FROM DYCK PATHS TO STANDARD YOUNG TABLEAUX

JUAN B. GIL, PETER R. W. MCNAMARA, JORDAN O. TIRRELL, AND MICHAEL D. WEINER

Abstract. The number of Dyck paths of semilength n is certainly not equal to the number of standard Young tableaux (SYT) with n boxes. We investigate several ways to add structure or restrict these sets so as to obtain equinumerous sets. Our most sophisticated bijective proof starts with Dyck paths whose k-ascents for $k > 1$ are labeled by connected matchings on $[k]$ and arrives at SYT with at most $2k - 1$ rows. Along the way, this bijection visits k-noncrossing and k-nonnesting partial matchings, oscillating tableaux and involutions with decreasing subsequences of length at most $2k-1$. In addition, we present bijections from eight other types of Dyck and Motzkin paths to certain classes of SYT.

1. Introduction

Two classic and well-studied sets in combinatorics are the set of Dyck paths and the set of standard Young tableaux (SYT). These sets are certainly not equinumerous. On the one hand, the number of Dyck paths with semilength n is the Catalan number C_n and, starting at C_0, gives the sequence [19, A000108]:

$$1, 1, 2, 5, 14, 42, 132, 429, \ldots.$$

Asymptotically,

$$C_n \sim \frac{4^n}{n^{\frac{3}{2}}\sqrt{\pi}}$$

by Stirling's formula. On the other hand, the number $\mathrm{SYT}(n)$ of standard Young tableaux with n boxes gives the sequence [19, A000085]:

$$1, 1, 2, 4, 10, 26, 76, 232, \ldots$$

and, asymptotically,

$$\mathrm{SYT}(n) \sim \left(\frac{n}{e}\right)^{\frac{n}{2}} \frac{e^{\sqrt{n}}}{(4e)^{\frac{1}{4}}}.$$

A proof, which also makes use of Stirling's formula, can be found in [11, Theorem 8].

We address the following question: in what ways can we add extra structure or restrictions to either set of objects to yield equinumerous sets? Roughly speaking, our goal is to reconcile the two numbers C_n and $\mathrm{SYT}(n)$.

One well-known way to perform such a reconciliation is to observe that the number of Dyck paths of semilength n equals the number of SYT of shape (n, n). Indeed, to map bijectively from an SYT of shape (n, n) to a Dyck path, locate the numbers in the SYT

2010 *Mathematics Subject Classification.* 05A19 (Primary); 05A18, 05A05 (Secondary).

Key words and phrases. Dyck path, Young tableau, perfect matching, noncrossing partition, nonnesting partition.

Peter McNamara was partially supported by grant #245597 from the Simons Foundation.

in increasing order, and let an appearance on the upper row (resp. lower row) in English notation give an up-step (resp. down-step) on the Dyck path.

Our main reconciliation goes in the other direction of adding extra structure to the Dyck paths, and involves some ubiquitous combinatorial objects, including perfect matchings, noncrossing partitions and nonnesting partitions, as well as their generalizations. Our approach is motivated by the noncrossing partition transform (see Callan [8]), which can be defined in terms of partial Bell polynomials as follows. For a sequence (x_n), define (y_n) by

$$y_0 = 1, \quad y_n = \sum_{k=1}^{n} \frac{1}{(n-k+1)!} B_{n,k}(1!x_1, 2!x_2, \dots) \text{ for } n \geq 1, \tag{1.1}$$

where $B_{n,k}$ denotes the (n,k)-th partial Bell polynomial defined as

$$B_{n,k}(z_1, \dots, z_{n-k+1}) = \sum_{\alpha \in \pi_k(n)} \frac{n!}{\alpha_1! \cdots \alpha_{n-k+1}!} \left(\frac{z_1}{1!}\right)^{\alpha_1} \cdots \left(\frac{z_{n-k+1}}{(n-k+1)!}\right)^{\alpha_{n-k+1}}$$

with $\pi_k(n)$ denoting the set of $(n-k+1)$-part partitions of k such that $\alpha_1 + 2\alpha_2 + \cdots + (n-k+1)\alpha_{n-k+1} = n$. As shown in [3], if (x_n) is a sequence of nonnegative integers, y_n gives the number of Dyck paths of semilength n such that each j-ascent may be labeled in x_j different ways. For example, we use $x_j = 0$ if j-ascents are to be avoided. Recall that an *ascent* of a Dyck path is a maximal consecutive sequence of up-steps, and it is a *j-ascent* if it consists of j up-steps. As expected, if (x_n) is the sequence of ones, then $y_n = C_n$. In general, (y_n) enumerates configurations obtained by adorning the ascents with structures whose elements are counted by (x_n). In this paper, we are interested in a combinatorial structure on Dyck paths that gives a configuration equinumerous to SYT. Rather than just numbers, our labels on the ascents will be combinatorial objects, which we describe in the next paragraph. A similar approach to counting labeled Catalan objects using partial Bell polynomials has been used in [4] for partitions of polygons, in [5] for weighted compositions, in [6] for rational Dyck paths, and the present paper can be seen as a new strand in this program of work.

Let $[n]$ denote the set $\{1, 2, \dots, n\}$. A graph on $[n]$ is a *partial matching* if every vertex has degree at most one. We will also refer to such graphs as *involutions* since they are clearly in bijection with self-inverse permutations of $[n]$. We will call vertices of degree zero *singletons*. A partial matching is a *perfect matching* if every vertex has degree exactly one; note that the existence of a perfect matching implies that n is even. We will represent partial matchings by graphs on the number line with the edges drawn as arcs, with these arcs always drawn above the number line, as in Figure 1.1. A partial matching is a *connected matching* if these arcs together with the n points on the number line form a connected set as a subset of the plane. For example, in Figure 1.1, the matching on the left is connected whereas the matching on the right has four connected components. Note that a partial matching on $[n]$ with $n > 1$ can only be connected if it is a perfect matching. When $n = 1$, we consider its unique partial matching (consisting of no arcs) to be connected.

Definition. A *cm-labeled Dyck path* is a Dyck path where each k-ascent is labeled by a connected matching on $[k]$.

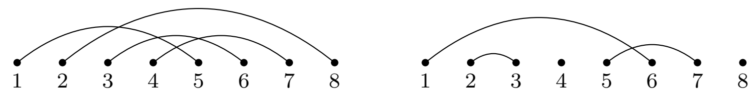
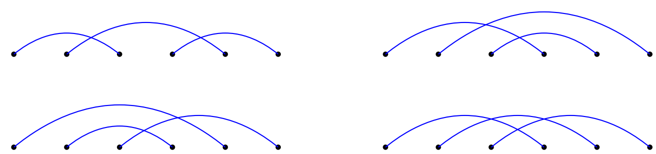


FIGURE 1.1. Two partial matchings, namely (15)(28)(36)(47) and (16)(23)(4)(57)(8).

