

# NONCROSSING HYPERTREES

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ABSTRACT. Hypertrees and noncrossing trees are well-established objects in the combinatorics literature, but the hybrid notion of a noncrossing hypertree has received less attention. In this article I investigate the poset of noncrossing hypertrees as an induced subposet of the hypertree poset. Its dual is the face poset of a simplicial complex, one that can be identified with a generalized cluster complex of type  $A$ . The first main result is that this *noncrossing hypertree complex* is homeomorphic to a piecewise spherical complex associated with the noncrossing partition lattice and thus it has a natural metric. The fact that the order complex of the noncrossing partition lattice with its bounding elements removed is homeomorphic to a generalized cluster complex was not previously known or conjectured.

The metric noncrossing hypertree complex is a union of unit spheres with a number of remarkable properties: 1) the metric subspheres and simplices in each dimension are both bijectively labeled by the set of noncrossing hypertrees with a fixed number of hyperedges, 2) the number of spheres containing the simplex labeled by the noncrossing tree  $\tau$  is the same as the number simplices in the sphere labeled by the noncrossing tree  $\tau$ , and 3) among the maximal spherical subcomplexes one finds every normal fan of a metric realization of the simple associahedron associated to the cluster algebra of type  $A$ . In particular, the poset of noncrossing hypertrees and its metric simplicial complex provide a new perspective on familiar combinatorial objects and a common context in which to view the known bijections between noncrossing partitions and the vertices/facets of simple/simplicial associahedra.

## INTRODUCTION

The properties of hypertrees are well-documented [MM96, War98, Kal99, BMMM01, MM04, JMM06, JMM07, Cha07, Oge13] as are the properties of noncrossing trees [Noy98, FN99, DN02, PP02, Pan03, Hou03, CY06, SW09, LP14], but the hybrid concept of a noncrossing hypertree has received much less attention. The poset of noncrossing

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hypertrees, viewed as an induced subposet of the better-known hypertree poset has upper intervals that are Boolean lattices and thus is (the dual of) the face lattice of a simplicial complex that I call the *noncrossing hypertree complex*.<sup>1</sup>

The first main result is to identify the topology of the noncrossing hypertree complex as that of the piecewise spherical metric simplicial complex that is the link of the long diagonal edge in the orthoscheme complex of the noncrossing partition lattice, which, for brevity, I refer to as the *noncrossing partition link*. Through this connection the noncrossing hypertree complex can be turned into a geometric object.

**Theorem A** (Topology and Geometry). *The noncrossing hypertree complex is naturally homeomorphic to the noncrossing partition link. As a consequence, the piecewise spherical metric on the latter induces a piecewise spherical metric on the former.*

One way to view the noncrossing hypertree complex is as a simpler simplicial structure on the noncrossing partition link. This simplified metric structure has a number of remarkable properties. For example, there is a duality between its simplices and its metric subspheres.

**Theorem B** (Simplices and Spheres). *Every spherical simplex in the metric noncrossing hypertree complex of any dimension is contained in a subcomplex isometric to a unit sphere of the same dimension. In fact, in the top dimension (and conjecturally in all dimensions) there is a natural map from simplices to spherical subcomplexes that establishes a bijection between these two sets.*

The spheres referred to in the theorem are called *special spheres*. Because the noncrossing partition link is a subcomplex of a spherical building, its top-dimensional simplices and top-dimensional spherical subcomplexes are sometimes called chambers and apartments, respectively. In the new simplicial structure of the noncrossing hypertree complex the top-dimensional spheres are the same as before so I still call them *apartments* and I call the top-dimensional simplices in the noncrossing hypertree complex *tree chambers*. They are amalgamations of the original chambers in the noncrossing partition link which,

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<sup>1</sup>There is an alternative encoding of noncrossing hypertrees as decompositions of even-sided polygons into even-sided subpolygons and as such the noncrossing hypertree complex can be identified with the generalized cluster complex of type  $A$  with  $m = 2$ . See §3 for details. In this guise it has been studied before but different aspects of its structure are visible when its simplex labels are viewed as noncrossing hypertrees. In particular the homeomorphism established in Theorem A is new and its proof relies heavily on the structure of the noncrossing hypertrees.

for the sake of clarity, I call *partition chambers*. In this language Theorem B states that the set of tree chambers and the set of apartments in the noncrossing hypertree complex are both bijectively labeled by noncrossing trees. The next result establishes some additional aspects of this duality.

**Theorem C** (Bijections). *For every noncrossing hypertree  $\tau$  there is a bijection between the number of special spheres containing the tree simplex labeled  $\tau$  as a top-dimensional simplex and the set of tree simplices contained in the special sphere labeled  $\tau$ . When  $\tau$  is a noncrossing tree, this means that there is a bijection between  $\{\sigma \mid \mathbf{Chamber}(\tau) \in \mathbf{Apart}(\sigma)\}$ , the set of apartments containing the tree chamber labeled  $\tau$  and  $\{\sigma \mid \mathbf{Chamber}(\sigma) \in \mathbf{Apart}(\tau)\}$ , the set of tree chambers in the apartment labeled  $\tau$ .*

As  $\tau$  varies, these numbers vary as well, from a minimum that is a power of 2 to a maximum, conjecturally, that is a Catalan number.<sup>2</sup> The structure of these extremal simplicial spheres is easy to describe. The minimum value corresponds to an orthoplex, the generalization of the octahedron also known as a cross-polytope, and the maximum value corresponds to a simplicial associahedron.

**Theorem D** (Associahedra). *Let  $\tau$  be a noncrossing tree. If the tree chamber labeled  $\tau$  consists of a single partition chamber then the apartment labeled  $\tau$  is a simplicial associahedron. In addition, the variety of simplicial associahedra produced in this way include all of the simplicial associahedra that are normal fans to the type A simple associahedra constructed by Hohlweg and Lange.*

The simplicial associahedra that this theorem produces are closely related to Reading's  $c$ -Cambrian fans [Rea06, RS09].

**Structure of the article.** The first three sections establish basic properties of the noncrossing hypertree complex and the next two sections establish basic properties of the noncrossing partition link. The proof of Theorem A is spread over the three sections after that. Once these foundations are in place, the remaining main theorems are proved in the final sections.

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<sup>2</sup>The Catalan numbers are the maximum values in all the cases where computer investigation is feasible, but I do not currently have a proof that this is the maximal value.

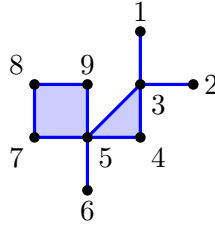


FIGURE 1. A hypertree on 9 vertices with hyperedges  $\{1, 3\}$ ,  $\{2, 3\}$ ,  $\{3, 4, 5\}$ ,  $\{5, 6\}$  and  $\{5, 7, 8, 9\}$ .

## 1. HYPERFORESTS AND HYPERTREES

A simple graph is a set of vertices and a collection of 2-element subsets called edges, a forest is simple graph with no cycles, and a tree is a connected forest. Hypergraphs, hyperforests and hypertrees are expanded versions of these notions.

**Definition 1.1** (Hypergraphs). A *hypergraph* is a collection of vertices, usually identified with the first few natural numbers, and a collection of subsets of the vertices called *hyperedges* where each hyperedge must contain at least 2 elements. Familiar graph definitions are slightly modified to accommodate this change. A *path of length  $k$*  in a hypergraph is an alternating sequence  $(v_0, e_1, v_1, \dots, e_k, v_k)$  of vertices  $v_i$  and hyperedges  $e_i$ , starting and ending with a vertex, where each hyperedge  $e_i$  contains the vertices  $v_{i-1}$  and  $v_i$ . Its *endpoints* are  $v_0$  and  $v_k$ , its length is  $k$  and paths of positive length are *nontrivial*. A path is *simple* if all of its edges and vertices are distinct, a *cycle* if its endpoints are equal, and a *simple cycle* if all of its edges are distinct and all of its vertices are distinct except that its endpoints are equal. A hypergraph is *connected* if every pair of vertices are the endpoints of a path, a *hyperforest* if there are no nontrivial simple cycles and a *hypertree* if it is a connected hyperforest. A pair of hyperedges are said to be *weakly disjoint* when they have at most one vertex in common and the hyperedges of a hyperforest are pairwise weakly disjoint because two hyperedges with two common vertices form a simple cycle. A hypergraph whose hyperedges are pairwise disjoint can be viewed as a *partition*. It has a block for every hyperedge and a singleton block for each vertex not contained in any hyperedge. A hypertree is shown in Figure 1.

Although the focus of this article is noncrossing hypertrees, hyperforests are useful when establishing results by induction. References to hyperforests are rare (one of few is [Knu05]) but they are definitely

worthy studying since the naturally defined bounded graded hyperforest poset includes trees, hypertrees, forests and partitions, as well as the noncrossing versions of these objects.

**Definition 1.2** (Hyperforest poset). The set of all hyperforests on a fixed number of vertices comes equipped with a natural partial order where  $\sigma \leq \tau$  if and only if every hyperedge of  $\sigma$  is a subset of a hyperedge of  $\tau$ .<sup>3</sup> The result is a bounded graded poset  $\mathbf{HF}_{n+1}$  where  $n + 1$  denotes the number of vertices. The unique minimal element is the *trivial hyperforest*  $t$  with no hyperedges. The unique maximal element has a single hyperedge containing all the vertices which, in anticipation of its later uses, I call the *Coxeter hypertree*. There are two types of covering relations. The *first type* adds a new edge, i.e. a hyperedge with only 2 vertices, whose vertices belong to distinct connected components. The *second type* involves replacing two hyperedges that share a vertex with the hyperedge that is their union. This process is called *merging* and its inverse is called *splitting*. Note that a hyperedge needs to contain at least 3 vertices in order to be splittable. From this description of the covering relations, it is easy to show that the *height* of a hyperforest  $\tau$  is  $2|V| - |E| - 2|C|$  where  $V$  is the set of vertices,  $E$  is the set of hyperedges and  $C$  is the set of connected components.

Let  $\tau$  be a hyperforest with  $n + 1$  vertices. It has height 0 if  $\tau$  is the trivial hyperforest  $t$ ,  $|E|$  if  $\tau$  is a forest,  $n$  if  $\tau$  is a tree,  $2n - |E|$  if  $\tau$  is a hypertree and  $2n - 1$  if  $\tau$  is the Coxeter hypertree. The choice of the variable  $n$  to denote the number of vertices minus one, rather than the number of vertices itself, is not an accident or a mistake. It is a notational irritant with potential future benefits.

**Remark 1.3** (Rank). The symmetric group  $\mathbf{Sym}_{n+1}$  naturally acts on  $\mathbb{R}^{n+1}$  by permuting coordinates and it also acts on the  $n$ -dimensional subspace of vectors where the sum of the coordinates is equal to 0. This action shows that the symmetric group  $\mathbf{Sym}_{n+1}$  is an example of a finite group of euclidean isometries generated by reflections, the reflections in this case being the transpositions  $(i, j)$  that switch the  $i^{\text{th}}$  and  $j^{\text{th}}$  coordinates pointwise fixing the hyperplane defined by the equation  $x_i = x_j$ . More specifically, the symmetric group  $\mathbf{Sym}_{n+1}$  is the finite Coxeter group of type  $A_n$ , where the  $n$  indicates, among other things, the size of the smallest reflection generating set, a number called its *rank*. All of the results presented here, conjecturally at least, are a “type  $A$ ” version of a more general theory. It is with these future

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<sup>3</sup>This ordering agrees with the standard ordering on partitions, but it is the dual of the standard ordering on hypertrees found in the literature.

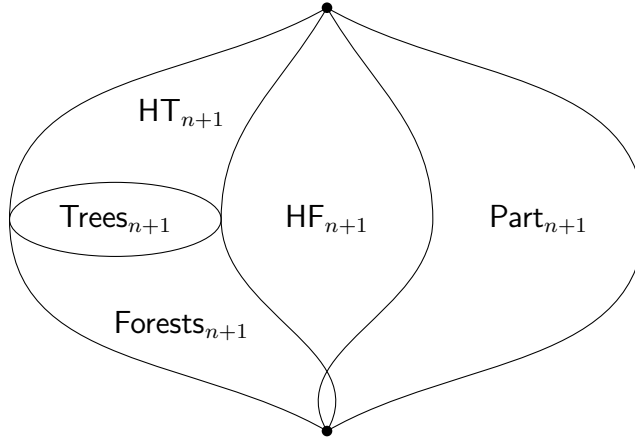


FIGURE 2. The poset  $\text{HF}_{n+1}$ , its subposets  $\text{Forests}_{n+1}$ ,  $\text{Part}_{n+1}$  and  $\text{HT}_{n+1}$  and its subset  $\text{Trees}_{n+1}$ .

developments in mind that the variable  $n$  is usually used to denote the rank of the associated Coxeter group. In a currently hypothetical general theory of hyperforests for arbitrary Coxeter groups, the set  $\text{HF}_{n+1}$  of all hyperforests on  $n + 1$  vertices would be denoted  $\text{HF}(A_n)$ .

The hyperforest poset and its various substructures are shown in Figure 2. A collection of pairwise disjoint edges defines both a forest and a partition, hence the partial overlap.

**Remark 1.4** (Forests, trees and hypertrees). The forests, trees and hypertrees inside the hyperforest poset can be identified by focusing on the two types of covering relations. Forests are connected to the trivial hyperforest by covering relations that add a new edge, hypertrees are connected to the Coxeter hypertree by covering relations that merge two hyperedges, and trees are on the boundary between these two subposets. Moreover, every maximal chain from the trivial hyperforest to the Coxeter hypertree contains exactly  $n$  covering relations that add edges and  $n - 1$  covering relations that merge hyperedges. If the covering relations are labeled by the endpoints of the new edge in the first case and the pair of merged hyperedges in the second, then the steps along any maximal chain can be reordered so that all of the edge additions take place before any of the hyperedge mergers and this reordering does not change the set of labels on the covering edges. The reordered maximal chain passes through a tree at height  $n$  with forests below and hypertrees above.

**Definition 1.5** (Hypertree poset). The set of all hypertrees on  $n + 1$  vertices form an induced subposet  $\text{HT}_{n+1}$  of the hyperforest poset.

The grading and the covering relations in this *hypertree poset* come from the hyperforest poset but all covering relations are of the second type. It is bounded above by the Coxeter hypertree and its minimal elements are trees. Kalikow [Kal99] and Warne [War98] proved that its size is given by the formula  $|\text{HT}_{n+1}| = \sum_k (n+1)^{k-1} S(n, k)$ , where  $S(n, k)$  is a Stirling number of the second kind. For more information see the sequence [oeis:A030019](#) in the Online Encyclopedia of Integer Sequences [Slo] and for its uses in geometric group theory, see [MM96, BMMM01, MM04, JMM06, JMM07, Pig12].

The *degree* of a vertex is the number of hyperedges containing it and the total number of hyperedges is determined by its vertex degrees.

**Proposition 1.6** (Vertex degrees). *If  $\tau$  is a hypertree with vertices  $V$  and hyperedges  $E$ , then  $\sum_{v \in V} (\deg(v) - 1) = |E| - 1$ . In other words, if one expects there to be one hyperedge and vertices to have degree 1, then the sum of the excess vertex degrees is the number of excess hyperedges.*

*Proof.* The equation holds for the Coxeter hypertree and it is preserved as hyperedges are split.  $\square$

The *size* of a hyperedge is the number of vertices it contains. For hypertrees the number of vertices is determined by its hyperedge sizes.

**Proposition 1.7** (Hyperedge sizes). *For a connected hypergraph  $\tau$  with vertices  $V$  and hyperedges  $E$ ,  $\sum_{e \in E} (\text{size}(e) - 1) = |V| - 1$  if and only if  $\tau$  is a hypertree.*

*Proof.* In a connected hypergraph it is possible to order the hyperedges so that each hyperedge except the first has at least one vertex in the union of the hyperedges earlier in the list. Thus for all connected hypergraphs  $\sum_{e \in E} (\text{size}(e) - 1) \geq |V| - 1$ . When  $\tau$  is a hypertree each hyperedge in this ordering has exactly one vertex with the union of hyperedges earlier in the list and the inequality is an equality. Conversely, suppose  $\tau$  contains a simple cycle. When the last hyperedge of the simple cycle is added, at least two of its vertices are not new and the inequality is strict.  $\square$

**Definition 1.8** (Partitions). Partitions form a bounded graded lattice, but it is important to note that its natural grading and covering relations are distinct from those of the hyperforest poset. A covering relation in the partition lattice involves joining two blocks into a bigger block, but this changes 1, 2 or 3 levels in hyperforest poset depending on whether neither, one or both of the blocks are more than a single element. One way to identify the partitions inside the hyperforest poset

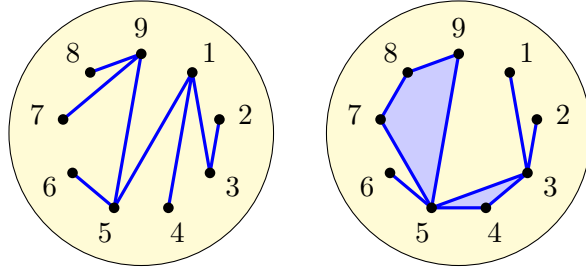


FIGURE 3. A noncrossing tree and a noncrossing hypertree.

is to restrict the partial order on  $\mathbf{HF}_{n+1}$  to the transitive closure of the second type of covering relation that merge two hyperedges sharing a vertex. The connected components in the resulting order each have a unique maximum element that is a partition and all partitions are maximal in their connected component. Thus the partitions index the connected components. For each hyperforest  $\tau$ , the unique partition  $\sigma$  above  $\tau$  in this restricted ordering is obtained from  $\tau$  by iteratively merging hyperedges with a vertex in common until no more mergers of this type are possible. It is called the *partition of  $\tau$*  and it is the partition determined by the connected components of  $\tau$ . The map sending hyperforests to their partitions is an example of a closure operator.

