$(an+b)$-COLOR COMPOSITIONS

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Abstract. For $a, b \in \mathbb{N}_0$, we consider $(an+b)$-color compositions of a positive integer ν for which each part of size n admits $an+b$ colors. We study these compositions from the enumerative point of view and give a formula for the number of $(an+b)$-color compositions of ν with k parts. Our formula is obtained in two different ways: 1) by means of algebraic properties of partial Bell polynomials, and 2) through a bijection to a certain family of weak compositions that we call *domino compositions*. We also discuss two cases when b is negative and give corresponding combinatorial interpretations.

1. Introduction

A composition of a positive integer ν with k parts is an ordered k-tuple $(j_1, \dots, j_k)$ of positive integers called parts such that $j_1 + \cdots + j_k = \nu$.

Given a sequence of nonnegative integers $w = (w_n)_{n\in\mathbb{N}}$, we define a w-color composition of ν to be a composition of ν such that part n can take on w_n colors. If $w_n = 0$, it means that we do not use the integer n in the composition. Such colored compositions have been considered by many authors and continue to be of current interest. For a comprehensive account on the subject, we refer to the book by S. Heubach and T. Mansour [2].

If we let W_n be the number of w-color compositions of n, Moser and Whitney [4] observed that the generating functions $w(t) = \sum_{n=1}^\infty w_n t^n$ and $W(t) = \sum_{n=1}^\infty W_n t^n$ satisfy the relation $W(t) = \frac{w(t)}{1-w(t)}$, which means that the sequence $(W_n)_{n\in\mathbb{N}}$ is the invert transform of $(w_n)_{n\in\mathbb{N}}$.

In this paper, we consider the sequence of colors $w_n = an + b$ for $n \ge 1$, with $a, b \in \mathbb{N}_0$. Thus $w(t) = \sum_{n=1}^\infty (an+b)t^n$, and we have

$$w(t) = \sum_{n=1}^{\infty} (an+b)t^n = \frac{at}{(1-t)^2} + \frac{bt}{1-t} = \frac{(a+b)t - bt^2}{(1-t)^2}.$$

Therefore, $W(t) = \dfrac{w(t)}{1-w(t)} = \dfrac{(a+b)t - bt^2}{1-(a+b+2)t + (b+1)t^2}$, and so the number W_ν of $(an+b)$-color compositions of ν satisfies the recurrence relation

$$W_\nu = (a+b+2)W_{\nu-1} - (b+1)W_{\nu-2} \quad \text{for } \nu > 2, \tag{1.1}$$

with initial conditions $W_1 = a + b$ and $W_2 = (a+b)^2 + (2a+b)$.

2. Colored compositions with k parts

Let $c_{n,k}(w)$ be the number of w-color compositions of n with exactly k parts. In [3], Hoggatt and Lind derived the formula

$$c_{n,k}(w) = \sum_{\pi_k(n)} \frac{k!}{k_1!\cdots k_n!}\, w_1^{k_1}\cdots w_n^{k_n}, \tag{2.1}$$

where the sum runs over all k-part partitions of n, i.e. over all solutions of

$$k_1 + 2k_2 + \cdots + nk_n = n \text{ such that } k_1 + \cdots + k_n = k$$

with $k_j \in \mathbb{N}_0$ for all j. Observe that the right-hand side of (2.1) is precisely the (n,k)-th partial Bell polynomial $B_{n,k}(1!w_1, 2!w_2, \dots)$ multiplied by the factor $k!/n!$. Thus (2.1) may be written as

$$c_{n,k}(w) = \frac{k!}{n!} B_{n,k}(1!w_1, 2!w_2, \dots), \tag{2.2}$$

and the total number of such compositions of n is $W_n = \sum_{k=1}^n c_{n,k}(w)$.

Proposition 1. *Let $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ be sequences of non-negative integers, and let $a, b \in \mathbb{Z}$. Letting $c_{0,0}(w) = 1$ and $c_{m,j}(w) = 0$ for $m < j$, we have*

$$c_{n,k}(ax + by) = \sum_{m=0}^{n}\sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} c_{m,j}(x) c_{n-m,k-j}(y).$$

