, eq. (160), in the proof section, part *C*.
A standard recurrence for *Sheffer* row polynomials ([22], p. 50, Corollary 3.7.2) leads, with $PS2(d,a;n,x) \;:=\; \sum_{m=0}^{n}S2(d,a;n,m)\,x^m$, to

$$PS2(d,a;n,x) \;=\; [a\;+\;d\,x\;+\;d\,\mathbf{E}_x]\,PS2(d,a;n-1,x)\,,\quad\text{for }n\,\in\,\mathbb{N}\,, \tag{19}$$

with input $PS2(d,a;0,x) \;=\; 1$.
The eq. (8) version of the *o.g.f.* is not convenient to find $EPS(d,a;n,t)$ by inverse *Laplace* transform because of the power $k\;+\;2$ instead of $k\;+\;1$. The solution is to consider first the *o.g.f.* of the powers (instead of the one of the sums of powers) which can be found analogously to the GPS case. Of course, if eq. (8) has been proved the *o.g.f.* for the first difference sequence follows immediately. This will later lead to another form of GPS which is amenable to find EPS.

$$GP(d,a;n,x) \;:=\; \sum_{m=0}^{\infty}(a\;+\;d\,m)^n\,x^m\,, \tag{20}$$

$$=\; \sum_{k=0}^{n}S2(d,a;n,k)\,k!\,\frac{x^k}{(1-x)^{k+1}}\;. \tag{21}$$

From this the *e.g.f.* can be computed directly based on $\mathcal{L}^{-1}\left[\frac{1}{(p-1)^{k+1}}\right] \;=\; \frac{t^k}{k!}\,e^t$, using the linearity of the inverse *Laplace* transform.

$$EP(d,a;n,t) \;=\; e^t\sum_{k=0}^{n}S2(d,a;n,k)\,t^k\,. \tag{22}$$

Continuing with the search for a more tractable form of $GPS(d,a;n,x)$ we apply another reordering identity on $GP(d,a;n,x)$, *viz*

$$\sum_{j=0}^{n} b_j^{(n)}\,\frac{x^j}{(1-x)^{j+1}} \;=\; \frac{1}{(1-\ x)^{n+1}}\,\sum_{i=0}^{n} a_i^{(n)}\,x^i\,,\quad n\,\in\,\mathbb{N}_0\,, \tag{23}$$

with

$$a_i^{(n)} \;=\; \sum_{j=0}^{i}(-1)^{i-j}\binom{n-j}{i-j}\,b_j^{(n)}, \tag{24}$$

$$b_j^{(n)} \;=\; \sum_{i=0}^{j}\binom{n-i}{j-i}\,a_i^{(n)}\,. \tag{25}$$

Note that this reordering identity can not be applied to $GPS(d,a;n,x)$ because of the wrong power in the denominator. But here it produces

$$GP(d,a;n,x) \;=\; \frac{1}{(1-\ x)^{n+1}}\,PrEu(d,a;n,x)\,,\ \text{with the polynomials} \tag{26}$$

$$PrEu(d,a;n,x) \;=\; \sum_{k=0}^{n} rEu(d,a;n,k)\,x^k\,,\ \text{where} \tag{27}$$

$$rEu(d,a;n,k) \;=\; \sum_{j=0}^{k}(-1)^{k-j}\binom{n-j}{k-j}\,S2(d,a;n,j)\,j!\,. \tag{28}$$

Here $rEu(d,a;n,k)$ are generalized *Euler*ian numbers, which constitute a number triangle (sometimes called *Euler* triangle) but compared with the usual *Euler*ian triangle for $[d,a]\,=\,[1,0]$, given in *Graham et al.* [9], Table 268, p. 268, or A173018, the rows are reversed. The row reversed number triangle is shown in A123125. This explains the r in front of Eu for *Euler*ian.
The inverse of the relation between $\mathbf{rEu}[d,a]$ and $\mathbf{S2}[d,a]$ is

$$S2(d.a;n,m)\,m!\;=\;\sum_{k=0}^{m}\binom{n-k}{m-k}\,rEu(d,a;n,k)\,. \tag{29}$$

From the explicit form of $\mathbf{S2}[d,a]$ in eq.(11) the one for $\mathbf{rEu}[d,a]$ follows by eq. (23).

$$rEu(d,a;n,k)\;=\;\sum_{j=0}^{k}(-1)^{k-j}\binom{n+1}{k-j}\,(a+d\,j)^n\,. \tag{30}$$

In terms of the usual *Eulerian* numbers one finds from eqs. (28) with (18) and eq. (29) for $[d,a]\,=\,[1,0]$

$$rEu(d,a;n,k)\;=\;\sum_{m=0}^{n}\binom{n}{m}\,a^{n-m}\,d^m\,\sum_{p=0}^{k}(-1)^{k-p}\binom{n-m}{k-p}\,rEu(m,p)\,. \tag{31}$$

Note that no formula analogous to eq. (18) holds due to the different binomial structure in $rEu[d,a]$ of eq. (28).
The three term recurrence for $\mathbf{rEu}[d,a]$ is

$$rEu(d,a;n,m)\;=\;(d\,(n-m)\;+\;(d-a))\,rEu(d,a;n-1,m-1)\;+\;(a\;+\;d\,m)\,rEu(d,a;n-1,m), \tag{32}$$

for $n\,\geq 1\,,\ m\,=\,0,\,1,\,...,\,n$, with $rEu(d,a;n,-1)\,=\,0$, $rEu(d,a;n,m)\,=\,0$ for $n\,<\,m$ and $rEu(d,a;0,0)\,=\,1$. The corresponding (ordinary, not exponential) row polynomials are

$$PrEu(d,a;n,x)\;:=\;\sum_{m=0}^{n} rEu(d,a;n,m)\,x^m\,,\quad n\,\in\,\mathbb{N}_0\,. \tag{33}$$

From eq. (21) and eqs. (26) with (27) follows a relation between these $\mathbf{rEu}[d,a]$ row polynomials and those of the number triangle with entries $S2fac(d,a;n,m)\;:=\;S2(d,a;n,m)\,m!$, named $PS2fac(d,a;n,x)$, *viz*

$$PrEu(d,a;n,x)\;=\;(1-\ x)^n\,PS2fac\left(d,a;n,\frac{x}{1-x}\right)\,. \tag{34}$$

It may be noted, in passing, that the transformation $y = \frac{x}{1 + x}$, or $x = \frac{y}{1 - x}$ is called *Euler*'s transformation (see, *e.g.*, [10], p. 191, last row).
From this preceding relation the *e.g.f.* (*i.e.*, the e.g.f. for the row reversed *Euler*ian triangle) follows:

$$\boxed{EPrEu(d, a; t, x)} = \frac{(1 - x)\, e^{a\,(1-x)\,t}}{1 - x\, e^{d\,(1-x)\,t}} . \tag{35}$$

This is not a *Sheffer* structure, not even one of the more general *Brenke* type $g(z)\,B(x\,z)$, [2], [4], p. 167.

The *e.g.f.* of the row sums ($x \to 1$) of $\mathbf{rEu}[d, a]$ is obtained *via l'Hôpital*'s rule as $\frac{1}{1 - d\,t}$, independently of a.

A one parameter k-family of generalized Eulerian polynomials $A_{n,k}(x)$ with coefficient triangles has been considered by *Luschny* [18]. The coefficients of $A_{n,k}(x)$ build $\mathbf{Eu}[k, 1] = \mathbf{rEu}[k, k-1]$.

Now the new form of $GPS(d, a; n, x)$ is simply obtained by multiplying $GPS(d, a; n, x)$ with $\frac{1}{1 - x}$ because this is the rule to obtain the *o.g.f.* for partial sums of a sequence from the *o.g.f.* of the sequence.

$$GPS(d, a; n, x) = \frac{1}{(1 - x)^{n+2}}\, PrEu(d, a; n, x) . \tag{36}$$

There is still this power $n+2$ but now we can use the reordering identity, eq. (23) with n replaced by $n+1$:

$$\sum_{j=0}^{n+1} b_j^{(n+1)} \frac{x^j}{(1 - x)^{j+1}} = \frac{1}{(1 - x)^{(n+1)+1}} \sum_{i=0}^{n+1} a_i^{(n+1)}\, x^i , \tag{37}$$

with (see eq. (30))

$$a_i^{(n+1)} = rEu(d, a; n, i) = \sum_{p=0}^{i} (-1)^{i-p} \binom{n+1}{i-p} (a + d\,p)^n . \tag{38}$$

Note that $a_{n+1}^{(n+1)} = 0$ because $PrEu(d, a; n, x)$ has degree n.

Note that now the x dependence is amenable for a later inverse *Laplace* transform. The calculation of $b_j^{(n+1)}$ is a bit lengthy but it turns out to have a nice form (we add the $[d, a]$ parameters).

$$b_j^{(n+1)}(d, a) = S2(d, a; n, j)\, j! + S2(d, a; n, j-1)\,(j-1)! =: \Sigma S2(d, a; n, j) , \tag{39}$$

leading finally to the result for the *e.g.f.*

$$\boxed{EPS(d, a; n, t)} = e^t\, \Sigma S2(d, a; n, j) \frac{t^j}{j!} , n \in \mathbb{N}_0 . \tag{40}$$

Let us recapitulate the detour we made in a diagram referring to eq. (23) for obtaining two versions of GP or GPS:

$$\begin{array}{ccccc} \mathbf{GPSv1} & \overset{(23)}{\nrightarrow} & \mathbf{GPSv2} & \overset{\mathcal{L}^{-1}}{\rightarrow} & \mathbf{EPS} \\ \downarrow & & \uparrow & & \\ \mathbf{GPv1} & \overset{(23)}{\rightarrow} & \mathbf{GPv2} . & & \end{array} \tag{41}$$

B) Generalized Faulhaber formula and Bernoulli polynomials
The next topic is to find for the power sum $PS(d, a; n, m)$ a formula in terms of *Bernoulli* polynomials evaluated appropriately. This formula has bee named *Faulhaber* formula for the ordinary $[d, a] = [1, 0]$ case by *Conway* and *Guy* [5], p. 106. *Faulhaber* used the numbers, later called *Bernoulli* numbers by *de Moivre* and *Euler* (see [6], [7]), already by 1631 before *Jakob I Bernoulli*. For this formula see [9], p. 367. eq. (7.79), and [13], p. 167 eq. (1). Here it is, with our definition of $PS(n, m) = PS(1, 0; n, m)$,

$$PS(n, m) = \delta_{n,0} + \frac{1}{n+1}\,(B(n+1, x = m+1) - B(n+1, x = 1)) , \quad n \in \mathbb{N}_0 , \tag{42}$$

where $\delta_{n,0} = [n = 0]$ is the *Kronecker* symbol: 1 if $n = 0$ and 0 otherwise. The *Bernoulli* numbers are defined recursively by (see [9], p. 284, eq. (6.79))

$$B(n) := \frac{1}{n+1}\left(\delta_{n,0} - \sum_{k=0}^{n-1}\binom{n+1}{k}B(k)\right) \text{ for } n \in \mathbb{N}, \text{ with } B(0) = 1 . \tag{43}$$

They have $B(1) = -\frac{1}{2}$ and are found in OEIS [21] under A027641 / A027642. The corresponding *Bernoulli* polynomials are

$$B(n,\, x) := \sum_{m=0}^{m}\binom{n}{m}B(n-m)\, x^m . \tag{44}$$

Their coefficient tables are given in A196838 / A196839 or A053382 / A053383 for rising or falling powers of x, respectively.

For the generalized case one finds for the power sums $PS(d,a;n,m)$ from eqs. (5) and (42) the following *Faulhaber* formula in terms of ordinary *Bernoulli* polynomials

$$\boxed{PS(d,a;n,m)} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}\, d^k\left[\delta_{k,0} + \frac{1}{k+1}\left(B(k+1,\, x=m+1) - B(k+1,\, x=1)\right)\right] . \tag{45}$$

But the idea is to find the analogon of formula (42) with generalized *Bernoulli* polynomials.

An obvious generalization of the *Bernoulli* numbers is

$$B(d,a;n) := \sum_{m=0}^{n}(-1)^m\frac{1}{m+1}\, S2(d,a;n,m)\, m! ,\; n \in \mathbb{N}_0 . \tag{46}$$

For $[d,a] = [1,0]$ see, *e.g.*, *Charalambides* [3], or the formula and Maple section of A027641.

From eq.(18) one finds $B[d,a]$ in terms of B.

$$B(d,a;n) := \sum_{m=0}^{n}\binom{n}{m}a^{n-m}\, d^m\, B(m) . \tag{47}$$

The *e.g.f.* of $\{B(d,a;n)\}_{n=0}^{\infty}$ is

$$EB(d,a;t) = \frac{d\,t\, e^{a\,t}}{e^{d\,t} - 1} . \tag{48}$$

The corresponding generalized *Bernoulli* polynomials are (compare with eq. (44))

$$B(d,a;n,x) := \sum_{m=0}^{n}\binom{n}{m}B(d,a;n-m)\, x^m , \tag{49}$$

and from eq. (47) they can also be written in terms of $\{B(m)\}_{m=0}^{n}$ as

$$B(d,a;n,x) := \sum_{m=0}^{n}\binom{n}{m}d^m\, B(m)\,(a + x)^{n-m} . \tag{50}$$

Their *e.g.f.* is, either from eq. (49) or (50),

$$EB(d,a;t,x) = \frac{d\,t\, e^{a\,t}}{e^{d\,t} - 1}\, e^{x\,t} , \tag{51}$$

identifying their coefficients as *Sheffer* arrays $\left(\frac{d\,z\, e^{a\,z}}{e^{d\,z} - 1},\, z\right)$. Such arrays are of the so called *Appell* type (compare *Roman* [22], pp. 26 - 28, with a different notation). It turns out that in order to obtain a generalized *Faulhaber* formula in terms of *Bernoulli* polynomials the $B[d,a]$ just introduced are not quite the ones needed. In fact, they are too general. One has to work with the polynomials depending only on d, *viz*

$$\boxed{B(d;n,x)} = \sum_{m=0}^{n}\binom{n}{m}B(d;n-m)\, x^m , \tag{52}$$

with the generalized *Bernoulli* numbers

$$B(d;n) \;:=\; B(d,a=0;n) = d^n\,B(n)\,,\;\; n\,\in\,\mathbb{N}_0\,. \tag{53}$$

They can also be obtained by exponential convolution of the more general ones with the sequence $\{-a^n\}_{n=1}^{\infty}$.

$$B(d;n) \;=\; \sum_{m=0}^{n}\binom{n}{m}\,B(d,a;n-m)\,(-a)^m\,. \tag{54}$$

For $a\,=\,0$ only $m\,=\,0$ survives and B(d,0;n) results. But also for non-vanishing a the a dependence drops out, as can be seen from the *e.g.f.* of the sequence on the *r.h.s.*, using eq. (48).

$$\frac{d\,t\,e^{a\,t}}{e^{d\,t}\,-\,1}\,e^{-a\,t} \;=\; \frac{d\,t}{e^{d\,t}\,-\,1} \;=\; EB(d,a=0;t) \;=:\; EB(d;t)\,. \tag{55}$$

For $B(d;n)$ with $d\,=\,2$, 3 and 4 see $(-1)^n$A239275(n)/A141459(n), A285863(n)/A285068(n) and A288873(n)/A141459(n).

