Hamiltonicity of token graphs of fan graphs

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Abstract

In this note we show that the token graphs of fan graphs are Hamiltonian. This result provides another proof of the Hamiltonicity of Johnson graphs and also extends previous results obtained by Mirajkar and Priyanka Y. B.

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1 Introduction.

Let G be a simple graph of order n and let k be an integer such that $1 \leq k \leq n-1$. The *k-token graph*, or *symmetric kth power*, of G is the graph $G^{\{k\}}$ whose vertices are the k-subsets of $V(G)$ and two vertices are adjacent in $G^{\{k\}}$ if their symmetric difference is an edge of G. A classical example is the Johnson graph $J(n,k)$, that is isomorphic to the k-token graph of the complete graph K_n.

The definition of k-token graphs (without a name) appeared in a work of Rudolph [17], in connection with problems in quantum mechanics and with the graph isomorphism problem. Rudolph presented examples of cospectral graphs G and H such that their corresponding 2-token graphs are not cospectral. Audenaert et al. [5], proved that the 2-token graphs of strongly regular graphs with the same parameters are cospectral, and suggested that for a given positive integer k there exists infinitely many pairs of non-isomorphic graphs with cospectral k-token graphs. This conjecture was proved by Barghi and Ponomarenko [7] and, independently, by Alzaga et al. [3]. Later, Fabila-Monroy et al. [10] reintroduced the k-token graphs as part of several models of swapping in the literature [12, 19], and studied some properties of these graphs: connectivity, diameter, cliques, chromatic number, Hamiltonian paths and Cartesian product of token graphs. This line of research was continued by Carballosa et al., [8] who studied regularity and planarity, de Alba et al., [2], who studied independence and matching numbers, and Mirajkar et al., [15], who studied some covering properties of token graphs. Finally, de Alba et. al., [1] classified the triangular graphs, that are the 2-token graphs of complete graphs, that are Cohen-Macaulay.

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The k-token graph of a complete graph K_n is isomorphic to the Johnson graph $J(n,k)$. This class of graphs is widely studied and has connections with coding theory [13, 9, 11, 14, 16] (another connection of token graphs with coding theory was showed in [18]). A Hamiltonian cycle of G is a cycle that contains all the vertices of G. A graph is Hamiltonian if it contains a Hamiltonian cycle. It is well known that $J(n,k)$ is Hamiltonian [6, 20], in fact, it is Hamiltonian connected [4]. As was noted in [10], the existence of a Hamiltonian cycle in G does not imply that $G^{(k)}$ contains a Hamiltonian cycle. For example, if k is even then $K_{m,m}^{(k)}$ is not Hamiltonian. We are interested in graphs G such that its token graphs are Hamiltonian. The fan graph F_n is the join of graphs K_1 and P_{n-1}. In this note we show that the token graphs of fan graphs is Hamiltonian. Our result provides another proof that $J(n,k)$ is Hamiltonian, and also extends some of the results obtained by Mirajkar and Priyanca Y. B. [15] about the Hamiltonicity of the token graphs of wheel graphs.

2 Main result

First we present some definitions and notations. For vertices u, v in graph G we write $u \sim v$ to mean that u and v are adjacent vertices in G. We write $G \simeq G'$ if G and G' are isomorphic graphs. If H is a subgraph of G then we say that G is a supergraph of H. A spanning subgraph of G is a subgraph H such that $V(H) = V(G)$. The following proposition is obvious.

Proposition 2.1. *If H is a spanning subgraph of G and H is Hamiltonian then G is Hamiltonian.*

One of the main properties of token graphs is that $G^{(1)}$ and G are isomorphic. Moreover, $G^{(k)} \simeq G^{(n-k)}$ for any $k \in \{1, \dots, n-1\}$. Another known property of token graphs is the following.

Proposition 2.2. *If H is a subgraph of G then $H^{(k)}$ is a subgraph of $G^{(k)}$. Even more, if H is a spanning subgraph of G then $H^{(k)}$ is a spanning subgraph of $G^{(k)}$.*

For a fan graph F_n we assume that the vertices of P_{n-1} are $\{1, \dots, n\}$ and the vertex in K_1 is labeled as n. For vertice $A = \{a_1, \dots, a_k\}$ of $F_n^{(k)}$ we use the convention that $a_1 < \dots < a_k$.

