Preprint, arXiv:1707.06223

SOME UNIVERSAL QUADRATIC SUMS OVER THE INTEGERS

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Abstract. Let $a, b, c, d, e, f \in \mathbb{N}$ with $a \geqslant c \geqslant e > 0$, $b \leqslant a$ and $b \equiv a \pmod 2$, $d \leqslant c$ and $d \equiv c \pmod 2$, $f \leqslant e$ and $f \equiv e \pmod 2$. If any nonnegative integer can be written as $x(ax+b)/2 + y(cy+d)/2 + z(ez+f)/2$ with $x, y, z \in \mathbb{Z}$, then the tuple (a, b, c, d, e, f) is said to be universal over $\mathbb{Z}$. Recently, Z.-W. Sun found all candidates of such universal tuples over $\mathbb{Z}$. In this paper, we use the theory of ternary quadratic forms to show that 38 concrete tuples (a, b, c, d, e, f) in Sun's list of candidates are indeed universal over $\mathbb{Z}$. For example, we prove the universality of $(16, 4, 2, 0, 1, 1)$ over $\mathbb{Z}$ which is related to the famous form $x^2 + y^2 + 32z^2$.

1. Introduction

Those $T_x = x(x+1)/2$ with $x \in \mathbb{Z}$ are called triangular numbers. In 1796 Gauss proved Fermat's assertion that each $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be expressed as the sum of three triangular numbers.

For polynomials $f_1(x), f_2(x), f_3(x)$ with $f_i(\mathbb{Z}) = \{f_i(x) : \ x \in \mathbb{Z}\} \subseteq \mathbb{N}$ for $i = 1, 2, 3$, if any $n \in \mathbb{N}$ can be written as $f_1(x) + f_2(y) + f_3(z)$ with $x, y, z \in \mathbb{Z}$ then we call the sum $f_1(x) + f_2(y) + f_3(z)$ *universal over* $\mathbb{Z}$. For example, $T_x + T_y + T_z$ is universal over $\mathbb{Z}$ by Gauss' result.

In 1862 Liouville (cf. [2, p. 82]) determined all universal sums $aT_x + bT_y + cT_z$ over $\mathbb{Z}$ with $a, b, c \in \mathbb{Z}^+$. Z.-W. Sun [23, 24] studied universal sums of the form $ap_i(x) + bp_j(y) + cp_k(z)$ with $a, b, c \in \mathbb{N}$ and $i, j, k \in \{3, 4, \ldots\}$, where $p_m(x)$ denotes the generalized polygonal number $(m-2)\binom{x}{2} + x$; see also [11, 19, 10, 18, 16] for subsequent work on some of Sun's conjectures posed in [23, 24]. In 2017 Sun [26] investigated universal sums $x(ax+1) + y(by+1) + z(cz+1)$ over $\mathbb{Z}$ with $a, b, c \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and also universal sums $x(ax+b) + y(ay+c) + z(az+d)$ with $a, b, c, d \in \mathbb{N}$ and $a \geqslant b \geqslant c \geqslant d$. Quite recently, Sun [27] investigated for what tuples (a, b, c, d, e, f) with $a \geqslant c \geqslant e \geqslant 1$, $b \equiv a \pmod 2$ and $|b| \leqslant a$, $d \equiv c \pmod 2$ and $|d| \leqslant c$,

2010 *Mathematics Subject Classification*. Primary 11E25; Secondary 11D85, 11E20.
Keywords. Universal sums, quadratic polynomials, ternary quadratic forms.
Supported by the National Natural Science Foundation of China (Grant No. 11571162).

$f \equiv e \pmod 2$ and $|f| \leqslant e$, the sum

$$\frac{x(ax+b)}{2}+\frac{y(cy+d)}{2}+\frac{z(ez+f)}{2}$$

is universal over $\mathbb{Z}$. Such tuples (a,b,c,d,e,f) are said to be universal over $\mathbb{Z}$. He showed such tuples with $|b|<a$, $|d|<c$, $|f|<e$, and $b \geqslant d$ if $a=c$, and $d \geqslant f$ if $c=e$, must be in his list of 12082 candidates (cf. [28, A286944]), and conjectured that all such candidates are indeed universal over $\mathbb{Z}$. Note that

$$\left\{\frac{x(x-1)}{2}:\ x\in\mathbb{Z}\right\}=\{T_x:\ x\in\mathbb{Z}\}=\{x(2x+1):\ x\in\mathbb{Z}\}.$$

Sun [27] proved that some candidates (a,b,c,d,e,f) are universal over $\mathbb{Z}$, e.g. $(5,1,3,1,1,1)$ (equivalent to $(5,1,4,2,3,1)$) is universal over $\mathbb{Z}$. Sun even conjectured that any $n\in\mathbb{N}$ can be written as $x(x+1)/2+y(3y+1)/2+z(5z+1)/2$ with $x,y,z\in\mathbb{N}$.

In this paper, via the theory of ternary quadratic forms, we establish the universality (over $\mathbb{Z}$) of 38 concrete tuples (a,b,c,d,e,f) in Sun's list of candidates.

Theorem 1.1. *The tuples*

$$\begin{gathered}(5,1,2,2,1,1),\ (6,0,3,3,3,1),\ (6,2,5,5,1,1),\\ (6,6,3,3,3,1),\ (8,2,3,1,1,1),\ (8,6,3,1,1,1),\ (8,8,3,1,1,1)\end{gathered}$$

are universal over $\mathbb{Z}$.

Remark 1.1. Our proof of Theorem 1.1 uses some special techniques. Sun [24] conjectured that any $n\in\mathbb{N}$ can be written as $T_x+2T_y+p_7(z)$ with $x,y,z\in\mathbb{N}$, and J. Ju, B.-K. Oh and B. Seo [16] proved that $T_x+2T_y+p_7(z)$ (or the tuple $(5,3,2,2,1,1)$) is universal over $\mathbb{Z}$.

Similarly to [27, Theorem 1.4], we observe that $\{T_x+p_5(y):\ x,y\in\mathbb{Z}\}=\{p_5(x)+3p_5(y):\ x,y\in\mathbb{Z}\}$ which can be easily proved.

Theorem 1.2. *The tuples*

$$(6,0,5,1,3,1),\ (6,0,5,3,3,1),\ (7,1,1,1,1,1),\ (7,1,2,0,1,1),$$
$$(7,1,2,2,1,1),\ (7,1,3,1,1,1),\ (7,1,3,3,1,1),\ (7,3,1,1,1,1),$$
$$(7,3,2,0,1,1),\ (7,3,2,2,1,1),\ (7,3,3,1,1,1),\ (7,3,3,3,1,1),$$
$$(7,5,1,1,1,1),\ (7,5,3,1,1,1),\ (7,5,3,3,1,1),\ (15,3,3,1,1,1),$$
$$(15,5,1,1,1,1),\ (15,5,3,1,2,0),\ (15,5,3,1,2,2),\ (15,9,3,1,1,1),$$
$$(21,7,3,1,2,2)$$

are universal over $\mathbb{Z}$.

Remark 1.2. Our proof of Theorem 1.2 involves the theory of genera of ternary quadratic forms. Sun [24] conjectured that any $n \in \mathbb{N}$ can be written as $T_x + y^2 + p_9(z)$ (or $T_x + 2T_y + p_9(z)$) with $x, y, z \in \mathbb{N}$, and Ju, Oh and Seo [16] proved that $T_x + y^2 + p_9(z)$ and $T_x + 2T_y + p_9(z)$ are universal over $\mathbb{Z}$, i.e., the tuples $(7,5,2,0,1,1)$ and $(7,5,2,2,1,1)$ are universal over $\mathbb{Z}$.

Theorem 1.3. (i) *The tuples* $(5,5,3,1,3,1)$, $(5,5,3,3,3,1)$, $(6,4,5,5,1,1)$ *and* $(7,7,3,1,1,1)$ *are universal over* $\mathbb{Z}$.

(ii) *All the five tuples*

$$(6,2,5,1,1,1), (6,2,5,5,1,1), (6,4,5,1,1,1), (15,5,6,2,1,1), (15,5,6,4,1,1)$$

are universal over $\mathbb{Z}$.

Remark 1.3. Our proof of Theorem 1.3(i) employs the Minkowski-Siegel formula (cf. [17, pp. 173–174]). Sun [24] conjectured that any $n \in \mathbb{N}$ can be written as $T_x + p_7(y) + 2p_5(z)$ (or $T_x + p_7(y) + p_8(z)$) with $x, y, z \in \mathbb{N}$, and Ju, Oh and Seo [16] proved that $T_x + p_7(y) + 2p_5(z)$ and $T_x + p_7(y) + p_8(z)$ are universal over $\mathbb{Z}$, i.e., the tuples $(6,2,5,3,1,1)$ and $(6,4,5,3,1,1)$ are universal over $\mathbb{Z}$.

Theorem 1.4. *The tuple* $(16,4,2,0,1,1)$ *is universal over* $\mathbb{Z}$. *In other words, any* $n \in \mathbb{N}$ *can be written as* $T_x + y^2 + 2z(4z+1)$ *with* $x, y, z \in \mathbb{Z}$.

Remark 1.4. This result is closely related to the famous form $x^2 + y^2 + 32z^2$. Sun [27] even conjectured that any $n \in \mathbb{N}$ can be written as $T_x + y^2 + 2z(4z-1)$ with $x, y, z \in \mathbb{N}$.

We will show Theorems 1.1-1.4 in Sections 2-5 respectively.

In view of Theorems 1.1-1.3, [27, Theorem 1.4], and some basic facts on regular quadratic forms, among those conjectural universal tuples (a,b,c,d,e,f) with $a=6\geqslant c\geqslant e\geqslant 2$, $b\in(-a,a)$, $d\in(-c,c)$, $f\in(-e,e)$ and $a-b,c-d,e-f$ all even listed in [28, A286944], only the universality of the tuples

$$\begin{aligned}&(6,0,5,1,4,2),\ (6,0,5,3,4,2),\ (6,2,5,3,4,0),\ (6,2,5,3,5,3),\\&(6,2,6,0,5,3),\ (6,2,6,2,5,3),\ (6,4,5,1,4,0),\ (6,4,5,1,5,1),\\&((6,4,5,3,2,0),\ (6,4,5,3,4,0),\ (6,4,5,3,5,3),\ (6,4,6,0,5,1),\\&(6,4,6,0,5,3)\end{aligned}$$

remains open.