First note there are no connected matchings on $[k]$ when k is odd and greater than 1, so all the ascents in a cm-labeled Dyck path must be of even length or length 1. Secondly, a cm-labeled Dyck path all of whose ascents are length 1, 2 or 4 is equivalent to its unlabeled version since there is a unique connected matching on $[k]$ when $k = 1, 2, 4$. The first interesting case is when a Dyck path has 6-ascents, because then there are 4 ways to label each 6-ascent:



For an explicit example of a cm-labeled Dyck path, see Figure 1.2.

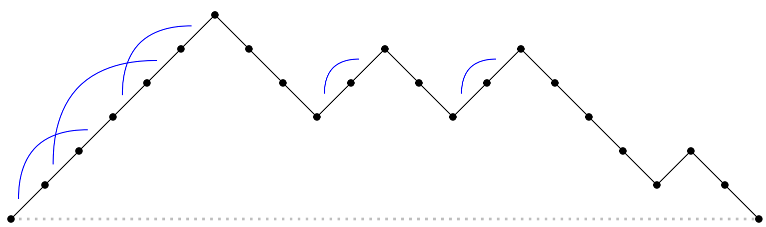


FIGURE 1.2. A cm-labeled Dyck path.

This brings us to our promised reconciliation between Dyck paths and standard Young tableaux.

Theorem 1.1. *The number of cm-labeled Dyck paths of semilength n equals* $\mathrm{SYT}(n)$.

In fact, we will prove a significant refinement of this theorem. Abusing terminology, we will use the term *singleton* for a 1-ascent. In a partial matching, two arcs (i, j) and (k, ℓ) form a *crossing* if $i < k < j < \ell$ or, equivalently, if the arcs cross in the graphical representation of the partial matching. A *k-crossing* is a set of k arcs in a partial matching M that are pairwise crossing, and the *crossing number* of M is the largest k such that M has a k-crossing. A partial matching is k-noncrossing if it has no k-crossings. For example, the partial matching (15)(28)(36)(47) on the left in Figure 1.1 is 4-noncrossing and has crossing number 3 due to the arcs (15)(36)(47).

Our main result is the following theorem, which we will prove bijectively in Section 2.

Theorem 1.2. *The number of cm-labeled Dyck paths of semilength n with s singletons and k-noncrossing labels equals the number of standard Young tableaux with n boxes, s columns of odd length, and at most $2k - 1$ rows.*

Letting k be sufficiently large in Theorem 1.2 and summing over s yields Theorem 1.1.

Example 1.3. The set of cm-labeled Dyck paths of semilength n with 2-noncrossing labels is precisely the set of Dyck paths of semilength n with ascents of length 1 or 2. Since this set is in bijection with the set of Motzkin paths of length n (see [7]), Theorem 1.2 with $k = 2$ gives the known correspondence to SYT with n boxes and at most 3 rows. ([23] includes an enumeration of these SYT; see [13] for a bijection from Motzkin paths to these SYT.)

The two ways we have mentioned so far to reconcile the number of Dyck paths and SYT share the feature that only one of these two sets needs to be modified to obtain a bijection. Specifically, the *full set* of Dyck paths of semilength n is in bijection with the SYT of shape (n, n), while Theorem 1.1 tells us that the *full set* of SYT with n boxes is in bijection with the Dyck paths of semilength n adorned with additional structure. There are further reconciliations obtained by restricting *both* the set of Dyck paths and the set of SYT. For example, the number of Dyck paths of semilength n without singleton ascents equals the number of SYT of "flag shape" with n boxes. Five such special cases with bijective proofs are the subject of Section 3, including one that arises from the Bell polynomial viewpoint.

Further connections between Dyck and Motzkin paths and SYT are examined in Section 4. In particular, we consider the affect of assigning colors to the singletons of a Dyck path. We obtain an elegant expression for the generating function for SYT of height at most $2k-1$ in terms of the generating function for k-noncrossing perfect matchings. We conclude with a result (Proposition 4.1) that shows that three classes of Motzkin paths are in bijection with SYT.

2. cm-Labeled Dyck paths to SYT

The goal of this section is to bijectively prove Theorem 1.2. Our bijection will actually be a composition of a series of bijections as illustrated in Figure 2.1.

The second step involving [10] is not needed if we just want to prove Theorem 1.1, but it is needed to control the number of rows as bounded by $2k-1$ as in Theorem 1.2. The reason, as we will see, is that the Robinson–Schensted–Knuth (RSK) algorithm behaves well with respect to nestings. Nestings are defined for partial matchings in an analogous way to crossings: two arcs (i, j) and (k, ℓ) form a *nesting* if $i < k < \ell < j$. A *k-nesting* is a set of k arcs in a partial matching that are pairwise nesting, with the *nesting number* and *k-nonnesting* defined in a way parallel to the analogous terms for crossings. For example, the partial matching (15)(28)(36)(47) on the left in Figure 1.1 has nesting number 2 due to the arcs (28)(36) but is 3-nonnesting.

2.1. Dyck paths to noncrossing partial matchings.

The first component of this bijection is based on one which Callan [8] says "is essentially due to Prodinger [22]," and maps from Dyck paths to noncrossing partitions. Prodinger takes plane trees as his domain, but could equally have worked in the language of Dyck paths (see [28, Theorem 1.5.1]).

Consider a cm-labeled Dyck path of semilength n with k-noncrossing labels. We number the up-steps in the following fashion. First number the down-steps with $[n]$ in increasing order from left-to-right. Then move each such label horizontally to the left until it meets its corresponding up-step, resulting in a labeling on the up-steps. See Figure 2.2 for an example.

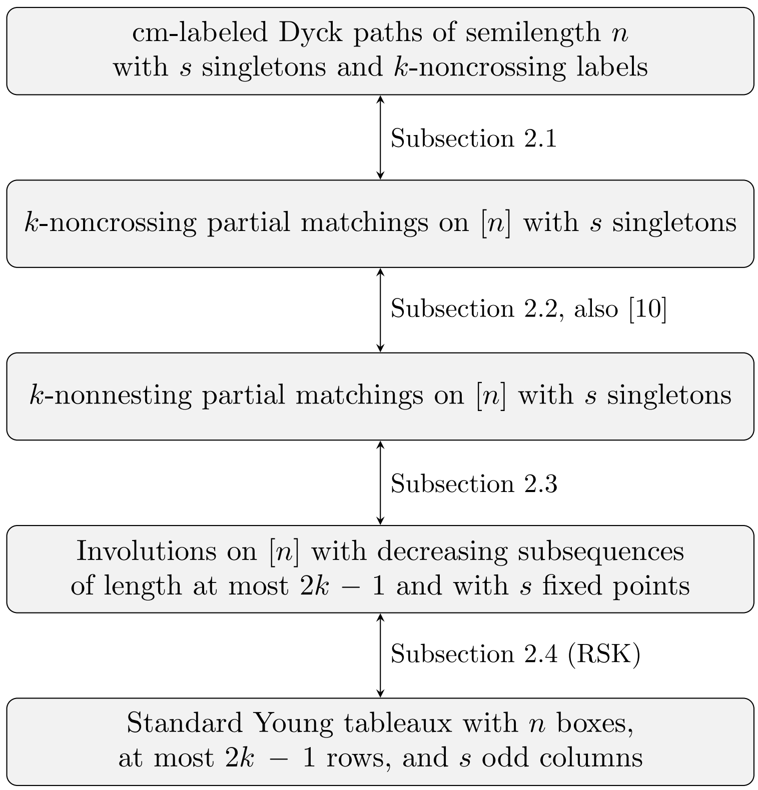


FIGURE 2.1. Structure of the proof of Theorem 1.2.