The main result of this note is the following.

Theorem 2.3. *Let n and k be positive integers with $n \geq 3$ and $1 \leq k \leq n-1$. Then $F_n^{\{k\}}$ is Hamiltonian.*

Proof. For $k = 1$, $F_n^{(1)} \simeq F_n$ that is Hamiltonian so in the rest of the proof we assume that $k \geq 2$. We will show that $F_n^{(k)}$ has a Hamiltonian cycle such that the vertices $\{n-k, n-k+1, \dots, n-2, n\}$ and $\{n-k, n-k+1, \dots, n-2, n-1\}$ are adjacent in the cycle. As $F_n^{(k)} \simeq F_n^{(n-k)}$ and $F_n^{(1)} \simeq F_n$, for $n \in \{3, 4, 5\}$ it is enough to prove the case $k = 2$. The sequence of vertices $\{1,3\}\{1,2\}\{2,3\}\{1,3\}$ is a Hamiltonian cycle in $F_3^{(2)}$. The sequence of vertices $\{1,3\}\{1,4\}\{1,2\}\{2,4\}\{2,3\}\{3,4\}\{1,3\}$ is a Hamiltonian cycle in $F_4^{(2)}$, and the sequence of vertices

$$\{1,4\}\{1,5\}\{1,3\}\{1,2\}\{2,5\}\{2,4\}\{2,3\}\{3,5\}\{3,4\}\{4,5\}\{1,4\},$$

is a Hamiltonian cycle in $F_5^{(2)}$.

The proof for $n \geq 6$ is by double induction. First we show the case $k = 2$ and $n \geq 6$. The sequence of vertices

$$\begin{array}{l}
\{1,n-1\}\{1,n\}\{1,n-2\}\{1,n-3\}\ldots\{1,3\}\{1,2\} \\
\{2,n\}\{2,n-1\}\{2,n-2\}\{2,n-3\}\ldots\{2,4\}\{2,3\} \\
\qquad\qquad\qquad\qquad\vdots \\
\{n-3,n\}\{n-3,n-1\}\{n-3,n-2\} \\
\{n-2,n\}\{n-2,n-1\} \\
\{n-1,n\} \\
\{1,n-1\},
\end{array}$$

is a Hamiltonian cycle in $F_n^{(2)}$, where vertices $\{n-2,n-1\}$ and $\{n-2,n\}$ are adjacent in the cycle. We assume as induction hypothesis that $F_{n'}^{(k')}$ satisfies the conditions whenever $k' < k$ and $n' > k'$, or $F_{n'}^{(k)}$ satisfies the conditions whenever $n' < n$ and $n' > k$.

Claim 2.4. *Let S_i be the subgraph of $F_n^{(k)}$ induced by the vertex set*

$$V_i = \left\{\{a_1,\ldots,a_k\} \in V(F_n^{(k)})\colon a_1 = i\right\},$$

with $1 \leq i \leq n-k$. Then S_i and $F_{n-i}^{(k-1)}$ are isomorphic.

Proof. Suppose that $V(F_{n-i}) = \{i+1,\ldots,n\}$ with $V(P_{n-i-1}) = \{i+1,\ldots,n-1\}$ and n the vertex of K_1. Then the function $A \mapsto A \setminus \{i\}$ is a graph isomorphism between S_i and $F_{n-i}^{(k-1)}$. □

We identify S_i with $F_{n-i}^{(k-1)}$ using the isomorphism given in the proof of the claim. By induction there exists a Hamiltonian cycle C_i in S_i, where vertices $X_i := \{i, n-k+1,\ldots,n-2,n-1\}$ and $Y_i := \{i,n-k+1,\ldots,n-2,n\}$ are adjacent in C_i, for $1 \leq i \leq n-k$. Let P_i be the Hamiltonian subpath of C_i from X_i to Y_i, for $1 \leq i \leq n-k$. Let Y_{n-k+1} denote the vertex $\{n-k+1,n-k+2,\ldots,n-1,n\}$. Therefore $V_{n-k+1} = \{Y_{n-k+1}\}$.