2. Proof of Theorem 1.1

Lemma 2.1. *Let V be a quadratic space. For any isometry $T\in O(V)$ of infinite order,*

$$V_T=\{x\in V: \text{there is a positive integer } k \text{ such that } T^k(x)=x\}.$$

is a subspace of V with dimension one, and $T(x)=\det(T)x$ for any $x\in V_T$.

Remark 2.1. Any unexplained notation in the theory of quadratic forms can be found in [4, 17, 20]. Lemma 2.1 is a known result, see, e.g., [18].

Lemma 2.2. (i) *For any $n\in\mathbb{N}$, we can write $12n+5$ as $x^2+y^2+(6z)^2$ with $x,y,z\in\mathbb{Z}$.*

(ii) *Let $n\in\mathbb{Z}^+$ and $\delta\in\{0,1\}$. Then we can write $6n+1$ as $x^2+3y^2+6z^2$ with $x,y,z\in\mathbb{Z}$ and $x\equiv\delta\pmod 2$.*

Remark 2.2. Lemma 2.2 is a known result due to the second author, see [24, Theorem 1.7(iii) and Lemma 3.3] and [26, Remark 3.1].

John S. Hsia, in a letter to Irving Kaplansky in 1993, proved that $x^2+y^2+10z^2$ represents all eligible numbers of the form $3m+2$ (cf. [14, pp. 12–14]). As all positive odd numbers are eligible by Hensel's lemma, we have the following lemma.

Lemma 2.3. *For each $n\in\mathbb{N}$, we can write $6n+5$ as $x^2+y^2+10z^2$ with $x,y,z\in\mathbb{Z}$.*

For $a,b,c\in\mathbb{Z}^+$, we define

$$E(a,b,c)=\{n\in\mathbb{N}:\ n\neq ax^2+by^2+cz^2 \text{ for all } x,y,z\in\mathbb{Z}\}.$$

L.E. Dickson [7, pp. 112-113] listed all the 102 primitive regular diagonal quadratic forms $ax^2 + by^2 + cz^2$ for which the structure of $E(a, b, c)$ is known explicitly. For example, the Gauss-Legendre theorem asserts that $E(1, 1, 1) = \{4^k(8l + 7) : k, l \in \mathbb{N}\}$.

In 1996 W. Jagy [12] investigated so-called *nearly regular* quadratic forms, and showed the following result (cf. [14, pp. 25–26]).

Lemma 2.4. *We have*

$$E(1, 4, 9) = \{2\} \cup \bigcup_{k,l\in\mathbb{N}} \{4^k(8l + 7),\ 8l + 3,\ 9l + 3\}.$$

Proof of Theorem 1.1. (i) Let $n \in \mathbb{N}$ and $r \in \{1, 3\}$. Apparently,

$$\begin{aligned} &n = T_x + y(y + 1) + \frac{z(5z + r)}{2} \\ \iff &40n + r^2 + 15 = 5(2x + 1)^2 + 10(2y + 1)^2 + (10z + r)^2. \end{aligned}$$

Since

$$E(1, 5, 10) = \{25^k m : k, m \in \mathbb{N} \text{ and } m \equiv 2, 3 \pmod 5\},$$

we have $40 + r^2 + 15 \in \{x^2 + 5y^2 + 10z^2 : x, y, z \in \mathbb{N}\}$. Thus we can write

$$40n + r^2 + 15 = (2^k x_0)^2 + 5(2^k y_0)^2 + 10(2^k z_0)^2 = 4^k(x_0^2 + 5y_0^2 + 10z_0)^2$$

with $k \in \mathbb{N}$, $x_0, y_0, z_0 \in \mathbb{Z}$, and x_0, y_0, z_0 not all even. In the case $k = 0$, if $2 \mid z_0$ then $x_0^2 + 5y_0^2 \equiv r^2 + 15 \equiv 0 \pmod 8$ and hence $x_0 \equiv y_0 \equiv 0 \pmod 2$ which contradicts that x_0, y_0, z_0 are not all even, thus $2 \nmid z_0$ and also $2 \nmid x_0y_0$ since $x_0^2 + 5y_0^2 \equiv r^2 + 15 - 10z_0^2 \equiv 6 \pmod 8$.

It is easy to verify the following new identity:

$$4^2(x^2+5y^2+10z^2) = (x-5y-10z)^2+5(x+3y-2z)^2+10(x-y+2z)^2. \quad (2.1)$$

If x, y, z are odd integers, then by (2.1) we have

$$4(x^2 + 5y^2 + z^2) = \bar{x}^2 + 5\bar{y}^2 + 10\bar{z}^2$$

with

$$\tilde{x} = \frac{x-y}{2} - 2y - 5z,\ \tilde{y} = \frac{x-y}{2} + 2y - z,\ \tilde{z} = \frac{x-y}{2} + z$$

all odd. Thus, if $2 \nmid x_0y_0z_0$ then

$$40n+r^2+15 = 4^k(x_0^2+5y_0^2+10z_0^2) \in \{x^2+5y^2+10z^2 : x,y,z \text{ are odd}\}. \quad (2.2)$$

If $x_0 \not\equiv y_0 \pmod 2$, then $x_0^2 + 5y_0^2 + 10z_0^2 \equiv 1 \pmod 2$ and $k \geqslant 2$ since $40n + r^2 + 15 \equiv 0 \pmod 8$, hence by (2.1) we have

$$4^2(x_0^2 + 5y_0^2 + 10z_0^2) = \bar{x}_0^2 + 5\bar{y}_0^2 + 10\bar{z}_0^2$$

with $\bar{x}_0 = x_0 - 5y_0 - 10z_0$, $\bar{y}_0 = x_0 + 3y_0 - 2z_0$ and $\bar{z}_0 = x_0 - y_0 + 2z_0$ all odd, and therefore (2.2) holds.

Now we suppose that $k > 0$, $2 \mid x_0y_0z_0$ and $x_0 \equiv y_0 \pmod 2$. By (2.1),

$$4(x_0^2 + 5y_0^2 + 10z_0^2) = x_1^2 + 5y_1^2 + 10z_1^2$$

with

$$x_1 = \frac{x_0 - y_0}{2} - 2y_0 - 5z_0,\ y_1 = \frac{x_0 - y_0}{2} + 2y_0 - z_0,\ z_1 = \frac{x_0 - y_0}{2} + z_0$$

If x_0 and y_0 are odd, then we may assume $x_0 \not\equiv y_0 - 2z_0 \pmod 4$ without loss of generality (otherwise we replace x_0 by $-x_0$), and hence x_1, y_1, z_1 are all odd. If $x_0, y_0, (x_0 - y_0)/2$ are all even, then z_0 is odd and so are x_1, y_1, z_1. If x_0 and y_0 are even with $x_0 \not\equiv y_0 \pmod 4$, then z_0 is odd and we may assume $z_0 \equiv (y_0 - x_0)/2 \pmod 4$ without loss of generality (otherwise we replace z_0 by $-z_0$), hence $z_1 \equiv 0 \pmod 4$, $y_1 = z_1 + 2(y_0 - z_0) \equiv 0 \pmod 2$ and $(x_1 - y_1)/4 \equiv -y_0 - z_0 \equiv 1 \pmod 2$, therefore by (2.1) we have

$$x_1^2 + 5y_1^2 + 10z_1^2 = x_2^2 + 5y_2^2 + 10z_2^2$$

with

$$x_2 = \frac{x_1 - 5y_1 - 10z_1}{4},\ y_2 = \frac{x_1 + 3y_1 - 2z_1}{4},\ z_2 = \frac{x_1 - y_1 + 2z_1}{4}$$

all odd. So we still have (2.2).

By the above, there always exist odd integers x, y, z such that $40n + r^2 + 15 = x^2 + 5y^2 + 10z^2$. Write $y = 2u + 1$ and $z = 2v + 1$ with $u, v \in \mathbb{Z}$. As $x^2 \equiv r^2 \pmod 5$, either x or $-x$ has the form $10w + r$ with $w \in \mathbb{Z}$. Therefore

$$40n + r^2 + 15 = (10w + r)^2 + 5(2u + 1)^2 + 10(2v + 1)^2$$

and hence $n = T_u + v(v + 1) + w(5w + r)/2$. This proves the universality of $(5, r, 2, 2, 1, 1)$ over $\mathbb{Z}$.

There is an alternative way using (2.1) and Lemma 2.1 with

$$T = \begin{pmatrix} 1/4 & -5/4 & -5/2 \\ 1/4 & 3/4 & -1/2 \\ 1/4 & -1/4 & 1/2 \end{pmatrix}$$

to explain that $40n + r^2 + 15 = x^2 + 5y^2 + 10z^2$ for some odd integers x, y, z.

(ii) Let $n \in \mathbb{N}$ and $r \in \{1, 3\}$. Apparently,

$$\begin{aligned} & n = T_x + \frac{y(3y + 1)}{2} + z(4z + r) \\ \iff & 48n + 3r^2 + 8 = 6(2x + 1)^2 + 2(6y + 1)^2 + 3(8z + r)^2. \end{aligned}$$

Since

$$E(2,3,6)=\{3q+1:\ q\in\mathbb{N}\}\cup\{4^k(8l+7):\ k,l\in\mathbb{N}\}$$

by Dickson [7, pp. 112-113], we see that $48n+3r^2+8=2x^2+3y^2+6z^2$ for some $x,y,z\in\mathbb{Z}$. Clearly, $y^2+2z^2\neq 0$, and hence by [24, Lemma 2.1] we have $y^2+2z^2=y_0^2+2z_0^2$ for some $y_0,z_0\in\mathbb{Z}$ not all divisible by 3. Thus, without any loss of generality, we simply assume that $3\nmid y$ or $3\nmid z$. Note that $3\nmid x$, $2\nmid y$, and $x\equiv z\pmod 2$ since $2(x^2+z^2)\equiv 2x^2+6z^2\equiv 3r^2+8-3y^2\equiv 0\pmod 4$. If $3\mid y$ and $3\nmid z$, then z or $-z$ is congruent to $x+y$ modulo 3. If $3\nmid y$ and $3\mid z$, then y or $-y$ is congruent to $x+z$ modulo 3. If $3\nmid yz$, then $\varepsilon_1 y\equiv\varepsilon_2 z\equiv x\pmod 3$ for some $\varepsilon_1,\varepsilon_2\in\{\pm 1\}$. So, without loss of generality, we may assume that $x+y+z\equiv 0\pmod 3$ (otherwise we may change signs of x,y,z suitably). Note that

$$48n+3r^2+8=2x^2+3y^2+6z^2=2a^2+3b^2+6c^2,$$

where $a=y+z$, $b=(2x-y+2z)/3$ and $c=(x+y-2z)/3$ are integers. If $x\equiv z\equiv 1\pmod 2$, then x,y,z are all odd. If $x\equiv z\equiv 0\pmod 2$, then a,b,c are all odd.