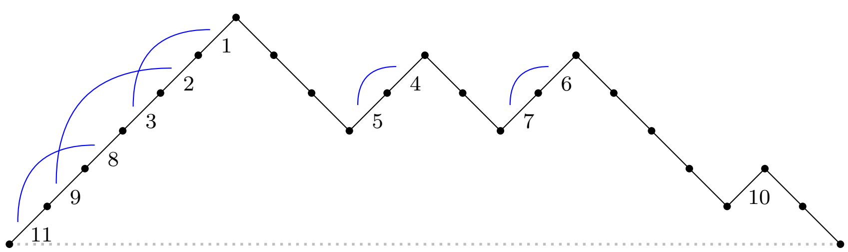


FIGURE 2.2. Numbering the up-steps of a cm-labeled Dyck path.

Our partial matching is then the one inherited from the cm-labeled Dyck path, with the connected matching on each ascent applied to the ascent's numbers. In our example, we get $(1,3)(2,9)(4,5)(6,7)(8,11)(10)$; see Figure 2.3. Since the cm-labels were k-noncrossing, so is the partial matching, and clearly the number of singletons on the Dyck path equals the number of singletons in the matching, as required.

To invert the map, each connected component of the partial matching corresponds to a particular ascent of the Dyck path. The ascent's numerical labels are the numbers of the connected component, and appear on the Dyck path in increasing order from top to bottom, while the ascent's cm-label is inherited directly from the partial matching. We

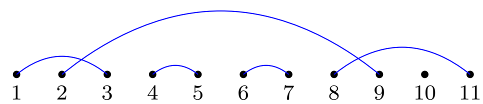


FIGURE 2.3. Partial matching $(1,3)(2,9)(4,5)(6,7)(8,11)(10)$.

order the ascents from left-to-right according to the sizes of their smallest numerical labels i. To complete the inverse map, we need to determine the length of each descent as follows. Let A_i denote the set of numerical labels appearing in the ascent with smallest label i or appearing in any ascent further to the left. The length of the descent following the up-step labeled with i will be the largest j such that $\{i, i+1, \ldots, i+j-1\} \subseteq A_i$. For example, in Figure 2.2 we have $A_6 = \{1,2,3,4,5,6,7,8,9,11\}$, so the descent following the up-step with $i=6$ is of length 4 since the numbers $6,7,8,9$ all appear in A_6 but 10 does not.

2.2. **Noncrossing to nonnesting partial matchings.** This step works by modifying a known bijection between perfect matchings and oscillating tableaux. We follow a technique from [10] by first mapping a partial matching to an oscillating tableau, then transposing the tableau, and then mapping the result back to a partial matching. As we will see, the results of [10] imply that the final partial matching will be k-nonnesting if and only if the initial one is k-noncrossing. Our maps will be constructed so as to preserve the number of singletons.

In fact, we will use a slight variant of an oscillating tableau, which we define next. We will think of a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ in terms of its Young diagram in English notation, meaning an array of left-justified rows of boxes with λ_i boxes in the ith row from the top.

Definition 2.1. A *weakly oscillating tableau* of shape λ and length n is a sequence of partitions $\Lambda = (\lambda^0, \lambda^1, \ldots, \lambda^n)$ such that:

(i) $\lambda^0 = \emptyset$, the empty partition;
(ii) $\lambda^n = \lambda$;
(iii) for $1 \leq i \leq n$, λ^i is obtained from λ^{i-1} by either doing nothing, adding a box, or deleting a box.

We obtain the standard definition of an oscillating tableau by removing the possibility that $\lambda^i = \lambda^{i-1}$. Oscillating tableaux were originally defined by Berele [2]. We restrict our attention to weakly oscillating tableaux of empty shape, i.e., $\lambda^n = \emptyset$.

The bijection we now present between partial matchings and weakly oscillating tableaux is a mild variant of a bijection of Stanley between perfect matchings and oscillating tableau of empty shape, which was extended to arbitrary shapes by Sundaram [31].

Given a partial matching M on $[n]$, represented as a graph on the number line as in Figure 2.4, we construct a sequence of tableaux $T^0, \ldots, T^n$ as follows. The resulting weakly oscillating tableau Λ will be the sequence of shapes $(\lambda^0, \ldots, \lambda^n)$, where λ^j is the shape of T^j. We work from T^n down to T^0, beginning by setting $T^n = \emptyset$, the empty tableau. For $n \geq j \geq 1$, construct T^{j-1} according to the following rules.

(1) If j is a singleton in M, then set $T^{j-1} = T^j$.
(2) If j is the right-hand endpoint of an arc (i,j) in M, then RSK insert[1] i into T^j.

[1] See [24, §3.1] or [30, §7.11] for an introduction to RSK insertion.

j	0	1	2	3	4	5	6	7	8	9
T^j	$\emptyset$	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3\\\hline 2 \\\cline{1-1}\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline 5\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 5\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 5\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 5\\\hline\end{array}$	$\emptyset$
λ^j	$\emptyset$	$\begin{array}{\|c\|}\hline \ \\\hline\end{array}$	$\begin{array}{\|c\|}\hline \ \\\hline \ \\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline \ & \ \\\hline \ \\\cline{1-1}\end{array}$	$\begin{array}{\|c\|}\hline \ \\\hline \ \\\hline\end{array}$	$\begin{array}{\|c\|}\hline \ \\\hline \ \\\hline \ \\\hline\end{array}$	$\begin{array}{\|c\|}\hline \ \\\hline \ \\\hline\end{array}$	$\begin{array}{\|c\|}\hline \ \\\hline\end{array}$	$\begin{array}{\|c\|}\hline \ \\\hline\end{array}$	$\emptyset$
M^j	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$(3,4)$	$(3,4)$	$(3,4)$ $(1,6)$	$(3,4)$ $(1,6)$ $(2,7)$	$(3,4)$ $(1,6)$ $(2,7)$	$(3,4)$ $(1,6)$ $(2,7)$ $(5,9)$

TABLE 1. The sequences constructed in Example 2.2.

(3) If j is the left-hand endpoint of an arc (j,k) in M, then remove j (and the box that contained j) from T^j.