The *e.g.f.* of the polynomial system $\{B(d;n,x)\}_{n=0}^{\infty}$ of eq. (52) is

$$EB(d;t,x) \;=\; \frac{d\,t}{e^{d\,t}\,-\,1}\,e^{x\,t} \;=\; EB(d,a=0;t,x)\,. \tag{56}$$

The *Appell* type *Sheffer* structure is obvious.

Now the stage is set for giving the result for the generalized *Faulhaber* formula in terms of the polynomials $B(d,n,x)$.

$$\boxed{PS(d,a;n,m)} \;=\; \frac{1}{d\,(n+1)}\,[B(d;n+1,x\,=\,a\,+\,d\,(m+1))\,-\,B(d;n+1,x\,=\,d)$$
$$-\;B(d;n+1,x\,=\,a)\,+\,B(d,n+1,x=0)\,+\,d\,\delta_{n,0}]\,. \tag{57}$$

Here $B(d,n+1,x=0) = B(d;n+1) \;=\; d^{n+1}\,B(n+1)$, and the *Kronecker* symbol enters because of our definition of $PS(d,a;n,m)$ where the sum starts with $j=0$, not with 1.

The generalized *Lah* numbers $\mathbf{L}[d,a]$ are discussed in the proof section 2, C) 4.

C) Generalized Stirling1 numbers

As elements of the *Sheffer* group the inverse of the (infinite, lower triangular) matrix $\mathbf{S2[d,a]}$ exists and is called $\mathbf{S1[d,a]}$. This is therefore a generalized *Stirling* number triangle of the first kind.

$$\mathbf{S2}[d,a]\cdot\mathbf{S1}[d,a] \;=\; \mathbf{1} \;=\; \mathbf{S1}[d,a]\cdot\mathbf{S2}[d,a]\,, \tag{58}$$

with the (infinite dimensional) identity matrix $\mathbf{1}$. For practical purposes it is sufficient to consider the finite dimensional case of $N\,\times\,N$ matrices. $\mathbf{S1}[d,a]$ is a signed matrix with fractional entries for $d\,\neq\,1$. Therefore, in order to have non-negative entries one considers $\mathbf{S1p}[d,a]$ with entries $S1p(d,a;n,m) \;:=\; (-1)^{n-m}\,S1(d,a;n,m)$. But in the combinatorial context also a scaling is needed to obtain a non-negative integer matrix $\widehat{\mathbf{S1p}}[d,a]$ with diagonal entries 1 (*i.e.*, monic row polynomials). This is done by scaling the $\mathbf{S1p}[d,a]$ rows n with d^n.

We then have the *Sheffer* structures

$$\mathbf{S1}[d,a] \;=\; \left(\frac{1}{(1\,+\,x)^{\frac{a}{d}}},\,\frac{1}{d}\,log(1\,+\,x)\right)\,, \qquad \text{and} \qquad \boxed{\widehat{\mathbf{S1p}}[d,a]} \;=\; \left(\frac{1}{(1\,-\,\mathrm{d\,x})^{\frac{\mathrm{a}}{\mathrm{d}}}},\,-\frac{1}{\mathrm{d}}\log(1\,-\,\mathrm{d\,x})\right)\,. \tag{59}$$

The $\widehat{\mathbf{S2}}[d,a]$ matrices (see eq. (15)) which have scaled matrix elements $\widehat{S2}(d,a;n,m) \;=\; S2(d,a;n,m)/d^m$ have the signed inverse matrices $\widehat{\mathbf{S1}}[d,a] \;=\; \left((1\,+\,d\,x)^{-\frac{a}{d}},\,\frac{1}{d}\,\log(1\,+\,d\,x)\right)$.

The signed $\widehat{\mathbf{S1}}[d,a]$ matrices have been considered by *Bala* [1] as $s_{(d,0,a)}$. In *Luschny* [17] the $SF-C$ matrices are our $\widehat{\mathbf{S1p}}[d,d-1]$, and the $SF-CS$ matrices are the unsigned inverse matrices of $\widehat{\mathbf{S2}}[d,d-1]$.

The *Sheffer* structure of $\widehat{\mathbf{S1p}}[d,a]$ means that the *e.g.f.* of column m is

$$\boxed{E\widehat{S1p}Col(d,a;t,m)} \;=\; \frac{1}{(1\,-\,d\,t)^{\frac{a}{d}}}\,\frac{1}{m!}\,\left(-\frac{1}{d}\,log(1\,-\,d\,t)\right)^m\,,\;\; m\,\in\,\mathbb{N}_0\,. \tag{60}$$

There seems not to exist a simple form for the corresponding *o.g.f.* .

The three term recurrence for the $\widehat{\mathbf{S1p}}[d,a]$ matrix entries is

$$\widehat{S1p}(d,a;n,m) \,=\, \widehat{S1p}(d,a;n-1,m-1) \,+\, (d\,n \,-\, (d-a))\,\widehat{S1p}(d,a;n-1,m), \quad \text{for } n \,\geq 1\,,\ m \,=\, 0,\,1,\,...,\,n\,, \tag{61}$$

with $\widehat{S1p}(d,a;n,-1) \,=\, 0$, $\widehat{S1p}(d,a;n,m) \,=\, 0$ for $n \,<\, m$ and $\widehat{S1p}(d,a;0,0) \,=\, 1$.

The usual transition from the monomial basis $\{x^n\}_{n=0}^{\infty}$ to the rising factorials (see [9], p. 263, eq. (6.11)) generalizes to the following identification of the row polynomials of $\widehat{\mathbf{S1p}}[da]$

$$\boxed{P\widehat{S1p}(d,a;n,x)} \,:=\, \sum_{m=0}^{n} \widehat{S1p}(d,a;n,m)\,x^m \,=\, risefac(d,a;x,n)\,, \tag{62}$$

with the generalized rising factorials (compare this with the generalized falling factorials eq. (17))

$$risefac(d,a;x,n) \,:=\, \prod_{j=0}^{n-1}(x \,+\, (a+j\,d)) \quad \text{with} \quad risefac(d,a;x,0) \,:=\, 1. \tag{63}$$

This can be rewritten also in terms of the usual rising factorial $x^{\overline{n}} \,:=\, \prod_{j=0}^{n}(x+j)$ for $n \,\in\, \mathbb{N}$ and $x^{\overline{0}} \,:=\, 1$ as $risefac(d,a;x,n) \,=\, d^n\left(\frac{x+a}{d}\right)^{\overline{n}}$. In terms of falling factorials this is $risefac(d,a;x,n) \,=\, (-d)^n\left(\frac{-(x+a)}{d}\right)^{\underline{n}}$.

This identification implies *via Vieta*'s theorem that the coefficients of the monic polynomial $P\widehat{S1p}(d,a;n,x)$ are the elementary symmetric functions $\sigma_{n-m}^{(n)}(a_0,a_1,...,a_{n-1})$ in the indeterminates $\{a_j\}_{j=0}^{n-1}$ given by $a_j \,:=\, a \,+\, j\,d$, with $\sigma_0^n \,:=\, 1$. Sometimes $\sigma_{n-m}^{(n)}[d,a]$ is used for these symmetric functions. Thus

$$\widehat{S1p}(d,a;n,m) \,=\, \sigma_{n-m}^{(n)}(a_0,a_1,...,a_{n-1}), \text{ with } a_j \,=\, a \,+\, j\,d\,. \tag{64}$$

If $d \,=\, 1$ (and $a \,=\, 0$) $a_0 \,=\, 0$ does not contribute and one can write $\widehat{S1p}(1,0;n,m) \,=\, S1p(n,\,m) \,=\, \sigma_{n-m}^{(n-1)}(1,\,2,\,...,\,n-1)$.

Sorting in falling powers of a one obtains the formula for $\widehat{S1p}(d,a;n,m)$ in terms of the usual unsigned *Stirling1* numbers $S1p(n,\,m)$ =A132393$(n,\,m)$.

$$\widehat{S1p}(d,a;n,m) \,=\, \sum_{j=0}^{n-m}\binom{n-j}{m}S1p(n,\,n-j)\,a^{n-m-j}\,d^j \,=\, \sum_{j=m}^{n}\binom{j}{m}S1p(n,\,j)\,a^{j-m}\,d^{n-j}\,. \tag{65}$$

This satisfies the recurrence relation eq. (61).

The standard *Sheffer* recurrence ([22], p. 50, Corollary 3.7.2) for these row polynomials boils down to

$$P\widehat{S1p}(d,a;n,x) \,=\, (x+a)\,P\widehat{S1p}(d,a;n \,-\, 1,x \,+\, d)\,, \quad n \,\in\, \mathbb{N}\,, \tag{66}$$

with input $P\widehat{S1p}(d,a;0,x) \,=\, 1$.

From the *Sheffer* property the *e.g.f.* of these row polynomials, *i.e.*, the *e.g.f.* of the number triangle $\widehat{\mathbf{S1p}}[d,a]$, is

$$\boxed{EP\widehat{S1p}(d,a;t,x)} \,=\, \frac{1}{(1 \,-\, d\,t)^{\frac{a}{d}}}\,exp\left(-x\,\frac{1}{d}\,log(1 \,-\, d\,t)\right) \,=\, \frac{1}{(1 \,-\, d\,t)^{\frac{a+x}{d}}}\,. \tag{67}$$

For the *Meixer* type recurrence see the proof section 2, C), 7.

A more involved problem is to find the generalization of the formula giving $\widehat{S1p}(d,a;n,m)$ in terms of the column scaled $\widehat{S2}(d,a;n,m)$ elements. The standard *Schlömilch* formula is (see, e.g., [3], p. 290, eq. (8.20) for the signed $S1$ entries)

$$S1p(n,\,m) \,=\, (-1)^{n-m}\sum_{k=0}^{n-m}(-1)^k\binom{n+k-1}{m-1}\binom{2\,n-m}{n-m-k}S2(n-m+k,\,k)\,. \tag{68}$$

The direct proof starts with eq. (65). Inserting the *Schlömilch* formula just given, then using the inverse of eq. (18) leads to

$$\boxed{\widehat{S1p}(d,a;n,m)} \;=\; a^{n-m}\sum_{j=m}^{n}\binom{j}{m}\sum_{k=0}^{n-j}\binom{n-k-1}{j-1}\binom{2\,n-j}{n-j-k}\,a^{k}\,* \\ *\sum_{l=0}^{n-j+k}(-1)^{l}\binom{n-j+k}{l}\,a^{-l}\,\widehat{S2}(d,a;l,k)\,,\ \text{for } n\ \geq m\ \geq 0\,. \qquad (69)$$

Note that this result also holds for $a = 0$ because then $a^{n-m+k-l}$ becomes $\delta_{0,n-m+k-l}$ (from $0^0 = 1$) leading to a collapse of the $l-$sum, and the remaining two sums produce $\widehat{S1p}(d,0;n,m) = d^{n-m}\,S1p(n,\,m)$.

Also the known result for $m = 0$ from eq. (62) is recovered, *viz* $\widehat{S1p}(d,a;n,0) = risefac(d,a;0,n) = d^{n}\,\left(\frac{a}{d}\right)^{\overline{n}}$.

For another formula, following from a proof along the lines of the ordinary formula in [3], p. 290, see the proof section *C*, eq. (175).

The inverse of the generalized *Lah* matrix $\mathbf{L}^{-1}[d,a]$ is discussed in the proof section 2, C) 4.

D) Combinatorial Interpretation

I) $\widehat{\mathbf{S2}}[d,a]$

The *o.g.f.* , eq. (14), divided by d^m, which generates the complete homogeneous symmetric functions $h^{(m+1)}_{n-m}[d,a]$ of degree $n-m$ of the $m+1$ symbols $a_j = a + d\,j$, $j = 0,\,1,\,...,\,m$, leads immediately to the following combinatorial interpretation of $\widehat{S2}(d,a;n,m) := \dfrac{1}{d^m}\,S2(d,a;n,m)$ (see eq. (15)).

$\widehat{S2}(d,a;n,m)$ is for $d \geq 2$ the (dimensionless) total volume of the $multichoose(m+1,\,n-m) = \binom{n}{m}$ hyper-cubes and hyper-cuboids (polytopes) of dimension $n-m$ which are build from the $n-m$ orthogonal $\mathbb{Z}^{n-m}$ vectors of lengths taken from the repertoire $a_j = a + d\,j$, $j = 0,\,1,\,...,\,m$.

For $d = 1$ (and $a = 0$), the standard *Stirling2* case $\mathbf{S2} = \mathbf{S2}[1,0] = \widehat{\mathbf{S2}}[1,0]$, $a_0 = 0$ does not contribute and the $n-m$ vectors are from the set $\{1,\,2,\,...,\,m\}$ for the $multichoose(m,\,n-m) = \binom{n-1}{m-1}$ polytopes.

Some examples:

a) $\widehat{S}2(1,0;3,2) = S2(3,2) = 3$ from the $\binom{2}{1} = 2$ polytopes of dimension 1 with basis lengths 1, 2, *i.e.*, two lines of length 1 and 2 with total length 3.

b) $\widehat{S}2(2,1;3,2) = 9$ (see A039755) from the $\binom{3}{2} = 3$ polytopes of dimension 1 with basis lengths 1, 3, 5, *i.e.*, three lines of total length 9.

c) $\widehat{S}2(3,2;3,1) = 39$ (see A225468) from the $\binom{3}{1} = 3$ polytopes of dimension 2 with basis lengths from the set $\{2,\,5\}$, *i.e.*, two squares of area 2^2 and 5^2 and a rectangle of area $2^1\,5^1$, giving total area $4 + 25 + 10 = 39$.

II) $\widehat{\mathbf{S1p}}[d,a]$

From eq. (62) for the row polynomials and the implied elementary symmetric function formula for $\widehat{S1p}(d,a;n,m)$ of eq. (64) one has the combinatorial interpretation.