## 2. NONCROSSING HYPERTREES

This section introduces noncrossing versions of all of the combinatorial structures defined in §1. The key definition is that of noncrossing and weakly noncrossing subsets of vertices of a convex polygon.

**Definition 2.1** (Noncrossing subsets). Two subsets of the vertices of a convex polygon in the plane are called *noncrossing* when their convex hulls are completely disjoint and *weakly noncrossing* when their convex hulls have at most one vertex, but no other points, in common.

**Definition 2.2** (Noncrossing hypergraphs). A hypergraph  $\tau$  whose vertices have been identified with those of a convex polygon in the plane is called a *noncrossing hypergraph* when its hyperedges, thought of as subsets of vertices, are pairwise weakly noncrossing. A *noncrossing tree, forest, hypertree, hyperforest, or partition* is a noncrossing hypergraph that lives in this subcategory. Note that because the hyperedges of a partition are completely disjoint, they are noncrossing in the strong sense and not merely weakly noncrossing. When describing noncrossing hypergraphs, the vertices of the convex  $k$ -gon are usually labeled with the set  $[k] := \{1, 2, \dots, k\}$  where these integers are viewed



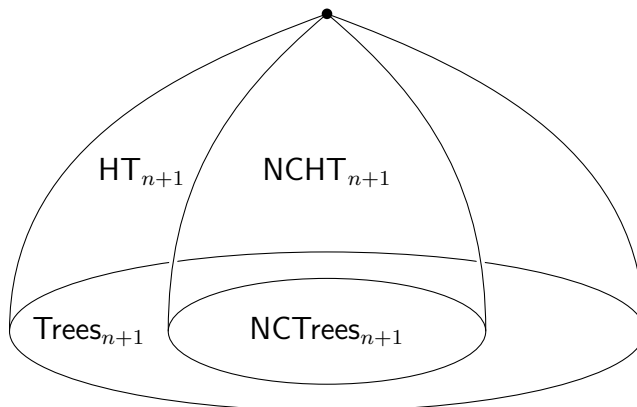


FIGURE 4. The poset  $\text{HT}_{n+1}$ , its subposet  $\text{NCHT}_{n+1}$  and its subsets  $\text{Trees}_{n+1}$  and  $\text{NCTrees}_{n+1}$ .

as representing the integers modulo  $k$ . They are placed in the order the vertices appear in the clockwise oriented boundary of the polygon.

A noncrossing tree and noncrossing hypertree are shown in Figure 3, and note that the noncrossing hypertree on the right is the vertex-labeled hypertree of Figure 1 redrawn inside a regular 9-gon. The noncrossing versions restrict the size of the various sets and posets but their internal structure remains much the same. The relationships among the sets  $\text{Trees}_{n+1}$  and  $\text{NCTrees}_{n+1}$  and the posets  $\text{HT}_{n+1}$  and  $\text{NCHT}_{n+1}$  are shown in Figure 4. A similar diagram could be drawn for all of the substructures of the hyperforest poset shown in Figure 2.

**Remark 2.3** (Noncrossing trees). The size of the set  $\text{NCTrees}_{n+1}$  of all noncrossing trees on  $n + 1$  vertices is the generalized Catalan number  $\frac{1}{2n+1} \binom{3n}{n}$ , also known as the Fuß-Catalan number  $\text{Cat}^{(2)}$  of type  $A_n$ . This was proved by Noy [Noy98] and Panholzer and Prodinger [PP02] among others. For additional information, see the sequence [oeis:A001764](https://oeis.org/A001764) in [Slo].

**Definition 2.4** (Noncrossing hypertree poset). The poset  $\text{NCHT}_{n+1}$  of noncrossing hypertrees on  $n + 1$  vertices is the induced subposet of  $\text{HT}_{n+1}$  restricted to those hypertrees that are noncrossing. The Coxeter hypertree remains the maximum element, the set  $\text{NCTrees}_{n+1}$  of noncrossing trees are the minimal elements, the new covering relations are a subset of the old covering relations and the grading is as before.

The covering relations in the noncrossing hypertree poset are particularly important. Since they correspond to covering relations in the hypertree poset, they involve merging two hyperedges that share a vertex, but there are restrictions.

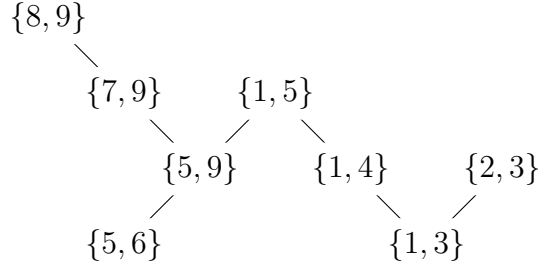


FIGURE 5. The Hasse diagram for the hyperedge poset of the noncrossing tree shown on the left in Figure 3. The hyperedges in the covering relations that lie in a straight line are locally linearly ordered by the vertex they share.

**Definition 2.5** (Local linear orderings). Let  $v$  be a vertex of a noncrossing hyperforest  $\tau$ . The  $\deg(v)$  hyperedges of  $\tau$  that contain  $v$  have a *local linear order* defined by standing at  $v$  facing towards the interior of the polygon and linearly ordering these hyperedges as they occur from left to right. Concretely, for distinct hyperedges  $e$  and  $e'$  containing  $v$ ,  $e < e'$  if and only if  $e$  is to the left of  $e'$  from the perspective of  $v$  looking towards the interior of the polygon. Note that when two hyperedges sharing a vertex in a noncrossing hyperforest are merged, the result remains a noncrossing hyperforest if and only if these hyperedges are adjacent in the local linear ordering of their shared vertex.

These local linear orderings can be combined into a single poset containing a lot of information.

**Definition 2.6** (Hyperedge posets). Let  $\tau$  be a noncrossing hyperforest. The *hyperedge poset* of  $\tau$  is a poset  $\mathbf{Poset}(\tau)$  whose elements are the hyperedges of  $\tau$  and whose partial order is the transitive closure of the covering relations from local linear orderings of the hyperedges at each vertex of  $\tau$ . Here are two examples to help clarify the definition. The noncrossing hypertree shown on the right in Figure 3 has an hyperedge poset that is linear. In particular, its hyperedges are ordered as follows:  $\{5, 6\} < \{5, 7, 8, 9\} < \{3, 4, 5\} < \{1, 3\} < \{2, 3\}$ . The noncrossing tree shown on the left has a more complicated hyperedge poset whose Hasse diagram is shown in Figure 5.

For an algebraic reason for this ordering, see Lemma 7.7. The examples illustrate key properties of hyperedge posets.

**Proposition 2.7** (Hyperedge posets). *For every noncrossing hypertree  $\tau$ , the Hasse diagram of its hyperedge poset  $\mathbf{Poset}(\tau)$ , viewed as an*

undirected graph, is a tree. Similarly, the Hasse diagram of the hyperedge poset of a noncrossing hyperforest, viewed as an undirected graph, is a forest. In both cases, the edges of the Hasse diagram correspond to pairs of hyperedges that share a vertex and are adjacent in its local linear ordering.

*Proof.* First note that the fact that the noncrossing hypertree  $\tau$  is connected means the Hasse diagram of  $\text{Poset}(\tau)$  must be connected. Next, from the way the partial order of  $\text{Poset}(\tau)$  is generated, the only possible covering relations are those from the local vertex linear orders. These are of the form described and there are  $\deg(v) - 1$  of these for each vertex  $v$ , but by Proposition 1.6, the total number of possible covering relations is the minimum number needed to connect the Hasse diagram. Thus, all of the possible cover relations really are cover relations and there are none left over to make the undirected graph underlying the Hasse diagram anything other than a tree. For noncrossing hyperforests consider one connected component at a time.  $\square$

The observation made in Definition 2.5 implies the following.

**Proposition 2.8** (Merging hyperedges). *Let  $\tau$  be a noncrossing hyperforest and let  $e$  and  $e'$  be two hyperedges that share a vertex. The hyperforest  $\tau'$  formed by merging  $e$  and  $e'$  remains noncrossing if and only if  $e$  and  $e'$  are the endpoints of a covering relation in the hyperedge poset  $\text{Poset}(\tau)$ . Moreover, when  $\tau'$  remains noncrossing, the Hasse diagram of its hyperedge poset  $\text{Poset}(\tau')$  is obtained from the Hasse diagram of  $\text{Poset}(\tau)$  by shrinking the covering edge with endpoints  $e$  and  $e'$  and labeling the new element by the merged hyperedge.*

Iteratively applying Proposition 2.8 proves the following.

**Proposition 2.9** (Partition of a noncrossing hyperforest). *The partition of a noncrossing hyperforest is a noncrossing partition.*

Proposition 2.8 also shows that the noncrossing hypertrees above a fixed noncrossing hypertree  $\tau$  correspond exactly to those obtained by systematically collapsing a forest of edges in the Hasse diagram of its hyperedge poset while merging the corresponding hyperedges. That the result is independent of the order in which they are collapsed and distinct for every distinct subset of edges in the Hasse diagram should already be clear, but it will be crystal clear once the collapsing/merging operation has been reinterpreted as a polygon dissection.

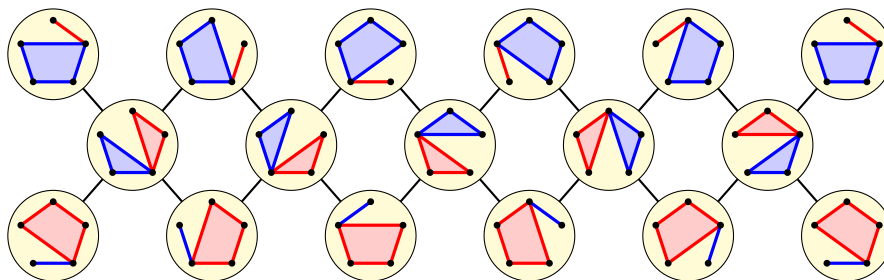


FIGURE 6. The 15 basic noncrossing hypertrees on 5 vertices. Although displayed in a horizontal strip, the hypertrees on the left and the right are identical and should be identified to create an arrangement on a cylinder. The lines indicate inclusions between their lower hyperedges or, equivalently, reverse inclusions between their upper hyperedges.

### 3. THE NONCROSSING HYPERTREE COMPLEX

This section establishes a bijection between noncrossing hypertrees and certain types of polygon dissections. The noncrossing hypertrees one step below the Coxeter hypertree are the key.

**Definition 3.1** (Basic noncrossing hypertrees). A hypertree with exactly two hyperedges is called a *basic hypertree* and the basic noncrossing hypertrees are easy to describe. A basic noncrossing hypertree  $\tau$  is completely determined by the vertex its two hyperedges have in common and the edge on the boundary of the polygon that is not included in the convex hull of either hyperedge. The only restriction is that the shared vertex cannot be an endpoint of the missing boundary edge. Thus there are exactly  $(n+1)(n-1) = n^2 - 1$  basic noncrossing hypertrees on  $n+1$  vertices. In each basic noncrossing hypertree, one of the two hyperedges is below the other in its hyperedge poset and it can be reconstructed from its lower hyperedge alone. In Figure 6 the 15 basic noncrossing hypertrees on 5 vertices are shown ordered according the ordering of their lower hyperedges under inclusion.

The poset of noncrossing hypertrees on a fixed number of vertices are going to bijectively correspond to the poset of all dissections of a polygon with twice as many vertices into even-sided subpolygons. The easy direction is from polygon dissections to noncrossing hypertrees.

**Remark 3.2** (From polygon dissections to noncrossing hypertrees). Given such a polygonal dissection of an even-sided polygon into even-sided subpolygons, one can create a noncrossing hypertree with half as

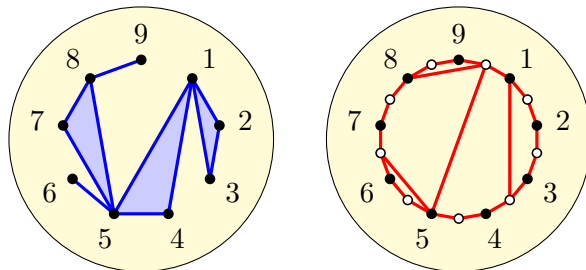


FIGURE 7. A noncrossing hypertree on 9 vertices and the corresponding dissection of an 18-gon into even-sided subpolygons.

many vertices as follows. First assume that the vertices of the polygon have been alternately colored black and white and select the convex hull of the black vertices as the polygon for the noncrossing hypertree. The convex hulls of the black vertices in each subpolygon are its hyperedges. See Figure 7 for an illustration.

This construction is bijective because the process is reversible.

**Remark 3.3** (From noncrossing hypertrees to polygon dissections). Let  $\tau$  be a noncrossing hypertree with hyperedge poset  $\text{Poset}(\tau)$ . To create an even-sided polygon dissection of a polygon with twice as many vertices, first add a white dot in the middle of each boundary edge. The diagonals of the dissection correspond to the covering relations of the hyperedge poset  $\text{Poset}(\tau)$  as follows. Let  $e$  and  $e'$  be hyperedges in  $\tau$  so that  $e < e'$  is a covering relation in  $\text{Poset}(\tau)$ . Using the Hasse diagram of  $\text{Poset}(\tau)$  as a guide, systematically shrink all of the other covering relations, which amounts to iteratively merging various hyperedges. By Proposition 2.8 the end result remains noncrossing and it now has only two hyperedges. As remarked in Definition 3.1, this basic noncrossing hypertree is determined by the (black) vertex the two resulting hyperedges share and the (white vertex in the middle of) the boundary edge that is missing. More concretely, for each vertex  $v$  and for each pair of adjacent hyperedges that contain  $v$  there is a unique midpoint of a boundary edge that is visible from  $v$  looking between these two hyperedges. The diagonal connecting this black vertex  $v$  to this visible white vertex in the middle of the boundary edge is added. See Figure 7. Because the endpoints have different colors the subpolygons are even-sided and these diagonals must be weakly noncrossing because when diagonals cross their black endpoints are in a different connected component from the black vertices between their white endpoints, contradicting the connected nature of the hypertree.

A polygon dissection is determined by the set of pairwise weakly noncrossing diagonals and these sets form a poset using inclusion. The constructions described above establish the following.

**Theorem 3.4** (Noncrossing hypertrees and polygon dissections). *There is a natural order-reversing bijection between the poset of noncrossing hypertrees on a fixed number of vertices and the poset of dissections of an even-sided polygon with twice as many vertices into even-sided subpolygons through the addition of pairwise noncrossing diagonals.*

Under this bijection the process of merging two adjacent hyperedges as described in Proposition 2.8 corresponds to removing a diagonal from the dissection. Since any subset of the diagonals can be removed, the upper intervals in the noncrossing hypertree poset are Boolean lattices and the dual of the noncrossing hypertree poset is the face poset of a simplicial complex.

**Definition 3.5** (Noncrossing hypertree complex). The dual of the noncrossing hypertree poset  $\text{NCHT}_{n+1}$  is the face lattice of an  $n - 2$  dimensional simplicial complex  $\text{Complex}(\text{NCHT}_{n+1})$  that I call the *noncrossing hypertree complex*. It has a vertex for each of the  $n^2 - 1$  basic noncrossing hypertrees, or equivalently, a vertex for every diagonal of an alternately colored  $(2n + 2)$ -sided polygon connecting a black vertex and a white vertex. Two basic noncrossing hypertrees are *compatible* and connected by an edge if and only if the corresponding diagonals are weakly noncrossing. More generally there is a simplex, naturally labeled by a noncrossing hypertree, for every collection of pairwise compatible basic noncrossing hypertrees. The maximal simplices correspond to noncrossing trees. A noncrossing tree on  $n + 1$  vertices has  $n$  edges, so its hyperedge poset has  $n$  vertices and  $n - 1$  covering relations and the maximal dimensional simplices, or *tree chambers* have  $n - 1$  vertices and dimension  $n - 2$ . Table 1 shows the  $f$ -vector and reduced euler characteristic of the noncrossing hypertree complex for small values of  $n$ . Recall that the number  $f_i$  counts the number of  $i$ -dimensional simplices (with the dimension of the empty set being  $-1$ ). For the noncrossing hypertree complex this number is equal to the number of noncrossing hypertrees with  $i + 2$  hyperedges. Also note that the reduced euler characteristic of this complex is a signed Catalan number. For further information see the sequence [oeis:A102537](#) in the Online Encyclopedia of Integer Sequences.

Theorem 3.4 also means that the noncrossing hypertree complex can be identified as a generalized cluster complex.

$n$	$f_{-1}$	$f_0$	$f_1$	$f_2$	$f_3$	$f_4$	$f_5$	$f_6$	$\widetilde{\chi}$
2	1	3							2
3	1	8	12						-5
4	1	15	55	55					14
5	1	24	156	364	273				-42
6	1	35	350	1400	2380	1428			132
7	1	48	680	4080	11628	15504	7752		-429
8	1	63	1197	9975	41895	92169	100947	43263	1430

TABLE 1. The  $f$ -vector and reduced euler characteristic of  $\text{Complex}(\text{NCHT}_{n+1})$  for small values of  $n$ .

**Remark 3.6** (Generalized cluster complexes). Generalized cluster complexes are a family of simplicial complexes with one complex defined for every finite Coxeter group and for every integer  $m \geq 1$ . They were introduced for types  $A$  and  $B$  by Tzanaki [Tza06] and for all types by Fomin and Reading [FR05]. Since their introduction in 2005 they have been studied extensively [AT06, Kra06, Tza08, AT08, FKT13]. The specific case of even-sided dissections of an even-sided polygon corresponds to the generalized cluster complex of type  $A$  with  $m = 2$ .

Although generalized cluster complexes have already been investigated, different aspects of the noncrossing hypertree complex are visible when its simplices are labeled by noncrossing hypertrees. In particular the homeomorphism established in §8 between the noncrossing hypertree complex and the noncrossing partition link is new and its proof relies heavily on the structure of noncrossing hypertrees.