Example 2.2. Consider the partial matching $(1,6)(2,7)(3,4)(5,9)(8)$ shown in Figure 2.4. The sequence $T^0,\ldots,T^9$ is given on Table 1. Recall that the construction of the T^j proceeds from right to left, and that T^{j-1} is determined by the properties of the number j, rather than of $j-1$. The resulting oscillating tableau is the sequence in the third row of the table, read from left to right.

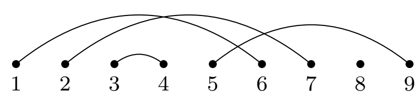


FIGURE 2.4. The partial matching of Example 2.2.

Let us next define the inverse map. Along the way, we will need to reverse an RSK insertion given the location of the new box that resulted from the insertion. The process for reverse insertion can be deduced by carefully inverting the steps of RSK insertion; see the proof of [24, Theorem 3.1.1] or [30, Theorem 7.11.5] for more details.

So suppose we have an oscillating tableau $\Lambda = (\lambda^0,\ldots,\lambda^n)$, and let us construct a sequence $(T^0,M^0),\ldots,(T^n,M^n)$, where T^j is a tableaux of shape λ^j, and M^j is a partial matching on $[n]$ such that M^n will be the result of the inverse map. We begin by setting $(T^0,M^0)=(\emptyset,\emptyset)$ and, for $1\le j\le n$, we construct (T^j,M^j) from left to right according to the following rules.

(1) If $\lambda^j=\lambda^{j-1}$, then set $(T^j,M^j)=(T^{j-1},M^{j-1})$.
(2) If $\lambda^j\subset\lambda^{j-1}$, then obtain T^j from T^{j-1} by reverse RSK insertion, starting with the entry k in the box in position $\lambda^j\setminus\lambda^{j-1}$. This will result in an entry $i\le k$ leaving T^{j-1}. Add the pair (i,j) to M^{j-1} to obtain M^j.
(3) If $\lambda^j\supset\lambda^{j-1}$, let T^j be obtained from T^{j-1} by adding the box $\lambda^j\setminus\lambda^{j-1}$ with entry j, and simply let $M^j=M^{j-1}$.

For an example of the inverse map, see Table 1, where the pairs in M^j are listed vertically for compactness; we see that M^9 is indeed the partial matching given in Figure 2.4.

To deduce that the maps above indeed give a bijection from partial matchings to oscillating tableau, there are two approaches we can take. The first is to observe that each of the three steps in the inverse map invert the same-numbered step in the forward map. The second approach is to refer to [31, Lemma 2.2], which proves the bijection in the case of perfect matchings. To see that this proof extends to partial matchings, we merely need to observe that the procedures are unchanged in the setting of partial matchings except for the addition of the harmless rules numbered (1) which say to change nothing.

With this bijection in place, we now construct the composite bijection from k-noncrossing partial matchings to k-nonnesting partial matchings. Let M be a k-noncrossing matching of $[n]$ with s singletons, which maps to an oscillating tableau $\Lambda = (\lambda^0, \ldots, \lambda^n)$. As we will justify, switching crossings to matchings requires nothing more than transposing/conjugating the partitions to obtain $\Lambda^t := \big((\lambda^0)^t, \ldots, (\lambda^n)^t\big)$. Applying the inverse map from above results in a partial matching, which we denote by M^t.

We leave it as an exercise for the reader to check that for the matching M of Example 2.2, we get $M^t = (1,9)(2,4)(3,7)(5,6)$. Observe that M has a 3-crossing and a 2-nesting, whereas M^t has a 2-crossing and a 3-nesting. Explaining why this is no coincidence is essentially our task for the remainder of this subsection.

FIGURE 2.5. The partial matchings M and M^t for Example 2.2.

First, though, let us observe that the number of singletons is the same for M and M^t. Indeed, suppose j is a singleton in M, which is exactly the condition that implies $T^{j-1} = T^j$, and equivalently $\lambda^{j-1} = \lambda^j$ and $(\lambda^{j-1})^t = (\lambda^j)^t$. Letting $(T^t)^j$ denote the tableau corresponding to $(\lambda^j)^t$ in the inverse map, we have that $(\lambda^{j-1})^t = (\lambda^j)^t$ if and only if $(T^t)^{j-1} = (T^t)^j$, which is exactly the requirement for j to be absent from the pairs of M^t and hence be a singleton in M^t.

To complete this subsection it remains to show that M is k-noncrossing if and only if M^t is k-nonnesting. The main results of [10] work in the more general setting of set partitions rather than matchings, and objects known as *vacillating tableaux* are the appropriate replacement for oscillating tableaux. However, as Chen et al. remark in their Section 5, their results can be restricted to the case of perfect matchings and oscillating tableaux. As before, singletons have no harmful effects on the bijections involved, so their results also apply to partial matchings and weakly oscillating tableaux. In this setting, [10, Theorem 3.2] states that for a partial matching M with corresponding weakly oscillating tableaux $\Lambda = (\lambda^0, \ldots, \lambda^n)$, the crossing (resp. nesting) number of M is the largest number of rows (resp. columns) in any λ^i. Thus the crossing number of M equals the nesting number of M^t, as required.

2.3. **Nonnesting partial matchings to involutions.** In this subsection, we justify the third bijection from the diagram given in Figure 2.1.

Partial matchings M of $[n]$ are clearly in bijection with involutions π of $[n]$, with the arc (i, j) corresponding to the transposition $(i\,j)$. As a working example, the partial matching in Figure 2.4 gives the permutation $(16)(27)(34)(59)(8) = 674391285$. The number of singletons in M clearly equals the number of fixed points of π. Observe that a k-nesting in M results in a decreasing subsequence of length $2k$ in π, and if the k-nesting additionally has a singleton under its middle arc, then π will have a decreasing subsequence of length $2k + 1$. Conversely, and less obviously, decreasing subsequences of maximum length in π give rise to k-nestings as stated precisely in Lemma 2.3 below. In our example, the nesting $(1, 6)(3, 4)$ corresponds to the decreasing subsequence 6431, while $(5, 9)(8)$ corresponds to the decreasing subsequence 985. Once we prove Lemma 2.3, we get the desired conclusion for this subsection: k-nonnesting partial matchings M of $[n]$ with s singletons are in bijection with involutions π of $[n]$ with maximal decreasing subsequence of length at most $2k-1$ and with s fixed points.

A stronger version of the result below was given by Post [21, Theorem 5.2]; a remark that follows Post's proof notes the existence of the method we use.

Lemma 2.3. *Suppose M is a partial matching of $[n]$ that corresponds to the involution $\pi = (\pi_1, \ldots, \pi_n)$ when the arcs of M are treated as transpositions in π. If the longest decreasing subsequence of π has length m then either:*

(1) *$m = 2k$ is even, and M has a k-nesting, or*
(2) *$m = 2k + 1$ is odd, and M has a k-nesting and a singleton that lies underneath the middle arc in the nesting.*

Proof. It will be helpful to represent π in an n-by-n grid in the customary fashion, with each $\pi_i = j$ corresponding to a dot in the ith column and jth row from the bottom. See Figure 2.6 for the diagram of the involution 674391285 of Figure 2.4.