$\widehat{S1p}(d,a;n,m)$ is for $d \geq 2$ the total volume of the $\binom{n}{n-m}$ hyper-cuboids of dimension $n-m$ formed from the $n-m$ orthogonal $\mathbb{Z}^{n-m}$ vectors with distinct (dimensionless) lengths from the n-set $\{a + d\,j\,|\,j = 0,\,1,\,...,\,n-1\}$. For $[d,a] = [1,0]$ the ordinary unsigned *Stirling1* number $S1p(n,\,m)$ (see A132393) gives the total volume of the $\binom{n-1}{n-m}$ hyper-cuboids of dimension $n-m$ formed from the $n-m$ orthogonal $\mathbb{Z}^{n-m}$ vectors with distinct (dimensionless) lengths from the $(n-1)$-set $\{1,\,2,\,...\,n-1\}$.

Some examples:

a) $\widehat{S1p}(1,0;4,2) = S1p(4,\,2) = 11$ (see A132393) from the $\binom{3}{2} = 3$ hyper-cuboids of dimension 2 with distinct basis vector lengths from the set $\{1,\,2,\,3\}$, *i.e.*, six rectangles of area $1\cdot 2$, $1\cdot 3$ and $2\cdot 3$, with total area $2 + 3 + 6 = 11$.

b) $\widehat{S1p}(2, 1; 4, 1) = 176$ (see A028338) from the $\binom{4}{3} = 4$ hyper-cuboids of dimension 3 with distinct basis vector lengths from the set $\{1, 3, 5, 7\}$, *i.e.*, four cuboids with volumes $1 \cdot 3 \cdot 5$, $1 \cdot 3 \cdot 7$, $1 \cdot 5 \cdot 7$ and $3 \cdot 5 \cdot 7$, adding to 176.

c) $\widehat{S1p}(3, 1; 4, 0) = 280$ (see A286718) from the $\binom{4}{4} = 1$ hyper-cuboid of dimension 4 with distinct basis vector lengths from the set $\{1, 4, 7, 10\}$, *i.e.*, the $4D$ hyper-cuboid with volume $1 \cdot 4 \cdot 7 \cdot 10 = 280$.

Two remarks: The first column sequences $\{\widehat{S1p}(d, 1; n, 0)\}_{n=1}^{\infty}$ have also an interpretation as numbers of $(d+1)$-ary rooted increasing trees with n vertices, including the root vertex. This is the sequence $\{S(k = d + 1; n, 1)\}$ of generalized *Stirling2* numbers with parameter k in the notation of [15], eq. (5). The reason is the *e.g.f.* called there $g2(k = d + 1; x) = -1 + (1 + (1 - (d + 1))\, x)^{\frac{1}{1-(d+1)}} = -1 + (1 - d\,x)^{-\frac{1}{d}}$ which is the *e.g.f.* of the $m = 0$ column of $\widehat{\mathbf{S1p}}[d, 1]$, viz $E\widehat{S1p}Col(d, 1; x, 0) = (1 - d)^{-\frac{1}{d}}$ (see eq. (60)) but with the $n = 0$ entry removed. See the instances A001147, A007559, A007696, A008548, ... for $d = 2, 3, 4, 5 ...$, respectively. They are, for $n >= 1$, the number of 3, 4, 5, 6, ...-ary increasing rooted trees.
Similarly, the first column sequences $\{\widehat{S1p}(d, d-1; n, 0)\}_{n=1}^{\infty}$ are related to another variety of increasing trees given by the *e.g.f.* $g2p(k = d - 1; x) = 1 - (1 - d\,x)^{\frac{1}{d}}$ for the sequence $\{|S(-k = 1 - d; n, 1)|\}_{n=0}^{\infty}$ of [15], eq. (6). This is related to the *e.g.f.* of the sequence $\{\widehat{S1p}(d, d-1, n, 0)\}_{n=0}^{\infty}$, *i.e.*, $E\widehat{S1p}Col(d, d-1; x, 0) = (1 - d\,x)^{-\frac{d-1}{d}}$ by integrating and adding 1: $\int dx\, g\widehat{S1p}(d, d-1; x) + 1 = g2p(d - 1; x)$.
Some instances are: A001147, A008544, A008545, A008546, ... for $d = 2, 3, 4, 5 ...$, respectively.

The combinatorial interpretation of $\mathbf{rEu}[da]$ should also be consodered.

2 Proofs

In the following proofs the setting of formal power series is used. No convergence issues are considered. Infinite sums are interchanged (one could use alternatively a large cutoff). Differentiation as well as integration will also be interchanged with infinite sums. Only statements which are not already obvious from the main text are proved here. Note that for binomials the definition of [9], p. 154, eq. (5.1) is taken. This is not the definition used by *Maple13* [19]. Also $0^0 := 1$. The symmetry of binomial coefficients is used *ad libitum* (but the upper number in the binomial has to be a non-negative integer).

A) Proofs of section 1 A

1. Proof of eqs. (8) to (11)

Lemma 1: With the notation of eq. (9):

$$(a\,\mathbf{1} + d\,\mathbf{E}_x)^n\, x^j = (a + d\,j)^n\, x^j\,. \tag{70}$$

Proof: Trivial, by induction over $n \in \mathbb{N}_0$ with $j \in \mathbb{N}_0$.
For the *o.g.f.* $GPS(d, a; n, x)$ from eq. (2) with eq. (1) one has, after an exchange of the two sums, inserting $x^j\, x^{-j}$ and application of *Lemma 1*:

$$\begin{aligned} GPS(d, a; n, x) &= \sum_{m=0}^{\infty} x^m \sum_{j=0}^{m} (a + d\,j)^n = \sum_{j=0}^{\infty} (a + d\,j)^n \sum_{m=j}^{\infty} x^m \\ &= \sum_{j=0}^{\infty} \left((a\,\mathbf{1} + d\,\mathbf{E}_x)^n\, x^j\right) \sum_{m=j}^{\infty} x^{m-j} = \sum_{j=0}^{\infty} \left((a\,\mathbf{1} + d\,\mathbf{E}_x)^n\, x^j\right) \frac{1}{1 - x}\,. \end{aligned} \tag{71}$$

After summing over j, the reordering of differentials from eq. (9), *i.e.*, the definition of $S2(d, a; n, m)$, is used:

$$\begin{aligned} &= (a\,\mathbf{1} + d\,\mathbf{E}_x)^n\, \frac{1}{(1 - x)^2} = \sum_{k=0}^{n} S2(d, a; n, k)\, x^k\, \mathbf{d}_x^k\, \frac{1}{(1 - x)^2} \\ &= \sum_{k=0}^{n} S2(d, a; n, k)\, k!\, \frac{x^k}{(1 - x)^{2+k}}\,. \end{aligned} \tag{72}$$

The three term recurrence eq. (10) of the number triangle $\{S2(d,a;n,k)\}$ follows from the definition eq. (9):

$$\sum_{m=0}^{n+1} S2(d,a;n+1,m)\,x^m\,\mathbf{d}_x^m \;=\; (a\,\mathbf{1}\,+\,d\,\mathbf{E}_x)\,(a\,\mathbf{1}\,+\,d\,\mathbf{E}_x)^n \;=\; (a\,\mathbf{1}\,+\,d\,\mathbf{E}_x)\sum_{m=0}^{n} S2(d,a;n,m)x^m\,\mathbf{d}_x^m$$

$$= \sum_{m=0}^{n} S2(d,a;n,m)\,a\,x^m\,\mathbf{d}_x^m + \sum_{m=0}^{n} S2(d,a;n,m)\,d\,\left(m\,x^m\,\mathbf{d}_x^m\,+\,x^{m+1}\,\mathbf{d}_x^{m+1}\right)$$

$$= \sum_{m=0}^{n+1} S2(d,a;n,m)\,(a\,+\,d\,m)\,x^m\,\mathbf{d}_x^m + \sum_{m=1}^{n+1} S2(d,a;n,m-1)\,d\,x^m\,\mathbf{d}_x^m\;. \quad (73)$$

In the second to last sum the $m = n+1$ term has been added due to the triangle condition $S2(d,a;n,m) = 0$ if $n < m$. In the last sum the lower index $m = 0$ can be added because of the input condition $S2(d,a;n,-1) = 0$. Comparing powers of $x^m\,\mathbf{d}_x^m$ then leads to the recurrence eq. (10) after the change $n \to n-1$.

The explicit form of $S(d,a;n,m)$ from eq. (11) satisfies the recurrence eq. (10) together with the inputs because

$$d\,S2(d,a;n-1,m-1) \,+\, (a\,+\,d\,m)\,S2(d,a;n-1,m)$$

$$= \sum_{k=0}^{m-1} \frac{d}{(m-1)!}\,(-1)^{m-k}\,(-1)\binom{m-1}{k}(a\,+\,d\,k)^{n-1} + \sum_{k=0}^{m}\frac{1}{m!}\,(-1)^{m-k}\binom{m}{k}(a\,+\,d\,m)\,(a\,+\,d\,k)^{n-1}$$

$$= \frac{1}{m!}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\left[-d\,m\frac{\binom{m-1}{k}}{\binom{m}{k}} \,+\, (a\,+\,d\,m)\right](a\,+\,d\,k)^{n-1}\,. \quad (74)$$

I the first term of the last sum the term $k = m$ does not contribute because of the binomial. Now the term within the bracket becomes

$$[...] = -d\,\frac{m}{m}\,(m\,-\,k) \,+\, (a\,+\,d\,m) \,=\, a\,+\,d\,k\,, \quad (75)$$

leading to the *l.h.s.* of the recurrence, viz $S2(d,a;n,m)$ of eq. (11)
For the instances of $S2[d,a]$ for $[d,a] = [1,0], [2,1], [3,1], [3,2], [4,1],]4,3]$ see A048993, A154537, A282629, A225466, A285061, A225467, respectively.

We give also the recurrence for $S2fac(d,a;n,m) := S2(d,a;n,m)\,m!$:

$$S2fac(d,a;n,m) = m\,d\,S2fac(d,a;n-1,m-1) + (a\,+\,d\,m)\,S2fac(d,a;n-1,m), \;\text{ for } n \geq 1\,,\; m = 0,\,1,\,...,\,n\,, \quad (76)$$

with $S2fac(d,a;n,-1) = 0$, $S2fac(d,a;n,m) = 0$ for $n < m$ and $S2fac(d,a;0,0) = 1$.

The *e.g.f.* of the row polynomials of $\mathbf{S2fac}[d,a]$ is $\frac{e^{a\,t}}{1\,-\,(e^{d\,t}-\,1)}$. This is seen after interchanging the two sums and using the *e.g.f.* of columns of $\mathbf{S2}[d,a]$.
For the instances $[d,a] = [1,0], [2,1], [3,1], [3,2], [4,1],]4,3]$ see A131689, A145901, A284861, A225472, A285066, A225473, respectively.

2. Proof of eq. (13), i.e., eq. (12)

The *Sheffer* structure eq. (12) means that the column *e.g.f.* of the $\mathbf{S2}[d,a]$ number triangle satisfies eq. (13). The column *e.g.f.* $ES2Col(d,a;t,m)$ is here named $E(t,m)$ for simplicity. The lower summation index in brackets can be used instead of the given one because of the triangle structure of $\mathbf{S2}[d,a]$. The recurrence is used in the first step.

$$E(t,m) \;:=\; \sum_{n=m(0)}^{\infty} S2(d,a;n,m)\frac{t^n}{n!}$$

$$= d\sum_{n=0(1)}^{\infty} S2(d,a;n-1,m-1)\frac{t^n}{n!} + (a\,+\,d\,m)\sum_{n=0(1)}^{\infty} S2(d,a;n-1,m)\,\frac{t^n}{n!}$$

$$= \int dt\,\left(d\sum_{n=1}^{\infty} S2(d,a;n-1,m-1)\frac{x^{n-1}}{(n-1)!} + (a\,+\,d\,m)\sum_{n=1}^{\infty} S2(d,a;n-1,m)\,\frac{t^{n-1}}{(n-1)!}\right)$$

$$= \int dt\,\left(d\,E(t,m-1) \,+\, (a\,+\,d\,m)\,E(t,m)\right)\;. \quad (77)$$

Differentiating both sides of the final equation produces a recurrence for $E(t,m)$:

$$\left(\frac{d}{dt} - (a + d\,m)\right) E(t,m) = d\,E(t,m-1)\,, \tag{78}$$

with the input $E(t,0) = e^{a\,t}$ because $S2(d,a;n,0) = a^n$ from the recurrence.
The solution of this differential difference equation, satisfying the input, is

$$E(t,m) \equiv ES2Col(d,a;t,m) = e^{a\,t}\,\frac{\left(e^{d\,t} - 1\right)^m}{m!}\,, \tag{79}$$

which is eq. (13).

3. Proof of eq. (14)

The *o.g.f.* of eq. (14) $GS2Col(d,a;x,m) \equiv G(x,m)$ for short in this proof, is shown to lead to the *e.g.f.* $ES2Col(d,a;t,m) \equiv E(t,m)$ of eq. (13) *via* the inverse *Laplace* transform, given in eq. (4).

Lemma 2:

$$\prod_{j=0}^{m} \frac{1}{x - (a + d\,j)} = \left(\frac{-1}{d}\right)^m \frac{1}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{1}{x - (a + d\,j)}\,, \quad \text{for } m \in \mathbb{N}_0\,. \tag{80}$$

Proof: This is a standard partial fraction decomposition for the rational function $\frac{1}{P(x)}$ with $P(x)$ a polynomial of degree $m+1$ with the simple roots $\alpha_j = a + d\,j,\ j = 0, 1, ..., m.$ $\frac{1}{P(x)} = \sum_{j=0}^{m} \frac{a_j}{x - (a + d\,j)}$. Here $a_j = \frac{1}{P'(\alpha_j)} = \prod_{k=0,\neq j}^{m} \frac{1}{\alpha_j - \alpha_k} = \frac{1}{d^m} \prod_{k=0,\neq j}^{m} \frac{1}{j-k} = \frac{1}{d^m}\frac{1}{j!\,(-1)^{m-j}\,(m-j)!} = \frac{1}{d^m}\frac{1}{m!}(-1)^{m-j}\binom{m}{j}$.
□

Now, due to the linearity of $\mathcal{L}^{-1}$, and the transform $\mathcal{L}^{-1}\left[\frac{1}{p - \alpha}\right] = e^{\alpha\,t}$, one finds after application of *Lemma 2*, and with the help of the binomial theorem:

$$\begin{aligned} E(t,m) &= \mathcal{L}^{-1}\left[\frac{1}{p}\,G\left(\frac{1}{p},m\right)\right] = \mathcal{L}^{-1}\left[\frac{d^m}{\prod_{j=0}^{m}(p - (a + d\,j))}\right] = \frac{(-1)^m}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\mathcal{L}^{-1}\left[\frac{1}{p - (a + d\,j)}\right] \\ &= \frac{(-1)^m}{m!}\,e^{a\,t}\sum_{j=0}^{m}\binom{m}{j}(-e^{d\,t})^j = \frac{e^{a\,t}}{m!}\,(1 - e^{d\,t})^m\,(-1)^m\,. \end{aligned} \tag{81}$$

which is indeed the *e.g.f.* given in eq. (13).