#### 4. BOOLEAN LATTICES, CUBES AND SPHERES

This section reviews how to construct a simplicial sphere from a Boolean lattice using the geometric shapes called orthoschemes. The material is elementary and well-known, but it is included to establish notation for later constructions. For a more detailed treatment of orthoschemes and links see [BM10].

**Definition 4.1** (Boolean lattices). Let  $S$  be a finite set of size  $n$ . Its subsets under inclusion form the *Boolean lattice of rank  $n$*  denoted  $\text{Bool}(S)$  or  $\text{Bool}_n$  when only the size of  $S$  is relevant. It contains, of course, many subposets that are isomorphic to smaller Boolean lattices and those with the same bounding elements are of particular interest.

For every partition of the set  $S$  into  $k$  blocks, there is a Boolean sublattice of type  $\mathbf{Bool}_k$  inside  $\mathbf{Bool}_n$  whose elements are unions of blocks of the partition. I call these *special Boolean sublattices*.

**Definition 4.2** (Chains and weak linear orderings). If  $S' \subset S''$  are two elements in  $\mathbf{Bool}(S)$ , then the *label* of this pair is the set of new elements, those in  $S''$  but not in  $S'$ . For every chain  $\emptyset = S_0 \subset S_1 \subset \dots \subset S_k = S$  in  $\mathbf{Bool}(S)$  between the bounding elements, the labels on the adjacent elements in the chain form a partition of  $S$ . In fact, the collection of all such chains is in bijection with the *ordered partitions of  $S$* , by which I mean a partition of  $S$  together with a linear ordering of its blocks. An alternative way to encode an ordered partition is as a function from  $S$  to any linearly ordered set such as the reals. The blocks of the partition are determined by the elements with the same image and the ordering of the blocks by the linear ordering of these images. Such a function is called a *weak linear ordering of the set  $S$*  and two such functions are considered to be the same weak linear ordering when they determine the same partition and the same ordering of its blocks. A (strict) *linear ordering of the set  $S$*  is a weak linear ordering where the function is injective, or equivalently the partition is trivial. Thus chains in  $\mathbf{Bool}(S)$  between the bounding elements bijectively correspond to weak linear orderings of  $S$  and the maximal chains in  $\mathbf{Bool}(S)$  correspond to the strict linear orderings.

The geometry of a Boolean lattice is essentially that of a cube.

**Definition 4.3** (Cubes and orthoschemes). Let  $S$  be a set with  $n$  elements and identify the elements of  $S$  with the standard orthonormal basis of  $\mathbb{R}^n$ . The elements of  $\mathbf{Bool}(S)$  can then be identified with the  $2^n$  corners of an  $n$ -dimensional cube in  $\mathbb{R}^n$  by sending each subset of  $S$  to the sum of the corresponding basis vectors. Thus the subset  $S'$  is sent to the vector where every coordinate is 0 or 1 and the locations of the 1's indicating the elements in the subset. One of these points is less than another in the ordering if and only if the vector from the first to the second has nonnegative coordinates. For each maximal chain, the convex hulls of the corresponding  $n + 1$  points in  $\mathbb{R}^n$  with its standard euclidean geometry is a shape called an *orthoscheme*, or more precisely a *standard  $n$ -orthoscheme* in the language of [BM10]. Figure 8 shows a 3-dimensional orthoscheme. Since there are  $n!$  ways to linearly order the elements of  $S$ , the  $n$ -dimensional cube is subdivided into  $n!$  standard  $n$ -orthoschemes.

The process that converts a Boolean lattice into a metric simplicial subdivided cube can be extended to other types of posets.



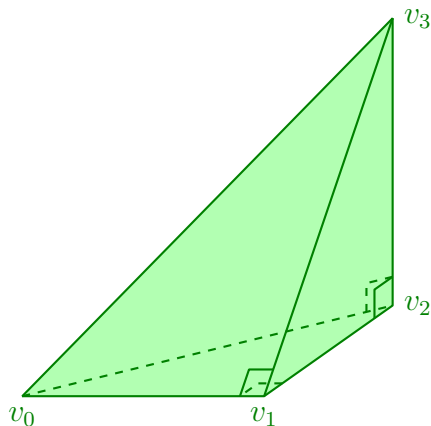


FIGURE 8. A 3-dimensional orthoscheme.

**Definition 4.4** (Orthoscheme complex). The *order complex* of a poset is the simplicial complex formed by its chains with subchains as subsimplices: the elements, viewed as 1-element chains are the vertices, the 2-element chains are the edges, etc. The *orthoscheme complex* of a bounded graded poset is a metric version of its order complex. The simplex corresponding to each maximal chain is given the same metric as on the maximal chains in the geometric version of the Boolean lattice, namely, that of an orthoscheme. Concretely, the length of the edge connecting two vertices is the square root of the difference in the levels of the corresponding elements of the poset and, more generally, the metric on a simplex labeled by a chain is that of the unique euclidean simplex with these specified edge lengths.

The orthoscheme complex of a Boolean lattice is, of course, the simplicially subdivided cube described in Definition 4.3. When considering the orthoscheme complex of a poset with one or more bounding elements, the resulting space is always contractible. Thus, it is usually more interesting topologically to consider the link of the vertex or edge spanned by the bounding element(s) of the poset.

**Definition 4.5** (Link of a poset). The *link of a face of a euclidean polytope* is the spherical polytope formed by the set of unit vectors based at a point in the interior of the face that (1) are perpendicular to the affine span of the face and such that (2) the point plus some small positive scalar multiple of the unit vector remains in the polytope. For example, the link of an edge in a euclidean tetrahedron is a spherical arc whose length is equal to the dihedral angle (in radians) along this edge. The spherical polytope that results is independent of the point

chosen in the interior of the face. When the euclidean polytope is a euclidean simplex, the links are spherical simplices. The *link of a simplex in a piecewise euclidean simplicial complex* is the piecewise spherical simplicial complex formed by gluing together the spherical simplices that are the link of this simplex in each of the euclidean simplices that contain it as a face in the obvious fashion. See [BM10] or [BH99] for more precise definitions. In the concrete case where  $P$  is a bounded graded poset, the edge connecting its two bounding elements in its orthoscheme complex is the unique longest edge and in [BM10] we called it the *long diagonal*. The *link of the poset  $P$*  is the piecewise spherical simplicial complex that is the link of this long diagonal edge in its orthoscheme complex.

From this construction, the following is immediate.

**Proposition 4.6** (Links and subposets). *If  $P$  is a bounded graded poset and  $Q$  is an induced subposet with the same bounding elements and the same grading, then  $\text{Link}(Q)$  is a simplicial subcomplex of  $\text{Link}(P)$ .*

The link of a Boolean lattice is a simplicial sphere also known as a Coxeter complex of type  $A$ .

**Definition 4.7** (Coxeter complex of type  $A$ ). From the geometry of its orthoscheme complex it is clear that  $\text{Link}(\mathbf{Bool}_n)$  is a metric simplicial sphere  $\mathbb{S}^{n-2}$  with  $n!$  top-dimensional simplices called *chambers*. That these simplices are all isometric can be seen from the action of the symmetric group on  $\mathbb{R}^n$  by permuting coordinates which acts transitively on the standard  $n$ -orthoschemes in the cube and transitively on the chambers in  $\mathbb{S}^{n-2}$ . Because the symmetric group  $\text{Sym}_n$  is the Coxeter group of type  $A_{n-1}$ , the sphere  $\text{Link}(\mathbf{Bool}_n)$  is called a *Coxeter complex of type  $A_{n-1}$*  and the common shape of its chambers is a spherical simplex also known as the *Coxeter shape of type  $A_{n-1}$* . Also, note that the simplices in this sphere correspond to chains in  $\mathbf{Bool}_n$  between its bounding elements and thus to weak linear orderings of the set  $S$  used to construct  $\mathbf{Bool}_n$ . In fact, the coordinates of a point can be reinterpreted as a function from  $S$  to the reals and thus as a weak linear ordering of the set  $S$ . The interiors of the simplices correspond to the sets of points in the sphere whose coordinate functions determine the same weak linear ordering of  $S$ .

These Coxeter complexes have many simplicial subspheres.

**Proposition 4.8** (Subspheres and special sublattices). *Let  $S$  be a set with  $n$  elements. For every partition  $\sigma$  of  $S$  with  $k$  blocks, there is a corresponding simplicial subcomplex of  $\text{Link}(\mathbf{Bool}_n)$  isometric to  $\mathbb{S}^{k-2}$ .*

*It has the simplicial structure of a Coxeter complex of type  $A_{k-1}$  but the metric might be slightly different. Moreover, the top-dimensional simplices in this subsphere are precisely those corresponding to the  $k!$  orderings of the blocks of  $\sigma$ .*

*Proof.* The partition  $\sigma$  determines a special Boolean sublattice of type  $\mathbf{Bool}_k$  and the corresponding vertices of the  $n$ -cube are precisely those living in the subspace defined by setting the coordinates  $x_i$  and  $x_j$  equal if and only if  $i$  and  $j$  belong to the same block  $\sigma$ . The intersection of this  $k$ -dimensional subspace with the Coxeter complex determines the  $\mathbb{S}^{k-2}$  sphere. As a simplicial complex it is the link of  $\mathbf{Bool}_k$  but the geometry of the subcomplex is slightly distorted when compared to the standard Coxeter complex of this type because the embedding into  $\mathbb{R}^n$  depends on the number of elements in each block. The final assertion follows from the description of simplices in terms of coordinates given in Definition 4.7.  $\square$

The simplicial subspheres of  $\mathbf{Link}(\mathbf{Bool}_n)$  determined by a partition in this way are called *special subspheres*. The following result is nearly immediate from the construction.

**Corollary 4.9** (Special subspheres). *Every simplex in  $\mathbf{Link}(\mathbf{Bool}_n)$  belongs to a unique special subsphere of the same dimension.*

*Proof.* Let  $S$  be the set of size  $n$  used to construct  $\mathbf{Bool}_n$ . A simplex of dimension  $k - 2$  corresponds to an ordered partition  $\sigma$  of  $S$  with  $k$  blocks and this partition then determines a special subsphere which includes the simplex. Uniqueness follows from the explicit description of the top-dimensional simplices in the special subsphere.  $\square$

The final remarks in this section concern the various hemispheres in the Coxeter complex and the ways in which they intersect.

**Definition 4.10** (Roots and hemispheres). Let  $S$  be a set of size  $n$  and let  $\mathbb{R}^n$  have a standard orthonormal basis  $\{\epsilon_i\}$  indexed by the elements  $i$  in  $S$ . The vector  $v_{ij} := \epsilon_i - \epsilon_j$  is called a *root* and the set of all roots is the *root system for  $\mathbf{Sym}_n$* . The span of the root system is the subspace  $\mathbb{R}^{n-1}$  containing the Coxeter complex  $\mathbb{S}^{n-2}$  on which  $\mathbf{Sym}_n$  acts. For each choice of  $i$  and  $j$  in  $S$  the inequality  $x_i \geq x_j$  defines a closed *half-space* in  $\mathbb{R}^n$  that becomes a closed *hemisphere*  $H_{ij}$  in  $\mathbb{S}^{n-2}$  characterized as the set of unit vectors in  $\mathbb{S}^{n-2}$  that form a nonobtuse angle with the root  $v_{ij}$ . In this context, the root  $v_{ij}$  is the *pole* of  $H_{ij}$ . The hemisphere  $H_{ij}$  is also the union of the chambers that correspond to linear orderings of  $S$  where  $i$  occurs before  $j$ .

**Proposition 4.11** (Intersecting hemispheres). *Let  $T$  be a directed tree with vertex set  $S$  of size  $n$  and directed edges. If for each directed edge from  $i$  to  $j$  in  $T$ ,  $H_{ij}$  is the corresponding hemisphere in  $\text{Link}(\text{Bool}(S))$  with pole  $v_{ij}$ , then the intersection of these hemispheres is a union of chambers that form a single spherical simplex.*

*Proof.* Because  $T$  has no cycles, the set of poles  $v_{ij}$  is linearly independent and because  $T$  is connected, it spans the subspace  $\mathbb{R}^{n-1}$  containing the sphere  $\mathbb{S}^{n-2}$ . In particular these roots form a basis of this root subspace. From this it quickly follows that the intersection of the closed hemispheres is a spherical simplex.  $\square$

## 5. THE NONCROSSING PARTITION LINK

This section establishes basic facts about the piecewise spherical simplicial complex that I call the noncrossing partition link with an emphasis on its connections to spherical buildings, noncrossing hypertrees and noncrossing hyperforests.

**Definition 5.1** (Noncrossing partition link). The *noncrossing partition link*  $\text{Link}(\text{NCPart}_{n+1})$  is the link of the long diagonal edge in the orthoscheme complex of the noncrossing partition lattice  $\text{NCPart}_{n+1}$  (Definition 4.4). As a simplicial complex it is the order complex of the noncrossing partition lattice with both bounding elements removed. The top-dimensional simplices in the noncrossing partition link are called *partition chambers*.

The noncrossing partition link can also be viewed as a subcomplex of the spherical building associated to a linear subspace poset. A spherical building is a very special type of highly symmetric simplicial complex with piecewise spherical metrics on its simplices. Even though the only spherical buildings used in this article are those of type  $A$  (whose structure can be directly described without mentioning the general theory), a few general remarks are in order.

**Definition 5.2** (Spherical buildings). A *spherical building*, roughly speaking, is a metric simplicial complex that can be viewed as a union of unit spheres of the same dimension where the simplicial structure on each sphere is that of a Coxeter complex for a fixed finite Coxeter group  $W$ . These top-dimensional spheres are called *apartments* and the top-dimensional simplices are called *chambers*. In a building, any two chambers belong to a common apartment and the isometry group of the building acts transitively on its chambers and transitively on its apartments. The *type of the building* is the type of the finite Coxeter

group whose Coxeter complex is the model for the simplicial structure on each apartment.

Since the link of a Boolean poset is a Coxeter complex of type  $A$ , spherical buildings of type  $A$  can be constructed from posets with lots of maximal Boolean subposets.

**Definition 5.3** (Maximal Boolean subposets). Let  $P$  be a bounded graded poset. A *maximal Boolean subposet of  $P$*  is a subposet  $Q$  isomorphic to a Boolean lattice  $\mathbf{Bool}_n$  with the same bounding elements and the same grading as  $P$ . By Proposition 4.6 each maximal Boolean subposet corresponds to a sphere in  $\text{Link}(P)$ . This also means that any special Boolean sublattice inside a maximal Boolean subposet inside  $P$  gives rise to a smaller dimensional sphere inside this sphere in the link of  $P$ .

Spherical buildings of type  $A$  are built out of linear subspaces.

**Definition 5.4** (Linear subspaces). Let  $V$  be a finite dimensional vector space over a field  $\mathbb{F}$ . The *linear subspace poset  $\mathbf{Linear}(V)$*  is the set of all linear subspaces of  $V$  under inclusion. It is a bounded graded self-dual poset and it has many maximal Boolean subposets. For example, a maximal Boolean subposet of  $\mathbf{Linear}(V)$  can be constructed from any basis  $\mathcal{B}$  by taking the span of every subset of  $\mathcal{B}$ . More precisely, this construction depends less on the basis itself than on the lines the basis vectors determine. The link of the linear subspace poset  $\mathbf{Linear}(V)$  is an example of a spherical building of type  $A$ . Its apartments are precisely those spheres derived from maximal Boolean subposets of  $\mathbf{Linear}(V)$  constructed from linearly independent spanning sets of lines. As an example, consider the 3-dimensional vector space  $V$  over the field  $\mathbb{F}_2$  with only two elements. The corresponding spherical building is the Heawood graph shown in Figure 9 where every edge is a metric spherical arc of length  $\frac{\pi}{3}$ . The various combinatorial hexagons in the graph are thus unit circles and these are its apartments.

The noncrossing partition lattice embeds into a linear subspace poset.

**Remark 5.5** (Noncrossing partitions and linear subspaces). If the vertices of a convex polygon are identified with the coordinates of a vector space, then every noncrossing partition (and in fact every partition) naturally describes a linear subspace defined by equating the coordinates belonging to each nontrivial block. For example, the partition  $\sigma = \{\{1, 3, 4\}, \{2\}, \{5, 6, 7, 8, 9\}\}$  is sent to the 3-dimensional subspace of tuples  $(x_1, x_2, \dots, x_9)$  such that  $x_1 = x_3 = x_4$  and  $x_5 = x_6 = x_7 = x_8 = x_9$ . This map embeds the noncrossing partition lattice  $\mathbf{NCPart}_{n+1}$

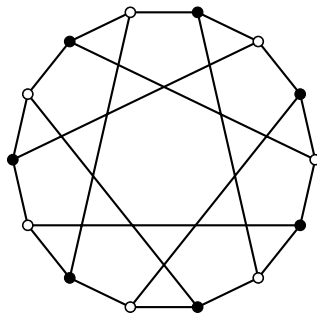


FIGURE 9. A spherical building of type  $A_2$ . The seven black dots correspond to the 1-dimensional subspaces of the 3-dimensional vector space over  $\mathbb{F}_2$  and the seven white dots to the 2-dimensional subspaces. The edges are meant to represent spherical arcs of length  $\frac{\pi}{3}$ .

into the *dual* of the linear subspace poset  $\mathbf{Linear}(\mathbb{F}^{n+1})$  because  $\sigma < \tau$  implies that the subspace associated to  $\sigma$  contains the subspace associated to  $\tau$ . The order reversal is of little consequence since both  $\mathbf{NCPart}_{n+1}$  and  $\mathbf{Linear}(\mathbb{F}^{n+1})$  are self-dual posets. Of more concern is the fact that the top element, the noncrossing partition that I call the Coxeter hypertree, is sent to a 1-dimensional subspace rather than the trivial subspace. To remedy this intersect each subspace with the codimension 1 subspace where vectors have coordinate sum 0. The new map embeds  $\mathbf{NCPart}_{n+1}$  into the dual of  $\mathbf{Linear}(\mathbb{F}^n)$ .