By the above, $48n+3r^2+8=2a^2+3b^2+6c^2$ for some odd integers a,b,c. Since $3b^2\equiv 3r^2+8-2a^2-6c^2\equiv 3r^2\pmod{16}$, we can write b or $-b$ as $8w+r$ with $w\in\mathbb{Z}$. Clearly, a or $-a$ has the form $6u+1$ with $u\in\mathbb{Z}$, and $c=2v+1$ for some $v\in\mathbb{Z}$. Therefore

$$48n+3r^2+8=2(6u+1)^2+3(8w+r)^2+6(2v+1)^2$$

and hence $n=u(3u+1)/2+T_v+w(4w+r)$. This proves the universality of $(8,2r,3,1,1,1)$ over $\mathbb{Z}$.

(iii) Let $n\in\mathbb{N}$. By Lemma 2.2(ii), we can write $6n+7$ in the form $x^2+3y^2+6z^2$ with $x,y,z\in\mathbb{Z}$ and $x\equiv n+1\pmod 2$. Clearly, $y\equiv n\pmod 2$. Since $6z^2\equiv 6n+7-(n+1)^2-3n^2\equiv 6\pmod 4$, we have $2\nmid z$. Hence

$$24n+28=4(6n+7)=4(x^2+3y^2+6z^2)=(x-3y)^2+3(x+y)^2+24z^2$$

with $x-2y$, $x+2y$ and z all odd. Note that $x-3y$ or $3y-x$ has the form $6w+1$ with $w\in\mathbb{Z}$. Write $x+y=2u+1$ and $z=2v+1$ with $u,v\in\mathbb{Z}$. Then

$$24n+28=(6w+1)^2+3(2u+1)^2+24(2v+1)^2$$

and hence $n=w(3w+1)/2+T_u+8T_v$. This proves the universality of $(8,8,3,1,1,1)$.

(iv) Let $n \in \mathbb{N}$. By Lemma 2.3, we can write $6n+5$ as $x^2+y^2+10z^2$ with $x,y,z \in \mathbb{Z}$. Clearly, $x \not\equiv y \pmod 2$. Since $x^2+y^2+z^2 \equiv 2 \pmod 3$, exactly one of x,y,z is divisible by 3. Without loss of generality, we may assume that $x+y+z \equiv 0 \pmod 3$ (other we adjust signs of x,y,z suitably to meet our purpose). Observe that

$$4(x^2+y^2+10z^2) = 2(x-y)^2 + 3\left(\frac{x+y+10z}{3}\right)^2 + 15\left(\frac{x+y-2z}{3}\right)^2.$$

So, $4(6n+5) = 2a^2+3b^2+15c^2$ for some odd integers a,b,c. As $3 \nmid a$, we may write a or $-a$ as $6w+1$ with $w \in \mathbb{Z}$. Write $b = 2u+1$ and $c = 2v+1$ with $u,v \in \mathbb{Z}$. Then

$$24n+20 = 2(6w+1)^2 + 3(2u+1)^2 + 15(2v+1)^2$$

and hence $n = T_u + 5T_v + w(3w+1)$. This proves the universality of $(6,2,5,5,1,1)$ over $\mathbb{Z}$.

(v) Let $n \in \mathbb{N}$. By Lemma 2.2(i), we can write $12n+5$ in the form $x^2+y^2+(6z)^2$ with $x,y,z \in \mathbb{Z}$. It follows that $24n+10 = (x+y)^2+(x-y)^2+72z^2$. As $(x+y)^2+(x-y)^2 \equiv 10 \equiv 2 \pmod 4$, both $x+y$ and $x-y$ are odd. Since $(x+y)^2+(x-y)^2 \equiv 10 \equiv 1 \pmod 3$, exactly one of $x+y$ and $x-y$ is divisible by 3. So $(x+y)^2+(x-y)^2 = (6u+1)^2+(6v+3)^2$ for some $u,v \in \mathbb{Z}$. Therefore

$$24n+10 = (6u+1)^2 + (6v+3)^2 + 72z^2$$

and hence $n = u(3u+1)/2 + 3T_v + 3z^2$. This proves the universality of $(6,0,3,3,3,1)$ over $\mathbb{Z}$.

By Lemma 2.4, we can write $12n+14$ in the form $x^2+4y^2+9z^2$ with $x,y,z \in \mathbb{Z}$. Since $x^2+z^2 \equiv 14 \pmod 4$, we have $2 \nmid xz$. Observe that

$$24n+28 = 2(x^2+4y^2+9z^2) = (x-2y)^2 + (x+2y)^2 + 18z^2$$

with $x \pm 2y$ and z all odd. Clearly, exactly one of $x-2y$ and $x+2y$ is divisible by 3. So, for some $u,v,w \in \mathbb{Z}$ we have

$$24n+28 = (6x+1)^2 + 9(2y+1)^2 + 18(2z+1)^2$$

and hence $n = x(3x+1)/2 + 3T_y + 6T_z$. This proves the universality of $(6,6,3,3,3,1)$ over $\mathbb{Z}$.

The proof of Theorem 1.1 is now complete. $\square$

3. Proof of Theorem 1.2

The following lemma is one of the most important theorems about integral representations of quadratic forms (cf. [4, pp.129]).

Lemma 3.1. *Let f be a nonsingular integral quadratic form and let m be a nonzero integer which is represented by f over the real field $\mathbb{R}$ and the ring $\mathbb{Z}_p$ of p-adic integers for each prime p. Then m is represented by some form f^* over $\mathbb{Z}$ where f^* is in the same genus of f.*

Lemma 3.2. (i) [24, Lemma 3.2] *If $x^2+3y^2 \equiv 4 \pmod 8$ with $x, y \in \mathbb{Z}$, then $x^2+3y^2 = u^2+3v^2$ for some odd integers u and v.*

(ii) [24, Lemma 3.6] *If $w = x^2+7y^2 > 0$ with $x, y \in \mathbb{Z}$ and $8 \mid w$, then $w = u^2+7v^2$ for some odd integers u and v.*

(iii) [27, Lemma 5.1] *If $w = 3x^2+5y^2 > 0$ with $x, y \in \mathbb{Z}$ and $8 \mid w$, then $w = 3u^2+5v^2$ for some odd integers u and v.*

Proof of Theorem 1.2. (i) Let $n \in \mathbb{N}$. Clearly,

$$n = T_x+T_y+5z(3z+1)/2 \iff 24n+11 = 3(2x+1)^2+3(2y+1)^2+5(6z+1)^2.$$

There are two classes in the genus of $3x^2+3y^2+5z^2$, and the one not containing $3x^2+3y^2+5z^2$ has the representative

$$\begin{aligned} 3x^2+2y^2+8z^2-2yz =& 3x^2+3\left(\frac{y}{2}+z\right)^2+5\left(\frac{y}{2}-z\right)^2 \\ =& 3x^2+3\left(\frac{y-3z}{2}\right)^2+5\left(\frac{y+z}{2}\right)^2 \end{aligned}$$

If $24n+11 = 3x^2+2y^2+8z^2-2yz$ with y odd and z even, then $3x^2 \equiv 11-2y^2 \equiv 9 \pmod 4$ which is impossible. Thus, if $24n+11 \in \{3x^2+2y^2+8z^2-2yz:\ x,y,z \in \mathbb{Z}\}$ then $24n+11 \in \{3x^2+3y^2+5z^2:\ x,y,z \in \mathbb{Z}\}$. With the help of Lemma 3.1, there are $x, y, z \in \mathbb{Z}$ such that $24n+11 = 3x^2+3y^2+5z^2$. As $5z^2 \not\equiv 11 \pmod 4$, x and y cannot be both even. Without loss of generality, we assume that $2 \nmid x$. Then $3y^2+5z^2 \equiv 11-3x^2 \equiv 0 \pmod 8$ and $3y^2+5z^2 \neq 0$. By Lemma 3.2(iii), $3y^2+5z^2 = 3y_0^2+5z_0^2$ for some odd integers y_0 and z_0. Write $x = 2u+1$ and $y_0 = 2v+1$ with $u, v \in \mathbb{Z}$. As $2 \nmid z_0$ and $3 \nmid z_0$, z_0 or $-z_0$ has the form $6w+1$ with $w \in \mathbb{Z}$. Thus $24n+11 = 3(2u+1)^2+3(2v+1)^2+5(6w+1)^2$ and hence $n = T_u+T_v+5w(3w+1)/2$. This proves the universality of $(15, 5, 1, 1, 1, 1)$ over $\mathbb{Z}$.

(ii) Let $n \in \mathbb{N}$ and $r \in \{1,3\}$. Obviously,

$$\begin{aligned} & n = T_x + \frac{y(3y+1)}{2} + 3\frac{z(5z+r)}{2} \\ \iff & 120n + 9r^2 + 20 = 15(2x+1)^2 + 5(6y+1)^2 + 9(10z+r)^2. \end{aligned}$$

There are two classes in the genus of $x^2+15y^2+5z^2$, and the one not containing $x^2+15y^2+5z^2$ has the representative

$$\begin{aligned} 4x^2+4y^2+5z^2+2xy &= \left(\frac{x}{2}+2y\right)^2 + 15\left(\frac{x}{2}\right)^2 + 5z^2 \\ &= \left(2x+\frac{y}{2}\right)^2 + 15\left(\frac{y}{2}\right)^2 + 5z^2. \end{aligned}$$

If $120n+9r^2+20 = 4x^2+4y^2+5z^2+2xy$ with $x,y,z \in \mathbb{Z}$, then $2xy \equiv 9r^2 - 5z^2 \equiv 0 \pmod 4$ and hence x or y is even. Thus, with the help of Lemma 3.1, we can always write $120n+9r^2+20 = x^2+15y^2+5z^2$ with $x,y,z \in \mathbb{Z}$. Since $x^2+5z^2 \equiv 20 \equiv 2 \pmod 3$, $x = 3x_0$ for some $x_0 \in \mathbb{Z}$. As $15y^2 \not\equiv 9r^2 \pmod 4$, x and z cannot be both even. If $2 \nmid x$, then $5(3y^2+z^2) \equiv 9r^2+20-x^2 \equiv 4 \pmod 8$ and hence by Lemma 3.2(i) we can write $3y^2+z^2$ as $3y_0^2+z_0^2$ with y_0 and z_0 both odd. If $2 \nmid z$, then $x^2+15y^2 \neq 0$ and $x^2+15y^2 = 3(3x_0^2+5y^2) \equiv 9r^2+20-5z^2 \equiv 0 \pmod 8$, hence by Lemma 3.2(iii) we can write $3x_0^2+5y^2$ as $3x_1^2+5y_1^2$ with x_1 and y_1 both odd.