Proof. Since $c_{n,k}(w) = \frac{k!}{n!} B_{n,k}(1!w_1, 2!w_2, \dots)$, we can use basic properties of the partial Bell polynomials (see e.g. [1, Sec. 3.3]) together with the notation $!w = (n!w_n)$ to get

$$\begin{aligned}
c_{n,k}(ax+by) &= \frac{k!}{n!} B_{n,k}(!(ax+by)) \\
&= \frac{k!}{n!} \sum_{m=0}^{n}\sum_{j=0}^{k} \binom{n}{m} B_{m,j}(!(ax)) B_{n-m,k-j}(!(by)) \\
&= \frac{k!}{n!} \sum_{m=0}^{n}\sum_{j=0}^{k} \binom{n}{m} a^j B_{m,j}(!x) b^{k-j} B_{n-m,k-j}(!y) \\
&= \frac{k!}{n!} \sum_{m=0}^{n}\sum_{j=0}^{k} \binom{n}{m} a^j b^{k-j} \frac{m!}{j!} c_{m,j}(x) \frac{(n-m)!}{(k-j)!} c_{n-m,k-j}(y) \\
&= \sum_{m=0}^{n}\sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} c_{m,j}(x) c_{n-m,k-j}(y).
\end{aligned}$$

□

Theorem 2. *The number of $(an+b)$-color compositions of ν with k parts is given by*

$$c_{\nu,k}(an+b) = \sum_{j=0}^{k} a^j b^{k-j} \binom{k}{j}\binom{\nu+j-1}{\nu-k}.$$

Thus the total number of $(an+b)$-color compositions of ν is

$$W_\nu = \sum_{k=1}^{\nu}\sum_{j=0}^{k} a^j b^{k-j} \binom{k}{j}\binom{\nu+j-1}{\nu-k}. \tag{2.3}$$

Proof. We use the above proposition with the sequences $x_n = n$ and $y_n = 1$. Then

$$\begin{aligned}
c_{\nu,k}(an+b) &= \sum_{m=0}^{\nu}\sum_{j=0}^{k}\binom{k}{j} a^j b^{k-j} c_{m,j}(n) c_{\nu-m,k-j}(1)\\
&= \sum_{j=0}^{k} a^j b^{k-j}\binom{k}{j}\sum_{m=j}^{\nu}\binom{m+j-1}{m-j}\binom{\nu-m-1}{k-j-1}\\
&= \sum_{j=0}^{k} a^j b^{k-j}\binom{k}{j}\sum_{\ell=0}^{\nu-j}\binom{\ell+2j-1}{\ell}\binom{\nu-\ell-j-1}{k-j-1}\\
&= \sum_{j=0}^{k} a^j b^{k-j}\binom{k}{j}\sum_{\ell=0}^{\nu-k}\binom{\ell+2j-1}{\ell}\binom{\nu-\ell-j-1}{\nu-k-\ell}\\
&= \sum_{j=0}^{k} a^j b^{k-j}\binom{k}{j}(-1)^{\nu-k}\sum_{\ell=0}^{\nu-k}\binom{-2j}{\ell}\binom{j-k}{\nu-k-\ell}\\
&= \sum_{j=0}^{k} a^j b^{k-j}\binom{k}{j}(-1)^{\nu-k}\binom{-j-k}{\nu-k}\\
&= \sum_{j=0}^{k} a^j b^{k-j}\binom{k}{j}\binom{\nu+j-1}{\nu-k}.
\end{aligned}$$

□

Example 3. For some values of a and b, (2.3) gives nice formulas for the following sequences, listed in the OEIS [5]:

Compositions	Sequence	Compositions	Sequence
n-color	A001906	$(2n-1)$-color	A003946
$(n+1)$-color	A003480	$2n$-color	A052530
$(n+2)$-color	A010903	$(2n+1)$-color	A060801
$(n+3)$-color	A010908	$(3n-1)$-color	A055841

3. Combinatorial interpretation: Domino compositions

In this section, an *n-domino* is a tile of the form

$$\boxed{\alpha \mid \beta} \quad \text{with } 0 < \alpha \le n \text{ and } 0 \le \beta \le n. \tag{3.1}$$

A domino with $\beta = 0$ will be called a *zero n-domino*. An n-domino composition of ν is a weak composition of ν using n-dominos of the form (3.1). For example,

$$\boxed{1 \mid 1}\,\boxed{4 \mid 0}\,\boxed{1 \mid 3}$$

is a domino composition of 10 with 4-dominos corresponding to the weak composition $(1, 1, 4, 0, 1, 3)$.

Definition 4. For $a, b \in \mathbb{N}_0$, $n, k \in \mathbb{N}$, and $j \le k$, let $T_j^{a,b}(n, k)$ be the set of n-domino compositions of $n + j$ with j nonzero n-dominos, available in a different colors, and $k - j$ zero n-dominos available in b different colors. Let $T^{a,b}(n, k) = \bigcup_{j=0}^{n} T_j^{a,b}(n, k)$.