There are three key observations to make about such diagrams. The first is that, since π is an involution, the positions of the dots are symmetric about the diagonal shown in the figure. Secondly, a decreasing subsequence of π is represented by a subset of dots of decreasing height from left to right. Thirdly, again since π is an involution, the dots above the diagonal can also be interpreted as the pairs of the partial matching M: the dot in column i and row j with $i < j$ denotes the arc (i, j). The singletons of M are clearly given by the dots along the diagonal. Under this interpretation, the k-nestings of M are then the subsets of k dots, all above the diagonal, that decrease in height from left to right. For example, the dots at (1,6) and (3,4) give one of the 2-nestings of Figure 2.4.

Let $D = (\pi_{i_1}, \ldots, \pi_{i_m})$ be a decreasing subsequence of maximum length m in π and suppose first that $m = 2k$. In the diagram of π suppose there are ℓ dots coming from D that appear above the diagonal. If $\ell = k$, then this immediately yields a k-nesting in M. For example, if $D = 7431$ in our running example, we obtain the nesting $(2, 7)(3, 4)$. If $\ell > k$, then we can produce a decreasing subsequence of length 2ℓ by taking these ℓ dots along with their reflective images across the diagonal. This yields a contradiction with D having maximum length $m = 2k$. If $\ell < k$, we can consider the $2k - \ell$ dots on or below the diagonal; since D is decreasing, at most one of these is on the diagonal. Taking these $2k - \ell$ dots along with their reflective images above the diagonal yields a decreasing subsequence of length at least $4k - 2\ell - 1$, which leads again to a contradiction.

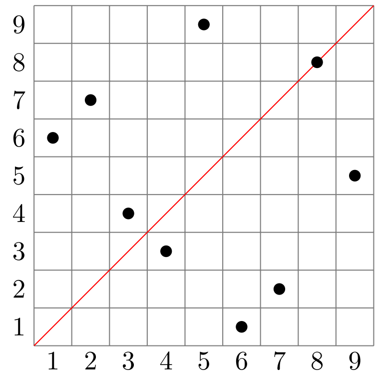


FIGURE 2.6. The diagram of the involution 674391285 = (16)(27)(34)(59)(8)

It remains to consider the odd case $m = 2k + 1$. If there are k dots appearing above the diagonal and one dot appearing on the diagonal, we get the desired conclusion: M has a k-nesting and a singleton that lies underneath the middle arc in the nesting. For the remaining cases, we get a contradiction by arguing as in the previous paragraph. □

2.4. **Involutions to SYT.** The final step of the proof of Theorem 1.2 is to put such involutions π in bijection with SYT T on $[n]$ with at most $2k-1$ rows and with s odd columns. The RSK algorithm bijectively maps a permutation σ of $[n]$ to a pair (P, Q) of SYT each with n boxes and of the same shape λ, and maps σ^{-1} to (Q, P). See [24, Chapter 3] or [30, Chpater 7] for an introduction to the RSK algorithm, and [24, Theorem 3.6.6] or [30, Theorem 7.13.1] for the result of Schützenberger [26] about σ^{-1}. Thus the RSK algorithm restricts to a bijection between involutions π of $[n]$ and SYT T with n boxes.

We finish the proof by direct application of two results from the literature. The first is Schensted's Theorem [25], [24, Theorem 3.3.2], [30, Theorem 7.23.17], which states that the number of rows of T equals the length of the longest decreasing subsequence of π. The second result [1, 27] gives that the number of fixed points in π equals the number of odd columns in T, as desired.

3. DYCK PATHS TO SYT OF HOOK AND FLAG SHAPE

In this section, we consider five bijections that map Dyck paths with certain restrictions to SYT of special shapes. The first two bijections have the same starting point: we define a map ϕ from the set of Dyck paths of semilength n to the set of partitions of $[n]$ as follows. First number the down-steps of the Dyck path with $[n]$ in increasing order from left to right. At each peak UD, label the up-step with the number already assigned to its paired down-step. (Here and elsewhere, U (resp. D) denotes an up-step (resp. down-step).) Working through the ascents from left to right, label the remaining up-steps from top to bottom on each ascent in a greedy fashion. The resulting labeling gives a partition of $[n]$ whose blocks are the labels on the ascents. For example, the path in Figure 3.1 gives the partition 1237-48-5-69. As in Figure 3.1, we will represent such a partition by a tableau-like array where the column entries are increasing from top to bottom and give the blocks of the partition while the top row is also increasing and contains the smallest entry from each block; we will call such an array a *modified tableau.*

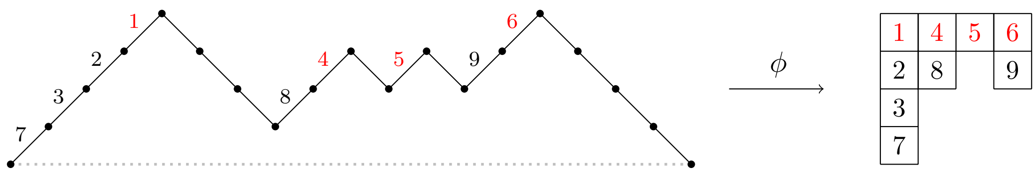


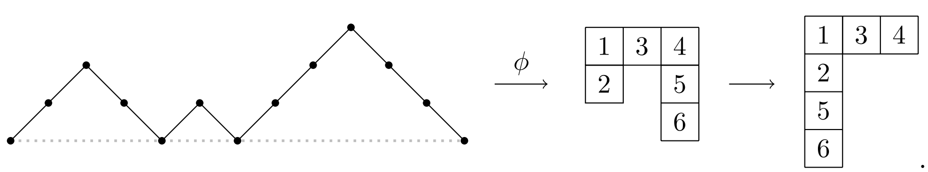
FIGURE 3.1. Example of the map ϕ, where the columns of the array on the right are the blocks of the resulting partition.

The map ϕ is clearly injective, and we will use it to obtain bijections between certain Dyck paths and SYT. Note that the difference of the smallest entries in two consecutive blocks gives the number of down-steps between the corresponding ascents on the path.

3.1. **Hook shapes.** An SYT is said to be of *hook shape* if its shape is $(k, 1^\ell)$ for some k and ℓ, where 1^ℓ denotes a sequence consisting of ℓ copies of 1. A Dyck path of semilength n with k peaks and k returns is a Dyck path of the form $\mathsf{U}^{j_1}\mathsf{D}^{j_1}\cdots\mathsf{U}^{j_k}\mathsf{D}^{j_k}$ with $j_1+\cdots+j_k=n$. There are $\binom{n-1}{k-1}$ such paths, which is the number of compositions of n with k parts as well as the number of SYT of shape $(k,1^{n-k})$. These equal cardinalities suggest a bijection involving the map ϕ.