By the above, there are odd integers x,y,z such that $120n+9r^2+20 = 9x^2+15y^2+5z^2$. Write $y = 2u+1$ with $u \in \mathbb{Z}$. As $3 \nmid z$, we can write z or $-z$ as $6v+1$ with $v \in \mathbb{Z}$. Since $x^2 \equiv r^2 \pmod 5$, we can write x or $-x$ as $10w+r$ with $w \in \mathbb{Z}$. Thus

$$120n+9r^2+20 = 15(2u+1)^2 + 5(6v+1)^2 + 9(10z+r)^2$$

and hence $n = T_x + y(3y+1)/2 + 3z(5z+r)/2$ with $x,y,z \in \mathbb{Z}$. This proves the universality of $(15,3r,3,1,1,1)$ over $\mathbb{Z}$.

(iii) Let $n \in \mathbb{N}$ and $r \in \{1,3\}$. Apparently,

$$\begin{aligned} & n = 3x^2 + \frac{y(3y+1)}{2} + \frac{z(5z+r)}{2} \\ \iff & 120n + 3r^2 + 5 = 360x^2 + 5(6y+s)^2 + 3(10z+r)^2. \end{aligned}$$

If $60n+(3r^2+5)/2 = 4x^2+4y^2+5z^2+2xy$ with $x,y,z \in \mathbb{Z}$, then x or y must be even. Thus, as in part (ii), $60n+(3r^2+5)/2 = x^2+5y^2+15z^2$ for some $x,y,z \in \mathbb{Z}$. Note that $x^2+y^2 \equiv z^2 \pmod 4$. If y is odd, then $2 \mid x$, $2 \nmid z$ and we may assume $y \not\equiv z \pmod 4$ (otherwise it suffices to change the sign of z), hence

$$y^2+3z^2=\left(\frac{y-3z}{2}\right)^2+3\left(\frac{y+z}{2}\right)^2$$

with $y_1=(y-3z)/2$ and $z_1=(y+z)/2$ both even. So, without loss of generality, we may simply assume that $2\mid y$ and $x\equiv z \pmod 2$. Observe that

$$120n+3r^2+5=2(x^2+5y^2+15z^2)=3a^2+5b^2+10y^2.$$

with $a=(x+5z)/2$ and $b=(x-3z)/2$ both integral. Since $3a^2+5b^2\equiv 5s^2+3t^2-10y^2\equiv 0 \pmod 8$ and $3a^2+5b^2>0$, by Lemma 3.2(iii) we can write $3a^2+5b^2=3c^2+5d^2$ with c and d both odd. Thus

$$120n+3r^2+5=3c^2+5d^2+40\left(\frac{y}{2}\right)^2.$$

As $(y/2)^2\equiv 5(1-d^2)\equiv d^2-1 \pmod 3$, we must have $3\nmid d$ and $3\mid y$. Write $y=6u$ with $u\in\mathbb{Z}$. Clearly, d or $-d$ has the form $6v+1$ with $v\in\mathbb{Z}$. Since $c^2\equiv r^2 \pmod 5$, we may write c or $-c$ as $10w+r$ with $w\in\mathbb{Z}$. Therefore

$$120n+3r^2+5=3(10w+r)^2+5(6v+1)^2+40(3u)^2$$

and hence $n=3u^2+v(3v+1)/2+w(5w+r)/2$. This proves the universality of $(6,0,5,r,3,1)$ over $\mathbb{Z}$.

(iv) Let $n\in\mathbb{N}$ and $\delta\in\{0,1\}$. Clearly,

$$\begin{aligned} n&=x(x+\delta)+\frac{y(3y+1)}{2}+5\frac{z(3z+1)}{2}\\ \iff 24n+6(\delta+1)&=6(2x+\delta)^2+(6y+1)^2+5(6z+1)^2.\end{aligned}$$

There are two classes in the genus of $x^2+5y^2+6z^2$, and the one not containing $x^2+5y^2+6z^2$ has the representative $3x^2+3y^2+4z^2-2yz+2zx$. If $24n+6(\delta+1)=3x^2+3y^2+4z^2-2yz+2zx$, then $u=(x+y)/2$ and $v=(x-y)/2$ are integers, and

$$24n+6(\delta+1)=6u^2+6v^2+4z^2+4vz=6u^2+5v^2+(v+2z)^2.$$

Thus, by Lemma 3.1, $24n+6(\delta+1)=x^2+5y^2+6z^2$ for some $x,y,z\in\mathbb{Z}$. Since $x^2\equiv -5y^2\equiv y^2 \pmod 3$, we may assume that $x\equiv y \pmod 3$ without loss of generality. If $z\not\equiv\delta \pmod 2$, then $x^2+5y^2\equiv 6(\delta+1)-6z^2\equiv 6(\delta+1)-6(1-\delta)\equiv 4\delta \pmod 8$, hence both x and y are even and $(x-y)/2\equiv\delta \pmod 2$, and thus

$$x^2+5y^2+6z^2=\left(z-\frac{5(x-y)}{6}\right)^2+5\left(\frac{x-y}{6}+z\right)^2+6\left(\frac{x-y}{6}+y\right)^2$$

with $(x-y)/6+y\equiv (x-y)/2\equiv\delta \pmod 2$.

By the above, $24n+6(\delta+1)=x^2+5y^2+6z^2$ for some $x,y,z\in\mathbb{Z}$ with $x,y,z\in\mathbb{Z}$ with $z\equiv\delta\pmod 2$. Since x^2+5y^2 is a positive multiple of 3, by [24, Lemma 2.1] we can write $x^2+5y^2=x_0^2+5y_0^2$ with $x_0y_0\in\mathbb{Z}$ and $3\nmid x_0y_0$. So, there are $x,y,z\in\mathbb{Z}$ with $x\equiv y\not\equiv 0\pmod 3$ and $z\equiv\delta\pmod 2$ such that $24n+6(\delta+1)=x^2+5y^2+6z^2$. Write $z=2w+\delta$ with $w\in\mathbb{Z}$. Since $x^2+5y^2\equiv 6\pmod 8$, both x and y are odd. Thus x or $-x$ has the form $6u+1$ with $u\in\mathbb{Z}$, and y or $-y$ has the form $6v+1$ with $v\in\mathbb{Z}$. Therefore

$$24n+6(\delta+1)=(6u+1)^2+5(6v+1)^2+6(2w+\delta)^2$$

and hence $n=w(w+\delta)+u(3u+1)/2+5v(3v+1)/2$. This proves the universality of $(15,5,3,1,2,2\delta)$ over $\mathbb{Z}$.

(v) Let $n\in\mathbb{N}$. Apparently,

$$\begin{aligned}&n=x(x+1)+\frac{y(3y+1)}{2}+7\frac{z(3z+1)}{2}\\ \iff& 24n+14=6(2x+1)^2+(6y+1)^2+7(6z+1)^2.\end{aligned}$$

There are two classes in the genus of $x^2+6y^2+7z^2$, and the one not containing $x^2+6y^2+7z^2$ has the representative

$$2x^2+5y^2+5z^2-4yz=2x^2+10u^2+10v^2-4(u+v)(u-v)=2x^2+6u^2+14v^2$$

with $u=(y+z)/2$ and $v=(y-z)/2$. If $24n+14=2x^2+6u^2+14v^2$ for some $x,u,v\in\mathbb{Z}$ with $x\not\equiv v\pmod 2$, then $14\equiv 2+6u^2\pmod 8$ which is impossible. If $24n+14=2x^2+6u^2+14v^2$ with $x,u,v\in\mathbb{Z}$ and $x\equiv v\pmod 2$, then

$$24n+14=6u^2+\left(\frac{x-7v}{2}\right)^2+7\left(\frac{x+v}{2}\right)^2.$$

By the above and Lemma 3.1, there are $x,y,z\in\mathbb{Z}$ such that $24n+14=6x^2+y^2+7z^2$. If $2\mid x$, then $y^2+7z^2\equiv 6-6x^2\equiv 6\pmod 8$ which is impossible. So $x=2u+1$ for some $u\in\mathbb{Z}$. Note that $y^2+7z^2\equiv 6-6x^2\equiv 0\pmod 8$ and $y^2+7z^2\neq 0$. Applying Lemma 3.2(ii) we can write y^2+7z^2 as $y_0^2+7z_0^2$ with y_0 and z_0 both odd. Note that $y_0^2+z_0^2\equiv y_0^2+7z_0^2\equiv 14\equiv 2\pmod 3$. So y_0 or $-y_0$ can be written as $6v+1$ with $v\in\mathbb{Z}$, and z_0 or $-z_0$ has the form $6w+1$ with $w\in\mathbb{Z}$. Thus

$$24n+14=6x^2+y_0^2+7z_0^2=6(2u+1)^2+(6v+1)^2+7(6w+1)^2$$

and hence $n=u(u+1)+v(3v+1)/2+7z(3z+1)/2$. This proves the universality of $(21,7,3,1,2,2)$.