4. Proof of eq. (16) with eq. (17)

Lemma 3: Sheffer transform of a sequence

If the *e.g.f.* of sequence $\{b_n\}_{n=0}^{\infty}$ is $\mathcal{B}(t)$, the *e.g.f.* of sequence $\{a_n\}_{n=0}^{\infty}$ is $\mathcal{A}(t)$, and the *Sheffer* transform of $\{a_n\}$ is $b_n = \sum_{m=0}^{n} S(n,\,m)\,a_n$, with S *Sheffer* of type $S = (g(t),\ f(t))$ then

$$\mathcal{B}(t) = g(t)\,\mathcal{A}(f(t))\ . \tag{82}$$

The **proof** uses an exchange of the two summations and the *e.g.f.* of column m of S, *i.e.*, $g(t)\,\frac{f(t)^m}{m!}$.

Corollary 1:

The row polynomials $PS(n,\,x)$ of a *Sheffer* matrix $\mathbf{S} = (g(t),\ f(t))$ have *e.g.f.* $EPS(t,\,x) = g(t)\,e^{x\,f(t)}$.
This is also called the *e.g.f.* of the triangle $\mathbf{S}$.

That the $fallfac(d,a;x,m)$ definition in eq. (17) can be rewritten in terms of the usual falling factorial $x^{\underline{n}}$ is trivial.

Lemma 4: E.g.f. of fallfac[d,a]

The *e.g.f.* of $fallfac(d,a;x,m)$ (see eq. (17)) is

$$F(d,a;x,t) := 1 + \sum_{m=1}^{\infty} fallfac(d,a;x,m)\,\frac{t^m}{m!} = (1 + d\,t)^{\frac{x-a}{d}} . \tag{83}$$

Proof: With the binomial theorem and the rewritten form of $fallfac[d,a]$ in terms of the ordinary falling factorial this is trivial.

Now eq. (16) is a *Sheffer* transform of the sequence $\{fallfac(d,a;x,m)\}_{m=0}^{\infty}$, therefore, with *Lemmata* 3 and 4 the *e.g.f.* of the *r.h.s.* of eq. (16) is, with the *e.g.f.* of $\widehat{\mathbf{S2}}$ given in connection with eq. (15),

$$e^{a\,t}\,F\left(d,a;t,\frac{1}{d}\,(e^{d\,t} - 1)\right) = e^{a\,t}\,\left(1 + (e^{d\,t} - 1)\right)^{\frac{x-a}{d}} = e^{a\,t}\,e^{d\,t\,\frac{x-a}{d}} = e^{t\,x}, \tag{84}$$

which is the *e.g.f.* of the sequence $\{x^n\}_{n=0}^{\infty}$ of the *l.h.s.* .

5. Meixner recurrence and recurrence for Sheffer row polynomials eq. (19)

a) Monic row polynomials $s(n,\,x)$ of the *Sheffer* type $(g(x),\,f(x))$ satisfy the *Meixner* [20], p. 9, eqs. (4.1) and (4.2), recurrence ($f^{[-1]}$ denotes the compositional inverse of f)

$$f^{[-1]}(\mathbf{d}_x)\,s(n,\,x) = n\,s(n-1,\,x), \quad \text{with input} \quad s(0,\,x) = 1. \tag{85}$$

For the proof see the original reference.

For $P\widehat{S2}(d,a;n,x) = \sum_{m=0}^{n} \widehat{S2}(d,a;n,m)\,x^m$ with $f^{[-1]}(y) = \frac{1}{d}\,log(1 + y)$ one has

$$\frac{1}{d}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\,\frac{d^k}{dx^k}\,P\widehat{S2}(d,a;n,x) = n\,P\widehat{S2}(d,a;n-1,x), \quad \text{with input} \quad P\widehat{S2}(d,a;0,x) = 1 . \tag{86}$$

b) The standard recurrence for *Sheffer* row polynomials $S(n,\,x)$ (not necessarily monic ones) is given in *Roman* [22], p. 50, Corollary 3.7.2, which in our notation is

Lemma 5: (the differentiation is with respect to t)

$$S(n,\,x) = \left[x + \left(log(g(f^{[-1]}(t)))\right)'\right]\,\frac{1}{f^{[-1]}(t)'}\Bigg|_{t\,=\,\mathbf{d}_x}\,S(n-1,\,x), \quad \text{for } n \in \mathbb{N}\,, \tag{87}$$

and input $S(0,\,x) = 1$.

For $PS2(d,a;n,x)$ of eq. (19) $f^{[-1]}(t) = \frac{1}{d}\,log(1 + t)$, $f^{[-1]}(t)' = \frac{1}{d}\,\frac{1}{1+\,t}$, $g(t) = e^{a\,t}$ and $\left(log(g(f^{[-1]}(t)))\right)' = \frac{a}{d}\,\frac{1}{1+\,t}$, leading to $PS2(d,a;n,x) = [x\,d\,(1 + t) + a]|_{t\,=\,\mathbf{d}_x}\,PS2(d,a;n-1,x)$, which his eq. (19),

6. Proof of eqs. (23) to (25)

Multiplication of eq. (23) with $(1 - x)^{n+1}$ and the binomial formula gives

$$\begin{aligned}\sum_{i=0}^{n} a_i^{(n)}\,x^i &= \sum_{j=0}^{n} b_j^{(n)}\,x^j\,(1 - x)^{n-j} = \sum_{j=0}^{n} b_j^{(n)}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}x^{k+j}\\ &= \sum_{i=0}^{n} x^i\left(\sum_{j=0} b_j^{(n)}\,(-1)^{i-j}\binom{n-j}{i-j}\right),\end{aligned} \tag{88}$$

where in the last step a new summation index $i = k + j$ has been used instead of k, and the upper summation index j is determined by the binomial as $\min(n,\,i) = i$, because $0 \leq i \leq n$. Comparing the coefficients of the powers x^i, for $i = 1,\,2,\,...,\,n$, leads then to eq. (24) for $a_i^{(n)}$.

The inverse relation, eq. (25), uses the following binomial identity (see [9], p. 169. eq. (5.24) with $k \to p$, $m = s = 0$, $l \to n-k$, $n \to n - j$)

$$\sum_{p\,\geq 0}(-1)^p\binom{n-k}{p}\binom{p}{n-j} = (-1)^{n-k}\binom{0}{k-j} = (-1)^{n-k}\,\delta_{k,j}\,, \tag{89}$$

with the *Kronecker* symbol $\delta_{j,k} = 1$ if $j = k$, and 0 otherwise.

In the *r.h.s.* of eq. (25), with $a_i^{(n)}$ from eq. (24) inserted, the two finite sums are interchanged, a new summation index $p = n - i$ is used instead of i, and finally the preceding binomial identity is employed.

$$\sum_{i=0}^{j} \binom{n-i}{j-i} a_i^{(n)} = \sum_{i=0}^{j} \binom{n-i}{j-i} \sum_{k=0}^{i} (-1)^{i-k} b_k^{(n)} \binom{n-k}{i-k} = \sum_{k=0}^{j} (-1)^{i-k} b_k^{(n)} \left(\sum_{i=k}^{j} \binom{n-i}{n-j} \binom{n-k}{n-i} \right)$$
$$= \sum_{k=0}^{j} (-1)^{n-k} b_k^{(n)} \left(\sum_{p=n-j}^{n-k} (-1)^p \binom{n-k}{p} \binom{p}{n-j} \right) = b_j^{(n)} . \tag{90}$$

7. Proof of eqs. (26) to (28)

This follows by using in eq. (21) eqs. (23) with $b_j^{(n)} = S2(d.a; n, j)\, j!$ and eq. (24). Then $a_k^{(n)} = rEu(d, a; n, k)$ is obtained as given in eq. (28).

8. Proof of eq. (29)

This is eq. (25) (replacing $i \to k$) with $b_j^{(n)}$ and $a_k^{(n)}$ as given in the previous proof.

9. Proof of eq. (30)

This uses the binomial identity (see [9], p. 169, eq. (5.26) with $k \to j$, $q \to 0$, $m \to n-k$, $l \to n$ and $n \to l$)

$$\sum_{j=l}^{k} \binom{n-j}{n-k} \binom{j}{l} = \binom{n+1}{n-k+l+1} = \binom{n+1}{k-l} . \tag{91}$$

Insertion of eq. (11) into eq. (28) followed by an interchange of the two sums and application of the binomial identity leads to

$$rEu(d,a;n,k) = \sum_{j=0}^{k} (-1)^{k-j} \binom{n-j}{k-j} \sum_{l=0}^{j} (-1)^{j-l} \binom{j}{l} (a + d\,l)^n = \sum_{l=0}^{k} (-1)^{k-l} (a + d\,l)^n \sum_{j=l}^{k} \binom{n-j}{n-k} \binom{j}{l}$$
$$= \sum_{l=0}^{k} (-1)^{k-l} (a + d\,l)^n \binom{n+1}{k-l} = \sum_{j=0}^{k} (-1)^{k-j} \binom{n+1}{k-j} (a + d\,j)^n . \tag{92}$$

For the **rEu**$[d, a]$ triangles with $[d, a] = [1, 0], [2, 1], [3, 2], [4, 3]$ see A123125, A060187, A225117, A225118, respectively. The case **rEu**$[3, 1]$ is the row reversed version of **rEu**$[3, 2]$, and **rEu**$[4, 1]$ is the row reversed version of **rEu**$[4, 3]$, In general this row reversion relation holds between **rEu**$[d, d-a]$ and **rEu**$[d, a]$, for $a = 1, ..., \left\lfloor \frac{d}{2} \right\rfloor$ for $\gcd(d-a, a) = 1$.

10. Proof of eq. (31)

Like eq. (6) one has for $GP(d, a; n, x)$ of eq. (20)

$$GP(d,a;n,x) = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, d^k\, GP(k,x) , \tag{93}$$

with $GP(k, x) = GP(1, 0; k, x)$. However, this does not lead immediately to the desired formula for $rEu(d, a; n, k)$ in terms of the usual *Euler*ian numbers $rEu(n, k)$ claimed in eq. (31). The proof is done by inserting **S2**$[d, a]$ from eq. (18) into eq. (28), then replacing the usual **S2** by **rEu** *via* eq. (29) for $[d, a] = [1, 0]$. Here two binomial identities are needed. The first is given in [9], p. 169, eq. (5.25) (with $k \to j$, $s \to m-i$, $n \to i$, $l \to n$, and $m \to n-k$). Here one needs $n - k \geq 0$ and the upper summation index which would be n can be replaced by k because for $j = k = 1, ..., n$ the first binomial vanishes because the upper non-negative number is then smaller than the lower one.

$$\sum_{j=i}^{k} (-1)^j \binom{n-j}{n-k} \binom{m-i}{j-i} = (-1)^k \binom{m-n+k-i-1}{k-i} = (-1)^k \binom{-(n-m-(k-i)+1)}{k-i} = (-1)^i \binom{n-m}{k-i} . \tag{94}$$

In the last step another identity, given in [9], p. 164, eq. (5.14), has been used.

$$\binom{-r}{p} \;=\; (-1)^p \binom{p-1+r}{p}\,. \tag{95}$$

Now

$$\begin{aligned}
rEu(d,a;n,k) &= \sum_{j=0}^{k} (-1)^{k-j} \binom{n-j}{k-j} S2(d,a;n,j)\, j! \,=\, \sum_{j=0}^{k} (-1)^{k-j} \binom{n-j}{k-j} \sum_{m=0}^{n} \binom{n}{m} a^{n-m}\, d^m\, S2(m,\,j)\, j!\\
&= \sum_{j=0}^{k} (-1)^{k-j} \binom{n-j}{k-j} \sum_{m=0}^{n} \binom{n}{m} a^{n-m}\, d^m \sum_{i=0}^{j} \binom{m-i}{j-i} rEu(m,\,i)\\
&= \sum_{m=0}^{n} \binom{n}{m} a^{n-m}\, d^m \sum_{i=0}^{k} (-1)^k\, rEu(m,\,i) \sum_{j=0}^{k} (-1)^j \binom{n-j}{n-k} \binom{m-i}{j-i}\\
&= \sum_{m=0}^{n} \binom{n}{m} a^{n-m}\, d^m \sum_{i=0}^{k} (-1)^k\, rEu(m,\,i)\, (-1)^i \binom{n-m}{k-i}\,,
\end{aligned} \tag{96}$$

which is eq. (31) with summation index $p \to i$.