Proposition 3.1. *For $1 \le k \le n$, Dyck paths of semilength n with k peaks and k returns are in bijection with SYT of shape $(k,1^{n-k})$.*

Proof. Given a Dyck path of the form $\mathsf{U}^{j_1}\mathsf{D}^{j_1}\cdots\mathsf{U}^{j_k}\mathsf{D}^{j_k}$ with $j_1+\cdots+j_k=n$, we apply the map ϕ to get the partition $[1,\dots,j_1][j_1+1,\dots,j_1+j_2]\cdots[n-j_k+1,\dots,n]$, which can be represented as a modified tableau. We then obtain an SYT of hook shape by pushing all the boxes below the first row into the first column. For example,



For the inverse, let $a_1, a_2, \dots, a_k$ be the entries of the first row of a given SYT of shape $(k,1^{n-k})$. Move the boxes that appear below the first row to the unique place such that the modified tableau T has columns with increasing consecutive entries. The length of column i in T is then the length of the ith ascent (from left to right) on the Dyck path, and for $i=1,\dots,k-1$, the difference $a_{i+1}-a_i$ gives the number of down-steps following the ith ascent. This uniquely determines a Dyck path with k peaks and k returns. $\square$

Corollary 3.2. *The number of Dyck paths of semilength n with as many peaks as returns equals the number of SYT of hook shape with n boxes.*

3.2. **Flag shapes.** We next consider results related to SYT of shape $(k, k, 1^{n-2k})$, which we will refer to as SYT of *flag shape*. Using the hook-length formula, one can check that the number of such tableaux is

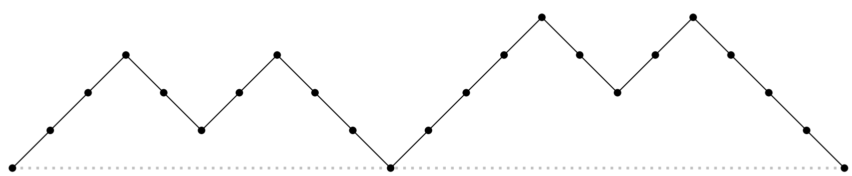
$$\frac{1}{n+1}\binom{n+1}{k}\binom{n-k-1}{k-1}, \tag{3.1}$$

which is [19, A033282].

For a fixed k, Stanley [29] gave a bijection from dissections of an $(n-k+2)$-gon using exactly $k-1$ diagonals to SYT of shape $(k, k, 1^{n-2k})$. We will give a bijection that extends this to the Dyck path setting. Analogous to the way that Narayana numbers refine Catalan numbers by considering the number of peaks, we get the following result.

Proposition 3.3. *For $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, Dyck paths of semilength n with k peaks and no singletons are in bijection with SYT of shape $(k, k, 1^{n-2k})$.*

Proof. We will present the bijection using the illustrative example:



We apply the map ϕ to a Dyck path of semilength n with no singletons and represent the resulting partition of $[n]$ as a modified tableau:

1	3	6	8
2	5	7	11
4		9	
		10	

The SYT of flag shape is then produced by pushing all the boxes below the second row into the first column:

1	3	6	8
2	5	7	11
4			
9			
10			

Conversely, given an SYT of shape $(k, k, 1^{n-2k})$, let us call the entries of the first row $a_1, a_2, \ldots, a_k$ from left to right, and let us use $b_1, b_2, \ldots, b_k$ for the entries in the second row. We rearrange the boxes that appear below the second row so that the result is in the image of ϕ as follows: move the box containing the number j into the unique column i whereby $b_i < j < b_{i+1}$ (where we let $b_{k+1} = n+1$). In other words, the columns of the resulting modified tableau form a partition P of $[n]$ with the property that when the smallest entry of each block of the partition is removed, the remaining entries form an increasing sequence of $n-k$ numbers. Finally, we construct a corresponding Dyck path D as follows. The length of block i in P will be the length of the ith ascent of the path (from left to right), and for $i = 1, \ldots, k-1$, the difference $a_{i+1} - a_i$ will be the number of down-steps following the ith ascent. By design, we have $P = \phi(D)$. □

Summing over $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$, we recover two manifestations of the sequence [19, A005043] of "Riordan numbers."

Corollary 3.4. *The number of Dyck paths of semilength n without singleton ascents equals the number of SYT of flag shape with n boxes.*

Remark 3.5. Note that, letting $n = 2k$ in Proposition 3.3, we obtain that C_k equals the number of Dyck paths of semilength $2k$ with k peaks and no singletons, an apparently new[2] interpretation of the Catalan number C_k.

Remark 3.6. There is a less direct way to construct a bijection proving Proposition 3.3 using results already in the literature. An *increasing tableau* is a semistandard Young tableau whose rows and columns are strictly increasing and the set of entries is an initial segment of the positive integers. In [20], Pechenik gives a bijection from SYT of shape $(k, k, 1^{n-2k})$ to increasing tableaux of shape $(n-k, n-k)$ whose maximum entry is at most n. He also provides a bijection from such increasing tableaux to noncrossing partitions of n into k blocks each of size at least 2. By the bijection from [8] mentioned at the start of Subsection 2.1, these noncrossing partitions are in bijection with Dyck paths of semilength n with k peaks and no singletons, as required.

Another connection between Dyck paths and SYT of flag shape begins with a result from [3]. A special case of the Dyck paths considered there is the set $\mathfrak{D}_n(1,1)$ which denotes the set of Dyck paths of semilength $2n$ created from strings of the form D and $\mathsf{U}^{2j}\mathsf{D}^j$ for $j = 1, \ldots, n$. In [3, Theorem 3.5], the number of such Dyck paths with exactly k peaks is shown to be

$$\binom{n+k}{k-1}\frac{(k-1)!}{n!}B_{n,k}(1!, 2!, \ldots). \tag{3.2}$$

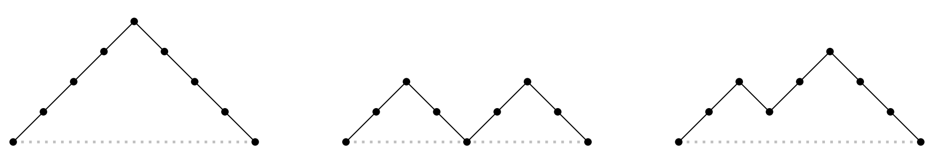
Since $B_{n,k}(1!, 2!, \ldots) = \frac{n!}{k!}\binom{n-1}{k-1}$ as shown, for example, in [12, §3.3, eqn. (3h)], the quantity in (3.2) is equal to

$$\frac{1}{n+k+1}\binom{n+k+1}{k}\binom{n-1}{k-1},$$

which is exactly the number of SYT of shape $(k, k, 1^{n-k})$ (cf. (3.1)). Thus we obtain the following reconciliation.