(vi) Let $r \in \{1,3,5\}$ and $n \in \mathbb{N}$. Clearly,

$$n = T_x + T_y + \frac{z(7z+r)}{2} \iff 56n+14+r^2 = 7(2x+1)^2+7(2y+1)^2+(14z+r)^2.$$

There are two classes in the genus of $x^2+7y^2+7z^2$, and the one not containing $x^2+7y^2+7z^2$ has the representative

$$\begin{aligned} 2x^2+4y^2+7z^2+2xy &= \left(\frac{x}{2}+2y\right)^2 + 7\left(\frac{x}{2}\right)^2 + 7z^2. \\ &= \left(\frac{x-3y}{2}\right)^2 + 7\left(\frac{x+y}{2}\right)^2 + 7z^2 \end{aligned}$$

If $56n+14+r^2 = 2x^2+4y^2+7z^2+2xy$ with x odd and y even, then $15 \equiv 14+r^2 \equiv 2x^2+7z^2 \equiv 9 \pmod 4$ which is impossible. Thus, if $56n+14+r^2 \in \{2x^2+4y^2+7z^2+2xy : \ x,y,z \in \mathbb{Z}\}$ then $56n+14+r^2 \in \{x^2+7y^2+7z^2 : \ x,y,z \in \mathbb{Z}\}$. With the help of Lemma 3.1, there are $x,y,z \in \mathbb{Z}$ such that $56n+14+r^2 = x^2+7y^2+7z^2$. As $x^2 \not\equiv 14+r^2 \equiv 15 \pmod 4$, y and z cannot be both even. Without loss of generality, we assume that $2 \nmid z$. Then $x^2+7y^2 \equiv 14+r^2-7z^2 \equiv 0 \pmod 8$ and $x^2+7y^2 \neq 0$. By Lemma 3.2(ii), $x^2+7y^2 = x_0^2+7y_0^2$ for some odd integers x_0 and y_0. Now $56n+14+r^2 = x_0^2+7y_0^2+7z^2$. Clearly, x_0 or $-x_0$ has the form $14w+r$ with $w \in \mathbb{Z}$. Write $y_0 = 2u+1$ and $z = 2v+1$ with $u,v \in \mathbb{Z}$. Then

$$56n+14+r^2 = (14w+r)^2 + 7(2u+1)^2 + 7(2v+1)^2$$

and hence $n = T_u + T_v + w(7w+r)/2$. This proves the universality of $(7,r,1,1,1,1)$ over $\mathbb{Z}$.

(vii) Let $n \in \mathbb{N}$, $s \in \{1,3\}$ and $t \in \{1,3,5\}$. Clearly,

$$\begin{aligned} & n = T_x + \frac{y(3y+s)}{2} + \frac{z(7z+t)}{2} \\ \iff & 168n+21+7s^2+3t^2 = 21(2x+1)^2 + 7(6y+s)^2 + 3(14z+t)^2. \end{aligned}$$

There are two classes in the genus of $3x^2+21y^2+7z^2$, and the one not containing $3x^2+21y^2+7z^2$ has the representative

$$\begin{aligned} 6x^2+12y^2+7z^2+6xy =& 3\left(\frac{x}{2}+2y\right)^2 + 21\left(\frac{x}{2}\right)^2 + 7z^2. \\ =& 3\left(\frac{x-3y}{2}\right)^2 + 21\left(\frac{x+y}{2}\right)^2 + 7z^2 \end{aligned}$$

If $168n+21+7s^2+3t^2 = 6x^2+12y^2+7z^2+6xy$ with x odd and y even, then $31 \equiv 21+7s^2+3t^2 \equiv 6x^2+7z^2 \equiv 13 \pmod 4$ which is impossible. Thus, if $168n+21+7s^2+3t^2 \in \{6x^2+12y^2+7z^2+6xy : \ x,y,z \in \mathbb{Z}\}$ then $168n+21+7s^2+3t^2 \in \{3x^2+21y^2+7z^2 : \ x,y,z \in \mathbb{Z}\}$. With the help of Lemma 3.1, there are $x, y, z \in \mathbb{Z}$ such that $168n + 21 + 7s^2 + 3t^2 = 3x^2 + 21y^2 + 7z^2$. As $21y^2 \not\equiv 21 + 7s^2 + 3t^2 \equiv 31 \pmod 4$, x and z cannot be both even. If $2 \nmid x$, then $21y^2 + 7z^2 \equiv 21 + 7s^2 + 3t^2 - 3x^2 \equiv 4 \pmod 8$ and hence by Lemma 3.2(i) we can write $3y^2 + z^2$ as $3y_0^2 + z_0^2$ with y_0, z_0 odd integers. Note that $x^2 + 7y^2 \neq 0$ since $7 \nmid t$. If $2 \nmid z$, then $3(x^2 + 7y^2) \equiv 21 + 7s^2 + 3t^2 - 7z^2 \equiv 0 \pmod 8$ and hence by Lemma 3.2(ii) $x^2 + 7y^2 = x_0^2 + 7y_0^2$ for some odd integers x_0 and y_0.

By the above, there are odd integers x, y, z such that $168n+21+7s^2+3t^2 = 3x^2 + 7y^2 + 21z^2$. Write $z = 2u + 1$ with $u \in \mathbb{Z}$. As $y^2 \equiv s^2 \pmod 3$, y or $-y$ has the form $6v + s$ with $v \in \mathbb{Z}$. Since $x^2 \equiv t^2 \pmod 7$, x or $-x$ has the form $14w + t$ with $w \in \mathbb{Z}$. Thus

$$168n + 21 + 7s^2 + 3t^2 = 3(14w + t)^2 + 7(6v + s)^2 + 21(2u + 1)^2$$

and hence $n = T_u + v(3v + s)/2 + w(7w + t)/2$. This proves the universality of $(7, t, 3, s, 1, 1)$ over $\mathbb{Z}$.

(viii) Let $\delta \in \{0, 1\}$ and $r \in \{1, 3, 5\}$. Clearly,

$$\begin{aligned} & n = T_x + y(y + \delta) + \frac{z(7z + r)}{2} \\ \iff & 56n + 14\delta + r^2 + 7 = 7(2x + 1)^2 + 14(2y + \delta)^2 + (14z + r)^2. \end{aligned}$$

There are two classes in the genus of $x^2+7y^2+14z^2$, the one not containing $x^2 + 7y^2 + 14z^2$ has the representative

$$2x^2 + 7y^2 + 7z^2 = 2x^2 + 14\left(\frac{y + z}{2}\right)^2 + 14\left(\frac{y - z}{2}\right)^2.$$

If $56n+14\delta+r^2+7 = 2x^2+14y^2+14z^2$ with $x, y, z \in \mathbb{Z}$ and $y \equiv z \pmod 2$, then $2x^2 \equiv 14\delta + r^2 + 7 \equiv 2\delta \pmod 4$, hence $x^2 \equiv \delta \pmod 4$ and $y \equiv z \equiv \delta \pmod 2$ since

$$-2(y^2 + z^2) \equiv 14(y^2 + z^2) \equiv 14\delta + r^2 + 7 - 2\delta \equiv -4\delta \pmod 8.$$

If $56n+14\delta+r^2+7 = 2x^2+14y^2+14z^2$ with $x, y, z \in \mathbb{Z}$ and $x \equiv y \pmod 2$, then

$$56n + 14\delta + r^2 + 7 = \left(\frac{x - 7y}{2}\right)^2 + 7\left(\frac{x + y}{2}\right)^2 + 14z^2.$$

In view of Lemma 3.1 and the above, there are $x, y, z \in \mathbb{Z}$ such that $56n + 14\delta + r^2 + 7 = x^2 + 7y^2 + 14z^2$. If $z \not\equiv \delta \pmod 2$, then

$$x^2 + 7y^2 \equiv 14\delta + r^2 + 7 - 14z^2 \equiv 14\delta - 14(1 - \delta) \equiv 2 \pmod 4$$

which is impossible. Thus $z \equiv \delta \pmod 2$ and $x^2+7y^2 \equiv r^2+7 \equiv 0 \pmod 8$. Note that $x^2 + 7y^2 \neq 0$ since $7 \nmid r$. Applying Lemma 3.2(ii) we can write

x^2+7y^2 as $x_0^2+7y_0^2$ with x_0 and y_0 both odd. Since $x_0^2\equiv r^2 \pmod 7$, either x_0 or $-x_0$ has the form $14w+r$ with $w\in\mathbb{Z}$. Write $y_0=2u+1$ and $z=2v+\delta$ with $u,v\in\mathbb{Z}$. Then

$$56n+14\delta+r^2+7\equiv(14w+r)^2+7(2u+1)^2+14(2v+\delta)^2$$

and hence $n=T_u+v(v+\delta)+w(7w+r)/2$. This proves the universality of $(7,r,2,2\delta,1,1)$ over $\mathbb{Z}$.

The proof of Theorem 1.2 is now complete. $\square$

4. Proof of Theorem 1.3

For a positive definite integral ternary quadratic form $f(x,y,z)$ and an integer n, as usual we define

$$r(n,f):=\{(x,y,z)\in\mathbb{Z}^3:\ f(x,y,z)=n\}|$$

and adopt the standard notation $r(n,\mathrm{gen}(f))$ introduced in [17, pp. 173–174].

Lemma 4.1. *Let f be a positive ternary quadratic form with determinant $d(f)$. Suppose that $m\in\mathbb{Z}^+$ is represented by the genus of f. Then, for each prime $p\nmid 2md(f)$, we have*

$$\frac{r(mp^2,\mathrm{gen}(f))}{r(m,\mathrm{gen}(f))}=p+1-\left(\frac{-md(f)}{p}\right). \tag{4.1}$$

Proof. By the Minkowski-Siegel formula [17, pp. 173–174],

$$r(mp^2,\mathrm{gen}(f))=2\pi\sqrt{\frac{mp^2}{d(f)}}\prod_q\alpha_q(mp^2,f),$$

where q runs over all primes and α_q is the local density. As $p\nmid 2md(f)$, by [29] we have

$$\begin{aligned}\alpha_p(mp^2,f)&=1+\frac1p-\frac1{p^2}+\left(\frac{-md(f)}{p}\right)\frac1{p^2},\\ \alpha_p(m,f)&=1+\left(\frac{-md(f)}{p}\right)\frac1p.\end{aligned}$$

Thus

$$\frac{r(mp^2,\mathrm{gen}(f))}{r(m,\mathrm{gen}(f))}=p\frac{\alpha_p(mp^2,f)}{\alpha_p(m,f)}=p+1-\left(\frac{-md(f)}{p}\right).$$

This concludes the proof. $\square$

Lemma 4.2. *Let $w=u^2+15v^2>0$ with $u,v\in\mathbb{Z}$ and $8\mid w$. Then $w=x^2+15y^2$ for some odd integers x and y.*

Proof. Let k be the 2-adic order of $\gcd(u,v)$, and write $u = 2^k u_0$ and $v = 2^k v_0$ with $u_0, v_0 \in \mathbb{Z}$ not all even. If $k = 0$, then both u_0 and v_0 are odd since w is even. Below we assume that $k > 0$.