11. Proof of eq. (32)

We show that eq. (30) satisfies the three term recurrence eq. (31). The *l.h.s.* of the recurrence is

$$rEu(d,a;n,m) \;=\; \sum_{j=0}^{m} (-1)^{m-j} \binom{n+1}{m-j} (a \,+\, d\,j)^n\,. \tag{97}$$

The *r.h.s.* of the recurrence is

$$\begin{aligned}
&(d\,(n-m) \,+\, (d \,-\, a)) \sum_{j=0}^{m-1} (-1)^{m-j-1} \binom{n}{m-1-j} (a \,+\, d\,j)^{n-1}\\
&+ (a \,+\, d\,m) \sum_{j=0}^{m} (-1)^{m-j} \binom{n}{m-j} (a \,+\, d\,j)^{n-1}\\
= \;& \sum_{j=0}^{m} (-1)^{m-j} \binom{n+1}{m-j} (a \,+\, d\,j)^{n-1} \left[-(d\,(n-m) \,+\, (d \,-\, a)) \frac{\binom{n}{m-1-j}}{\binom{n+1}{m-j}} \,+\, (a \,+\, d\,m)\, \frac{\binom{n}{m-j}}{\binom{n+1}{m-j}} \right].
\end{aligned} \tag{98}$$

In the first sum the upper index $m-1$ has been extended to m because the extra term vanishes due to the binomial. The terms in the bracket are shown to become $a \,+\, d\,j$ as follows.

$$\begin{aligned}
[...] &= -(d\,(n-m) \,+\, (d \,-\, a)) \frac{n!\,(m-j)!}{(m-1-j)!\,(n+1)!} \,+\, (a \,+\, d\,m) \frac{n!(n+1-m+j)!}{(n-m+j)!\,(n+1)!}\\
&= \frac{1}{n+1} \left(-(d\,(n-m) \,+\, (d \,-\, a))\,(m-j) \,+\, (a \,+\, d\,m)\,(n-m+j+1)\right)\\
&= \frac{1}{n+1} \left(d\,j\,(n+1) \,+\, a\,(n+1)\right) \,=\, a + d\,j\,.
\end{aligned} \tag{99}$$

12. Proof of eq. (34)

From eq. (21) and the row polynomial definition of $PS2fac$ we have

$$GP(d,a;n,x) \,=\, \frac{1}{1-x} \sum_{k=0}^{n} S2(d,a;n,k)\, k! \left(\frac{x}{1-x}\right)^k \,=\, =:\, \frac{1}{1-x}\, PS2fac\left(d,a;n,\frac{x}{1-x}\right)\,. \tag{100}$$

Therefore, from eq. (26),

$$\frac{1}{1-x}\, PS2fac\left(d,a;n,\frac{x}{1-x}\right) \;=\; \frac{1}{(1-x)^{n+1}}\, PrEu(d,a;n,x)\,, \tag{101}$$

which is eq. (34) for $x \neq 1$. But eq. (34) holds also for $x = 1$, with the row sums $PrEu(d,a;n,1) = S2fac(d,a;n,n)$. From eq. (14) one has $S2fac(d,a;n,n) = [x^n]\,(n!\,GS2Col(d,a;x,n)) = d^n\,n!$ with *e.g.f.* $\frac{1}{1\,-\,d\,t}$, independently of a. These sequences are, for $d = 1, 2, ..., 5$, A000142, A000165, A032031, A047053, A052562.

13. Proof of eq. (35)

With eq. (34) and the *Sheffer* structure of $\mathbf{S2}[d,a]$ (eq. (13) for the column *e.g.f.* , here in the variable $(1-x)\,z$)

$$
\begin{aligned}
EPrEu(d,a;t,x) \quad &:= \sum_{n=0}^{\infty} \frac{t^n}{n!}\,PrEu(d,a;n,x) = \sum_{n=0}^{\infty} \frac{(t\,(1-x))^n}{n!} \sum_{m=0}^{n} S2(d,a;n,m)\,m!\left(\frac{x}{1-x}\right)^m \\
&= \sum_{m=0}^{\infty}\left(\frac{x}{1-x}\right)^m \sum_{n=m(0)}^{\infty} \frac{(t\,(1-x))^n}{n!}\,S2(d,a;n,m)\,m! \\
&= e^{a\,(1-x)\,t} \sum_{m=0}^{\infty}\left(\frac{x}{1-x}\right)^m \frac{(e^{d\,(1-x)\,t}\,-\,1)^m}{m!}\,m! \\
&= e^{a\,(1-x)\,t}\,\frac{1}{1\,-\,\frac{x}{1-x}\,(e^{d\,(1-x)\,t}\,-\,1)} = \frac{(1-x)\,e^{a\,(1-x)\,t}}{1\,-\,x\,e^{d\,(1-x)\,t}}\,. \qquad (102)
\end{aligned}
$$

The limit $x \to 1$ *via l'Hôpital*'s rule leads to the *e.g.f.* $\frac{1}{1\,-\,d\,t}$ for the row sums of $\mathbf{rEu}[d,a]$, as found also in the preceding proof.

14. Proof of eq. (39)

One obtains $b_j^{(n+1)}$ of eq. (37) from eq. (25) (with $n \to n+1$) with $a_i^{(n+1)}$ given in eq. (38). We omit the (d,a) labels here. Remember that $a_{n+1}^{(n+1)} = 0$ (but $b_{n+1}^{(n+1)}$ does not vanish).

$$
b_j^{(n+1)} = \sum_{i=0}^{j}\binom{n+1-i}{j-i}\,a_i^{(n+1)} = \sum_{i=0}^{j}\binom{n+1-i}{j-i}\sum_{p=0}^{i}(-1)^{i-p}\binom{n+1}{i-p}\,(a\,+\,d\,p)^n\,. \qquad (103)
$$

It is convenient to consider the case $j = 0$ separately.

$$
b_0^{(n+1)} = a_0^{(n+1)} = a^n, \quad n \in \mathbb{N}_0\,. \qquad (104)
$$

For $j = 1, 2, ..., n+1$ we have after exchange of the sums

$$
b_j^{(n+1)} = \sum_{p=0}^{j}(-1)^p\,(a\,+\,d\,p)^n \sum_{i=p\,(0)}^{j}(-1)^i\binom{n+1-i}{n+1-j}\binom{n+1}{i-p}\,. \qquad (105)
$$

Because of the second binomial one could start the sum with $i = 0$. Now the binomial identity used already above in the first step of eq. (95), is employed with $j \to i$, $i \to p$, $n \to n+1$, $k \to j$, $m \to p+n+1$. In the first binomial the lower number is non-negative and the original upper sum index $n+1$ can be replaced by j because for $i = j+1, ..., n+1$ the upper non-negative number in the first binomial becomes smaller then the lower one.

$$
b_j^{(n+1)} = \sum_{p=0\,(1)}^{j}(-1)^p\,(a\,+\,d\,p)^n\,(-1)^j\binom{j-1}{j-p}\,. \qquad (106)
$$

Because $j \geq 1$ one can use the symmetry of the binomials (this is why we have separated the $j = 0$ case).

$$
b_j^{(n+1)} = \sum_{p=0}^{j}(-1)^{j-p}\binom{j-1}{p-1}\,(a\,+\,d\,p)^n\,. \qquad (107)
$$

The identification with the $\Sigma S2[d,a]$ as claimed in eq. (39) is achieved by using the *Pascal* recurrence (see A007318).

$$
\begin{aligned}
&\sum_{p=0}^{j}(-1)^{j-p}\binom{j-1}{p-1}\,(a\,+\,d\,p)^n = -\sum_{p=0}^{j}(-1)^{j-p}\binom{j-1}{p}\,(a\,+\,d\,p)^n + \sum_{p=0}^{j}(-1)^{j-p}\binom{j}{p}\,(a\,+\,d\,p)^n \\
&= +\,S2(d,a;n,j-1)\,(j-1)! + \; S2(d,a;n,j)\,j!\,, \qquad (108)
\end{aligned}
$$

where eq (11) has been used. Now the $j = 0$ result a^n is also covered because $S2(d,a;n,0) = a^n$ and $S2(d,a;n,-1)*(-1)!$ is taken as vanishing ($S2(d,a;n,-1) = 0$ from the recurrence eq. (10)).

15. Proof of eq. (40)

Putting things together, the *e.g.f.* of $\{PS(d, a; n, m)\}_{m=0}^{\infty}$ (see eq. (1)) becomes *via* inverse *Laplace* transform of eq. (2)

$$EPS(d, a; n, t) := \sum_{m=0}^{\infty} PS(d, a; n, m) \frac{t^m}{m!} = \mathcal{L}^{-1}\left[\frac{1}{p} GPS\left(d, a; n, \frac{1}{p}\right)\right], \tag{109}$$

and from eq. (36) with eqs. (27), (38) and (37)

$$\frac{1}{p} GPS\left(d, a; n, \frac{1}{p}\right) = \sum_{j=0}^{n+1} b_j^{(n+1)}(d, a) \frac{1}{p} \frac{1}{p^j} \frac{1}{\left(1 - \frac{1}{p}\right)^{j+1}} = \sum_{j=0}^{n+1} b_j^{(n+1)}(d, a) \frac{1}{(p - 1)^{j+1}}. \tag{110}$$

Thus, by linearity of $\mathcal{L}^{-1}$, and the formula before eq. (22), one obtains

$$EPS(d, a; n, t) = e^t \sum_{j=0}^{n+1} b_j^{(n+1)}(d, a) \frac{t^j}{j!}, \tag{111}$$

which become finally eq. (40) after insertion of $b_j^{(n+1)}(d, a)$ from eq. (39).

B) Proofs of section 1 B

1. Proof of eq. (47)

Insertion of eq. (18) into eq. (46) leads after interchange of the two finite sums to

$$B(d, a; n) = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} d^k \sum_{m=0}^{n\ (k)} (-1)^m \frac{1}{m+1} S2(k, m)\, m!, \tag{112}$$

where in the second sum the upper index can be taken as k instead of n because $S2(k, m) = 0$ for $m > k$, and $k \leq n$ from the first sum. Then the second sum is equal to $B(k)$ by eq. (46) for $[d, a] = [1, 0]$. This is then eq. (47). For the $B[d, a]$ numbers for $[d, a] = [1, 0]$, $[2, 1]$, $[3, 1]$, $[4, 1]$, $[5, 1]$, $[5, 2]$ see A027641/A027642, A157779/A141459, A157799/A285068, A157817/A141459, A157866/A288872, A157833/A288872, respectively.
$B[3, 2](n) = (-1)^n B[3, 1](n)$, $B[4, 3](n) = (-1)^n B[4, 1](n)$, $B[5, 3](n) = (-1)^n B[5, 2](n)$, and $B[5, 4](n) = (-1)^n B[5, 1](n)$.

2. Proof of eq. (48)

The *e.g.f.* of $\{B(d, a; n)\}_{n=0}^{\infty}$ is obtained from the defining eq. (46) recognizing, after exchange of the sums, the *e.g.f.* $ES2Col(d, a; t, m)$ of eq. (13):

$$\begin{aligned} EB(d, a; t) &:= \sum_{n=0}^{\infty} B(d, a; n) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{m=0}^{n} (-1)^m \frac{m!}{m+1} S2(d, a; n, m) \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{m!}{m+1} \sum_{n=m}^{\infty} \frac{t^n}{n!} S2(d, a; n, m) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m+1} e^{a\,t} (e^{d\,t} - 1)^m \\ &= e^{a\,t} \frac{1}{y} \sum_{m=0}^{\infty} \frac{y^{m+1}}{m+1} \ (\text{with } y := 1 - e^{d\,t}) = e^{a\,t} \frac{1}{y} \int dy \sum_{m=0}^{\infty} y^m = e^{a\,t} \frac{1}{y} \int \frac{dy}{1-y} \\ &= e^{a\,t} \frac{1}{-y} log(1-y) = \frac{d\,t\, e^{a\,t}}{e^{d\,t} - 1}. \end{aligned} \tag{113}$$

3. Proof of eq. (50)

This follows from inserting into eq. (49) (with $n - m = p$) eq, (47), using the binomial identity (see [9], p. 174. Table 174. the trinomial revision formula) $\binom{n}{p}\binom{p}{m} = \binom{n}{m}\binom{n-m}{p-m}$ and interchange of the sums. Then the binomial formula is used.

$$
\begin{aligned}
B(d,a;n,x) &= \sum_{p=0}^{n}\binom{n}{p} B(d,a;p)\,x^{n-p} = \sum_{p=0}^{n}\binom{n}{p} x^{n-p}\sum_{m=0}^{p}\binom{p}{m} a^{p-m}\,d^m\,B(m)\\
&= \sum_{p=0}^{n}\sum_{m=0}^{p}\binom{n}{m}\binom{n-m}{p-m} x^{n-p}\,a^{p-m}\,d^m\,B(m) = \sum_{m=0}^{n}\binom{n}{m} a^{-m}\,d^m\,B(m)\sum_{p=m}^{n}\binom{n-m}{p-m} x^{n-p}\,a^p\\
&= \sum_{m=0}^{n}\binom{n}{m} a^{-m}\,d^m\,B(m)\sum_{p=0}^{n-m}\binom{n-m}{p} x^{n-(p+m)}\,a^{p+m}\\
&= \sum_{m=0}^{n}\binom{n}{m} d^m\,B(m)\sum_{p=0}^{n-m}\binom{n-m}{p} x^{(n-m)-p}\,a^p = \sum_{m=0}^{n}\binom{n}{m} d^m\,B(m)\,(a\,+\,x)^{n-m}\,. \qquad (114)
\end{aligned}
$$

4. Proof of eq. (51)

Eq. (49) is an exponential (also called binomial) convolution of the sequences $\{B(d,a;n)\}_{n=0}^{\infty}$ and $\{x^n\}_{n=0}^{\infty}$, hence the product of their *e.g.f.*s is with eq. (48) $EB(d,a;t)\,e^{x\,t} = \dfrac{d\,t\,e^{(a+x)\,t}}{e^{d\,t}\,-\,1}$.

Alternatively one can take the exponential convolution of eq. (50) of $\{d^m\,B(m)\}_{m=0}^{\infty}$ with *e.g.f.* $\dfrac{d\,t}{e^{d\,t}\,-\,1}$ and $\{(a\,+\,x)^n\}_{n=0}^{\infty}$ with *e.g.f.* $e^{(a+x)\,t}$, and their product is eq. (51).

5. Proof of eq. (57)

This proof will be rather lengthy. It will need the following simple *Lemma*.

Lemma 6: If the *e.g.f.* of the sequence $\{C_n\}_{n=0}^{\infty}$ is $\mathcal{C}(t)$ then the *e.g.f.* of the sequence $\left\{\dfrac{C_{n+1}}{n+1}\right\}_{n=0}^{\infty}$ is $\dfrac{1}{t}\,(\mathcal{C}(t) - C_0)$.

Proof:

$$
\sum_{n=0}^{\infty}\frac{C_{n+1}}{n+1}\frac{t^n}{n!} = \frac{1}{t}\sum_{n=0}^{\infty} C_{n+1}\,\frac{t^{n+1}}{(n+1)!} = \frac{1}{t}\,(\mathcal{C}(t)\,-\,a_0)\,. \qquad (115)
$$

We compute the *o.g.f.* of the first two terms of the claimed *Faulhaber* formula eq. (57) multiplied by $d\,(n+1)$

$$
d\,(n+1)\,G(d,a;n,x) := \sum_{m=0}^{\infty} x^m\,\{B(d;n+1,x=a+d\,(m+1))\,-\,B(d;n+1,x=d)\}\,, \qquad (116)
$$

with the polynomials $B(d;n,x)$ from eq. (52) which are inserted with summation index $k\,=\,n-m$ instead of m. The two terms with $k\,=\,n+1$ will be separated and they cancel. Then the sums will be interchanged.

$$
\begin{aligned}
d\,(n+1)\,G(d,a;n,x) &= \sum_{m=0}^{\infty} x^m\left\{\sum_{k=0}^{n}\binom{n+1}{k} B(d;k)\,(a\,+\,d\,(m+1))^{n+1-k} - \sum_{k=0}^{n}\binom{n+1}{k} B(d;k)\,d^{n+1-k}\right\}\\
&= \sum_{k=0}^{n}\binom{n+1}{k} B(d;k)\left\{\left(\sum_{m=0}^{\infty}(a\,+\,d\,(m+1))^{n+1-k}\,x^m\right) - \frac{1}{1-x}\,d^{n+1-k}\right\}\,. \qquad (117)
\end{aligned}
$$

The last term in the curly bracket simplifies with $B(d;k)\,=\,d^k\,B(k)$ from eq. (53), and with eq. (43) rewritten as

$$
\sum_{k=0}^{n}\binom{n+1}{k} B(k)\,=\,\delta_{n,0}\,, \qquad (118)
$$

to $-\dfrac{d}{1\,-\,x}\,\delta_{n,0}$.