This embedding means that the noncrossing partition link can be viewed as a simplicial subcomplex of the corresponding spherical building (Proposition 4.6). In fact, Tom Brady and I noticed around 2003 that the noncrossing partition link can be viewed as a union of apartments in this spherical building. The first published proof of this result is in [HKS, Proposition 3.25].

**Proposition 5.6** (Noncrossing apartments). *The maximal Boolean subposets of the noncrossing partition lattice, and thus the only apartments contained in the noncrossing partition link, are in natural bijection with the set of noncrossing trees.*

*Proof.* Since the apartments of the spherical building inside the noncrossing partition lattice correspond, by definition, to its maximal Boolean subposets, it is sufficient to focus on these. Let  $\tau$  be a noncrossing tree. For every subset of edges of  $\tau$ , the resulting noncrossing forest determines a noncrossing partition (Proposition 2.9) and these

elements form a maximal Boolean subposet. Conversely, in any maximal Boolean subposet of  $\mathbf{NCPart}_{n+1}$  there are exactly  $n$  elements that cover the bottom element, each of which is represented by a single edge connecting two vertices of the convex polygon. These edges are necessarily weakly noncrossing because crossing edges have a join that is higher than it should be in a maximal Boolean subposet. And since the join of all  $n$  elements is the top element of  $\mathbf{NCPart}_{n+1}$ , these edges must connect all  $n + 1$  vertices. Since  $n$  is the minimum number of edges needed to do so, there are none left over to form a cycle and their union is a tree. Finally, these two procedures are inverses of each other, thus establishing the bijection.  $\square$

**Definition 5.7** (Apartments and spheres). For each noncrossing tree  $\tau$  the maximal Boolean subposet constructed in Proposition 5.6 corresponds to a top-dimensional sphere in the noncrossing partition link called a *noncrossing apartment* and denoted  $\mathbf{Apart}(\tau)$ . A similar construction can be used with noncrossing hypertrees. For each noncrossing hypertree  $\tau$ , there is a special Boolean sublattice constructed from the noncrossing partitions that are the convex hulls of the connected components of the noncrossing hyperforests formed by subsets of the hyperedges of  $\tau$ . And this special Boolean subposet corresponds to a subsphere in the link called the *sphere of  $\tau$*  and denoted  $\mathbf{Sphere}(\tau)$ .

The fact that the noncrossing partition link is a union of these noncrossing apartments will be an immediate consequence of Lemma 8.4. The next remark relates the construction of an apartment from a noncrossing tree to the construction of an apartment in the spherical building from a basis of the vector space. It is not needed later, but it is included to clarify how the constructions correspond.

**Remark 5.8** (Noncrossing apartments and bases). Let  $\tau$  be a noncrossing tree on  $n + 1$  vertices. Subsets of the edges of  $\tau$  determine a maximal Boolean subposet of  $\mathbf{NCPart}_{n+1}$  which is sent to a maximal Boolean subposet of  $\mathbf{Linear}(\mathbb{F}^n)$ . Because of the order reversal of the embedding, the lines in the vector space  $V = \mathbb{F}^n$  that determine the corresponding apartment come from the elements of the Boolean subposet in  $\mathbf{NCPart}_{n+1}$  covered by the top element, the noncrossing partitions with exactly 2 blocks determined by removing a single edge from  $\tau$ . Let  $\sigma_e$  be the noncrossing partition formed when the edge  $e$  is removed and let  $A$  and  $B$  be the blocks of sizes  $k$  and  $\ell$  it contains. From  $\sigma_e$  we construct a vector  $v_e$  with a value of  $\ell$  in each of the coordinates associated with the  $k$  elements in block  $A$  and a value of  $-k$  in each of the coordinates associated with the  $\ell$  elements in block  $B$ . This vector  $v_e$  in  $\mathbb{F}^{n+1}$  has coordinate sum 0 and the line it spans is

the image of  $\sigma_e$  under the embedding. Note that if the roles of blocks  $A$  and  $B$  are reversed, the vector produced is  $-v_e$  and the line remains the same. The lines associated to these 2 block partitions are linearly independent and span the subspace of vectors with coordinate sum 0. In fact, since  $v_e$  is the only vector that assigns different values to the coordinates associated with its endpoints, its contribution to a vector with coordinate sum 0, is determined by the difference between these coordinate values. Subtracting off these necessary contributions leaves a vector where all coordinates are equal and the coordinate sum is unchanged.

## 6. PERMUTATIONS AND REDUCED PRODUCTS

Ever since the work of Brady and Watt [BW02] and Bessis [Bes03] more than a decade ago, the modern way to view the noncrossing partition lattice is as a portion of the symmetric group between the identity and a Coxeter element with respect to reflection length, and the proof of the homeomorphism between the noncrossing partition link and the noncrossing hypertree complex uses these ideas. The first step is to turn hyperedges into irreducible permutations and chains in the noncrossing partition lattice into reduced products.

**Definition 6.1** (Hyperedges and irreducible permutations). An *irreducible permutation* is one represented by a single cycle of length 2 or more. The name refers to the fact that these permutations are Coxeter elements for irreducible subroot systems of the root system of the symmetric group. There is a map from irreducible permutations to hyperedges that sends the permutation to the set of elements moved by its single cycle, but it is obviously not one-to-one since  $(k-1)!$  irreducible permutations are sent to each  $k$ -element hyperedge. In the other direction, there is an injective map that sends each hyperedge to the unique irreducible permutation where the elements in the single cycle are listed in the order they appear in the boundary cycle of their convex hull when traversed clockwise. When the standard vertex labeling is used this corresponds to their natural linear order as integers. Under these maps the irreducible permutation  $(1, 3, 5, 2)$  is sent to the hyperedge  $\{1, 2, 3, 5\}$  and this hyperedge is sent to the permutation  $(1, 2, 3, 5)$  (when the standard vertex labeling is used). The image of the unique hyperedge in the Coxeter hypertree on  $n+1$  vertices with the standard vertex labeling is the permutation  $c = (1, 2, \dots, n+1)$ , a notation that reflects the role of  $c$  as a *Coxeter element* in symmetric group  $\text{Sym}_{n+1}$ .



**Definition 6.2** (Noncrossing permutations). If  $\sigma$  is a noncrossing partition, the irreducible permutations of its hyperedges pairwise commute and their well-defined product is called a *noncrossing permutation*. Each noncrossing partition is identified with its noncrossing permutation. The partition  $\sigma = \{\{1, 3, 4\}, \{2\}, \{5, 6, 7, 8, 9\}\}$ , for example, becomes the permutation  $(1, 3, 4)(5, 6, 7, 8, 9)$  when the standard vertex order is used. With this identification, the noncrossing partition of a noncrossing hyperforest (Proposition 2.9) can now be viewed as the *noncrossing permutation of a noncrossing hyperforest*.

**Definition 6.3** (Reduced products). Let  $W$  be a Coxeter group and let  $R$  denote its set of reflections. The *reflection length* of a element  $\sigma \in W$  is the length  $\ell_R(\sigma)$  of the smallest list of reflections whose product is  $\sigma$ . Some parts of the literature call this the *absolute length* of  $\sigma$ . In the symmetric group  $\text{Sym}_{n+1}$ , the Coxeter group of type  $A_n$ , the reflections are the transpositions, and in the spirit of Proposition 1.6 if you expect every cycle to have length 1 then the reflection length of a permutation is the sum of its excess cycle lengths. A product  $\sigma_1\sigma_2\cdots\sigma_k = \sigma$  is called a *reduced product* and a *reduced factorization of  $\sigma$*  if the reflection length  $\ell_R(\sigma)$  of the product is equal to the sum  $\sum_{i=1}^k \ell_R(\sigma_i)$  of the reflection lengths of the factors. This leads to a partial *reflection order* on the elements of  $W$ :  $\sigma' < \sigma$  if and only if  $\sigma'$  is a factor in a reduced factorization of  $\sigma$ .

Reduced products are the same as what Vivian Ripoll calls “block factorizations” [Rip11, Rip12]. That term is not used here because the word “block” might mistakenly lead one to assume that each factor corresponds to a single block of a partition, but this is not part of the definition and is only true when the factors are irreducible permutations. Reduced products have many easy-to-prove properties.

**Proposition 6.4** (Reduced products). *If  $\sigma_1\sigma_2\cdots\sigma_k = \sigma$  is a reduced product, then so is the portion of the factorization  $\sigma_i\sigma_{i+1}\cdots\sigma_j = \sigma'$  for all  $1 \leq i \leq j \leq k$  and so is the factorization of  $\sigma$  where the portion from  $\sigma_i$  to  $\sigma_j$  is replaced by its product  $\sigma'$ .*

In the definition of the partial order on elements of  $W$ , the element  $\sigma$  can be required to be the first or last factor in a reduced factorization of  $\sigma$  without changing the partial order because of the following easy fact that follows from the closure of the set of reflections under conjugation.

**Proposition 6.5** (Rewriting reduced products). *Let  $\sigma_1\sigma_2\cdots\sigma_k$  be a reduced factorization of  $\sigma$  in a Coxeter group  $W$ . For any selection  $1 \leq i_1 < i_2 < \cdots < i_j \leq k$  of positions there is a length  $k$  reduced*

factorization of  $\sigma$  whose first  $j$  factors are  $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_j}$  and another length  $k$  reduced factorization of  $\sigma$  where these are the last  $j$  factors in the factorization.

The set of permutations below the Coxeter element  $c$  in the reflection order is exactly the set of noncrossing permutations and their poset structure in this partial order is isomorphic to that of the noncrossing partition lattice  $\mathbf{NCPart}_{n+1}$  [BW02, Bes03]. This immediately implies the following result.

**Proposition 6.6** (Factors in reduced products). *Let  $\sigma_1\sigma_2\cdots\sigma_k$  be a reduced factorization of a permutation  $\sigma$ . When  $\sigma$  is a noncrossing permutation, every  $\sigma_i$  is a noncrossing permutation and, equivalently, when one  $\sigma_i$  is not noncrossing,  $\sigma$  is not noncrossing.*

Another way to state this connection between partitions and permutations is that the Hasse diagram of the noncrossing partition lattice is a portion of the Cayley graph of the symmetric group. More precisely, the Hasse diagram of the poset  $\mathbf{NCPart}_{n+1}$  is the same as the union of the directed geodesic paths from the identity to the Coxeter element  $c = (1, 2, \dots, n+1)$  in the right Cayley graph of the symmetric group  $\mathbf{Sym}_{n+1}$  generated by the set  $R$  of all reflections. In particular, if  $c = r_1r_2\cdots r_n$  is a reduced factorization of the Coxeter element  $c$  into reflections, then there is a path of length  $n$  in the Hasse diagram of  $\mathbf{NCPart}_{n+1}$  from the identity element at the bottom to the Coxeter element at the top whose edges are labeled  $r_1, r_2, \dots, r_n$  in that order. More generally, for every pair of noncrossing permutations  $\tau < \tau'$  there is a well-defined *label* that is the unique noncrossing permutation  $\sigma$  solving the equation  $\tau\sigma = \tau'$ , and the sequence of labels on a chain in the noncrossing partition lattice describes a reduced factorization of the label assigned to its pair of endpoints.

**Proposition 6.7** (Reduced products and chains). *For every pair of elements  $\tau < \tau'$  in the noncrossing partition lattice  $\mathbf{NCPart}_{n+1}$ , the chains of length  $k$  from  $\tau$  to  $\tau'$  are in one-to-one correspondence with the reduced factorizations of the unique element  $\sigma$  such that  $\tau\sigma = \tau'$  into  $k$  factors. Concretely the chain  $\tau = \tau_0 < \tau_1 < \cdots < \tau_k = \tau'$  is paired with the reduced factorization  $\sigma_1\sigma_2\cdots\sigma_k = \sigma$  if and only if the  $\sigma_i$ 's and  $\tau_i$ 's satisfy the equations:  $\tau_0\sigma_1\cdots\sigma_i = \tau_i$  for  $i = 1, \dots, k$ .*

*Proof.* Given the endpoints and the  $\sigma_i$ 's one can solve for the  $\tau_i$ 's. Conversely, given the  $\tau_i$ 's one can solve for the  $\sigma_i$ 's.  $\square$

## 7. REDUCED PRODUCTS AND NONCROSSING HYPERFORESTS

When the permutations of the hyperedges of a noncrossing hyperforest  $\tau$  are multiplied together, the result depends on the order of multiplication unless, of course,  $\tau$  is a noncrossing partition whose hyperedge permutations pairwise commute. This section explores the impact that this order has on the result with a goal of establishing a bijection between noncrossing hyperforests whose hyperedges have been “properly ordered” in a sense made precise in Definition 7.2 and reduced products of noncrossing permutations into irreducible permutations (Theorem 7.8). As a first step consider the product of two irreducible permutations that are weakly disjoint but not disjoint.

**Lemma 7.1** (Two irreducible permutations). *If  $\sigma'$  and  $\sigma''$  are irreducible permutations whose single cycles have exactly one element in common, then  $\sigma'\sigma''$  is a reduced product and the product  $\sigma$  is an irreducible permutation. Moreover, the product  $\sigma$  is a noncrossing permutation if and only if the factors  $\sigma'$  and  $\sigma''$  are noncrossing permutations that form a noncrossing hyperforest with 2 hyperedges where  $\sigma'$  is to the left of  $\sigma''$  when viewed from the perspective of their common vertex.*

*Proof.* Let  $k + 1$  and  $\ell + 1$  be the lengths of the nontrivial cycles of  $\sigma'$  and  $\sigma''$  respectively and let  $a$  be the element they have in common. If we write  $(a, b_1, \dots, b_k)$  for  $\sigma'$  and  $(a, c_1, \dots, c_\ell)$  for  $\sigma''$ , then  $(a, b_1, \dots, b_k) \cdot (a, c_1, \dots, c_\ell) = (a, b_1, \dots, b_k, c_1, \dots, c_\ell) = \sigma$ . All of the assertions follow immediately from this explicit computation.  $\square$

When analyzing more complicated products it is convenient to attach a linear ordering to the hyperforest under consideration.

**Definition 7.2** (Ordered hyperforests). An *ordered hyperforest* is a hyperforest  $\tau$  together with a linear ordering of its hyperedges. The product of the hyperedge permutations of  $\tau$  in this prescribed order is called the *permutation of  $\tau$* . This should not be confused with the noncrossing permutation of a noncrossing hyperforest which is the permutation of the noncrossing partition formed from the blocks that are its connected components. The relationship between these two permutations is explained in Lemma 7.7. When the hyperforest  $\tau$  is a noncrossing hyperforest, it has a hyperedge poset (Definition 2.6) and the ordering on  $\tau$  is *proper* and  $\tau$  is *properly ordered* when the linear ordering of the hyperedges of  $\tau$  is a linear extension of the partial ordering of its hyperedges recorded in its hyperedge poset.

The next result extends Lemma 7.1 to all ordered hyperforests.

**Lemma 7.3** (Ordered hyperforest permutations). *The permutation of an ordered hypertree is an irreducible permutation of all of its vertices and the permutation of an ordered hyperforest has a cycle type determined by the sizes of its connected components. In both cases these products are reduced products of the hyperedge permutations.*

*Proof.* For hypertrees this follows from Lemma 7.1 and an easy induction. For hyperforests it is enough to note that the hyperedge permutations in distinct connected components pairwise commute.  $\square$

The next two lemmas focus on turning a reduced product of irreducible permutations into a properly ordered noncrossing hyperforest.

**Lemma 7.4** (Reduced products and hyperforests). *If  $\sigma_1\sigma_2\cdots\sigma_k = \sigma$  is a reduced factorization of a permutation into irreducible permutations, the hyperedges of the factors form a hyperforest  $\tau$ .*

*Proof.* Let  $\tau$  be the hypergraph whose hyperedges correspond to the factors  $\sigma_i$ . For each connected component of  $\tau$ , it is possible to use Proposition 6.5 and Proposition 6.4 to find a reduced product that only contains the factors  $\sigma_i$  corresponding to the hyperedges in this component. Let  $\tau'$  be the connected hypergraph with a vertex set restricted to this component and with only these hyperedges. The equation defining a reduced product implies that equality holds in the equation in Proposition 1.7. Therefore  $\tau'$  is a hypertree and  $\tau$  is a hyperforest.  $\square$

**Lemma 7.5** (Weakly noncrossing and properly ordered). *Let  $\tau$  be an ordered hyperforest whose vertices have been identified with the vertices of a convex polygon in the plane. If the permutation of  $\tau$  is a noncrossing permutation then its hyperedges are pairwise weakly noncrossing and they are properly ordered. In particular, if the permutation of  $\tau$  is a noncrossing permutation then  $\tau$  is a properly ordered noncrossing hyperforest.*

*Proof.* Let  $\sigma = \sigma_1\sigma_2\cdots\sigma_k$  be the product of the hyperedge permutations of  $\tau$  in the specified order. By Lemma 7.3 this is a reduced factorization of  $\sigma$ . By Proposition 6.5 for any two hyperedge permutations  $\sigma_i$  and  $\sigma_j$  with  $i < j$  there is another reduced factorization of  $\sigma$  into  $k$  permutations where  $\sigma_i$  and  $\sigma_j$  are its first two factors. Next, in the altered reduced factorization of  $\sigma$  replace  $\sigma_i\sigma_j$  with their product  $\sigma'$ . The result stays reduced (Proposition 6.4) and by Proposition 6.6  $\sigma'$  must be a noncrossing permutation. Because  $\tau$  is a hyperforest, the hyperedges of  $\sigma_i$  and  $\sigma_j$  are either disjoint or have exactly one vertex

in common. If they are disjoint,  $\sigma_i$  and  $\sigma_j$  commute, their order is irrelevant and the fact that  $\sigma'$  is a noncrossing permutation means that their convex hulls must be noncrossing in the strong sense of being completely disjoint. If the hyperedges of  $\sigma_i$  and  $\sigma_j$  have a vertex in common, then Lemma 7.1 and the fact that  $\sigma'$  is noncrossing imply that  $\sigma_i$  and  $\sigma_j$  are weakly noncrossing and properly ordered.  $\square$

The final two lemmas are used to turn a properly ordered noncrossing hyperforest into a reduced product of irreducible permutations.