We observe the identity

$$4^2(x^2+15y^2) = (x-15y)^2 + 15(x+y)^2.$$

If $u_0 \not\equiv v_0 \pmod 2$, then $k \geqslant 2$ (since $8 \mid w$) and $4^2(u_0^2+15v_0^2) = s^2+15t^2$ with $s = u_0-15v_0$ and $t = u_0+v_0$ both odd. For $j \in \mathbb{N}$, if $4^j(u_0^2+15v_0^2) = u_j^2+15v_j^2$ for some odd integers u_j and v_j, then we may assume $u_j \equiv v_j \pmod 4$ without loss of generality (otherwise we may replace v_j by $-v_j$), and hence

$$4^{j+1}(u_0^2+15v_0^2) = 4(u_j^2+15v_j^2) = u_{j+1}^2 + 15v_{j+1}^2$$

with $u_{j+1} = (u_j - 15v_j)/2$ and $v_{j+1} = (u_j+v_j)/2$ both odd. Thus, for some odd integers u_k and v_k, we have

$$w = 4^k(u_0^2+15v_0^2) = u_k^2 + 15v_k^2.$$

This concludes the proof. □

Proof of Theorem 1.3(i). (a) We first prove that $(7,7,3,1,1,1)$ is universal over $\mathbb{Z}$. Let $n \in \mathbb{N}$. Clearly,

$$\begin{aligned} n = T_x + 7T_y + \frac{z(3z+1)}{2} \\ \iff 24n+25 = 3(2x+1)^2 + 21(2y+1)^2 + (2z+1)^2. \end{aligned}$$

There are two classes in the genus of $x^2+3y^2+21z^2$ and the one not containing $x^2+3y^2+21z^2$ has the representative

$$\begin{aligned} x^2+6y^2+12z^2-6yz =& x^2 + 3\left(\frac{y}{2}-2z\right)^2 + 21\left(\frac{y}{2}\right)^2 \\ =& x^2 + 3\left(\frac{y+3z}{2}\right)^2 + 21\left(\frac{y-z}{2}\right)^2. \end{aligned} \tag{4.2}$$

If $24n+25 = x^2+6y^2+12z^2-6yz$ with $x,y,z \in \mathbb{Z}$, then the equality modulo 4 yields $y(y-z) \equiv 0 \pmod 2$. Thus, by (4.2) and Lemma 3.1, we have

$$24n+25 \in \{x^2+3y^2+21z^2 : \ x,y,z \in \mathbb{Z}\}. \tag{4.3}$$

Now we claim that $24n+25 = x^2+3y^2+21z^2$ for some $x,y,z \in \mathbb{Z}$ with $y^2+7z^2 > 0$. This holds by (4.3) if $24n+25$ is not a square. Suppose that $24n+25 = m^2$ with $m \in \mathbb{Z}^+$. Let p be any prime divisor of m. Clearly, $p \geqslant 5$. Note that $r(7^2, x^2+3y^2+21z^2) > 2$ since $7^2 = (\pm5)^2+3\times(\pm1)^2+21\times(\pm1)^2$. If $p \neq 7$ and $r(p^2, x^2+6y^2+12z^2-6yz) > 2$, then $p^2 = x^2+6y^2+12z^2-6yz$ for some $x,y,z \in \mathbb{Z}$ with $2 \mid y(y-z)$ and $y^2+z^2 > 0$, hence by (4.2) we have $p^2 = x^2 + 3u^2 + 21v^2$ for some $x, u, v \in \mathbb{Z}$ with $u^2 + 7v^2 > 0$, and thus $r(p^2, x^2 + 3y^2 + 21z^2) > 2$. By Lemma 4.1, if $p \neq 7$ then

$$\frac{r(p^2, \mathrm{gen}(x^2 + 3y^2 + 21z^2))}{r(1, \mathrm{gen}(x^2 + 3y^2 + 21z^2))} = p + 1 - \left(\frac{-7}{p}\right)$$

and hence

$$r(p^2, x^2+3y^2+21z^2)+r(p^2, x^2+6y^2+12z^2-6yz) = 4\left(p + 1 - \left(\frac{-7}{p}\right)\right) > 4.$$

So we still have $r(p^2, x^2+3y^2+21z^2) > 2$ if $r(p^2, x^2+6y^2+12z^2-6yz) \leqslant 2$. As $r(m^2, x^2 + 3y^2 + 21z^2) \geqslant r(p^2, x^2 + 3y^2 + 21z^2) > 2$, we can write $24n + 25 = m^2$ as $x^2 + 3y^2 + 21z^2$ with $x, y, z \in \mathbb{Z}$ and $y^2 + 7z^2 > 0$. This proves the claim.

By the claim, there are $x, y, z \in \mathbb{Z}$ such that $24n + 25 = x^2 + 3y^2 + 21z^2$ and $y^2 + 7z^2 > 0$. As $3y^2 \not\equiv 25 \equiv 1 \pmod 4$, either x or z is odd. If $2 \nmid x$, then $3(y^2 + 7z^2) \equiv 25 - x^2 \equiv 0 \pmod 8$ and hence by Lemma 3.2(ii) we can write $y^2 + 7z^2$ as $y_0^2 + 7z_0^2$ with y_0 and z_0 both odd. If $2 \nmid z$, then $x^2 + 3y^2 \equiv 25 - 21z^2 \equiv 4 \pmod 8$ and hence by Lemma 3.2(i) we can write x^2+3y^2 as $x_1^2+3y_1^2$ with x_1 and y_1 both odd. Thus $24n+25 = a^2+3b^2+21c^2$ for some odd integers a, b, c. As $3 \nmid a$, either a or $-a$ has the form $6w + 1$ with $w \in \mathbb{Z}$. Write $b = 2u + 1$ and $c = 2v + 1$ with $u, v \in \mathbb{Z}$. Then

$$24n + 25 = (6w + 1)^2 + 3(2u + 1)^2 + 21(2v + 1)^2$$

and hence $n = T_u + 7T_v + w(3w + 1)/2$. This proves the universality of $(7, 7, 3, 1, 1, 1)$ over $\mathbb{Z}$.

(b) Let $n \in \mathbb{N}$ and $r \in \{1, 3\}$. Clearly,

$$\begin{aligned} & n = 5T_x + \frac{y(3y + 1)}{2} + \frac{z(3z + r)}{2} \\ \iff & 24n + r^2 + 16 = 15(2x + 1)^2 + (6y + 1)^2 + (6z + r)^2. \end{aligned}$$

There are two classes in the genus of $x^2 + y^2 + 15z^2$, and the one not containing $x^2 + y^2 + 15z^2$ has the representative

$$\begin{aligned} x^2 + 4y^2 + 4z^2 - 2yz =& x^2 + \left(\frac{y}{2} - 2z\right)^2 + 15\left(\frac{y}{2}\right)^2 \\ =& x^2 + \left(2y - \frac{z}{2}\right)^2 + 15\left(\frac{z}{2}\right)^2. \end{aligned} \tag{4.4}$$

If $24n + r^2 + 16 = x^2 + 4y^2 + 4z^2 - 2yz$ with $x, y, z \in \mathbb{Z}$, then $2 \nmid x$ and $2 \mid yz$. Thus, in view of (4.4) and Lemma 3.1, we have

$$24n + r^2 + 16 \in \{x^2 + y^2 + 15z^2 : \ x, y, z \in \mathbb{Z}\}. \tag{4.5}$$

We claim that $24n + r^2 + 16 = x^2 + y^2 + 15z^2$ for some $x, y, z \in \mathbb{Z}$ with $(x^2 + 15z^2)(y^2 + 15z^2) > 0$. This holds by (4.5) if $24n + r^2 + 16$ is not a square. Now suppose that $24n + r^2 + 16 = m^2$ with $m \in \mathbb{Z}^+$. Let p be any prime divisor of m. Clearly, $p \geqslant 5$. Note that $r(5^2, x^2 + y^2 + 15z^2) > 4$ since

$$5^2 = (\pm 5)^2 + 0^2 + 15 \times 0^2 = 0^2 + (\pm 5)^2 + 15 \times 0^2 = (\pm 3)^2 + (\pm 4)^2 + 15 \times 0^2.$$

If $r(p^2, x^2 + 4y^2 + 4z^2 - 2yz) > 2$, then $p^2 = x^2 + 4y^2 + 4z^2 - 2yz$ for some $x, y, z \in \mathbb{Z}$ with $2 \mid yz$ and $y^2 + z^2 > 0$, hence by (4.4) $p^2 = x^2 + u^2 + 15v^2$ for some $x, u, v \in \mathbb{Z}$ with $(x^2 + 15v^2)(u^2 + 15v^2) > 0$, and thus $r(p^2, x^2 + y^2 + 15z^2) > 4$. When $p > 5$, by Lemma 4.1 we have

$$\frac{r(p^2, \mathrm{gen}(x^2 + y^2 + 15z^2))}{r(1, \mathrm{gen}(x^2 + y^2 + 15z^2))} = p + 1 - \left(\frac{-15}{p}\right)$$

and hence

$$r(p^2, x^2+y^2+15z^2)+2r(p^2, x^2+4y^2+4z^2-2yz) = 8\left(p + 1 - \left(\frac{-15}{p}\right)\right) > 50.$$

Thus we still have $r(p^2, x^2+y^2+15z^2) > 4$ if $r(p^2, x^2+4y^2+4z^2-2yz) \leqslant 2$. As $r(m^2, x^2+y^2+15z^2) \geqslant r(p^2, x^2+y^2+15z^2) > 4$, we can write $24n+r^2+16$ as $x^2 + y^2 + 15z^2$ with $(x^2 + 15z^2)(y^2 + 15z^2) > 0$. This proves the claim.