Lemma 5.

$$\left|T_j^{a,b}(n, k)\right| = a^j b^{k-j} \binom{k}{j} \binom{n + j - 1}{n - k}.$$

Proof. Having k dominos, there are $\binom{k}{j}$ ways to choose the j nonzero dominos. Once the dominos are chosen, there are $2j + (k - j) = k + j$ spaces to place positive numbers whose sum is $n + j$. These are compositions of $n + j$ with $k + j$ parts and there are $\binom{n+j-1}{k+j-1} = \binom{n+j-1}{n-k}$ of them. Since the nonzero dominos come in a colors and the zero dominos in b colors, we need to multiply by $a^j b^{k-j}$ to account for all of the possibilities. □

Theorem 6. *For any given $a, b \in \mathbb{N}_0$, there is a bijection φ between $T^{a,b}(\nu, k)$ and the set of $(an + b)$-color compositions of ν with k parts.*

Proof. We start by discussing the case when $a = 1$. Let $(D_1, \dots, D_k)$ be an element of $T^{1,b}(\nu, k)$ with j nonzero ν-dominos. For a nonzero domino D, we define $\varphi(D)$ by

$$\boxed{\alpha \mid \beta} \longrightarrow (\alpha + \beta - 1)_\beta,$$

where the notation $(i)_\ell$ means part i with color ℓ. For a zero domino D with color $\delta \le b$, we define $\varphi(D)$ by

$$\boxed{\alpha \mid 0}_\delta \longrightarrow (\alpha)_\ell, \text{ where } \ell = \alpha + \delta.$$

If we denote the nonzero dominos by $(\alpha_1, \beta_1), \dots, (\alpha_j, \beta_j)$, and the zero dominos by $(\alpha_{j+1}, 0), \dots, (\alpha_k, 0)$, then by definition $(\alpha_1 + \beta_1) + \cdots + (\alpha_j + \beta_j) + \alpha_{j+1} + \cdots + \alpha_k = \nu + j$, and therefore, $(\alpha_1 + \beta_1 - 1) + \cdots + (\alpha_j + \beta_j - 1) + \alpha_{j+1} + \cdots + \alpha_k = \nu$. In other words, $(\varphi(D_1), \dots, \varphi(D_k))$ is an $(n + b)$-color composition of ν with k parts.

Conversely, let $\big((i_1)_{\ell_1},\dots,(i_k)_{\ell_k}\big)$ be an $(n+b)$-color composition of ν such that j of its parts are of the form $(i)_\ell$ with $\ell \le i$. If $(i)_\ell$ is such a part, then we define $\psi((i)_\ell)$ by

$$(i)_\ell \longrightarrow \boxed{\alpha_i \mid \ell}\,, \text{ where } \alpha_i = i-\ell+1,$$

and if part $(i)_\ell$ is such that $\ell = i+\delta > i$, then we define $\psi((i)_\ell)$ by

$$(i)_\ell \longrightarrow \boxed{i \mid 0}_{\,\delta}.$$

In particular, the components of a domino $\psi((i)_\ell)$ add to $i+1$ if $\ell \le i$ or they add to i if $\ell > i$. Since $i_1+\dots+i_k=\nu$, we get that $\big(\psi((i_1)_{\ell_1}),\dots,\psi((i_k)_{\ell_k})\big)$ is a ν-domino composition in $T_j^{1,b}(\nu,k)$. Clearly, ψ is the inverse of φ.

For $a>1$ the argument is similar. In this case, for a nonzero domino $D_\gamma \in T^{a,b}(\nu,k)$ with color $1 \le \gamma \le a$, we define $\varphi(D_\gamma)$ by

$$\boxed{\alpha \mid \beta}_{\,\gamma} \longrightarrow (\alpha+\beta-1)_\ell, \text{ where } \ell = (\alpha+\beta-1)(\gamma-1)+\beta,$$

and for a zero domino D_δ with color $1 \le \delta \le b$, we define $\varphi(D_\delta)$ by

$$\boxed{\alpha \mid 0}_{\,\delta} \longrightarrow (\alpha)_\ell, \text{ where } \ell = a\alpha+\delta.$$