**Lemma 7.6** (Proper orderings and commutations). *Let  $\tau$  is a properly ordered noncrossing hyperforest. If there are disjoint hyperedges in  $\tau$  that are consecutive in the ordering, then they are incomparable in the hyperedge poset of  $\tau$  and the new ordered noncrossing hyperforest obtaining by switching the order on these two is also proper.*

*Proof.* If  $e$  and  $e'$  are two hyperedges of  $\tau$  that are disjoint but comparable in the ordering of the hyperedge poset, say with  $e < e'$ , it is because there is a sequence of covering relations  $e = e_0 < e_1 < \dots < e_\ell = e'$  with  $\ell > 1$ . In particular, in the linear ordering of the hyperedges of  $\tau$ ,  $e$  and  $e'$  are not consecutive because  $e_1$  must occur between them. Thus disjoint hyperedges that are consecutive in the ordering are incomparable in the ordering of the hyperedge poset and the rest is clear.  $\square$

The final lemma highlights the algebraic significance of being properly ordered.

**Lemma 7.7** (Proper orderings and noncrossing permutations). *The permutation of a properly ordered noncrossing hyperforest is a noncrossing permutation.*

*Proof.* Let  $\sigma_1\sigma_2\cdots\sigma_k = \sigma$  be the permutation of a properly ordered noncrossing hyperforest  $\tau$  with  $k$  hyperedges. When  $\tau$  is a noncrossing partition there is nothing to prove, and note that this includes the case where  $k = 1$ . So suppose it is true for all properly ordered noncrossing hyperforests with fewer hyperedges and that  $\tau$  is not a noncrossing partition. Let  $\sigma_i$  and  $\sigma_j$  be factors that do not commute because they contain a vertex in common and select  $i < j$  so that they are “innermost” in the sense that all other pairs of factors in the portion  $\sigma_i\sigma_{i+1}\cdots\sigma_j$  are disjoint and commute. Use the commutations to reorder the factors so that  $\sigma_i$  and  $\sigma_j$  are adjacent. By Lemma 7.1 the product  $\sigma_i\sigma_j$  is an irreducible permutation  $\sigma'$  and by Lemma 7.4 and Lemma 7.5 the reordered factorization of  $\sigma$  with  $\sigma'$  in place of  $\sigma_i\sigma_j$  has factors that correspond to the hyperedges of a properly ordered noncrossing hyperforest. By Lemma 7.1, factoring  $\sigma'$  back into  $\sigma_i$  and  $\sigma_j$

splits the hyperedge of  $\sigma'$  into a pair of properly ordered hyperedges  $\sigma_i$  and  $\sigma_j$ . Finally, undoing the commutations only changes the ordering and not the underlying noncrossing hyperforest and by Lemma 7.6 it remains properly ordered.  $\square$

**Theorem 7.8** (Reduced products and noncrossing hyperforests). *For each noncrossing permutation  $\sigma$  there is a natural bijection between the set of reduced factorizations of  $\sigma$  into irreducible permutations and the set of properly ordered noncrossing hyperforests  $\tau$  with  $\sigma$  as its noncrossing permutation.*

*Proof.* Let  $\sigma_1\sigma_2\cdots\sigma_k = \sigma$  be a reduced factorization of a noncrossing permutation  $\sigma$  into irreducible permutations. By Proposition 6.6 the factors  $\sigma_i$  are noncrossing permutations. In particular they are the permutations of the hyperedges to which they correspond. By Lemma 7.4 these hyperedges are form a hyperforest  $\tau$  that by Lema 7.5 is noncrossing and properly ordered. In the other direction let  $\tau$  be a properly ordered noncrossing hyperforest. By definition, its permutation  $\sigma$  is a product of irreducible permutations, by Lemma 7.3 this product is reduced, and by Lemma 7.7 the permutation  $\sigma$  is a noncrossing permutation.  $\square$

The following corollary is an immediate consequence of Theorem 7.8 applied to the case where  $\sigma$  is the Coxeter element  $c$  and the reduced factorizations considered have maximal length. These correspond to maximal chains in the noncrossing partition lattice and to the top-dimensional simplices in the noncrossing partition link that I call partition chambers. Note that the labels on a maximal chain are reflections which are automatically irreducible.

**Corollary 7.9** (Chambers and noncrossing trees). *The partition chambers in the noncrossing partition link  $\text{Link}(\text{NCPart}_{n+1})$  are bijectively labelled by properly ordered noncrossing trees on  $n + 1$  vertices.*

Theorem 7.8 can also be used to create labels on all the simplices in the noncrossing partition link, but the labels on arbitrary chains between the bounding elements of the noncrossing partition lattice are merely noncrossing permutations and not necessarily irreducible ones. To accommodate this one can expand the set of linear orderings.

**Definition 7.10** (Weak proper orderings). A weak linear ordering of a set  $S$  was defined in Definition 4.2 as a function  $f$  from a set  $S$  to some linearly ordered set. When the set  $S$  is replaced by a poset  $P$  there are more distinctions to be made depending on the extent to which the function respects the structure of the poset. When the

function  $f$  is injective and  $p < p'$  implies  $f(p) < f(p')$ , this is the usual notion of a *linear extension* of  $P$ . When the injectivity assumption is dropped but  $p < p'$  still implies  $f(p) < f(p')$  it is called a *weak linear extension* of  $P$ . Finally, when the only condition satisfied is that  $p \leq p'$  implies  $f(p) \leq f(p')$  it is called an *extremely weak linear extension* of  $P$ . A *weakly ordered hyperforest* is a hyperforest with a weak linear ordering of its set of hyperedges. A *weakly properly ordered noncrossing hyperforest* is a weakly ordered hyperforest where the weak ordering is a weak linear extension of its hyperedge poset.

With these definitions in place, the following is immediate.

**Corollary 7.11** (Simplices and noncrossing hypertrees). *The simplices in the noncrossing partition link  $\text{Link}(\text{NCPart}_{n+1})$  are bijectively labelled by weakly properly ordered noncrossing hypertrees on  $n + 1$  vertices.*

The noncrossing hypertree with a weak proper ordering of its hyperedges is called the *standard name* of the corresponding simplex in the noncrossing partition link. As a final remark, I should note that many of the results in this section are well-known although they are usually stated in a very different language. They are closely related to the various results that descend from the early work of Goulden and Jackson [GJ92], from [GP93] to the recent article by Du and Liu [DL15], that discuss permutation factorizations using multi-noded trees, as well as the article by Irving [Irv09] that discusses minimal transitive factorizations. Both of these concepts are closely related to hypertrees. The closest match is to the article [BMN] which includes drawings of noncrossing hypertrees.

## 8. THEOREM A: TOPOLOGY AND GEOMETRY

This section completes the proof of Theorem A by constructing a natural homeomorphism between the noncrossing hypertree complex and the noncrossing partition link. For the sake of readability, the noncrossing hypertree complex  $\text{Complex}(\text{NCHT}_{n+1})$  is denoted  $C$  and the noncrossing partition link  $\text{Link}(\text{NCPart}_{n+1})$  is denoted  $L$  throughout this section. At this point it should be quite clear that there is some sort of a relationship between  $C$  and  $L$  since the top-dimensional simplices in  $C$  are labeled by noncrossing trees and the top-dimensional simplices in  $L$  are labeled by properly ordered noncrossing trees and more general simplices are labeled by noncrossing hypertrees and weakly properly ordered noncrossing hypertrees. The first step in the construction of a map from  $C$  to  $L$  is to decide where to send the vertices of  $C$ .

**Definition 8.1** (Vertex images). Recall that the  $(n + 1)(n - 1)$  basic noncrossing hypertrees on  $n + 1$  vertices (Definition 3.1) are the vertices of the noncrossing hypertree complex  $C$ . Let  $\tau$  be one of these basic noncrossing hypertrees. Its two hyperedges have a unique proper ordering and it thus corresponds to a unique vertex in the noncrossing partition link. The basic noncrossing hypertree  $\tau$  also corresponds to a chain in the noncrossing partition lattice with a unique element other than its endpoints at the bounding elements and this unique point is the noncrossing partition corresponding to the lower hyperedge. In particular, the ordering of basic noncrossing hypertrees in an annular arrangement defined in Definition 3.1 and shown in Figure 6 is none other than the ordering of the corresponding elements inside the noncrossing partition lattice.

The other simplices in  $C$  are sent to unions of simplices in  $L$ , and the rough idea is to send each simplex in  $C$  labeled by a noncrossing hypertree  $\tau$  to the union of the simplices in  $L$  labeled by all proper orderings of this same noncrossing hypertree  $\tau$ . What needs to be shown is that these unions form simplicial shapes, that they overlap in the required fashion, and that the resulting continuous map is bijective. As an introduction to the general proof, consider the two simplicial complexes when there are 4 vertices and both complexes are graphs.

**Example 8.2** (Noncrossing hypertrees and partitions with 4 vertices). The metric graph shown in Figure 10 can be viewed as either the noncrossing partition link  $\text{NCPart}_4$  when the 4 white vertices of degree 2 are included, or as the noncrossing hypertree complex  $\text{NCHT}_4$  when they are ignored. In the noncrossing partition link there are 12 vertices and 16 edges all length  $\frac{\pi}{3}$  and it is a subgraph of the spherical building shown in Figure 9. When viewed as the noncrossing hypertree complex there are only 8 vertices and 12 edges of varying lengths. The 8 edges around the outside have length  $\frac{\pi}{3}$  and the four diagonal edges have length  $\frac{2\pi}{3}$ . The noncrossing hypertree labels for the vertices and edges on the boundary have been drawn. The diagonal edges are labeled by the four possible zig-zag trees. The vertical diagonal, for example, is labeled by the tree that looks like the letter ‘Z’. The noncrossing partition labels correspond to weak proper orders on these hypertrees. For every vertex and edge on the outside there is a unique such order and it is always a proper order. The more interesting possibilities occur along the diagonal edges. Let  $\tau$  be the ‘Z’ tree labeling the vertical diagonal. The diagonal edge of the ‘Z’ is below both other edges in its hyperedge poset and thus  $\tau$  has two proper orderings depending on which of the other two edges is last, and this accounts for the two edges



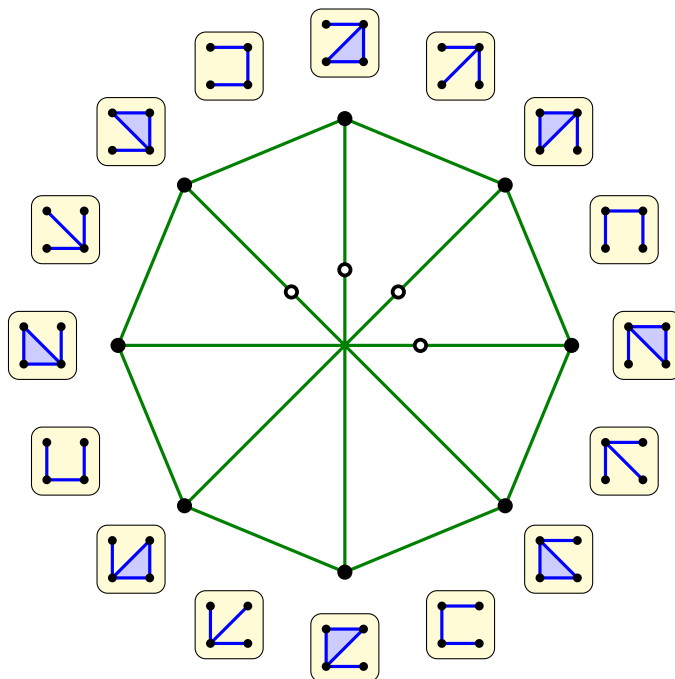


FIGURE 10. A graph that can be viewed as either  $\text{Link}(\text{NCPart}_4)$  or  $\text{Complex}(\text{NCHT}_4)$  depending on whether or not the white vertices are included.

of the noncrossing partition link that are amalgamated to form the single edge in the noncrossing hypertree complex. When a weakly proper order is chosen instead where both nondiagonal edges are last, the resulting simplex is the white vertex inbetween the two edges labeled by the proper orderings of  $\tau$ .

The previous sections have discussed two distinct ways of producing a simplex in the noncrossing partition link  $L$  from a noncrossing hypertree. The next remark compares them.

**Remark 8.3** (Two procedures). Definition 5.7 described how to turn a noncrossing hypertree  $\tau$  with  $k$  hyperedges into a simplicial sphere  $\text{Sphere}(\tau)$  inside  $L$  of dimension  $k - 2$  by considering all possible orderings of its hyperedges. In particular, for each ordering, the hyperedges are added one at a time and the noncrossing partition of the noncrossing hyperforest formed by hyperedges added so far is recorded. This essentially finds the join in the noncrossing partition lattice of the elements corresponding to the individual hyperedges. The resulting chain in the noncrossing partition lattice describes a simplex in  $L$ . The ordered noncrossing hypertree that produces this simplex can

be considered a *nonstandard name of the simplex* it produces. It is merely a name because the process of converting ordering noncrossing hypertrees into simplices in  $L$  is far from injective. The *standard name* for each simplex comes from Corollary 7.11. It creates a weakly properly ordered noncrossing hypertree  $\tau'$  from the labels of the adjacent elements in the chain that produces the simplex.

One consequence of Lemma 7.7 is that when  $\tau$  is a properly ordered noncrossing hypertree, the two procedures agree.

**Lemma 8.4** (Proper orderings and spheres). *Let  $\tau$  be a noncrossing hypertree with  $k$  hyperedges. For any proper ordering of its hyperedges, the simplex produced as part of the construction of  $\mathbf{Sphere}(\tau)$  is the same as the simplex whose standard label is this ordered noncrossing hypertree. In particular, every simplex whose standard label is  $\tau$  with some proper ordering of its hyperedges belongs to  $\mathbf{Sphere}(\tau)$ .*

*Proof.* Let  $\sigma_1\sigma_2\cdots\sigma_k = c$  be the reduced factorization of the Coxeter element  $c$  that corresponds to the chain in  $\mathbf{NCPart}_{n+1}$  and the simplex in  $L$  labeled by the properly ordered noncrossing hypertree  $\tau$ . By Lemma 7.7 for each  $j$  the product  $\sigma_1\cdots\sigma_j$  is equal to the noncrossing permutation of the noncrossing hyperforest formed by the first  $j$  hyperedges of  $\tau$ , which means that both procedures produce the same chain and the same simplex.  $\square$

As a consequence of Lemma 8.4, the noncrossing partition link is a union of its noncrossing apartments. Now that the proper orderings of a noncrossing hypertree  $\tau$  live in a common sphere, it is easy to analyze how that they fit together. The first step is to note which simplices form hemispheres.

**Lemma 8.5** (Hemispheres). *Let  $\tau$  be a noncrossing hypertree and let  $e$  and  $e'$  be hyperedges of  $\tau$ . The union of the simplices in  $\mathbf{Sphere}(\tau)$  that are constructed from the orderings of  $\tau$  in which  $e$  comes before  $e'$  form a hemisphere.*

*Proof.* This follows immediately from the description of the hemispheres in Definition 4.10.  $\square$

In fact, the proper orderings fit together to form a spherical simplex.

**Lemma 8.6** (Proper orderings and simplices). *For every noncrossing hypertree  $\tau$ , the union of the closed simplices whose standard name is  $\tau$  with a proper ordering of its hyperedges form a closed spherical simplex inside  $\mathbf{Sphere}(\tau)$ . As a consequence, the number of top-dimensional simplices in this union is equal to the number of linear extensions of an associated hyperedge poset  $\mathbf{Poset}(\tau)$ .*

*Proof.* Let  $k$  be the number of hyperedges in  $\tau$ . By Lemma 8.4 every proper ordering of  $\tau$  is the standard name of a simplex in  $\mathbf{Sphere}(\tau)$  and by Lemma 8.5, the  $k - 1$  covering relations in the hyperedge poset of  $\tau$  determine  $k - 1$  hemispheres inside the  $k - 2$  dimensional sphere  $\mathbf{Sphere}(\tau)$ . A proper ordering on  $\tau$  produces a simplex that lives in their intersection. In fact, any simplex in the intersection of dimension  $k - 2$  must come from a proper ordering because its position implies that all covering relations are properly ordered. Finally, the fact that the intersection of these hemispheres is a closed spherical simplex follows immediately from Proposition 4.11. Note that the Hasse diagram of the hyperedge poset  $\mathbf{Poset}(\tau)$  corresponds to the directed tree used in the proof of Proposition 4.11.  $\square$

For each noncrossing hypertree  $\tau$  let  $\mathbf{Simplex}(\tau)$  be the subcomplex of  $L$  formed by the union of the closed simplices labeled by the proper orderings of the hyperedges of  $\tau$ . This is the (subdivided) *simplex of  $\tau$*  and the homeomorphism between  $L$  and  $C$  simply removes the subdivisions. In the language of the literature on factorizations into cycles, two simplices are merged if and only if they correspond to *equivalent* reduced factorizations into cycles, the equivalence relation being the one generated by permuting commuting adjacent cycles.

**Remark 8.7** (Interior simplices and facets). The standard names of the simplices of  $L$  are not merely the proper orderings of noncrossing hypertrees, but weak proper orderings as well. Let  $\tau$  be a noncrossing hypertree. Every weak proper ordering of the hyperedges of  $\tau$  can be turned into a proper ordering by taking the noncrossing permutation that labels each adjacent pair in the corresponding chain in  $\mathbf{NCPart}_{n+1}$  and factoring it into irreducible permutations in some order. In  $L$  this places the simplex corresponding to the weak proper ordering in the boundary of several simplices with proper orderings. In particular, the weak proper ordering labels a simplex in the interior of  $\mathbf{Simplex}(\tau)$ . The simplices in the boundary of  $\mathbf{Simplex}(\tau)$ , on the other hand, are labeled by (weak) proper orderings of noncrossing hypertrees obtained by merging hyperedges of  $\tau$ .