By the claim, there are $x, y, z \in \mathbb{Z}$ such that $24n+r^2+16 = x^2+y^2+15z^2$ and $(x^2 + 15z^2)(y^2 + 15z^2) > 0$. Since $15z^2 \not\equiv r^2 \equiv 1 \pmod 4$, either x or y is odd. Without any loss of generality, we assume that $2 \nmid x$. Since $y^2 + 15z^2 > 0$ and $y^2 + 15z^2 \equiv r^2 - x^2 \equiv 0 \pmod 8$, by Lemma 4.2 we can write $y^2 + 15z^2 = y_0^2 + 15z_0^2$ with y_0 and z_0 both odd. Now, $24n + r^2 + 16 = x^2 + y_0^2 + 15z_0^2$. Since $x^2 + y_0^2 \equiv r^2 + 1 \pmod 3$, one of x^2 and y_0^2 is congruent to r^2 modulo 3 and the other one is congruent to 1 modulo 3. Thus $x^2 + y_0^2 = (6u + r)^2 + (6v + 1)^2$ for some $u, v \in \mathbb{Z}$. Write $z_0 = 2w + 1$ with $v \in \mathbb{Z}$. Then

$$24n + r^2 + 16 = (6u + r)^2 + (6v + 1)^2 + 15(2w + 1)^2$$

and hence $n = u(3u+r)/2 + v(3v+1)/2 + 5T_w$. This proves the universality of $(5, 5, 3, r, 3, 1)$ over $\mathbb{Z}$.

(c) Let $n \in \mathbb{N}$. Apparently,

$$\begin{aligned} & n = T_x + 5T_y + z(3z + 2) \\ \Longleftrightarrow\ & 24n + 26 = 3(2x + 1)^2 + 15(2y + 1)^2 + 2(6z + 2)^2. \end{aligned}$$

There are two classes in the genus of $2x^2+3y^2+15z^2$, and the one not containing $2x^2+3y^2+15z^2$ has the representative

$$g(x,y,z)=2x^2+5y^2+11z^2+2yz+2x(y-z)=2\,(x+v)^2+3(u-2v)^2+15u^2 \tag{4.6}$$

with $u=(y+z)/2$ and $v=(y-z)/2$. If $24n+26=g(x,y,z)$ with $x,y,z\in\mathbb{Z}$, then $y\equiv z\pmod 2$, and hence by (4.6) we have $24n+26=2a^2+3b^2+15c^2$ for some $a,b,c\in\mathbb{Z}$. So, in view of Lemma 3.1, we always have

$$24n+26\in\{2x^2+3y^2+15z^2:\ x,y,z\in\mathbb{Z}\}. \tag{4.7}$$

We claim that $24n+26=2x^2+3y^2+15z^2$ for some $x,y,z\in\mathbb{Z}$ with $y^2+5z^2>0$. This holds by (4.7) if $12n+13$ is not a square. Now suppose that $12n+13=m^2$ with $m\in\mathbb{Z}^+$. Let p be any prime divisor of m. Clearly, $p\geqslant 5$. Note that $r(2\times 5^2,2x^2+3y^2+15z^2)>2$ since

$$2\times 5^2=2\times(\pm5)^2+3\times 0^2+15\times 0^2=2(\pm1)^2+3(\pm4)^2+30\times 0^2.$$

If $r(2p^2,g(x,y,z))>2$, then $2p^2=g(x,y,z)$ for some $x,y,z\in\mathbb{Z}$ with $y^2+z^2>0$, hence by (4.6) $2p^2=2x^2+3b^2+15c^2$ for some $x,b,c\in\mathbb{Z}$ with $b^2+c^2>0$, and thus $r(2p^2,2x^2+3y^2+15z^2)>2$. When $p>5$, by Lemma 4.1 we have

$$\frac{r(2p^2,\mathrm{gen}(2x^2+3y^2+15z^2))}{r(2,\mathrm{gen}(2x^2+3y^2+15z^2))}=p+1-\left(\frac{-5}{p}\right)$$

and hence

$$r(2p^2,2x^2+3y^2+15z^2)+2r(2p^2,g(x,y,z))=6\left(p+1-\left(\frac{-5}{p}\right)\right)>40.$$

Thus we still have $r(2p^2,2x^2+3y^2+15z^2)>2$ if $r(2p^2,g(x,y,z))\leqslant 2$. As $r(2m^2,2x^2+3y^2+15z^2)\geqslant r(2p^2,2x^2+3y^2+15z^2)>2$, we can write $24n+26$ as $2x^2+3y^2+15z^2$ with $y^2+5z^2>0$. This proves the claim.

By the claim, there are $x,y,z\in\mathbb{Z}$ such that $24n+26=2x^2+3(y^2+5z^2)$ and $y^2+5z^2>0$. By [24, Lemma 2.1], $y^2+5z^2=y_0^2+5z_0^2$ for some integers y_0 and z_0 not all divisible by 3. Without any loss of generality, we simply assume that $3\nmid y$ or $3\nmid z$. Note that $3\nmid x$ and $y\equiv z\pmod 2$. If $3\nmid yz$, then $\varepsilon_1 y\equiv\varepsilon_2 z\equiv x\pmod 3$ for some $\varepsilon_1,\varepsilon_2\in\{\pm1\}$. If $3\mid y$ and $3\nmid z$ then $x+y+\varepsilon z\equiv 0\pmod 3$ for some $\varepsilon\in\{\pm1\}$; similarly, if $3\nmid y$ and $3\mid z$ then $x+\varepsilon y+z\equiv 0\pmod 3$. So, without loss of generality we may suppose that $x+y+z\equiv 0\pmod 3$ (otherwise we adjust signs of x,y,z suitably to meet our purpose). If $y\equiv z\equiv 0\pmod 2$, then $2x^2\equiv 26\pmod 4$, hence $2\nmid x$

and $y \equiv z \pmod 4$ since $y^2 + 5z^2 \equiv 0 \pmod 8$, therefore

$$2x^2 + 3y^2 + 15z^2 = 2\left(\frac{y-5z}{2}\right)^2 + 3\left(\frac{2x+5y+5z}{6}\right)^2 + 15\left(\frac{2x-y-z}{6}\right)^2 \tag{4.8}$$

with $(2x+5y+5z)/6$ and $(2x-y-z)/6$ both odd.

By the above, $24n+26 = 2a^2+3b^2+15c^2$ for some $a,b,c \in \mathbb{Z}$ with $2 \nmid bc$. As $3 \nmid a$ and $2a^2 \equiv 26-3-15 \equiv 0 \pmod 8$, a or $-a$ has the form $2(3w+1)$ with $w \in \mathbb{Z}$. Write $b = 2u+1$ and $c = 2v+1$ with $u, v \in \mathbb{Z}$. Then

$$24n + 26 = 2(2(3w+1))^2 + 3(2u+1)^2 + 15(2v+1)^2$$

and hence $n = T_u + 5T_v + w(3w+2)$. This proves the universality of $(6,4,5,5,1,1)$ over $\mathbb{Z}$. $\square$

Proof of Theorem 1.3(ii). (a) Let $n \in \mathbb{N}$ and $r \in \{1,2\}$. Apparently,

$$\begin{aligned} & n = T_x + 5\frac{y(3y+1)}{2} + z(3z+r) \\ \iff & 24n + 2r^2 + 8 = 3(2x+1)^2 + 5(6y+1)^2 + 2(6z+r)^2. \end{aligned}$$

As mentioned Part (b) in the proof of Theorem 1.3(i), there are two classes in the genus of $x^2+y^2+15z^2$, and the one not containing $x^2+y^2+15z^2$ has the representative $x^2+4y^2+4z^2-2yz$. If $12n+r^2+4 = x^2+4y^2+4z^2-2yz$ with $x,y,z \in \mathbb{Z}$, then $2 \mid yz$ since $r^2 \not\equiv x^2 - 2 \pmod 4$. Thus, in view of (4.4) and Lemma 3.1, $12n + r^2 + 4 = x^2 + y^2 + 15z^2$ for some $x,y,z \in \mathbb{Z}$. If $x \equiv y \pmod 2$, then $z \equiv r \pmod 2$, $x^2 + y^2 \equiv r^2 - 15z^2 \equiv 2r^2 \pmod 4$ and hence $x \equiv y \equiv r \equiv z \pmod 2$. So, x or y has the same parity with z. Without loss of generality we may assume that $y \equiv z \pmod 2$. Since $y^2 + 15z^2 \equiv 0 \pmod 4$, we have $x \equiv r \pmod 2$. If $r = 2$ and $y^2 + 15z^2 = 0$, then $12n + r^2 + 4 = 0^2 + x^2 + 15 \times 0^2$ with $x \equiv 0 \equiv r \pmod 2$ and $x^2 + 15 \times 0^2 > 0$. If $r = 1$, then $12n^2 + r^2 + 4 = 12n + 5$ is congruent to 2 modulo 3 and hence not a square. Thus, without loss of generality we may assume that $y^2 + 15z^2 > 0$.

Observe that

$$24n + 2r^2 + 8 = 2(x^2 + y^2 + 15z^2) = 2x^2 + 3u^2 + 5v^2$$

with $u = (y+5z)/2$ and $v = (y-3z)/2$ both odd. Since $3u^2 + 5v^2 \equiv 2r^2 - 2x^2 \equiv 0 \pmod 8$ and $2(3u^2+5v^2) = y^2 + 15z^2 > 0$, by Lemma 3.2(iii) we can write $3u^2 + 5v^2$ as $3y_0^2 + 5z_0^2$ with y_0 and z_0 both odd. As $2(x^2+z_0^2) \equiv 2x^2+5z_0^2 \equiv 2r^2+8 \pmod 3$, we have $x^2+z_0^2 \equiv r^2+1 \equiv 2 \pmod 3$

and hence we may write x or $-x$ as $6u+r$, z_0 or $-z_0$ as $6v+1$, and $y_0=2w+1$, where u,v,w are integers. Therefore

$$24n+2r^2+8=2x^2+3y_0^2+5z_0^2=2(6u+r)^2+3(2w+1)^2+5(6v+1)^2$$

and hence $n=u(3u+r)/2+5v(3v+1)/2+T_w$. This proves the universality of $(15,5,6,2r,1,1)$ over $\mathbb{Z}$.