In the remaining double sum one uses the *o.g.f.* of sums of powers (see eqs. (20) and (21)) after an index shift $m\,\to\,m-1$, then one adds and subtracts the new $m=0$ term. Thus,

$$
\begin{aligned}
d\,(n+1)\,G(d,a;n,x) &= \sum_{k=0}^{n}\binom{n+1}{k} B(d;k)\,\frac{1}{x}\left(GP(d,a;n+1-k,x)\,-\,a^{n+1-k}\right) - \frac{d}{1\,-\,x}\,\delta_{n,0}\\
&= \sum_{k=0}^{n}\binom{n+1}{k} B(d;k)\left[-\frac{a^{n+1-k}}{x} + \sum_{m=0}^{n+1-k} S2(d,a;n+1-k,m)\,m!\,\frac{x^{m-1}}{(1\,-\,x)^{m+1}}\right]\\
&\quad - \frac{d}{1\,-\,x}\,\delta_{n,0}\,. \qquad (119)
\end{aligned}
$$

The term with $\mathbf{S2}[d,a]$ will now be treated separately as $d\,(n+1)\,G1(d,a;n,x)$ and the remainder $d\,(n+1)\,G2(d,a;n,x)$ will be added later. In $d\,(n+1)\,G1(d,a;n,x)$ a new summation index $k' \,=\, n+1-k$ is used (called then again k), and the $m=0$ sum term will be separated in order to have in both sums the same offset 1.

$$\begin{aligned} d\,(n+1)\,G1(d,a;n,x) &= \sum_{k=1}^{n+1} \binom{n+1}{k} B(d;n+1-k) \left(\sum_{m=1}^{k} S2(d;a;k,m)\,m!\,\frac{x^{m-1}}{(1\,-\,x)^{m+1}} \,+\, \frac{a^k}{x\,(1\,-\,x)} \right) \\ &=: \; d\,(n+1)\,G11(d,a;n,x) \,+\, d\,(n+1)\,G12(d,a;n,x)\,. \end{aligned} \tag{120}$$

The $m=0$ term $d\,(n+1)\,G12(d,a;n,x)$ will be added later, and the for the first term we have the following *Lemma*.

Lemma 7:

$$G11(d,a;n,x) \,=\, GPS(d,a;n,x)\,, \tag{121}$$

which is the *o.g.f.* given in eq. (2) of the object of desire $PS(d,a;n,m)$, *i.e.*, the *l.h.s.* of the *Faulhaber* formula eq. (57).

Proof: The two sums are exchanged, and in the m-sum a shift $m \,\to\, m+1$ will be applied.

$$\begin{aligned} G11(d,a;n,x) &= \frac{1}{d\,(n+1)} \sum_{m=1}^{n+1} \frac{x^{m-1}}{(1-x)^{m+1}} \sum_{k=m}^{n+1} \binom{n+1}{k} B(d;n+1-k)\,S2(d,a;k,m)\,m! \\ &= \sum_{m=0}^{n} \frac{x^m}{(1-x)^{m+2}} \frac{1}{d\,(n+1)} \sum_{k=m+1}^{n+1} \binom{n+1}{k} B(d;n+1-k) S2(d,a;k,m+1)(m+1)!\,. \end{aligned} \tag{122}$$

The k-sum will be called $C_{n+1} \,\equiv\, C(d,a;n+1,m+1)$. Now *Lemma 6* is used to compute the *e.g.f.* of $\left\{ \dfrac{C_{n+1}}{d\,(n+1)} \right\}_{n=0}^{\infty}$.

Because $C_n \,=\, \sum_{k=0}^{n} \binom{n}{k} B(d;n-k) S2(d,a;k,m+1)(m+1)!$ (the sum can start with $k=0$ because $S2(d,a,k,m+1)$ vanishes for $k \,<\, m+1$) is an exponential convolution, the *e.g.f.* of $\{C_n\}_{n=0}^{\infty}$ is the product of $EB(d;t)$ from eq. (55) and the *e.g.f.* $ES2Col(d,a;t,m+1)$ multiplied by $(m+1)!$, hence

$$\frac{d\,t}{e^{d\,t}\,-\,1} \cdot e^{a\,t}\,(e^{d\,t}\,-\,1)^{m+1} \,=\, d\,t\,e^{a\,t}\,(e^{d\,t}\,-\,1)^{m}\,. \tag{123}$$

Thus, the *e.g.f.* of $\left\{ \dfrac{C_{n+1}}{d\,(n+1)} \right\}_{n=0}^{\infty}$ is, by *Lemma 6*, $e^{a\,t}\,(e^{d\,t}-1)^m$ because $C_0 \,=\, C(d,a;0,m+1) \,=\, B(d;0)\,S2(d,a;0,m+1)(m+1)! \,=\, 0$, since $S2(d,a;0,m+1) \,=\, 0$ for $m \geq 0$. But this is the *e.g.f.* of $\{S2(d,a;n,m)\,m!\}_{n=0}^{\infty}$, and therefore

$$G11(d,a;n,x) \,=\, \sum_{m=0}^{n} \frac{x^m}{(1-x)^{m+2}}\,S2(d,a;n,m)\,m! \,=\, GPS(d,a;n,x)\,. \tag{124}$$

In the last step eq. (8) was used. □

If all terms of $G1$ and $G2$ are added we have

$$\begin{aligned} G(d,a;n,x) \,=\, GPS(d,a;n,x) \;+\; \frac{1}{d\,(n+1)} \Bigg[\frac{1}{x\,(1-x)} \sum_{k=1}^{n+1} \binom{n+1}{k} B(d;n+1-k)\,a^k \\ -\sum_{k=0}^{n} \binom{n+1}{k} B(d;k)\,\frac{a^{n+1-k}}{x} \,-\, \frac{d}{1-x}\,\delta_{n,0} \Bigg]\,. \end{aligned} \tag{125}$$

In the second sum an index change $k' \,=\, n+1-k$ leads to

$$\begin{aligned} G(d,a;n,x) - GPS(d,a;n,x) &= \frac{1}{d\,(n+1)} \left[\left(\frac{1}{x\,(1-x)} \,-\, \frac{1}{x} \right) \sum_{k=1}^{n+1} \binom{n+1}{k} B(d;n+1-k)\,a^k \,-\, \frac{d}{1-x}\delta_{n,0} \right] \\ &= \frac{1}{d\,(n+1)}\,\frac{1}{1-x} \left[\sum_{k=1}^{n+1} \binom{n+1}{k} B(d;n+1-k)\,a^k \,-\, d\,\delta_{n,0} \right] \\ &= \frac{1}{d\,(n+1)}\,\frac{1}{1-x} \left[\sum_{k=0}^{n+1} \binom{n+1}{k} B(d;n+1-k)\,a^k \,-\, B(d;n+1) \,-\, d\,\delta_{n,0} \right] \\ &= \frac{1}{d\,(n+1)}\,\frac{1}{1-x} \left[B(d;n+1,x=a) \,-\, B(d;n+1,x=0) \,-\, d\,\delta_{n,0} \right]. \end{aligned} \tag{126}$$

In the last step the polynomials of eq. (52) have been identified. If now the definition $G(d,a;n,x)$ in eq. (116) is remembered, and the coefficient $[x^m]$ of this *o.g.f.* is picked one finds after this *tour de force* the *Faulhaber* formula eq. (57).

C) Proofs of section 1 C

1. Proof of eq. (59)

In the *Sheffer* group of (infinite) lower triangular matrices the inverse element of $\mathbf{S} = (g(x),\, f(x)) \equiv (g,\, f)$ is

$$\mathbf{S}^{-1} \;=\; \left(\frac{1}{g(f^{[-1]}(y))},\, f^{[-1]}(y)\right) \;\equiv\; \left(\frac{1}{g\circ f^{[-1]}},\, f^{[-1]}\right) \tag{127}$$

with the compositional inverse $f^{[-1]}$ of f, *i.e.*, $f(f^{[-1]}(y)) \,=\, y$, or $f^{[-1]}(f(x)) \,=\, x$, identically. Here $f(x) \,=\, x\,\hat{f}(x)$ with $\hat{f}(0) \,\neq\, 0$.

For the *Sheffer* matrix $\mathbf{S2}[d,a]$ one has $g(x) \,=\, e^{a\,x}$ and $f(x) \,=\, e^{d\,x} \,-\, 1$, hence $f^{[-1]}(y) \,=\, \dfrac{1}{d}\log(1\,+\,y)$, and

$$\mathbf{S1}[d,a] \;:=\; (\mathbf{S2}[d,a])^{-1} \;=\; \left((1\,+\,y)^{-\frac{a}{d}},\, \frac{1}{d}\log(1\,+\,y)\right)\,. \tag{128}$$

This matrix has in general fractional integer entries. The unsigned matrix $\mathbf{S1p}[d,a] \,\equiv\, |\mathbf{S1}[d,a]|$ (p for non-negative) has elements $S1p(d,a;n,m) \,=\, (-1)^{n-m}\, S1(d,a;n,m)$ because then the *e.g.f.* for column m becomes

$$ES1p(d,a;t,m) \;=\; (1-t)^{-\frac{a}{d}}\,\frac{\left(-\frac{1}{d}\log(1\,-\,t)\right)^m}{m!}\,, \tag{129}$$

and both (formal) power series have non-negative elements which are in general fractional numbers.

For combinatorial considerations one is interested in non-negative integer matrices. Therefore, a scaling of the rows is performed: $\widehat{S1p}(d,a;n,m) \,:=\, d^n\, S1p(d,a;n,m)$ which leads to diagonal elements 1, and the *Sheffer* matrix is

$$\widehat{\mathbf{S1p}}[d,a] \;=\; \left((1\,-\,d\,y)^{-\frac{a}{d}},\, -\frac{1}{d}\log(1\,-\,d\,y)\right)\,, \tag{130}$$

because the scaling leads to $t \,\to\, d\,t$ in $ES1p(d,a;t,m)$. The new power series generate exponentially non-negative integers, because $\left[\dfrac{x^n}{n!}\right](1\,-\,d\,y)^{-\frac{a}{d}} \,=\, \left(\dfrac{a}{d}\right)^{\overline{n}} d^n \,=\, \prod_{j=0}^{n-1}(a\,+\,d\,j) \,=\, risefac(d,a;0,n)$ (see eq. (63) for the $risefac$ definition), and $\left[\dfrac{x^n}{n!}\right]\left(-\dfrac{1}{d}\log(1\,-\,d\,y)\right) \,=\, (n-1)!\,d^{n-1}$ for $n\geq 1$ (and 0 for $n=0$).

2. Proof of eq. (61)

The three term recurrence of the $\widehat{\mathbf{S1p}}[d,a]$ can be obtained from the *e.g.f.* of their column sequences $E\widehat{S1p}Col(d,a;t,m)$ given in eq. (60) which we abbreviate for this proof as $Ep(t,m)$.

Lemma 8:

$$(1-d\,t)\,\frac{d}{dt}\,Ep(t,m) \;=\; a\,Ep(t,m) \,+\, Ep(t,m-1) \quad \text{for } m\,\in\,\mathbb{N}\,, \tag{131}$$

and the input is $Ep(t,0) \,=\, (1\,-\,d\,t)^{-\frac{a}{d}}$.

Proof: This is elementary with eq. (60).

Now the recurrence eq. (61) is seen to satisfy this *Lemma*.

$$\begin{aligned} Ep(t,m) \;&=\; \sum_{n=0}^{\infty}\frac{t^n}{n!}\,\widehat{S1p}(d,a;n,m) \\ &=\; \sum_{n=0\;(1)}^{\infty}\frac{t^n}{n!}\,\widehat{S1p}(d,a;n-1,m-1) \,+\, \sum_{n=0\;(1)}^{\infty}\frac{t^n}{n!}\,(d\,n\,-\,(d-a))\,\widehat{S1p}(d,a;n-1,m) \\ &=\; \sum_{n=0}^{\infty}\frac{t^{n+1}}{(n+1)!}\,\widehat{S1p}(d,a;n,m-1) \,+\, \sum_{n=0}^{\infty}\frac{t^{n+1}}{(n+1)!}\,(d\,(n+1)\,-\,(d-a))\,\widehat{S1p}(d,a;n,m) \\ &=\; \int dt\,Ep(t,m-1) \,+\, d\,t\,Ep(t,m) \,-\, (d-a)\int dt\,Ep(t,m)\,. \end{aligned} \tag{132}$$

In the second line the two sums actually start with $n = 1$ because $\widehat{\mathbf{S1p}}[d,a]$ vanishes for negative row indices. This is a integral-difference equation with input $Ep(t,0)$ as given in the *Lemma*.

$$(1 - d\,t)\,Ep(t,m) = \int dt\,(Ep(t,m-1) - (d - a)\,Ep(t,m))\ . \tag{133}$$

Differentiation produces precisely the equation of the *Lemma*.

3. Proof of eq. (62)

The row polynomials of *Sheffer* triangles are a *Sheffer* transform of the monomials $\{x^n\}_{n=0}^{\infty}$. Therefore, with *Lemma 3*, eq. (82), the *e.g.f.* of the (ordinary) row polynomials of $\widehat{\mathbf{S1p}}[d,a]$ is obtained from the *e.g.f.* of $\widehat{\mathbf{S1p}}[d,a]$ given by eq. (59) and $e^{x\,t}$, *i.e.*,

$$EP\widehat{S1p}(d,a;t,x) := \sum_{n=0}^{\infty} P\widehat{S1p}(d,a;n,x)\,\frac{t^n}{n!} = (1 - d\,t)^{-\frac{a}{d}}\,exp(x\,(-\frac{1}{d}\,\log(1-d\,t))) = (1 - d\,t)^{-\frac{x+a}{d}}\ . \tag{134}$$

Then the binomial theorem leads to

$$EP\widehat{S1p}(d,a;t,x) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,(-d)^n\left(-\frac{x+a}{d}\right)^{\underline{n}} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,risefac(d,a;x,n)\,, \tag{135}$$

where in the first equation the usual falling factorial $x^{\underline{n}} := \prod_{j=0}^{n-1}(x-j)$ appeared, and in the second equation the definition eq. (63) for $risefac(d,a;x,n)$ has been used in the rewritten form using ordinary falling factorials.