An example should help clarify these observations.

**Example 8.8** (Amalgamating simplices). Consider the noncrossing tree  $\tau$  shown in Figure 11 on the left. It has 5 vertices and 4 edges. Its hyperedge poset, shown on the right, has the 4 edges of  $\tau$  as elements and 3 covering relations. There are exactly 5 linear extensions of this hyperedge poset, or equivalently, 5 proper orderings of  $\tau$ . These are  $abcd$ ,  $abdc$ ,  $bacd$ ,  $badc$  and  $bdac$ . Each of these 5 labels a maximal

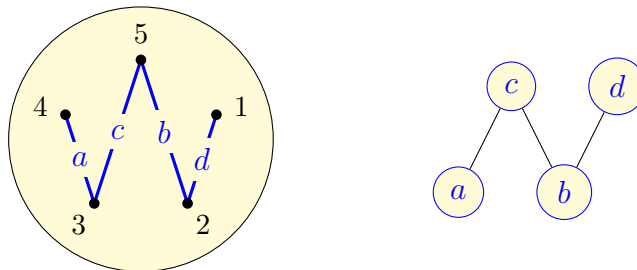


FIGURE 11. A noncrossing tree  $\tau$  with 5 vertices and 4 edges (left) and its hyperedge poset  $\text{Poset}(\tau)$  (right).

chain in  $\text{NCPart}_5$  and a spherical triangle in  $L$ . Their union is shown in Figure 12. The dotted lines and the vertex in the interior of this larger spherical triangle have standard labels that are weak proper orderings of  $\tau$ . The sides of the triangle correspond to  $\text{Simplex}(\tau')$  where  $\tau'$  is one of the three noncrossing hypertrees formed by merging two hyperedges of  $\tau$  as part of a covering relations in the noncrossing hypertree poset. The vertices arise from two such mergers. The hypertrees that label these features are included in Figure 12.

**Theorem A** (Topology and Geometry). *The noncrossing hypertree complex is naturally homeomorphic to the noncrossing partition link. As a consequence, the piecewise spherical metric on the latter induces a piecewise spherical metric on the former.*

*Proof.* The map between the noncrossing hypertree complex and the noncrossing partition link begins with the vertices (Definition 8.1) and then it proceeds up through the skeleta. When it is time to send the simplex of  $C$  labeled by the noncrossing hypertree  $\tau$  to  $L$ , the map on its boundary has already been defined and it is sent to the boundary of  $\text{Simplex}(\tau)$  inside  $L$  (Lemma 8.6 and Remark 8.7). Simply extend this map to the interior of the simplex labeled  $\tau$  in  $C$  to the interior of the simplex shaped subcomplex  $\text{Simplex}(\tau)$  in  $L$ . By construction the completed map is a continuous bijection and thus a homeomorphism.  $\square$

One way to restate this result is that the noncrossing hypertree complex is a simplicial coarsening of the noncrossing partition link. This is similar in spirit to the polyhedral coarsening discussed by Nathan Reading in [Rea12]. In order to distinguish between the two simplicial structures, when both are under discussion, the simplices in  $L$  are called *partition simplices* and the simplices in  $C$  are called *tree simplices*.

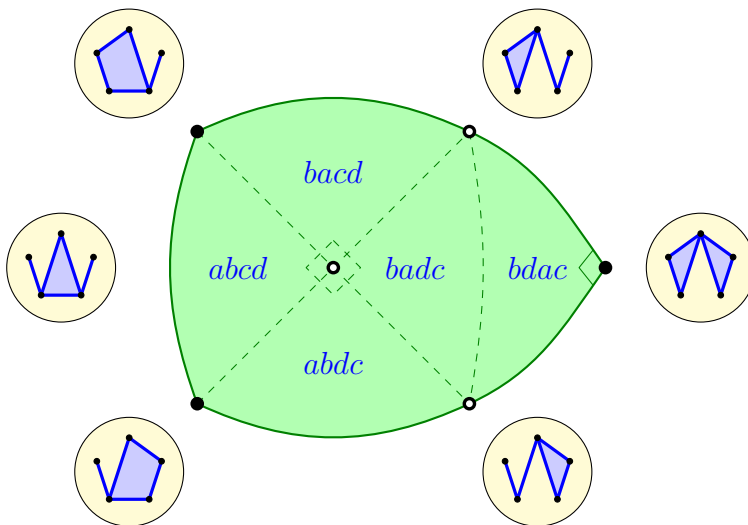


FIGURE 12. The hyperedge poset of the tree  $\tau$  shown in Figure 11 has 5 linear extensions, or equivalently there are 5 proper orderings of  $\tau$ :  $abcd$ ,  $abdc$ ,  $bacd$ ,  $badc$  and  $bdac$ . Each of these 5 labels a spherical triangle in the noncrossing partition link  $\text{NCPart}_5$  with angles  $\frac{\pi}{2}$ ,  $\frac{\pi}{3}$  and  $\frac{\pi}{3}$ . These 5 triangles fit together as shown to form a large spherical triangle with angle  $\frac{2\pi}{3}$ ,  $\frac{2\pi}{3}$  and  $\frac{\pi}{2}$ . The labels are the noncrossing hypertree labels of the 3 vertices and 3 sides of the large simplex.

For example, if  $\tau$  is a noncrossing hypertree, then  $\text{Simplex}(\tau)$  is a single tree simplex but it is a union of partition simplices. The relative efficiency of  $C$  as a simplicial structure over that of  $L$  is clear once we compare the number of vertices and chambers in each.

**Remark 8.9** (Relative sizes). The noncrossing partition link  $L = \text{Link}(\text{NCPart}_{n+1})$  has  $\text{Cat}_{n+1} = \frac{1}{n+2} \binom{2n+2}{n+1}$  many vertices and  $(n+1)^{n-1}$  many partition chambers. By contrast there are only  $(n+1)(n-1)$  many vertices and  $\text{Cat}^{(2)}(A_n) = \frac{1}{2n+1} \binom{3n}{n}$  many tree chambers in the noncrossing hypertree complex  $C = \text{Complex}(\text{NCHT}_{n+1})$ . In other words, exponentially many vertices and a superexponential number of chambers are amalgamated into a quadratic number of vertices and a mere exponential number of chambers. See Table 2.

## 9. THEOREM B: SIMPLICES AND SPHERES

This section completes the proof the Theorem B, but since each tree simplex is labeled by a noncrossing hypertree  $\tau$  and a sphere containing

$n$	$d$	$ V $	chambers
3	1	8	12
4	2	15	55
5	3	24	273
6	4	35	1,428
7	5	48	7,752
8	6	63	43,263
9	7	80	246,675

$n$	$d$	$ V $	chambers
3	1	12	16
4	2	40	125
5	3	130	1,296
6	4	427	16,807
7	5	1,428	262,144
8	6	4,860	4,782,969
9	7	16,794	100,000,000

TABLE 2. The number of vertices and chambers for the noncrossing hypertree complex  $\text{Complex}(\text{NCHT}_{n+1})$  (left) and the noncrossing partition link  $\text{Link}(\text{NCPart}_{n+1})$  (right) in low dimensions.

it inside the noncrossing partition link has already been identified for every noncrossing hypertree  $\tau$ , it suffices to show that  $\text{Sphere}(\tau)$  is a union of tree simplices and thus a subcomplex in the new cell structure. This follows from an understanding of the process of standardizing the names of simplices labeled by ordered hypertrees.

**Lemma 9.1** (Standard names). *Let  $\tau$  be an ordered hypertree with hyperedge permutations  $\sigma_1, \sigma_2, \dots, \sigma_k$  where the subscripts indicates the ordering. If  $\tau'$  is the standard name of the simplex corresponding to  $\tau$  then  $\tau'$  is the properly ordered noncrossing hypertree  $\tau'$  with the same number of hyperedges permutations  $\sigma'_1, \sigma'_2, \dots, \sigma'_k$  and the same multi-set of hyperedge sizes. Moreover, for any permutation  $\pi$  of  $k$  elements such that  $\sigma_{\pi(1)}\sigma_{\pi(2)} \cdots \sigma_{\pi(k)}$  is a proper ordering of the hyperedge permutations of  $\tau$ , the permutations  $\sigma'_j$  in the standard name satisfy the equations  $\sigma_j\beta_j = \beta_j\sigma'_j$  where  $\beta_j$  is the product of the permutations  $\sigma_i$  with  $i < j$  that occur to the right of  $\sigma_j$  in the chosen proper ordering of  $\tau$  in the order they occur.*

*Proof.* Let  $e_i$  be the hyperedge of  $\tau$  corresponding to the hyperedge permutation  $\sigma_i$  and for each  $j$  define three permutations. Let  $\tau_j$ ,  $\alpha_j$  and  $\beta_j$  be the noncrossing permutations of the noncrossing hyperforests whose hyperedges are (1) those  $e_i$  with  $i \leq j$ , (2) those  $e_i$  with  $i < j$  and  $\sigma_i$  to the left of  $\sigma_j$  in the chosen proper ordering of  $\tau$  and (3) those  $e_i$  with  $i < j$  and  $\sigma_i$  to the right of  $\sigma_j$  in the chosen proper ordering of  $\tau$ , respectively. Because the order is proper, the product of the  $\sigma_i$ 's in this order is a reduced factorization of the Coxeter element  $c = \tau_k$  (Lemma 7.7). Next, use Proposition 6.5 to find a new reduced factorization whose first  $j$  factors, as a set, are the permutations  $\sigma_i$

with  $i \leq j$ . Since the order in which they occur is a proper ordering of the subhyperforest with only these  $j$  hyperedges, the product of these  $j$  permutations in this order is  $\tau_j$ . Similarly, the product of the permutations that occur before and after  $\sigma_j$  in this ordering are  $\alpha_j$  and  $\beta_j$ , respectively. In other words  $\tau_j = \alpha_j \sigma_j \beta_j$  and by Proposition 6.4 this is a reduced factorization of  $\tau_j$ . If  $\sigma'_j$  is defined as the unique permutation that solves the equation  $\sigma_j \beta_j = \beta_j \sigma'_j$ , then  $\tau_j = \alpha_j \beta_j \sigma'_j$  and since  $\alpha_j \beta_j$  is the properly ordered product of all  $\sigma_i$  with  $i < j$ , it is equal to  $\tau_{j-1}$ . Thus  $\tau_j = \tau_{j-1} \sigma'_j$ . By Proposition 6.7 these labels  $\sigma'_j$  are the unique labels on the chain  $t = \tau_0 < \tau_1 < \dots < \tau_k = c$  and the corresponding hyperedges form a weakly properly ordered noncrossing hypertree  $\tau'$ . But since conjugation does not change the cycle type of a permutation,  $\sigma_j$  irreducible implies that  $\sigma'_j$  is also irreducible. As a consequence the ordering on the hyperedges on  $\tau'$  is a proper order and not merely weakly proper order.  $\square$

Since the new hyperedge permutations  $\sigma'_j$  are independent of the chosen proper ordering of  $\tau$ , different proper orderings can be chosen for each  $j$  to minimize the number of conjugations required. For example it is sufficient to focus on inversions in the hypertree poset.

**Definition 9.2** (Inversions in a partial order). Let  $P$  be a finite poset with  $k$  elements  $e_1, e_2, \dots, e_k$  where the subscripts indicate a linear ordering of the elements of  $P$ . If  $i < j$  as integers but  $e_i > e_j$  in the poset ordering of  $P$  then  $e_i$  *inverts*  $e_j$ .

**Remark 9.3** (Fewer conjugations). Let  $\tau$  be an ordered noncrossing hypertree with hyperedges  $e_1, e_2, \dots, e_k$  and hyperedge permutations  $\sigma_1, \sigma_2, \dots, \sigma_k$  where the subscripts indicate the linear order. For each  $j$  there is a proper ordering of  $\tau$  where the only hyperedges that occur after  $e_j$  are those that are above  $e_j$  in the hyperedge poset of  $\tau$ . By choosing this proper order in Lemma 9.1, it is clear that the hyperedge permutation  $\sigma'_j$  in the standardized hypertree  $\tau'$  satisfies the equation  $\sigma_j \beta_j = \beta_j \sigma'_j$  where  $\beta_j$  is a product of the hyperedge permutations of the hyperedges that invert  $e_j$  in  $\text{Poset}(\tau)$  in any order that is proper for the noncrossing hyperforest that they form.

The number of necessary conjugations can be further reduced but this is sufficient for our purposes.

**Example 9.4** (Standard names). Let  $e_1 = \{3, 4, 5\}$ ,  $e_2 = \{5, 6\}$ ,  $e_3 = \{2, 3\}$ ,  $e_4 = \{1, 3\}$  and  $e_5 = \{5, 7, 8, 9\}$  be a linear ordering of the 5 hyperedges of the noncrossing tree shown in on the right-hand side of Figure 3. As noted in Definition 2.6 the unique proper

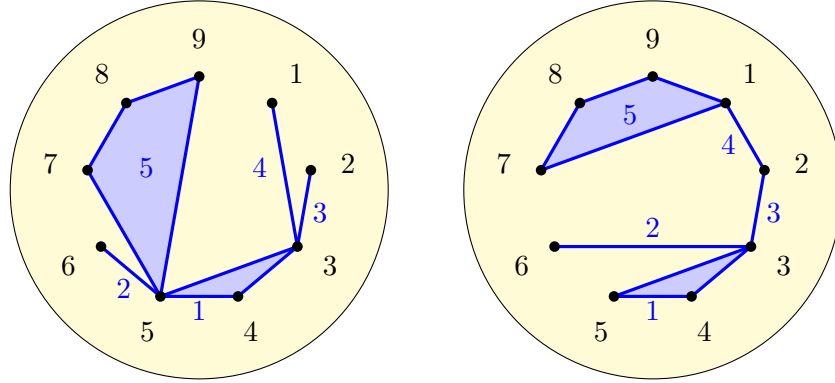


FIGURE 13. The ordered noncrossing hypertree  $\tau$  on the left corresponds to the properly ordered noncrossing hypertree  $\tau'$  on the right.

ordering of these 5 hyperedges is  $e_2 < e_5 < e_1 < e_4 < e_3$ . The hyperedge  $e_3$  is not inverted by any other hyperedge, so  $\sigma'_3 = \sigma_3 = (2, 3)$ . On the other hand,  $e_5$  is inverted by  $e_1$ ,  $e_4$  and  $e_3$  so  $\beta_5 = \sigma_1\sigma_4\sigma_3 = (3, 4, 5)(1, 3)(2, 3) = (1, 2, 3, 4, 5)$  and  $\sigma'_5$  is  $(1, 7, 8, 9)$  because  $(5, 7, 8, 9)(1, 2, 3, 4, 5) = (1, 2, 3, 4, 5, 7, 8, 9) = (1, 2, 3, 4, 5)(1, 7, 8, 9)$ . In Figure 13 the original ordered noncrossing hypertree  $\tau$  is shown on the left and the properly ordered noncrossing hypertree  $\tau'$  to which it corresponds is shown on the right.

**Lemma 9.5** (Facets and proper orderings). *If a simplex  $S$  labeled by a properly ordered noncrossing hypertree  $\tau$  has a facet  $F$  labeled by a weak proper ordering of  $\tau$ , then this facet is a facet of exactly one other simplex  $S'$  with a properly ordered noncrossing tree as its label. Moreover, the label on  $S'$  is another proper ordering of  $\tau$  and the new ordering differs from the old by a single transposition of the order on a pair of hyperedges adjacent in the original ordering and incomparable in the hyperedge poset.*

*Proof.* Recall that when passing to a facet a pair of adjacent hyperedge permutations are multiplied. In order for the result to be a weak proper ordering (as opposed to an extremely weak one), the corresponding hyperedges involved must be disjoint and the product of the two permutations is a noncrossing permutation with exactly two nontrivial blocks. When undoing this process, one noncrossing permutation label is split/factored. The only way that the factored labeling can correspond to a proper ordering of a hypertree is if the recently formed noncrossing permutation with two nontrivial blocks is the one that is factored and the factoring splits it into two irreducible permutations.



There are only two such factorings. One returns to the ordering of  $\tau$  and the other to a different ordering of  $\tau$ .  $\square$

From this it quickly follows that the simplicial spheres already identified in the noncrossing partition link remain simplicial spheres in the new cell structure.

**Lemma 9.6** (Union of tree simplices). *Let  $\tau$  and  $\tau'$  be noncrossing hypertrees with the same number of hyperedges. If one proper ordering of  $\tau'$  labels a partition simplex in  $\mathbf{Sphere}(\tau)$ , then every proper ordering of  $\tau'$  describes a partition simplex in  $\mathbf{Sphere}(\tau)$ . As a consequence, the simplicial sphere  $\mathbf{Simplex}(\tau)$  with its  $k!$  partition simplices can be viewed instead as a union of tree simplices.*

*Proof.* Consider two top-dimensional partition simplices  $S$  and  $S'$  in  $\mathbf{Simplex}(\tau')$  that share a facet  $F = S \cap S'$  and assume that  $S$  is a partition simplex in  $\mathbf{Sphere}(\tau)$ . Note that  $S$  is necessarily top-dimensional in  $\mathbf{Sphere}(\tau)$  because of the equality of the number of hyperedges. Since spheres are manifolds, there is a unique simplex  $S''$  in  $\mathbf{Sphere}(\tau)$  that shares the facet  $F$  and with  $S$ , by Lemma 9.1 it is labeled by a properly ordered noncrossing hypertree and by Lemma 9.5  $S'$  and  $S''$  must be one and the same. Finally, since the union of the simplices labeled by all of the proper orderings of  $\tau'$  form a spherical simplex, this type of facet-sharing adjacency can be used to conclude that all of the partition simplices in  $\mathbf{Simplex}(\tau')$  lie in  $\mathbf{Sphere}(\tau)$ .  $\square$

Theorem B is now immediate.