The vertices of S_{n-k} are

$$
\begin{aligned}
V_{n-k} = \Big\{ & \{n-k, n-k+1, n-k+2, \dots, n-3, n-2, n-1\}, \\
& \{n-k, n-k+1, n-k+2, \dots, n-3, n-2, n\}, \\
& \{n-k, n-k+1, n-k+2, \dots, n-3, n-1, n\}, \\
& \qquad \vdots \\
& \{n-k, n-k+1, n-k+3, \dots, n-2, n-1, n\}, \\
& \{n-k, n-k+2, n-k+3, \dots, n-2, n-1, n\} \Big\}.
\end{aligned}
$$

Let $D_i = \{n-k, n-k+1, \dots, n-1, n\} \setminus \{i\}$, with $n-k+1 \leq i \leq n$. Then $V_{n-k} = \{D_n, D_{n-1}, \dots, D_{n-k+1}\}$, $X_{n-k} = D_n$ and $Y_{n-k} = D_{n-1}$. Let

$$Q = D_{n-2} D_{n-3} \dots D_{n-k+2} D_{n-k+1},$$

that, in fact, is a path in S_{n-k} because $D_i \triangle D_{i-1} = \{i-1, i\}$, for $n-k+2 \leq i \leq n-2$. Now,

$$
\begin{aligned}
X_{n-k} \triangle D_{n-2} &= \{n-2, n\} \\
Y_{n-k} \triangle D_{n-2} &= \{n-2, n-1\} \\
Y_{n-k+1} \triangle D_{n-k+1} &= \{n-k, n-k+1\}
\end{aligned}
$$

and hence

$$
\begin{aligned}
& X_{n-k} \sim D_{n-2}, \\
& Y_{n-k} \sim D_{n-2}, \\
& D_{n-k+1} \sim Y_{n-k+1},
\end{aligned}
$$

in $F_k(A_n)$. Notice that $X_i \triangle X_{i+1} = \{i, i+1\}$, $Y_i \triangle Y_{i+1} = \{i, i+1\}$, for $1 \leq i \leq n-k-1$, and $X_1 \triangle Y_{n-k+1} = \{1, n\}$. Therefore we can define a Hamiltonian cycle $\mathcal{C}$ in $F_k(A_n)$ as

$$X_1 \underset{P_1}{\longrightarrow} Y_1 Y_2 \underset{P_2}{\longrightarrow} X_2 \dots X_{(n-k-1)} \underset{P_{n-k-1}}{\longrightarrow} Y_{(n-k-1)} Y_{(n-k)} X_{(n-k)} D_{n-2} \underset{Q}{\longrightarrow} D_{n-k+1} Y_{(n-k+1)} X_1,$$

if $n-k$ is even, and

$$X_1 \underset{P_1}{\longrightarrow} Y_1 Y_2 \underset{P_2}{\longrightarrow} X_2 \dots Y_{(n-k-1)} \underset{P_{n-k-1}}{\longrightarrow} X_{(n-k-1)} X_{(n-k)} Y_{(n-k)} D_{n-2} \underset{Q}{\longrightarrow} D_{n-k+1} Y_{(n-k+1)} X_1,$$

if $n-k$ is odd. Furthermore

$$\{n-k, n-k+1, ..., n-2, n-1\} = X_{n-k} \sim Y_{n-k} = \{n-k, n-k+1, ..., n-2, n\},$$

in $\mathcal{C}$, as desired.

$\square$

Corollary 2.5. *The matching number of $F_n^{\{k\}}$ is $\lfloor \binom{n}{k}/2 \rfloor$.*

The wheel graph W_n is the joint graph of K_1 and C_{n-1}.

Corollary 2.6. *If G is a supergraph of F_n such that $V(G) = V(H)$ then $G^{(k)}$ is Hamiltonian. In particular the Johnson graphs and the k-token graphs of wheel graphs are Hamiltonian.*

Proof. As F_n is a spanning subgraph of G then $H^{(k)}$ is a spanning subgraph of $G^{(k)}$ by Proposition 2.2. The k-token graph of F_n is Hamiltonian by Theorem 2.3 and hence $G^{(k)}$ is Hamiltonian by Proposition 2.1. In particular F_n is an spanning subgraph of W_n and K_n. □

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