(b) Let $n\in\mathbb{N}$, $s\in\{1,3,5\}$ and $t\in\{1,2\}$ with $(s,t)\neq(5,2)$. Apparently,

$$\begin{aligned}&n=T_x+\frac{y(5y+s)}{2}+z(3z+t)\\ \iff& 120n+3s^2+10t^2+15=15(2x+1)^2+3(10y+s)^2+10(6z+t)^2.\end{aligned}$$

There are two classes in the genus of $3x^2+10y^2+15z^2$, and the one not containing $3x^2+10y^2+15z^2$ has the representative

$$\begin{aligned}g(x,y,z)=&7x^2+7y^2+12z^2+6(x+y)z+4xy\\ =&3\left(\frac{x+y}{2}+2z\right)^2+10\left(\frac{x-y}{2}\right)^2+15\left(\frac{x+y}{2}\right)^2.\end{aligned}\tag{4.9}$$

If $120n+3s^2+10t^2+15=g(x,y,z)$ with $x,y,z\in\mathbb{Z}$, then we obviously have $x\equiv y\pmod 2$. Thus, in view of (4.9) and Lemma 3.1, $120n+3s^2+10t^2+15=3x^2+10y^2+15z^2$ for some $x,y,z\in\mathbb{Z}$. If $x=z=0$, then $120n+3s^2+10t^2+15=10y^2$, hence $(s,t)=(5,1)$ and $y^2=12n+10\equiv 2\pmod 4$ which is impossible. So $x^2+5z^2>0$, and hence by [24, Lemma 2.1] we can rewrite x^2+5z^2 as $x_0^2+5z_0^2$ with $x_0,z_0\in\mathbb{Z}$ not all divisible by 3. Without loss of generality, we simply assume that $3\nmid x$ or $3\nmid z$. Note that $3\nmid y$ since $3\nmid t$. If $3\nmid xz$, then $\varepsilon_1x\equiv y\equiv\varepsilon_2z$ for some $\varepsilon_1,\varepsilon_2\in\{\pm1\}$. If $3\mid x$ and $3\nmid z$, then $x+y+\varepsilon z\equiv 0\pmod 3$ for some $\varepsilon\in\{\pm1\}$. If $3\nmid x$ and $3\mid z$, then $\varepsilon x+y+z\equiv 0\pmod 3$ for some $\varepsilon\in\{\pm1\}$. Without loss of generality, we just assume that $x+y+z\equiv 0\pmod 3$ (otherwise we may adjust signs of x,y,z suitably). Note that $x\equiv z\pmod 2$ and we have the identity

$$3\left(\frac{x+10y-5z}{6}\right)^2+10\left(\frac{x+z}{2}\right)^2+15\left(\frac{x-2y-5z}{6}\right)^2=3x^2+10y^2+15z^2\tag{4.10}$$

with $x_1=(x+10y-5z)/6$, $y_1=(x+z)/2$ and $z_1=(x-2y-5z)/6$ all integral.

If $x\equiv z\equiv 1\pmod 2$, then $10y^2=120n+3s^2+10t^2+15-3x^2-15z^2\equiv 10t^2\pmod 4$ and hence $y\equiv t\pmod 2$.

Now suppose that $x \equiv z \equiv 0 \pmod 2$. Then $2y^2 \equiv 10y^2 \equiv 3s^2 + 10t^2 + 15 \equiv 2(t^2+1) \pmod 4$ and hence $y \not\equiv t \pmod 2$. Observe that

$$2t^2+2 \equiv 120n+3s^2+10t^2+15 = 3x^2+10y^2+15z^2 \equiv x^2+z^2+2(t+1)^2 \pmod 8$$

and hence

$$y_1 = \frac{x+z}{2} \equiv \left(\frac{x}{2}\right)^2 + \left(\frac{z}{2}\right)^2 = \frac{x^2+z^2}{4} \equiv t \pmod 2.$$

Thus

$$z_1 = x_1 - 2y \equiv x_1 \equiv \frac{x+z}{2} - 3z + 5y \equiv t + y \equiv 1 \pmod 2.$$

In view of the above, there are integers $x, y, z \in \mathbb{Z}$ with $x \equiv z \equiv 1 \pmod 2$ and $y \equiv t \pmod 2$ such that $120n + 3s^2 + 10t^2 + 15 = 3x^2 + 10y^2 + 15z^2$. Clearly, y or $-y$ has the form $6v + t$ with $v \in \mathbb{Z}$. Write $z = 2w + 1$ with $w \in \mathbb{Z}$. Since $x^2 \equiv s^2 \pmod 5$, we can write x or $-x$ as $10u+s$ with $w \in \mathbb{Z}$. Therefore

$$120n + 3s^2 + 10t^2 + 15 = 3(10u+s)^2 + 10(6v+t)^2 + 15(2w+1)^2$$

and hence $n = T_w + u(5u+s)/2 + v(3v+t)$. This proves the universality of $(6, 2t, 5, s, 1, 1)$ over $\mathbb{Z}$. $\square$

5. Proof of Theorem 1.4

B.W. Jones and G. Pall [15] proved the following celebrated result.

Lemma 5.1. *Let $n \in \mathbb{N}$ with $8n+1$ not a square. Then*

$$\begin{aligned}&|\{(x,y,z) \in \mathbb{Z}^3 :\ x^2+y^2+z^2 = 8n+1 \ \&\ 4 \mid x\}| \\ =&|\{(x,y,z) \in \mathbb{Z}^3 :\ x^2+y^2+z^2 = 8n+1 \ \&\ x \equiv 2 \pmod 4\}| > 0.\end{aligned}$$

A. G. Earnest [8, 9] showed the following useful result.

Lemma 5.2. *Let c be a primitive spinor exceptional integer for the genus of a positive ternary quadratic form $f(x,y,z)$, and let S be a spinor genus containing f. Let s be a fixed positive integer relatively prime to $2d(f)$ for which cs^2 can be primitively represented by S. If $t \in \mathbb{Z}^+$ is relatively prime to $2d(f)$, then ct^2 can be primitively represented by S if and only if*

$$\left(\frac{-cd(f)}{s}\right) = \left(\frac{-cd(f)}{t}\right).$$

Proof of Theorem 1.4. Fix $n \in \mathbb{N}$. Clearly,

$$n = T_x + y^2 + 2z(4z+1) \iff 8n+2 = (2x+1)^2 + 8y^2 + (8z+1)^2.$$

So, it suffices to show that $8n+2 = x^2+y^2+8z^2$ for some $x, y, z \in \mathbb{Z}$ with $x \equiv \pm 1 \pmod 8$.

Case 1. n is not twice a triangular number.

In this case, $4n+1$ is not a square. If $2 \mid n$, then by Lemma 5.1 we can write $4n+1$ as $x^2+y^2+z^2$ with $2 \nmid x$, $2 \mid y$ and $z \equiv 2 \pmod 4$. If $2 \nmid n$, then there are $x, y, z \in \mathbb{Z}$ with $2 \nmid x$ and $y \equiv z \equiv 0 \pmod 2$ such that $4n+1 = x^2+y^2+z^2$ and hence $y \not\equiv z \pmod 4$ since $y^2+z^2 \equiv 5-x^2 \equiv 4 \pmod 8$. So we can always write $4n+1 = x^2+y^2+z^2$ with $2 \nmid x$, $2 \mid y$ and $z \equiv 2n-2 \pmod 4$, hence

$$8n+2 = 2(x^2+y^2+z^2) = (x+y)^2 + (x-y)^2 + 8\left(\frac{z}{2}\right)^2$$

with $z/2 \equiv n-1 \pmod 2$, thus

$$(x+y)^2 + (x-y)^2 \equiv 8n+2-8(n-1) = 10 \not\equiv 3^2+3^2 \pmod{16}$$

and hence $x + \varepsilon y \equiv \pm 1 \pmod 8$ for some $\varepsilon \in \{\pm 1\}$.

Case 2. $n = 2T_m$ with $m \in \mathbb{N}$, and $2m+1$ has no prime factor of the form $4k+3$.

In this case, $2m+1$ can be expressed as the sum of two squares. If $4 \mid m$, then

$$8n+2 = 2(2m+1)^2 = (2m+1)^2 + (2m+1)^2 + 8 \times 0^2$$

with $2m+1 \equiv 1 \pmod 8$. If $4 \nmid m$, then $2m+1 = u^2 + (2v)^2$ for some odd integers u and v, and hence

$$\begin{aligned} 8n+2 =& 2(u^2+4v^2)^2 = 2((u^2-4v^2)^2 + (4uv)^2) \\ =& (u^2-4v^2+4uv)^2 + (u^2-4v^2-4uv)^2 + 8 \times 0^2 \end{aligned}$$

with $u^2 - 4v^2 \pm 4uv \equiv 1 \pmod 8$.

Case 3. $n = 2T_m$ with $m \in \mathbb{N}$, and $2m+1$ has a prime factor $p \equiv 3 \pmod 4$.

By Lagrange's four-square theorem, we can write $p = a^2+b^2+c^2+d^2$, where a is an even number and b, c, d are odd numbers. Thus

$$\begin{aligned} p^2 =& (a^2+b^2-c^2-d^2)^2 + 4(a^2+b^2)(c^2+d^2) \\ =& (a^2+b^2-c^2-d^2)^2 + (2ac+2bd)^2 + (2ad-2bc)^2 \end{aligned}$$

and hence $(2m+1)^2 = x^2 + (2y)^2 + (2z)^2$ for some odd integers x, y, z. Observe that

$$8n + 2 = 2(2m+1)^2 = (x+2y)^2 + (x-2y)^2 + 8z^2$$

and $(x+2y)^2 + (x-2y)^2 \equiv 2 - 8z^2 \equiv 10 \not\equiv 3^2 + 3^2 \pmod{16}$. So one of $x+2y$ and $x-2y$ is congruent to 1 or -1 modulo 8.

Now we give an advanced approach to Case 3. There are three classes in the genus of $x^2+y^2+32z^2$ with the three representatives

$$\begin{aligned} f_1(x,y,z) =& x^2+y^2+32z^2, \\ f_2(x,y,z) =& 2x^2+2y^2+9z^2+2yz-2zx, \\ f_3(x,y,z) =& x^2+4y^2+9z^2-4yz. \end{aligned}$$

f_1 and f_2 constitute a spinor genus while another spinor genus in the genus has the representative f_3. Since 2 is a a primitive spinor exceptional integer for this genus, by Lemma 5.2 we can write $2p^2$ as

$$f_3(u,v,w) = u^2 + 4v^2 + 9w^2 - 4vw = u^2 + (2v-w)^2 + 8w^2$$

with $u, v, w \in \mathbb{Z}$. Since $2 \nmid uw$, we see that $8n+2 = 2(2m+1)^2 = a^2+b^2+8c^2$ for some odd integers a, b, c. As $a^2+b^2 \equiv 2 - 8c^2 \equiv 10 \not\equiv 3^2+3^2 \pmod{16}$, a or b is congruent to 1 or -1 modulo 8. This concludes our discussion of Case 3.

In view of the above, we have completed the proof of Theorem 1.4. □

Remark 5.1. $f_3(x,y,z)$ in the proof of Theorem 1.4 is one of the very few spinor regular forms that are not regular. For more details, see [1].

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