**Theorem B** (Simplices and Spheres). *Every spherical simplex in the metric noncrossing hypertree complex of any dimension is contained in a subcomplex isometric to a unit sphere of the same dimension. In fact, in the top dimension (and conjecturally in all dimensions) there is a natural map from simplices to spherical subcomplexes that establishes a bijection between these two sets.*

*Proof.* The first assertion is immediate since each simplex in the noncrossing hypertree complex is labeled by a noncrossing hypertree  $\tau$  and  $\mathbf{Sphere}(\tau)$  is a sphere inside the noncrossing partition link of the same dimension containing the simplex labeled  $\tau$  which, by Lemma 9.6, remains a simplicial sphere in the simplified cell structure. Finally, by Proposition 5.6 the apartments labeled by noncrossing trees are the only metric spheres of this dimension inside the noncrossing partition link and therefore the only metric spheres of this dimension inside the noncrossing hypertree complex.  $\square$

The bijection established above between spheres and simplices in the top dimension (i.e. between tree simplices and apartments) is one that I firmly believe extends to all dimensions, but I do not currently have a proof of this conjecture. To do so would involve two steps. The first relatively easy step is showing that the map from simplices to spheres sending  $\mathbf{Simplex}(\tau)$  to  $\mathbf{Sphere}(\tau)$  for each noncrossing hypertree  $\tau$  is injective. The second more difficult step would be to show that no other metric spheres exist as subcomplexes of the metric noncrossing hypertree complex. Both steps are true for the small values of  $n$  where explicit checks are feasible. Since it has not been conclusively proved that these are the only spheres in the noncrossing hypertree complex, the known spheres deserve a name. For the remainder of the article, the spheres of the form  $\mathbf{Sphere}(\tau)$  for some noncrossing hypertree  $\tau$  are called *special spheres*.

## 10. THEOREM C: BIJECTIONS

In this section Theorem C is proved using some elementary automorphisms of the noncrossing hypertree complex.

**Remark 10.1** (Automorphisms). Let  $\mathbf{Dih}(k)$  denote the *dihedral group* of order  $2k$  that arises as the isometry group of a regular  $k$ -gon. The *natural automorphisms* of the poset of noncrossing hypertrees  $\mathbf{NCHT}_k$  and the noncrossing partition lattice  $\mathbf{NCPart}_k$  (and the corresponding simplicial automorphisms of the noncrossing hypertree complex  $\mathbf{Complex}(\mathbf{NCHT}_k)$  and the noncrossing partition link  $\mathbf{Link}(\mathbf{NCPart}_k)$ ) are the automorphisms that arise from the dihedral group  $\mathbf{Dih}(k)$  of isometries of the underlying  $k$ -gon in the plane used in their definition. These form half of the full automorphism group. The other half become visible when (the dual of) the noncrossing hypertree poset is identified with poset of dissections of a polygon with twice as many sides into even-sided subpolygons. The obvious automorphisms in this context form a dihedral group  $\mathbf{Dih}(2k)$  with twice as many elements with the previous automorphisms being those that preserve the bipartite coloring of the vertices as black and white. A representative new automorphism is the rotation by  $\frac{2\pi}{2k} = \frac{\pi}{k}$ . In the context of the noncrossing partition lattice this becomes the well-known Kreweras map sending each noncrossing permutation to its right complement. In the context of the noncrossing hypertree poset, it sends each noncrossing hypertree  $\tau$  to its “opposite” defined by picking a proper ordering of  $\tau$ , reversing it, and then finding the properly ordered hypertree  $\tau'$  that corresponds to this ordering of  $\tau$ . As in the noncrossing partition context, this map is not an involution. Doing it twice produces a hypertree  $\tau''$  that is a  $\frac{2\pi}{k}$

rotation of  $\tau$  as is easy to see from the polygon dissection viewpoint. This is a map with many interesting properties. They are not explored here but are worthy of further study.

The only automorphism used below is the natural one induced by a reflection of the underlying polygon.

**Lemma 10.2** (Reflection of spheres). *If  $\text{Refl}$  is an involution on the poset of noncrossing hypertrees induced by a fixed reflection of a polygon in the plane used to define them, then for any noncrossing hypertree  $\tau$ ,  $\text{Sphere}(\tau)$  and  $\text{Sphere}(\text{Refl}(\tau))$  are isomorphic as simplicial spheres and there is a natural bijection between their top dimensional tree simplices.*

*Proof.* This is nearly immediate since the reflection on the polygon induces an automorphism of the noncrossing partition lattice that sends the special Boolean subposet that corresponds to  $\text{Sphere}(\tau)$  to the special Boolean subposet that corresponds to  $\text{Sphere}(\text{Refl}(\tau))$ . This is because the reflection of the noncrossing partition constructed from the connected components of the noncrossing hyperforest formed by a subset of hyperedges of  $\tau$  is the noncrossing partition constructed from the connected components of the noncrossing hyperforest formed by the corresponding subset of the reflected hyperedges of  $\tau$ .  $\square$

Reflections also produce a kind of duality between simplices and special spheres.

**Lemma 10.3** (Reflections and inclusions). *Let  $\text{Refl}$  be an involution on the set of noncrossing hypertrees induced by a fixed reflection of a polygon in the plane used to define them. If  $\tau$  and  $\tau'$  are noncrossing hypertrees with the same number of hyperedges such that the simplex labeled  $\tau'$  belongs to the sphere labeled  $\tau$ , then the simplex labeled  $\text{Refl}(\tau)$  belongs to the sphere labeled  $\text{Refl}(\tau')$ . In particular, for any fixed noncrossing hypertree  $\tau$  there is a bijection between the tree simplices of top dimension inside  $\text{Sphere}(\text{Refl}(\tau))$  and the special spheres containing  $\text{Simplex}(\tau)$  as a simplex of top-dimension.*

*Proof.* When working with the reflected versions it is convenient to relabel the vertices of the underlying polygon so that the new vertex  $i$  is the image of  $i$  under the fixed reflection of the polygon. This makes the two situations easier to compare. The effect that this has is that the vertices are proceeding in the opposite direction around the boundary, the new Coxeter element is  $c^{-1}$ , where  $c$  is the old Coxeter element. Similarly, the hyperedge permutations of  $\text{Refl}(\tau)$  are the inverses of the hyperedge permutations of  $\tau$  and the poset  $\text{Poset}(\text{Refl}(\tau))$  is the dual of

$\text{Poset}(\tau)$ . More precisely, because of the vertex relabeling, the hyperedges of  $\tau$  and  $\text{Refl}(\tau)$  are identical, but the local linear orderings are reversed. Moreover, if  $\sigma_1\sigma_2\cdots\sigma_k$  is a reduced factorization of  $c$  coming from a properly ordered noncrossing hypertree  $\tau$ , then  $\sigma_k^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1}$  is the reduced factorization of  $c^{-1}$  that corresponds to a proper ordering of the noncrossing hypertree  $\text{Refl}(\tau)$  in the relabeled polygon. With this as background, the main idea of the proof is that when the simplex labeled  $\tau'$  belongs to the sphere labeled  $\tau$  then there are two orderings of  $\tau$ , one proper and one improper so that when Lemma 9.1 is applied to the improper ordering on  $\tau$  it leads to a proper ordering of  $\tau'$ . The two orderings together define a permutation  $\pi$  of  $k$  elements, where  $k$  is the common number of hyperedges in  $\tau$  and  $\tau'$ . The subscripts of the permutations in the corresponding reduced factorization of  $c^{-1}$  coming from this corresponding proper ordering of  $\text{Refl}(\tau')$  can be permuted using the permutation  $\pi$  and the global order-reversing permutation so that the process of standardizing this improper ordering of  $\text{Refl}(\tau')$  undoes precisely those conjugations that produced  $\tau'$  from  $\tau$ . Concretely relabel  $(\sigma'_i)^{-1}$  as  $\rho_j$  with  $j = \text{rev}(\pi(i))$  where  $\text{rev}$  is the function that globally reverses the ordering on the numbers 1 through  $k$ . The standard name that results is image of the standard name of  $\tau$  in the original proper ordering under the map  $\text{Refl}$ .  $\square$

The following is an extended concrete example that illustrates how this relabeling causes the conjugations to be undone.

**Example 10.4** (Reversing direction). This example continues working with the ordered hypertree  $\tau$  given in Example 9.4 and properly ordered hypertree  $\tau'$  that labels the simplex to which  $\tau$  corresponds. All of the notation established there for the hyperedge permutations is carried over here. In particular, note that this is an instance where the tree simplex labeled by  $\tau'$  belongs to the sphere labeled by  $\tau$ . One advantage is that both  $\tau$  and  $\tau'$ , viewed as unordered hypertrees, have a unique proper ordering, thereby eliminating a potential source of confusion.

The factorization of  $c$  corresponding to the unique proper ordering of the hyperedges of  $\tau$  is

$$\sigma_2\sigma_5\sigma_1\sigma_4\sigma_3 = (5, 6)(5, 7, 8, 9)(3, 4, 5)(1, 3)(2, 3).$$

Thus the permutation  $\pi$  used in Lemma 9.1 has  $\pi(1) = 2$ ,  $\pi(2) = 5$ ,  $\pi(3) = 1$ ,  $\pi(4) = 4$  and  $\pi(5) = 3$ . The standardization process leads to the reduced factorization

$$\sigma'_1\sigma'_2\sigma'_3\sigma'_4\sigma'_5 = (3, 4, 5)(3, 6)(2, 3)(1, 2)(1, 7, 8, 9)$$

of  $c$  corresponding to the unique proper ordering of  $\tau'$ . See Figure 13. For the reflected hypertree  $\text{Refl}(\tau')$ , with its relabeled vertices, the reduced factorization of  $c^{-1}$  is  $(\sigma'_5)^{-1}(\sigma'_4)^{-1}(\sigma'_3)^{-1}(\sigma'_2)^{-1}(\sigma'_1)^{-1}$ . Concretely this is the reduced product

$$(9, 8, 7, 1)(2, 1)(3, 2)(6, 3)(5, 4, 3) = (9, 8, 7, 6, 5, 4, 3, 2, 1)$$

Relabel these 5 permutations as follows. Let  $\rho_4 = (\sigma'_5)^{-1} = (9, 8, 7, 1)$ ,  $\rho_2 = (\sigma'_4)^{-1} = (2, 1)$ ,  $\rho_1 = (\sigma'_3)^{-1} = (3, 2)$ ,  $\rho_5 = (\sigma'_2)^{-1} = (6, 3)$  and  $\rho_3 = (\sigma'_1)^{-1} = (5, 4, 3)$ . The subscripts  $\rho_j = (\sigma'_i)^{-1}$  are chosen so that  $j = \text{rev}(\pi(i))$  where  $\text{rev}$  is the function that globally reverses the ordering on the numbers 1 through  $k$ , with  $k = 5$  in this example. In this case it switches 1 and 5, switches 2 and 4 and fixes 3. For example,  $\pi(5) = 2$  and  $\text{rev}(2) = 4$  so  $(\sigma'_5)^{-1}$  is relabeled  $\rho_4$ . See Figure 14.

When Lemma 9.1 is applied to this ordering  $\rho_4\rho_2\rho_1\rho_5\rho_3$  of the hyperedges of  $\text{Refl}(\tau')$ ,  $\rho_5 = (6, 3)$  is inverted by  $\rho_3 = (5, 4, 3)$  so  $\rho'_5 = (6, 5)$  because  $(6, 3)(5, 4, 3) = (6, 5, 4, 3) = (5, 4, 3)(6, 5)$ , but  $\rho_3$  and  $\rho_1$  are not inverted at all so that  $\rho'_3 = \rho_3 = (5, 4, 3)$  and  $\rho'_1 = \rho_1 = (3, 2)$ . The permutation  $\rho_2$  is inverted by  $\rho_1$  so  $\rho'_2 = (3, 1)$ , because  $(2, 1)(3, 2) = (3, 2, 1) = (3, 2)(3, 1)$ . Finally,  $\rho_4$  is inverted by  $\rho_2$ ,  $\rho_1$  and  $\rho_3$ . Thus  $\rho'_4 = (9, 8, 7, 6)$  because  $\rho_2\rho_1\rho_3 = (2, 1)(3, 2)(5, 4, 3) = (5, 4, 3, 2, 1)$  and  $(9, 8, 7, 1)(5, 4, 3, 2, 1) = (9, 8, 7, 5, 4, 3, 2, 1) = (5, 4, 3, 2, 1)(9, 8, 7, 5)$ . A close examination of these steps shows that they are undoing exactly the conjugations that were done in Example 9.4. The end result of this standardization is

$$\rho'_1\rho'_2\rho'_3\rho'_4\rho'_5 = (3, 2)(3, 1)(5, 4, 3)(9, 8, 7, 5)(6, 5)$$

and this is the reduced factorization of  $c^{-1}$  corresponding to the unique proper ordering of the hyperedges of  $\text{Refl}(\tau)$ .

These lemmas combine to prove the theorem.

**Theorem C** (Bijections). *For every noncrossing hypertree  $\tau$  there is a bijection between the number of special spheres containing the tree simplex labeled  $\tau$  as a top-dimensional simplex and the set of tree simplices contained in the special sphere labeled  $\tau$ . When  $\tau$  is a noncrossing tree, this means that there is a bijection between the set  $\{\sigma \mid \text{Chamber}(\tau) \in \text{Apart}(\sigma)\}$  of apartments containing the tree chamber labeled  $\tau$  and the set  $\{\sigma \mid \text{Chamber}(\sigma) \in \text{Apart}(\tau)\}$  of tree chambers in the apartment labeled  $\tau$ .*

*Proof.* Let  $\text{Refl}$  the involution on the set of noncrossing hypertrees induced by a fixed reflection of a polygon in the plane used to define them. Let  $A$ ,  $C$  and  $C'$  be the set of special spheres containing  $\text{Simplex}(\tau)$  as a top-dimensional simplex, the set of top-dimensional

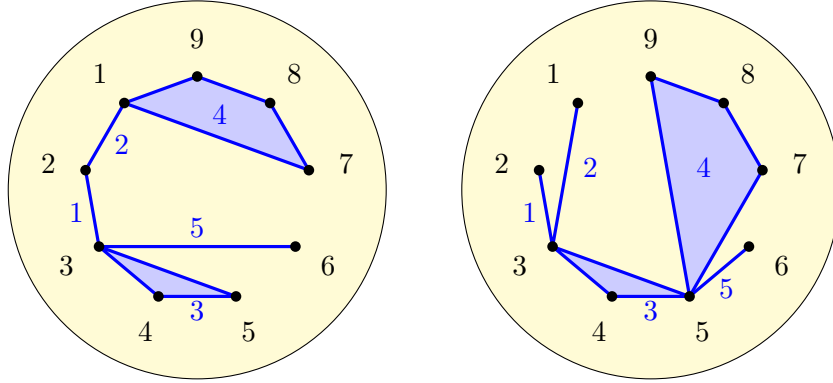


FIGURE 14. The ordered noncrossing hypertree  $\text{Refl}(\tau')$  on the left that corresponds to the properly ordered noncrossing hypertree  $\text{Refl}(\tau)$  on the right.

simplices inside  $\text{Sphere}(\tau)$  and the set of top-dimensional simplices inside  $\text{Sphere}(\text{Refl}(\tau))$ . Lemma 10.3 gives a bijection between  $A$  and  $C'$  and Lemma 10.2 gives a bijection between  $C'$  and  $C$ . The final assertion is merely the special case where the simplices and spheres under discussion have the largest possible dimension.  $\square$

## 11. THEOREM D: ASSOCIAHEDRA

This section completes the proof of the final main theorem, Theorem D, as part of an investigation of the simplicial structure of special sphere and especially apartments. The special spheres with the least number of tree simplices turn out to be cross polytopes and the apartments that seem to have the most turn out to be simplicial associahedra. The cross polytope case is considered first since it is much easier to establish. It is also less surprising because of the following observation.

**Lemma 11.1** (Subspheres). *If  $\tau$  and  $\tau'$  are noncrossing hypertrees with  $k$  and  $k - 1$  hyperedges, respectively, such that  $\tau > \tau'$  is a covering relation in the noncrossing hypertree poset, then the simplicial sphere  $\text{Sphere}(\tau)$  contains  $\text{Sphere}(\tau')$  as a subcomplex and this equatorial subsphere divides  $\text{Sphere}(\tau)$  into two equal hemispheres. As a consequence, the  $k - 1$  facets of  $\text{Simplex}(\tau)$  divide  $\text{Sphere}(\tau)$ , isometric to  $\mathbb{S}^{k-2}$ , into at least  $2^{k-1}$  top dimensional tree simplices.*

*Proof.* This is nearly immediate from the various constructions. If  $e$  and  $e'$  are the hyperedges labeling the endpoints of the covering relation in  $\text{Poset}(\tau)$  that are merged to form  $\tau'$ , then  $e$  and  $e'$  share a vertex  $v$ .

Any ordering of the hyperedges of  $\tau'$  can be turned into a ordering of  $\tau$  in two different minimal ways by splitting the hyperedge back into  $e$  and  $e'$  and then breaking the tie between them by placing one before the other. If the ordering is thought of as a map to the reals, with the merged hyperedges being sent to a real number  $r$ , then breaking the tie means sending  $e$  and  $e'$  to  $r + \epsilon$  and  $r - \epsilon$  or to  $r - \epsilon$  and  $r + \epsilon$ . The tree simplex in  $\mathbf{Sphere}(\tau')$  labeled by this ordering of  $\tau'$  is the facet between the two tree simplices in  $\mathbf{Sphere}(\tau)$  labeled by the two associated orderings of  $\tau$ .  $\square$

Let  $\tau$  be a noncrossing hypertree with  $k$  hyperedges and note that the simplicial structure determined by the intersections of the  $k - 1$  subspheres of  $\mathbf{Sphere}(\tau)$  coming from the facets of  $\tau$  is already that of a cross-polytope. This arrangement is called a *metric cross polytope* because it has additional structure: the simplicial structure is that of a cross polytope but also every simplex lives in a metric subsphere. The following is an easy consequence of Lemma 11.1.