Proposition 3.7. *For $1 \le k \le n$, Dyck paths in $\mathfrak{D}_n(1,1)$ with k peaks are in bijection with SYT of shape $(k, k, 1^{n-k})$.*

Example 3.8. For $n = 2$, the three elements of $\mathfrak{D}_2(1,1)$ are



[2] At least this interpretation does not appear among the 214 interpretations in [28].

and the three SYT are

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}$$

We can also bijectively prove Proposition 3.7: starting with a element of $\mathfrak{D}_n(1,1)$ with k peaks, replace each building block $\mathsf{U}^{2j}\mathsf{D}^j$ by $\mathsf{U}^{j+1}\mathsf{D}$ to obtain a Dyck path of semilength $n+k$ with k peaks and no singleton ascents, and then apply the bijection of Proposition 3.3.

Remark 3.9. Note that in the case $k=n$ of Proposition 3.7, we have yet another (presumably new) interpretation of the Catalan numbers as Dyck paths of semilength $2n$ with exactly n peaks and all ascents of even length such that an ascent of length $2j$ is followed by a descent of length at least j.

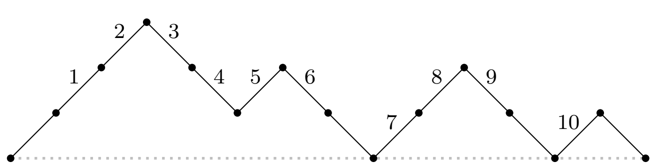
3.3. **SYT with two rows.** We finish collecting the five bijections of this section by mentioning two that map to SYT with two rows. In [17], Gudmundsson studies certain families of Dyck paths, SYT, and pattern avoiding permutations. The main result in [17] related to our work is the following theorem for which the author provides a bijective proof.

Theorem ([17]). *Let $d=k+p$. The class of Dyck paths of semilength n that begin with at least k successive up-steps, end with at least p successive down-steps, and touch the x-axis at least once somewhere between the endpoints is equinumerous with the class of SYT of shape $(n, n-d)$.*

Here is a different connection with the same class of SYT.

Proposition 3.10. *For $0 \le d \le n$, Dyck paths of semilength $n+1$ having exactly $d+1$ returns are in bijection with SYT of shape $(n, n-d)$.*

The bijection is defined as follows. Given a Dyck path of semilength $n+1$ with exactly $d+1$ returns, number each step from left to right ignoring the first up-step and skipping every down-step that touches the x-axis. Then create the SYT of shape $(n, n-d)$ by placing the labels of the n up-steps in the first row and the labels of the $n-d$ labeled down-steps in the second row. For example:



$$\longrightarrow\quad\begin{array}{|c|c|c|c|c|c|}\hline 1&2&5&7&8&10\\\hline 3&4&6&9\\\cline{1-4}\end{array}$$

We leave it to the reader to check that this map is indeed bijective.

4. Further remarks and other connections

4.1. **Colored singletons.** Let a_j denote the number of all possible cm-labels for an ascent of length $2j$. This is the number of connected matchings on $[2j]$ and is given by the sequence [19, A000699]

$$1, 1, 4, 27, 248, 2830, 38232, 593859, 10401712, \ldots$$

Therefore, if we define the sequence (x_n) by

$$x_1 = 1,$$
$$x_{2n+1} = 0 \text{ and } x_{2n} = a_n \text{ for } n \geq 1,$$

then from Theorem 1.1 and equation (1.1) we deduce that

$$\mathrm{SYT}(n) = \sum_{\ell=1}^{n} \frac{1}{(n-\ell+1)!} B_{n,\ell}(1!, 2!a_1, 0, 4!a_2, 0, \dots). \tag{4.1}$$

Observe that $\mathrm{SYT}(n)$ is a special case of the sequence

$$y_n^{(\alpha)} = \sum_{\ell=1}^{n} \frac{1}{(n-\ell+1)!} B_{n,\ell}(1!\alpha, 2!a_1, 0, 4!a_2, 0, \dots) \tag{4.2}$$

that counts the number of cm-labeled Dyck paths of semilength n, where singletons (ascents of length 1) may be colored in $\alpha \in \mathbb{N}_0$ different ways. The case $\alpha = 0$ means that no singletons are allowed. In this case, $y_{2n-1}^{(0)} = 0$ for all $n \geq 1$ while $y_{2n}^{(0)}$ gives the number of perfect matchings on $[2n]$, which are counted by the double factorials $(2n-1)!!$. Moreover, in the latter case, the bijection discussed in Subsection 2.1 maps cm-labeled Dyck paths of semilength n with k-noncrossing labels to k-noncrossing perfect matchings on $[2n]$.

Another interesting instance of (4.2) is when $\alpha = 2$, i.e. each singleton may be colored in two ways. In this case, (4.2) gives the sequence [19, A005425] whose nth term can be defined as the number of involutions on $[n]$ whose fixed points can each be colored in two different ways. In fact, for any positive integer α, the maps from Subsections 2.1–2.3 just need trivial modifications to get a bijection between cm-labeled Dyck paths whose singletons can each be colored in α ways to involutions whose fixed points can each be colored in α ways. As is the case for $\alpha = 0$ or 1, this bijection will preserve the k-noncrossing property of the labels in the Dyck paths.

4.2. **Generating functions.** Let $A(t)$ be the the generating function for the number of connected matchings on $[2n]$, and let $Y(t)$ be the corresponding function that enumerates SYT with n boxes. Equation (4.1) implies that $Y(t)$ is the noncrossing partition transform of $X(t) = t + A(t^2)$. Thus, in terms of generating functions, this means (cf. Callan [8, §4])

$$tY(t) = \left(\frac{t}{1+X(t)}\right)^{\langle -1 \rangle},$$

where ${}^{\langle -1 \rangle}$ denotes compositional inverse. In other words,

$$Y(t) - 1 = X(tY(t)), \text{ or equivalently, } (1-t)Y(t) = 1 + A(t^2Y(t)^2). \tag{4.3}$$

Further, if $P(t)$ is the generating function for the number of perfect matchings on $[2n]$, then $P(t^2)$ is the noncrossing partition transform of $A(t^2)$, and we have

$$1 + P(t^2) = \frac{1}{t}\left(\frac{t}{1+A(t^2)}\right)^{\langle -1 \rangle}.$$

This implies

$$\big(t(1+P(t^2))\big)^{\langle -1\rangle} = \frac{t}{1+A(t^2)}, \text{ or equivalently, } P\left(\frac{t^2}{(1+A(t^2))^2}\right) = A(t^2).$$

Combining this identity with (4.3), we obtain

$$P\left(\frac{t^2Y(t)^2}{(1-t)^2Y(t)^2}\right) = A(t^2Y(t)^2) = (1-t)Y(t) - 1,$$

which implies

$$Y(t) = \frac{1+P(t^2/(1-t)^2)}{1-t}.$$

While this formula is known [19, A001006], our approach using the noncrossing partition transform gives the same identity when restricted to k-noncrossing perfect matchings on $[2n]$ and SYT with n boxes and height at most $2k-1$. In other words, if $P_k(t)$ denotes the generating function for the number of k-noncrossing perfect matchings on $[2n]$, and if $Y_k(t)$ enumerates SYT with n boxes and height at most $2k-1$, then

$$Y_k(t) = \frac{1+P_k(t^2/(1-t)^2)}{1-t}.$$

This is the elegant expression we promised in the Introduction. For some values of k, these sequences are listed in [19] as follows:

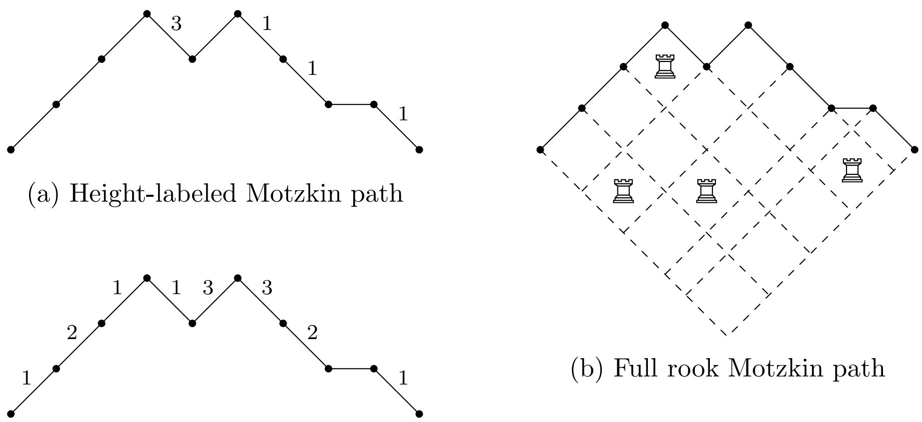
k	k-noncrossing matchings	SYT of height $\leq 2k-1$
2	A000108	A001006
3	A005700	A049401
4	A136092	A007578
5	A251598	A212915

4.3. **Labeled Motzkin paths enumerated by** SYT(n)**.** Our main bijection in Section 2 adds extra structure to Dyck paths of length $2n$ to obtain objects equinumerous to SYT with n boxes. In this section we will discuss other equinumerous sets which instead are obtained by adding extra structure to Motzkin paths of length n.

Proposition 4.1. *The following objects, defined by Motzkin paths of length n with s flat steps and some additional structure, are in bijection with partial matchings on $[n]$ having s singletons (and thus also with SYT with n boxes and s odd columns):*

- Height-labeled Motzkin paths, *meaning each down-step starting at height i is given a label from $[i]$.*
- Full rook Motzkin paths, *which have rooks placed in their lower shape such that there is exactly one in the "row" beneath each up-step and exactly one in the "column" beneath each down-step, where "row" and "column" refer to the* 45° *rotation.*
- Yamanouchi-colored Motzkin paths *which can be defined by their simple correspondence with weakly oscillating tableaux. Up, down, and flat steps correspond to adding, removing, or leaving as-is (respectively), and the label specifies the row in which to add or remove a box.*

For instance, for the partial matching $(1,6)(2,7)(3,4)(5,9)(8)$ discussed in Example 2.2, we have the labeled Motzkin paths in Figure 4.1.



(a) Height-labeled Motzkin path

(b) Full rook Motzkin path

(c) Yamanouchi-colored Motzkin path
(compare λ^j in Table 1)

FIGURE 4.1. Motzkin paths corresponding to $(1,6)(2,7)(3,4)(5,9)(8)$.

In contrast, the corresponding cm-labeled Dyck path is given in Figure 4.2.

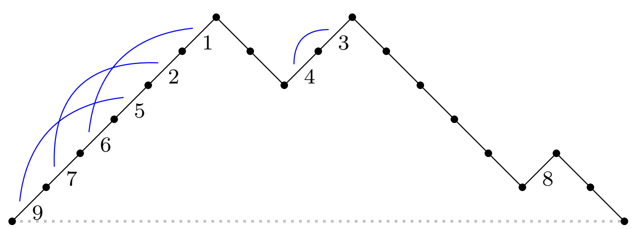


FIGURE 4.2. cm-labeled Dyck path corresponding to $(1,6)(2,7)(3,4)(5,9)(8)$.

Before proving Proposition 4.1, let us put it in context with related results in the literature. The bijection with height-labeled Motzkin paths is somewhat well known. The other two bijections are simple extensions of the better-known case when $s = 0$. Height-labeled Motzkin paths are a case of the *histoires* of orthogonal polynomials. This bijection is due to Françon and Viennot [15, 16]. In the Dyck path case ($s = 0$), height-labeled paths appear in Callan's survey of double factorials [9], and are also called *Hermite histoires*. Again for the case when $s = 0$, full rook Motzkin paths are better known as full rook placements in Ferrers shapes. These were used by Krattenthaler [18] to extend the work of Chen et al. [10]. For a reader already familiar with Fomin growth diagrams, full rook Motzkin paths are a simple intermediate step in the bijection between height-labeled and Yamanouchi-colored Motzkin paths. Yamanouchi-colored Motzkin paths were introduced by Eu et al. [14], who gave a definition and bijection using the language of Motzkin paths.

Proof of Proposition 4.1. First, there is a simple bijection between partial matchings and full rook Motzkin paths. Each pair (i, j) in the matching with $i < j$ indicates an up-step at step i and a down-step at step j. A singleton at i indicates a flat-step at step i. We then draw the path from left to right according to these steps and place rooks at the positions determined by the matching, as in Figure 4.2(b). For the reverse map, simply match the two steps diagonal from each rook, and leave the flats as singletons.

To make the bijection between height-labeled and full rook Motzkin paths easier to state, we use the terms "row" and "column" for the shape beneath the full rook path by considering the result of rotating it 45° counterclockwise. We assign height-labels to each down-step starting at height i (from left to right) according to the height of the rook in the column below, ignoring any rows with a rook in an earlier column. For example, in Figure 4.1, the first down-step in (a) has label 3 because in (b) the rook is at height 3 in the column beneath this down-step. A more interesting case is the third column, where the down-step has label 1 because it has a column of four beneath it, but ignoring the row with the rook already placed, there are three places available and the rook is in the first. Observe that the number of places available is always the starting height of the down-step, so we do indeed arrive at a height-labeled Motzkin path. Clearly, this map is easily reversed.

Finally, we defined Yamanouchi-colored Motzkin paths by their correspondence with weakly oscillating tableaux, so the bijection with partial matchings is simply the one we have already seen in Subsection 2.2. □

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Penn State Altoona, 3000 Ivyside Park, Altoona, PA 16601, USA
E-mail address: `jgil@psu.edu`

Department of Mathematics, Bucknell University, Lewisburg, PA 17837, USA
E-mail address: `peter.mcnamara@bucknell.edu`

Department of Mathematics and Statistics, Mount Holyoke College, South Hadley, MA 01075, USA
E-mail address: `jtirrell@mtholyoke.edu`

Penn State Altoona, 3000 Ivyside Park, Altoona, PA 16601, USA
E-mail address: `mdw8@psu.edu`