In this way we have found, as a corollary, the *e.g.f.* of $\{risefac(d,a;x,n)\}_{n=0}^{\infty}$ to be $(1 - d\,t)^{-\frac{x+a}{d}}$.

The $fallfac[d,a]$ analogon is obtained from inverting eq. (16) using the inverse of the scaled $\widehat{\mathbf{S2}}[d,a]$ *Sheffer* matrix, *i.e.*, the signed $\widehat{\mathbf{S1}}[d,a]$ matrix.

$$fallfac(d,a;x,n) = \sum_{m=0}^{n} \widehat{S1}(d,a;n,m)\,x^m\,. \tag{136}$$

These are the row polynomials of $\widehat{\mathbf{S1}}[d,a]$.

4. Lah[d,a]

It is tempting to give here the generalized unsigned *Lah* matrix $\mathbf{L}[d,a]$ as transition matrix between $risefac[d,a]$ and $fallfac[d,a]$.

For the ordinary $[d,a] = [1,0]$ *Lah* triangle see A271703 (or A008297 with $n \geq m \geq 1$) and [9], exercise 31, p. 312, solution p. 552.

The generalization is

$$risefac(d,a;x,n) = \sum_{m=0}^{n} L(d,a;n,m)\,fallfac(d,a;x,m)\ . \tag{137}$$

From eq. (62) and eq. (16) one has, in matrix notation

$$\mathbf{L}[d,a] = \widehat{\mathbf{S1p}}[d,a]\cdot\widehat{\mathbf{S2}}[d,a]\,. \tag{138}$$

We quote a *Lemma* on the multiplication law of the *Sheffer* group.

Lemma 9:

If the product of two *Sheffer* matrices with $\mathbf{S1} = (g1,\,f1)$ and $\mathbf{S2} = (g2,\,f2)$ is $\mathbf{S3} = \mathbf{S1}\cdot\mathbf{S2}$ with $\mathbf{S3} = (g3,\,f3)$ then

$$g3 = g1\,(g2\circ f1)\ ,\ f3 = (f2\circ f1)\ ,\ \text{i.e.,}\ g3(t) = g1(t)\,g2(f1(t)))\ ,\ f3(t) = f2(f1(t))\,. \tag{139}$$

This is standard *Sheffer* lore.

With eq. (59) and the statement just before eq. (15) this implies the *Sheffer* structure

$$\mathbf{L}[d,a] = \left((1 - d\,t)^{-\frac{2a}{d}},\ \frac{t}{1 - d\,t}\right)\,. \tag{140}$$

The proof of an explicit form along the lines of the mentioned exercise in [9] does not immediately lead to an explicit form for $L(d,a;n,m)$ if $a \neq 0$. See also the complicated form of eq. (69) for $\widehat{\mathbf{S1p}}[d,a]$. Of course the matrix product can be written with the help of eq. (64) or eq. (65) and eq. (11).
The *e.g.f.* of the column sequences is

$$ELCol(d,a;t,m) \,=\, (1\,-\,d\,t)^{-\frac{2\,a}{d}}\,\frac{1}{m!}\left(\frac{t}{1\,-\,d\,t}\right)^m\,,\quad m\,\in\,\mathbb{N}_0\,. \tag{141}$$

From the so called $a-$ and $z-$sequences for *Sheffer* matrices (see the link [16], where also references are given. This link is found also in A006232) one finds recurrence relations. The *e.g.f.*s of these sequences are (g and f are those of eq. (140)

$$\begin{aligned} a(y) &= \frac{y}{f^{[-1]}(y)} \,=\, 1\,+\,d\,y \,=\, a(d;y)\,,\\ z(y) &= \frac{1}{f^{[-1]}(y)}\left(1\,-\,\frac{1}{g(f^{[-1]}(y))}\right) \,=\, \frac{1\,+\,d\,y}{y}\left(1\,-\,(1\,+\,d\,y)^{-\frac{2\,a}{d}}\right) \,=\, z(d,a;y)\,. \end{aligned} \tag{142}$$

This means that there is always a three term recurrence for the matrix entries $L(d,a;n,m)$ for $n\,\geq\,m\,\geq 1$ because the $a-$sequence is $\{1,\,d,\,\text{repeat}(0)\}$ *i.e.*,

$$L(d,a;n,m) \,=\, \frac{n}{m}\,L(d,a;n-1,m-1)\,+\,n\,L(d,a;n-1,m)\quad n\,\in\,\mathbb{N},\,m\,=\,1,\,2,\,...,\,n\,, \tag{143}$$

where the input from column $m\,=\,0$, besides $L(d,a;0,0)\,=\,1$, can be taken form the *e.g.f.* eq. (141).
In general one can use the $z-$sequence for column $m = 0$ in combination with the given recurrence eq. (143). For $[d,a]\,=\,[1,0]$ where the $z-$sequence vanishes, the $m\,=\,0$ column becomes directly $\{1,\,\text{repeat}(0)\}$. For $[d,a]\,=\,[2,1]$ the $z-$sequence becomes $\{2,\,\text{repeat}(0)\}$, and the column $m = 0$ is also given directly as A000165. All other cases need also lower row entries with $m\,\geq 1$.
In this special $\mathbf{L}[d,a]$ case one can, however, derive from the column *e.g.f.* eq. (114) a four term recurrence, *i.e.*,

$$\begin{aligned} L(d,a;n,m) \,=\, L(d,a;n-1,m-1) \;&+\; 2\,(a\,+\,d\,(n-1))\,L(d,a;n-1,m)\\ &-\; d\,(n-1)\,(2\,a\,+\,d\,(n-2))\,L(d,a;n-2,m)\,, \end{aligned} \tag{144}$$

with inputs $L(d,a;0,0)\,=\,1$, $L(d,a;n,-1)\,=\,0$, $L(d,a;-1,m)\,=\,0$, and $L(d,a;n,m)\,=\,0$ if $n\,<\,m$.

Proof: This uses the definition of $ELCol(d,a;t,m) \,:=\, \sum_{n=m\,(0)}^{\infty} L(d,a;n,m)\,\frac{t^n}{n!}$, and the trivial result

$$(1-d\,t)^2\,\frac{d}{dt}\,ELCol(d,a;t,m) \,=\, 2\,a\,(1\,-\,d\,t)\,ELCol(d,a;t,m)\,+\,ELCol(d,a;t,m-1)\,. \tag{145}$$

The recurrence follows then by comparing powers of $\frac{t^n}{n!}$, sending $n\,\to\,n-1$.

The *Meixner* type recurrence for the row polynomials (see eq. (85)) is

$$\frac{\mathbf{d}_x}{1\,+\,d\,\mathbf{d}_x}\,PL(d,a;n,x) \,=\, n\,PL(d,a;n-1,x)\,,\quad n\,\in\,\mathbb{N}\,, \tag{146}$$

and input $PL(d,a;0,x)\,=\,1$. The series terminates and this becomes

$$\sum_{k=0}^{n-1}(-1)^d\,d^k\,\mathbf{d}_x^{k+1}\,PL(d,a;n,x) \,=\, n\,PL(d,a;n-1,x)\,. \tag{147}$$

The general *Sheffer* polynomial recurrence (see eq. (87) for the rewritten *Roman* corollary) is

$$PL(d,a;n,x) \,=\, \left((2\,a\,+\,x)\,\mathbf{1}\,+\,2\,d\,(a\,+\,x)\,\mathbf{d}_x\,+\,d^2\,x\,\mathbf{d}_x^2\right)\,PL(d,a;n-1,x)\,,\quad n\,\in\,\mathbb{N}, \tag{148}$$

and input $PL(d,a;0,x)\,=\,1$.

The inverse matrix of $\mathbf{L}[d,a]$ is also of interest:

$$fallfac(d,a;x,n) \,=\, \sum_{m=0}^{n} L^{-1}(d,a;n,m)\, risefac(d,a;x,m)\;. \tag{149}$$

From eq. (127) one finds the *Sheffer* structure

$$\mathbf{L}^{-1}[d,a] \,=\, \left(\frac{1}{(1+d\,t)^{\frac{2\,a}{d}}},\, \frac{t}{1\,+\,d\,t}\right) \,=\, (gL(-t),\, -fL(-t))\,, \tag{150}$$

where gL and fL are taken from eq. (140).

This means, by looking at the column *e.g.f.*s of *Sheffer* matrices, that the inverse matrix is just obtained by properly signing the $\mathbf{L}[d,a]$ matrix entries.

$$L^{-1}(d,a;n,\,m) \,=\, (-1)^{n-m}\, L(d,a;n,\,m)\,, \quad n\,\geq m\,\geq 0\,. \tag{151}$$

The explicit from of $gL^{-1}[d,a]$ and $fL^{-1}[d,a]$ shows that one has to replace in the $\mathbf{L}[d,a]$ recurrence formulae $a\,\to\,-a$ and $d\,\to\,-d$.

The $a-$ and $z-$sequences are then $aL^{-1}(d) \,=\, \{1,\,-d,\,\text{repeat}(0)\}$ and the *e.g.f.* for zL^{-1} is $zL^{-1}(d,a;y) \,=\, \dfrac{1\,-\,d\,y}{y}\left(1\,-\,(1\,-\,d\,y)^{-\frac{2\,a}{d}}\right) \,=\, -z(d,a;-y)$ (with $z(d,a;y)$ from eq. (142)). This gives a three term recurrence for $L^{-1}(d,a;n,m)$ for $n\,>\,m\,>\,1$ with the column $m=0$ as input.

The recurrence derived like above from the column *e.g.f.* is just eq. (144) with replacements $a\,\to\,-a$ and $d\,\to\,-d$

$$\begin{aligned} L^{-1}(d,a;n,m) \,=\, L^{-1}(d,a;n-1,m-1) \;&-\; 2\,(a\,+\,d\,(n-1))\,L^{-1}(d,a;n-1,m) \\ &-\; d\,(n-1)\,(2\,a\,+\,d\,(n-2))\,L^{-1}(d,a;n-2,m)\,, \end{aligned} \tag{152}$$

with inputs $L^{-1}(d,a;0,0) \,=\, 1$, $L^{-1}(d,a;n,-1) \,=\, 0$, $L^{-1}(d,a;-1,m) \,=\, 0$, and $L^{-1}(d,a;n,m) \,=\, 0$ if $n\,<\,m$.

The *Meixner* type recurrence for the row polynomials of $\mathbf{L}^{-1}$ is like the one in eq. (147) wit replacement $d\,\to\,-d$

$$\sum_{k=0}^{n-1} d^k\, \mathbf{d}_x^{k+1}\, PL^{-1}(d,a;n,x) \,=\, n\, PL^{-1}(d,a;n-1,x)\,, \tag{153}$$

with input $PL^{-1}(d,a;0,x) \,=\, 1$.

The general *Sheffer* recurrence is like eq. (148) with replacement $a\,\to\,-a$ and $d\,\to\,-d$.

$$PL^{-1}(d,a;n,x) \,=\, \left((x\,-\,2\,a)\,\mathbf{1}\,-\,2\,d\,(x\,-\,a)\,\mathbf{d}_x\,+\,d^2\,x\,\mathbf{d}_x^2\right)\, PL^{-1}(d,a;n-1,x)\,, \quad n\,\in\,\mathbb{N}, \tag{154}$$

and input $PL^{-1}(d,a;0,x) \,=\, 1$.

5. Proof of eq.(64)

It is well known (*Vieta*'s theorem) that the coefficients of a monic polynomial $P(n,\,x) \,=\, \sum_{m=0}^{n} p_m\, x^m$ of degree n are given in terms of the n zeros $x_j,\, j\,=\,1,\,...,\,n,$ of P by $p_m \,=\, (-1)^{n-m}\,\sigma_{n-m}^{(n)}(x_1,x_2,...,x_n) \,=\, -\sigma_{n-m}^{(n)}(-x_1,\,-x_2,\,...,\,-x_n)$ with the elementary symmetric functions σ_{n-m}^{n} of degree $n\,-\,m$, and $\sigma_0^{(n)} \,=\, 1$. For the $risefac(d,a;x,n)$ polynomials eq. (63) the zeros are $x_j \,=\, -(a\,+\,(j-1)\,d) \,=\, -a_{j-1},\; j\,=\,1,\,...,\,n$ proving eq. (64) for the coefficients $\widehat{S1p}(d,a;n,m)$.

6. Proof of eq. (65)

The second version is proved by using $risefac(d,a;x,n) \,=\, d^n \left(\dfrac{x\,+\,a}{d}\right)^{\overline{n}}$ (see eq. (63)) and the known result (*e.g.*, , [9], p. 263, eq. (6.11)) for $\mathbf{S1p}$ as transition matrix

$$x^{\overline{n}} \,=\, \sum_{k=0}^{n} S1p(n,\,k)\, x^k, \quad k\,\in\,\mathbb{N}_0\,. \tag{155}$$

Now, with the binomial formula,

$$\begin{aligned} risefac(d,a;x,n) \;&=\; d^n \sum_{k=0}^{n} S1p(n,\,k) \left(\frac{x\,+\,a}{d}\right)^k \,=\, \sum_{k=0}^{n} d^{n-k}\, S1p(n,\,k) \sum_{m=0}^{k} \binom{k}{m} a^{k-m}\, x^m \\ &=\; \sum_{m=0}^{n} x^m \sum_{k=m}^{n} \binom{k}{m} S1p(n,\,k)\, a^{k-m}\, d^{n-k}\,, \end{aligned} \tag{156}$$

and the coefficient of x^m is $\widehat{S1p}(d,a;n,m)$ given in the second version of eq. (65) (with summation index $k \to j$). The first version of eq. (65) is obtained by changing $j \to n - j'$, using then again j as summation index.

An alternative proof can be given using the recurrence $risfac(d,a;x,n) = (x + (n-1)\,d + a)\, risefac(d,a;x,n-1)$ with input $risefac(d,a;x,0) = 1$. Then the *Pascal* recurrence (see A007318) and the known **S1p** recurrence (given by eq. (61) for $[d,a] = [1,0]$) are used.