**Lemma 11.2** (Cross-polytopes). *For each noncrossing hypertree  $\tau$  with  $k$  hyperedges, the special sphere  $\mathbf{Sphere}(\tau)$  has at least  $2^{k-1}$  top-dimensional tree simplices and when this minimum value is achieved, the structure of  $\mathbf{Sphere}(\tau)$  is that of a metric cross-polytope.*

*Proof.* As noted above, the simplicial structure determined by the intersections of the  $k - 1$  subspheres of  $\mathbf{Sphere}(\tau)$  coming from the facets of  $\tau$  is already that of a metric cross-polytope, dividing the sphere into  $2^{k-1}$  top-dimensional simplices. Since every top-dimensional tree simplex must live in one of these pieces, the inequality and the consequence of equality are immediate.  $\square$

Many noncrossing hypertrees achieve this theoretical minimum.

**Theorem 11.3** (Cross-polytopes). *If  $\tau$  is noncrossing hypertree with  $k$  hyperedges such that every hyperedge is either a minimum element or a maximum element in its hyperedge poset, then  $\mathbf{Sphere}(\tau)$  contains exactly  $2^{k-1}$  top-dimensional tree simplices and, as a consequence,  $\mathbf{Sphere}(\tau)$  is a metric cross-polytope.*

*Proof.* Since by Lemma 11.2 there are at least  $2^{k-1}$  top-dimensional tree simplices in  $\mathbf{Sphere}(\tau)$ , it suffices to show that under these conditions  $2^{k-1}$  is also an upper bound. If  $e$  is a hyperedge of  $\tau$  that labels a maximum element in  $\mathbf{Poset}(\tau)$ , then by Remark 9.3, it remains unchanged by the standardization process under any ordering of the hyperedges of  $\tau$ . When  $e$  is a hyperedge that labels a minimum element in  $\mathbf{Poset}(\tau)$  that  $e$  has degree  $d$  in the tree that is the Hasse diagram of this poset,

then by Remark 9.3 there are at most  $2^d$  possibilities for the new hyperedge permutation  $\sigma'$  derived from  $\sigma$  in the new tree created by the standardizing process. Since the sum of the degrees of the minimal elements in  $\text{Poset}(\tau)$  is equal to  $k - 1$ , the number of covering relations it has, the total number of possible noncrossing hypertrees in  $\text{Sphere}(\tau)$  that result from the standardizing process is at most  $2^{k-1}$ , the product of these choices over the various minimal elements. The final assertion follows from Lemma 11.2.  $\square$

The noncrossing trees that satisfy this condition produce apartments that are cross-polytopes.

**Remark 11.4** (Zig-zag trees). If  $\tau$  is a noncrossing tree satisfying the hypothesis of Theorem 11.3, then  $\tau$  has the structure of a single path that zig-zags back and forth across the polygon so that every edge has one of two possible slopes (assuming that the underlying convex polygon is regular). The edges with one slope are the maximal elements in  $\text{Poset}(\tau)$  and the edges with the other slope are the minimal elements in  $\text{Poset}(\tau)$ . In addition to  $\text{Apart}(\tau)$  having the fewest possible number of tree chambers, the tree chamber  $\text{Chamber}(\tau)$  contains the maximum number of partition chambers. These arise from the orderings known as zig-zag permutations and they correspond to linear extensions of the zig-zag poset that is  $\text{Poset}(\tau)$ . The number of these linear extensions is given by the sequence `oeis:A000111` which starts 1, 1, 2, 5, 16, 61, 272, 1385, 7936, a set of numbers that also occur in the exponential generating function for  $\sec(x) + \tan(x)$ . For further information about zig-zag posets and zig-zag permutations see [Sta12] or the references in listed in the Online Encyclopedia of Integer Sequences [Slo]. In a context very closely related to the results presented here, these numbers also appear in an article by K. Saito [Sai07].

The remainder of the section focuses on certain apartments in the noncrossing hypertree complex that have the simplicial structure of a simplicial associahedron. That there are at least some simplicial associahedra in the noncrossing hypertree complex is clear because of its identification as a generalized cluster complex of type  $A$  (Remark 3.6).

**Remark 11.5** (Associahedra and generalized cluster complexes). In their foundational article on generalized cluster complexes [FR05], Fomin and Reading prove the generalized cluster complexes are nested in the following sense. If  $\Phi$  is a root system and  $m \geq m'$  are two positive integers then, in their notation, the generalized cluster complex  $\Delta^m(\Phi)$  contains the generalized cluster complex  $\Delta^{m'}(\Phi)$  as a full subcomplex determined by a subset of vertex set. In the case where  $m = 2$ ,  $m' = 1$



and  $\Phi$  is a type  $A$  root system, this means that the noncrossing hypertree complex contains a simplicial associahedron as a subcomplex.

The noncrossing trees that label apartments with a simplicial associahedral structure are those whose simplex is as small as possible. If the tree chamber labeled by a noncrossing tree  $\tau$  contains only a single partition chamber, it is because the hyperedge poset  $\text{Poset}(\tau)$  has a unique linear extension, which in turn means that  $\text{Poset}(\tau)$  is already a linear ordering. Two extreme cases with this property are *stars* where all edges in  $\tau$  share a common vertex, and *border trees* that consist of all but one edge of the boundary cycle of the underlying polygon. The full set of trees with a linear hyperedge poset can be characterized as those whose structure is that of a caterpillar.

**Definition 11.6** (Caterpillars). A noncrossing tree  $\tau$  is called a *caterpillar* if the set of edges in  $\tau$  that live in the boundary of the polygon form a connected subtree. The name comes from viewing the edges in the boundary path as its *backbone* and the other edges are its *legs*. Stars and border trees are the extreme cases where the caterpillar has as many or as few legs as possible. For every caterpillar and for every  $i$ , the noncrossing partition of the hyperforest formed by the first  $i$  edges of  $\tau$  in the unique proper ordering of  $\tau$ , has a single nontrivial block. In addition, these blocks are nested so that caterpillars correspond to the maximal chains in image of the annular poset formed by the basic hypertrees (see the example shown in Figure 6) when it is interpreted as a subposet of the noncrossing partition lattice using only the lower hyperedge of each basic hypertree. The number of such maximal chains, and thus the number of caterpillars is  $(n + 1)2^{n-1}$ . To see this pick a starting point in the bottom level and then move to the left or to the right at successive step.

In this language Theorem D asserts that the apartment of a caterpillar is a simplicial associahedron.

**Theorem D** (Associahedra). *Let  $\tau$  be a noncrossing tree. If the tree chamber labeled  $\tau$  consists of a single partition chamber, then the apartment labeled  $\tau$  is a simplicial associahedron. In addition, the variety of simplicial associahedra produced in this way include all of the simplicial associahedra that are normal fans to the type  $A$  simple associahedra constructed by Hohlweg and Lange.*

*Proof sketch.* <sup>4</sup> Let  $\tau$  be a caterpillar and number the vertices of the polygon as follows. Label the endpoint of the unique minimal edge in  $\text{Poset}(\tau)$  that is a leaf in  $\tau$  with the number 0, and then label the unique vertex that belongs to the nontrivial block at stage  $i$  but not at stage  $i - 1$  with the number  $i$ . This numbering corresponds to the vertex numbering used by Hohlweg and Lange in [HL07] to construct their various realizations of the classical type  $A$  simple associahedron. The normal fans of these simple associahedra are the  $c$ -Cambrian fans of Reading, which have the desired structure of a simplicial associahedron [Rea06]. The identifications of partition chambers made by Reading ultimately correspond to the identifications of partitions chambers into tree chambers made here.  $\square$

Now that these foundational results are in place, there are many avenues for future research that might be pursued. One obvious route is to try and extend these results to the other types of finite Coxeter groups, and another would be to re-examine earlier work relating noncrossing partitions and associahedra to see whether the arguments can be recast as processes that take place completely within the noncrossing hypertree complex and/or the noncrossing partition link. Both the bijective maps between associahedra and noncrossing partitions and the type-free proofs of the lattice property might benefit from this type of re-examination.

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## REFERENCES

- [AT06] Christos A. Athanasiadis and Eleni Tzanaki, *On the enumeration of positive cells in generalized cluster complexes and Catalan hyperplane arrangements*, J. Algebraic Combin. **23** (2006), no. 4, 355–375. MR 2236611 (2007c:20095)
- [AT08] ———, *Shellability and higher Cohen-Macaulay connectivity of generalized cluster complexes*, Israel J. Math. **167** (2008), 177–191. MR 2448023 (2009h:05214)

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<sup>4</sup>In a certain sense, the proof of this theorem is a matter of chasing definitions through the literature and the portion currently included here is merely a sketch of the main ideas. The final version of this article will contain more details.

- [Bes03] David Bessis, *The dual braid monoid*, Ann. Sci. École Norm. Sup. (4) **36** (2003), no. 5, 647–683. MR 2032983 (2004m:20071)
- [BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 1744486
- [BM10] Tom Brady and Jon McCammond, *Braids, posets and orthoschemes*, Algebr. Geom. Topol. **10** (2010), no. 4, 2277–2314. MR 2745672 (2012e:06003)
- [BMMM01] Noel Brady, Jon McCammond, John Meier, and Andy Miller, *The pure symmetric automorphisms of a free group form a duality group*, J. Algebra **246** (2001), no. 2, 881–896. MR 1872129 (2002i:20036)
- [BMN] G. Berkolaiko, J.M. Morrison, and M. Novaes, *On inequivalent factorizations of a cycle*, Available at [arXiv:0809.3476](https://arxiv.org/abs/0809.3476) [math.CO].
- [BW02] Thomas Brady and Colum Watt,  *$K(\pi, 1)$ 's for Artin groups of finite type*, Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I (Haifa, 2000), vol. 94, 2002, pp. 225–250. MR 1950880 (2004i:20066)
- [Cha07] F. Chapoton, *Hyperarbres, arbres enracinés et partitions pointées*, Homology, Homotopy Appl. **9** (2007), no. 1, 193–212. MR 2280292 (2007k:05225)
- [CY06] William Y. C. Chen and Sherry H. F. Yan, *Noncrossing trees and noncrossing graphs*, Electron. J. Combin. **13** (2006), no. 1, Note 12, 8 pp. (electronic). MR 2240755 (2007f:05040)
- [DL15] Rosena R. X. Du and Fu Liu, *Factorizations of cycles and multi-noded rooted trees*, Graphs Combin. **31** (2015), no. 3, 551–575. MR 3338018
- [DN02] Emeric Deutsch and Marc Noy, *Statistics on non-crossing trees*, Discrete Math. **254** (2002), no. 1-3, 75–87. MR 1909861 (2003g:68095)
- [FKT13] Susanna Fishel, Myrto Kallipoliti, and Eleni Tzanaki, *Facets of the generalized cluster complex and regions in the extended Catalan arrangement of type A*, Electron. J. Combin. **20** (2013), no. 4, Paper 7, 21. MR 3139392
- [FN99] Philippe Flajolet and Marc Noy, *Analytic combinatorics of non-crossing configurations*, Discrete Math. **204** (1999), no. 1-3, 203–229. MR 1691870 (2000c:05012)
- [FR05] Sergey Fomin and Nathan Reading, *Generalized cluster complexes and Coxeter combinatorics*, Int. Math. Res. Not. (2005), no. 44, 2709–2757. MR 2181310 (2006g:05230)
- [GJ92] I. P. Goulden and D. M. Jackson, *The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group*, European J. Combin. **13** (1992), no. 5, 357–365. MR 1181077 (93g:05148)
- [GP93] I. P. Goulden and S. Pepper, *Labelled trees and factorizations of a cycle into transpositions*, Discrete Math. **113** (1993), no. 1-3, 263–268. MR 1212884 (93k:05044)
- [HKS] Thomas Haettel, Dawid Kielak, and Petra Schwer, *The 6-strand braid group is CAT(0)*, Available at [arXiv:1304.5990](https://arxiv.org/abs/1304.5990) [math.GR].

- [HL07] Christophe Hohlweg and Carsten E. M. C. Lange, *Realizations of the associahedron and cyclohedron*, Discrete Comput. Geom. **37** (2007), no. 4, 517–543. MR 2321739 (2008g:52021)
- [Hou03] David S. Hough, *Descents in noncrossing trees*, Electron. J. Combin. **10** (2003), Note 13, 5 pp. (electronic). MR 2014541 (2004i:05009)
- [Irv09] John Irving, *Minimal transitive factorizations of permutations into cycles*, Canad. J. Math. **61** (2009), no. 5, 1092–1117. MR 2554234 (2011d:05014)
- [JMM06] Craig Jensen, Jon McCammond, and John Meier, *The integral cohomology of the group of loops*, Geom. Topol. **10** (2006), 759–784. MR 2240905 (2007c:20121)
- [JMM07] ———, *The Euler characteristic of the Whitehead automorphism group of a free product*, Trans. Amer. Math. Soc. **359** (2007), no. 6, 2577–2595. MR 2286046 (2007k:57003)
- [Kal99] Louis H. Kalikow, *Enumeration of parking functions, allowable permutation pairs, and labeled trees*, ProQuest LLC, Ann Arbor, MI, 1999, Thesis (Ph.D.)—Brandeis University. MR 2699150
- [Knu05] Donald E. Knuth, *The art of computer programming. Vol. 4, Fasc. 3*, Addison-Wesley, Upper Saddle River, NJ, 2005, Generating all combinations and partitions. MR 2251472 (2007f:68004c)
- [Kra06] Christian Krattenthaler, *The  $F$ -triangle of the generalised cluster complex*, Topics in discrete mathematics, Algorithms Combin., vol. 26, Springer, Berlin, 2006, pp. 93–126. MR 2249265 (2007g:05194)
- [LP14] Lun Lv and Sabrina X. M. Pang, *A decomposition algorithm for noncrossing trees*, Electron. J. Combin. **21** (2014), no. 1, Paper 1.5, 14. MR 3177500
- [MM96] Darryl McCullough and Andy Miller, *Symmetric automorphisms of free products*, Mem. Amer. Math. Soc. **122** (1996), no. 582, viii+97. MR 1329943 (96k:20069)
- [MM04] Jon McCammond and John Meier, *The hypertree poset and the  $l^2$ -Betti numbers of the motion group of the trivial link*, Math. Ann. **328** (2004), no. 4, 633–652. MR 2047644 (2005b:20104)
- [Noy98] Marc Noy, *Enumeration of noncrossing trees on a circle*, Proceedings of the 7th Conference on Formal Power Series and Algebraic Combinatorics (Noisy-le-Grand, 1995), vol. 180, 1998, pp. 301–313. MR 1603749 (98j:05080)
- [Oge13] Bérénice Oger, *Action of the symmetric groups on the homology of the hypertree posets*, J. Algebraic Combin. **38** (2013), no. 4, 915–945. MR 3119365
- [Pan03] Alois Panholzer, *The height distribution of nodes in non-crossing trees*, Ars Combin. **69** (2003), 19–32. MR 2007638 (2004f:05084)
- [Pig12] Adam Piggott, *The symmetries of McCullough-Miller space*, Algebra Discrete Math. **14** (2012), no. 2, 239–266. MR 3099973
- [PP02] Alois Panholzer and Helmut Prodinger, *Bijections for ternary trees and non-crossing trees*, Discrete Math. **250** (2002), no. 1-3, 181–195. MR 1905973 (2003c:05056)
- [Rea06] Nathan Reading, *Cambrian lattices*, Adv. Math. **205** (2006), no. 2, 313–353. MR 2258260 (2007g:05195)

- [Rea12] ———, *Coarsening polyhedral complexes*, Proc. Amer. Math. Soc. **140** (2012), no. 10, 3593–3605. MR 2929028
- [Rip11] Vivien Ripoll, *Submaximal factorizations of a Coxeter element in complex reflection groups*, 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011), Discrete Math. Theor. Comput. Sci. Proc., AO, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2011, pp. 813–824. MR 2820762 (2012j:05463)
- [Rip12] ———, *Lyashko-Looijenga morphisms and submaximal factorizations of a Coxeter element*, J. Algebraic Combin. **36** (2012), no. 4, 649–673. MR 2984160
- [RS09] Nathan Reading and David E. Speyer, *Cambrian fans*, J. Eur. Math. Soc. (JEMS) **11** (2009), no. 2, 407–447. MR 2486939 (2011a:20102)
- [Sai07] Kyoji Saito, *Principal  $\Gamma$ -cone for a tree*, Adv. Math. **212** (2007), no. 2, 645–668. MR 2329315 (2008f:52017)
- [Slo] N.J.A. Sloane, *The Online Encyclopedia of Integer Sequences*, Available at <https://oeis.org>.
- [Sta12] Richard P. Stanley, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012. MR 2868112
- [SW09] Yidong Sun and Zhiping Wang, *String pattern avoidance in generalized non-crossing trees*, Discrete Math. Theor. Comput. Sci. **11** (2009), no. 1, 79–93. MR 2513875 (2010f:05044)
- [Tza06] Eleni Tzanaki, *Polygon dissections and some generalizations of cluster complexes*, J. Combin. Theory Ser. A **113** (2006), no. 6, 1189–1198. MR 2244140 (2007c:05201)
- [Tza08] ———, *Faces of generalized cluster complexes and noncrossing partitions*, SIAM J. Discrete Math. **22** (2008), no. 1, 15–30. MR 2383226 (2009f:20059)
- [War98] David Michael Warme, *Spanning trees in hypergraphs with applications to Steiner trees*, ProQuest LLC, Ann Arbor, MI, 1998, Thesis (Ph.D.)—University of Virginia. MR 2697775