The inverse map is obtained as follows. For a part i with color ℓ, $1 \le \ell \le ai+b$, write $\ell = qi+r$ with $0 < r \le i$ and define a ν-domino as follows:

$$\begin{aligned} &\text{if } q<a: \quad (i)_\ell \longrightarrow \boxed{\alpha_i \mid r}_{\,q+1} \text{ with } \alpha_i = i-r+1,\\ &\text{if } q=a: \quad (i)_\ell \longrightarrow \boxed{i \mid 0}_{\,r}, \end{aligned}$$

where the subscript outside the domino indicates its color. ☐

Example 7. In the context of $(n+2)$-color compositions, we have

$$\begin{array}{lll}
1_1 \leftrightarrow \boxed{1 \mid 1} & 2_1 \leftrightarrow \boxed{2 \mid 1} & 3_1 \leftrightarrow \boxed{3 \mid 1}\\
1_2 \leftrightarrow \boxed{1 \mid 0}_{\,1} & 2_2 \leftrightarrow \boxed{1 \mid 2} & 3_2 \leftrightarrow \boxed{2 \mid 2}\\
1_3 \leftrightarrow \boxed{1 \mid 0}_{\,2} & 2_3 \leftrightarrow \boxed{2 \mid 0}_{\,1} & 3_3 \leftrightarrow \boxed{1 \mid 3}\\
 & 2_4 \leftrightarrow \boxed{2 \mid 0}_{\,2} & 3_4 \leftrightarrow \boxed{3 \mid 0}_{\,1}\\
 & & 3_5 \leftrightarrow \boxed{3 \mid 0}_{\,2}
\end{array}$$

For example, the composition $(3_5, 1_2, 3_2)$ of 7 corresponds to

$$\boxed{3 \mid 0}_{\,2}\ \boxed{1 \mid 0}_{\,1}\ \boxed{2 \mid 2}.$$

As a direct consequence of Theorem 6 and Lemma 5, we obtain:

Corollary 8. *The number of $(an+b)$-color compositions of ν with k parts is given by*

$$c_{\nu,k}(an+b)=\sum_{j=0}^{k}a^j b^{k-j}\binom{k}{j}\binom{\nu+j-1}{\nu-k}.$$

4. Other examples

We finish with two examples related to $(n-1)$-color and $(n-2)$-color compositions.

Example 9 $(a=1,\, b=-1)$**.** In this case, we have that

$$c_{\nu,k}(n-1)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{\nu+j-1}{\nu-k}$$

is the number of compositions of ν with k parts with no part 1 and such that each part $i>1$ may be colored in $i-1$ different ways. This is also the number of n-color compositions of ν with k parts and no color 1.

Example 10 $(a=1,\, b=-2)$**.** Let $\mathscr{C}_{n-2}(\nu,k)$ be the set of compositions of ν with k parts such that:

- there is no part 2
- each part $i>2$ maybe colored in $i-2$ different ways.

If $\mathscr{C}_{n-2}^{1,\text{even}}(\nu,k)$ denotes the set of compositions in $\mathscr{C}_{n-2}(\nu,k)$ with an even number of 1's, and $\mathscr{C}_{n-2}^{1,\text{odd}}(\nu,k)$ is the set of compositions with an odd number of 1's, then we have

$$c_{\nu,k}(n-2)=\left|\mathscr{C}_{n-2}^{1,\text{even}}(\nu,k)\right|-\left|\mathscr{C}_{n-2}^{1,\text{odd}}(\nu,k)\right|,$$

which implies

$$c_{\nu,\nu}(n-2)=(-1)^{\nu},$$
$$c_{\nu,k}(n-2)=\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}\binom{\nu-k-1}{2j-1}\ \text{ for } k<\nu.$$

Moreover, by (1.1), the sequence defined by $W_\nu=\sum_{k=1}^{\nu}c_{\nu,k}(n-2)$ satisfies the recurrence relation

$$W_1=-1,\quad W_2=1,$$
$$W_\nu=W_{\nu-1}+W_{\nu-2}\ \text{ for } \nu>2.$$

In other words, W_ν is the Fibonacci number $F_{\nu-3}$ and we get the identity

$$F_{\nu-3}=\sum_{k=1}^{\nu}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}\binom{\nu-k-1}{2j-1}.$$

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References

[1] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, D. Reidel Publishing Co., Dordrecht, 1974.

[2] S. Heubach and T. Mansour, *Combinatorics of Compositions and Words*, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2010.

[3] V. E. Hoggatt, D. A. Lind, Fibonacci and binomial properties of weighted compositions, *J. Combin. Theory* **4** (1968), 121–124.

[4] L. Moser and E. L. Whitney, Weighted compositions, *Canad. Math. Bull.* **4** (1961), 39–43.

[5] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

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