7. Meixner type recurrence, and proof of eq. (66)

The *Meixner* type recurrence for the monic row polynomials $P\widehat{S1p}(d,a;n,x)$ uses the compositional inverse of the *Sheffer* f function which is $\frac{1}{d}\,(1-e^{-d\,y})$ (see eq. (85)). Therefore, $f^{[-1]}(\mathbf{d}_x)\, P\widehat{S1p}(d,a;n,x) = n\, P\widehat{S1p}(d,a;n-1,x)$ becomes

$$\sum_{k=1}^{n} (-1)^{k-1}\, \frac{d^{k-1}}{k!}\, (\mathbf{d}_x)^k\, P\widehat{S1p}(d,a;n,x) \;=\; n\, P\widehat{S1p}(d,a;n-1,x)\,, \tag{157}$$

with input $P\widehat{S1p}(d,a;0,x) = 1$.

The general *Sheffer* recurrence (see *Lemma* 5, eq. (87), also for the *Roman* reference) uses, with the *Sheffer* g and the given $f^{[-1]}$ function,

$$g(f^{[-1]}(t)) \;=\; e^{a\,t}, \quad \frac{d}{dt}\, g(f^{[-1]}(t)) \;=\; a\, e^{a\,t}, \quad \frac{d}{dt}\, f^{[-1]}(t) \;=\; e^{-d\,t}\,, \tag{158}$$

leading to

$$P\widehat{S1p}(d,a;n,x) \;=\; (x \,+\, a)\, e^{d\,\mathbf{d_x}}\, P\widehat{S1p}(d,a;n-1,x) \;=\; (x \,+\, a)\, P\widehat{S1p}(d,a;n-1,x+d), \tag{159}$$

by *Taylor*'s theorem, and this proves eq. (66).

Proof of eq, (67)

This is covered by *Corollary 1* (after eq. (82)).

Proof of eq. (69)

The direct way uses the inversion of eq. (18).

Lemma 10:

$$S2(n,\, m) \;=\; \left(\frac{-a}{d}\right)^{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k}\, a^{-k}\, S2(d,a;k,m)\,, \quad \text{for} \;\; n \geq m \geq 0\;. \tag{160}$$

Proof: From the exponential convolution eq. (18) one has for the column *e.g.f.* $ES2Col(d,a;t,m) = e^{a\,t}\, ES2Col(d\,t,m)$ (see eq. (13)). This means (for $d \in \mathbb{N}$) that

$$ES2Col(t,m) \;=\; e^{-\frac{a}{d}\,t}\, ES2Col\left(d,a;\frac{t}{d},m\right)\,, \tag{161}$$

which gives for $S2(n,\, m)$ the exponential convolution leading to the assertion. □

Then the proof of eq. (69) starts by inserting $S1p(n\,j)$, in the second version of eq. (65), by *Schlöhmilch*'s formula, eq. (68), with $S2(n-j+k,\, k)$ from the *Lemma*, eq. (160):

$$\begin{aligned} \widehat{S1p}(d,a;n,m) \;&=\; \sum_{j=m}^{n} \binom{j}{m}\, a^{j-m}\, d^{n-j}\, (-1)^{n-j} \sum_{k=0}^{n-j} (-1)^k \binom{n+k-1}{j-1} \binom{2\,n-j}{n-j-k}\; * \\ &\quad * \left(\frac{-a}{d}\right)^{n-j+k} \sum_{l=0}^{n-j+k} (-1)^l \binom{n+k-j}{l}\, a^{-l}\, d^k\, \widehat{S2}(d,a;l,k)\,, \end{aligned} \tag{162}$$

where $\mathbf{S2}[d,a]$ has been replaced by $\widehat{\mathbf{S2}}[d,a]$ (see the line before eq. (15)). Collecting a and d powers and the signs leads to eq. (69).

Another more complicated proof follows the one of the usual *Schlöhmilch* formula given in [3], p. 290. This uses the *Lagrange* inversion theorem for powers of a (formal) power series.

Lemma 11: **Lagrange theorem and inversion** [8], p. 523, eq. (29), [24], p. 133.
a) With $\tilde{f}(x) = f(y(x))$, $y(x) = a + x\,\varphi(y)$ (here as formal power series)

$$\tilde{f}(x) = f(a) + \sum_{n=1}^{\infty} \frac{x^n}{n!} \frac{d^{n-1}}{da^{n-1}} \left[\varphi^n(a)\, f'(a)\right] . \tag{163}$$

b) With $a = 0$, $y = x\,\psi(x)$, and $f(y) = y^k$, $k \in \mathbb{N}_0$ and $x(y) = y^{[-1]}(y)$ (compositional inverse)

$$\begin{aligned} x^k(y) &= \delta_{k,0} + k \sum_{n=1}^{\infty} \frac{y^n}{n!} \frac{d^{n-1}}{dt^{n-1}} \left[\left(\frac{1}{\psi(t)}\right)^n t^{k-1}\right]\Bigg|_{t=0} \\ &= \delta_{k,0} + \sum_{n=1}^{\infty} \frac{y^n}{n!} \sum_{j=0}^{n-1} \binom{n-1}{j} k^{\underline{n-j}} \left[\frac{d^j}{dt^j} \left(\frac{1}{\psi(t)}\right)^n\right] t^{k-n+j}\Bigg|_{t=0} \\ &= \delta_{k,0} + k! \sum_{m=k}^{\infty} \frac{y^m}{m!} \binom{m-1}{m-k} \left[\frac{d^{m-k}}{dt^{m-k}} \left(\frac{1}{\psi(t)}\right)^m\right]\Bigg|_{t=0} . \end{aligned} \tag{164}$$

Proof: Part **a)** is the standard *Lagrange* theorem.

The first equation of part **b)** follows by exchanging the rôle of y and x, using $\varphi(x) = \frac{1}{\psi(x)}$ and $0^k = \delta_{k,0}$. (See [8], pp. 524-525 for the case $k = 1$). The second equation uses the *Leibniz* rule. Then only $j = n-k \geq 0$ survives after evaluation at $t = 0$, and in the last formula the summation index has been changed for later purposes from n to m.

Corollary 2:

$$\frac{d^n}{dy^n} \left[\frac{1}{k!} (y^{[-1]}(y))^k\right]\Bigg|_{y=0} = \binom{n-1}{n-k} \frac{d^{n-k}}{dt^{n-k}} \left[\frac{1}{\psi^n(t)}\right]\Bigg|_{t=0} , \quad \text{for } n \geq k \in \mathbb{N}. \tag{165}$$

The δ term now disappeared for $k \geq 1$.
This coincides with the inversion formula of *Lagrange* given in [3], Theorem 11.11, p. 435, used in the proof on p. 290.

Another formula is needed to convert later the negative powers of ψ in the *Corollary* into positive ones. This is given in [3], as *Remark 11.5*, p. 432, also used in the proof on p. 290.

$$\frac{d^m}{dt^m} \left[(h(t))^s\right]\Bigg|_{t=0} = \sum_{r=0}^{m} \binom{s}{r} \binom{m-s}{m-r} \frac{d^m}{dt^m} \left[(h(t))^r\right]\Bigg|_{t=0} , \quad \text{for } h(0) = 1,\ m \in \mathbb{N}_0,\ s \in \mathbb{R} . \tag{166}$$

Now we start with the derivation of a generalized *Schlöhmilch* formula (we use here the column index k).

$$\widehat{S1p}(d, a; n, k) = \frac{d^n}{dy^n} \left[(1 - d\,y)^{-\frac{a}{d}} \frac{1}{k!} \left(-\frac{1}{d} \log(1 - d\,y)\right)^k\right]\Bigg|_{y=0} \quad \text{for } 0 \leq k \leq n . \tag{167}$$

The result for $k = 0$ is known from eq. (62) to be $risefac(d, a; x = 0, n)$. The *Leibniz* rule is applied (because there is no closed formula for the inverse of the product in the solid brackets, written as $k-$th power). The derivatives of the first factor are known (see the $k = 0$ result) as

$$\frac{d^{n-m}}{dy^{n-m}} \left[(1 - d\,y)^{-\frac{a}{d}}\right]\Bigg|_{y=0} = risefac(d, a; x = 0, n - m) . \tag{168}$$

For the second factor the *Corollary* is applied with $n \to m$, and the known compositional inverse of $y^{[-1]}(d; y) = -\frac{1}{d} \log(1 - d\,y)$ *viz* $y(d; t) = t\,\psi(d; t) = \frac{1}{d}(1 - e^{-d\,t})$, is used.

$$\frac{d^m}{dy^m} \left[\frac{1}{k!} \left(-\frac{1}{d} \log(1 - d\,y)\right)^k\right]\Bigg|_{y=0} = \binom{m-1}{m-k} \frac{d^{m-k}}{dt^{m-k}} \left[\left(\frac{1 - e^{-d\,t}}{d\,t}\right)^{-m}\right]\Bigg|_{t=0} . \tag{169}$$

The negative power on the *r.h.s.* is now converted with the help of eq. (166) with $s \to -m$ and $m \to m-k$. Then the binomial with negative upper entry is rewritten as $\binom{-m}{r} = (-1)^r \binom{r+m-1}{r}$ (see eq. (95)), and we use the abbreviation $\psi(d;t) = \frac{1 - e^{-d\,t}}{d\,t}$ from above.

$$\frac{d^m}{dy^m}\left[\frac{1}{k!}\left(-\frac{1}{d}\,\log(1 - d\,y)\right)^k\right]\Bigg|_{y=0} = \binom{m-1}{m-k}\sum_{r=0}^{m-k}(-1)^r\binom{r+m-1}{r}\binom{2\,m-k}{m-k-r}\frac{d^{m-k}}{dt^{m-k}}\,(\psi(d;t)^r)\Big|_{t=0}. \tag{170}$$

With the *e.g.f.* $E\widehat{S2}Col(d,a;x,r) = e^{a\,x}\,\frac{\left(\frac{1}{d}\,(e^{d\,x} - 1)\right)^r}{r!}$ for column r of $\widehat{\mathbf{S2}}[\mathbf{d},\mathbf{a}]$ from its *Sheffer* structure one obtains after multiplication with x^{-r} and a sign flip $x = -t$

$$\psi(d;t)^r = \left(\frac{1 - e^{-d\,t}}{d\,t}\right)^r = e^{a\,t}\sum_{n=r}^{\infty}(-1)^{n-r}\,\widehat{S2}(d,a;n,r)\,r!\frac{t^{n-r}}{n!} =: e^{a\,t}\,A(d,a;t,r). \tag{171}$$

For the $(m-k)$-th derivative evaluated at $t = 0$ the *Leibniz* rule is again applied with a summation index p. The exponential factor leads to a^{m-k-p}. The p-th derivative w.r.t. t of $A(d,a;t,r)$ is evaluated at $t = 0$ after the index shift $n - r = s$ in A. This leads to the collapse of the $s-$sum due to the $t = 0$ evaluation, whence $s = p$, and $p^{\underline{p}} = p!$. The result is

$$\frac{d^p}{dt^p}\,A(d,a;t,r)\Big|_{t=0} = \frac{r!\,p!}{(p+r)!}\,(-1)^p\,\widehat{S2}(d,a;p+r,r)\,. \tag{172}$$

Thus

$$\frac{d^{m-k}}{dt^{m-k}}\,(\psi(d;t)^r)\Big|_{t=0} = \sum_{p=0}^{m-k}\binom{m-k}{p}\,a^{m-k-p}\,(-1)^p\,\frac{1}{\binom{p+r}{p}}\,\widehat{S2}(d,a,p+r,r)\,,\quad \text{for}\quad m \geq k\,. \tag{173}$$

This leads, with the abbreviation $risefac(d,a,x=0,n-m) = (d,a)^{\overline{n-m}}$, to

$$\begin{aligned}\widehat{S1p}(d,a;n,k) &= \sum_{m=k}^{n}\binom{n}{m}\,(d,a)^{\overline{n-m}}\binom{m-1}{m-k}\sum_{r=0}^{m-k}(-1)^r\binom{r+m-1}{r}\binom{2\,m-k}{m+r} * \\ &\quad * \sum_{p=0}^{m-k}(-1)^p\,\frac{\binom{m-k}{p}}{\binom{p+r}{r}}\,a^{m-k-p}\,\widehat{S2}(d,a;p+r,r)\,,\quad \text{for } n \geq m \geq 1,\end{aligned} \tag{174}$$

A new summation index $m - k = m'$ is used, and the sum over the triangular array, with rows indexed by m and columns by r, is reordered by summing first the columns $r = 0, \ldots, n-k$ and then the rows $m = r, \ldots, n-k$. The binomials in these two sums are rewritten and the final result is (using column index m instead of k)

$$\begin{aligned}\boxed{\widehat{S1p}(d,a;n,m)} &= \frac{n!}{(m-1)!}\sum_{r=0}^{n-m}\frac{(-1)^r}{r!}\sum_{k=r}^{n-m}(d,a)^{\overline{n-k-m}}\binom{2\,k+m}{k+m}\frac{1}{k+m+r}\,\frac{1}{(n-m-k)!}\,\frac{1}{(k-r)!} * \\ &\quad * \sum_{p=0}^{k}(-1)^p\,\frac{\binom{k}{p}}{\binom{p+r}{r}}\,a^{k-p}\,\widehat{S2}(d,a;p+r,r)\ \text{ for } n \geq m \geq 1,\end{aligned} \tag{175}$$

and $\widehat{S1p}(d,a;n,0) = risefac(d,a;x=0,n)$.

The binomials could be rewritten by building some binomials, but no essential simplification seems to be possible. Both generalizations of the *Schlöhmilch* formula involve three sums, but the direct version given in eq. (69) looks simpler than the second version eq. (175).

The statements in *section D* are obvious.

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2010 Mathematics Subject Classification: Primary 11B68, 11B73, Secondary 05A15, 11B25, 11B37.

Keywords: Sums of powers of positive integers, arithmetic progression, generating functions, Stirling numbers, Lah numbers, Eulerian numbers, Bernoulli numbers.

OEIS [21] A numbers:

A000142, A000165, A001147, A006232, A007559, A007318, A007559, A007696, A008297, A008544, A008545, A008546, A008548, A027641, A027642, A028338, A032031, A039755, A047053, A048993, A052562, A053382, A053383, A060187, A123125, A131689, A132393, A141459, A143395, A145901, A154537, A157779, A157799, A157817, A157833, A157866, A173018, A196838, A196839, A225117, A225118, A225466, A225467, A225468, A225472, A225473, A239275, A271703, A282629, A284861, A285061, A285066, A285068, A285863, A286718, A288872, A288873.
