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On Generating functions of Diagonals Sequences of Sheffer and Riordan Number Triangles

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Abstract

The exponential generating function of ordinary generating functions of diagonal sequences of general *Sheffer* triangles is computed by an application of *Lagrange*'s theorem. For the special *Jabotinsky* type this is already known. An analogous computation for general *Riordan* number triangles leads to a formula for the logarithmic generating function of the ordinary generating functions of the product of the entries of the diagonal sequence of *Pascal*'s triangle and those of the *Riordan* triangle. For some examples these ordinary generating functions yield in both cases coefficient triangles of certain numerator polynomials.

1 Introduction and Summary

The study of the diagonal sequences of *Sheffer* number triangles (exponential, also known as binomial, lower triangular convolution matrices) is interesting. The name exponential *Riordan* arrays is sometimes used for these triangles. The *Sheffer* structure immediately leads to the exponential generating functions (*e.g.f.*s) of the column sequences. It is more difficult to obtain information about these functions for diagonal sequences. *Bala* [1] has shown, following *Drake* [3], for a special type of *Sheffer* triangles, called Jabotinsky triangles by *Knuth* [6], that the *e.g.f.* of the ordinary generating functions (*o.g.f.*s) of the diagonal sequences can be computed from *Lagrange*'s inversion theorem. We present in the first part the result for general *Sheffer* triangles and give some examples. They lead to other number triangles providing the coefficients of the numerator polynomials of the *o.g.f.*s of the diagonal sequences. In the second part the same analysis is done for general *Riordan* number triangles (ordinary lower triangular convolution matrices). However, one does not obtain information about the diagonal sequences themselves but on certain products of the diagonal entries with other numbers. We will give the result for the logarithmic generating function of the *o.g.f.*s of the sequences of the product of the entries of the diagonals of the *Riordan* and the *Pascal* triangle. (The *Pascal* triangle is a special *Riordan* triangle, and also a special *Sheffer* triangle). Also in this case special examples lead to coefficient triangles for the numerator polynomials of these *o.g.f.*s .

For *Sheffer* and *Riordan* triangles see [11], [12] and the *W. Lang* link [7] in *OEIS* [10] A006232 (henceforth we will omit the *OEIS* reference for A-numbers). There also references can be found.

Proofs for not obvious or not standard *Sheffer* or *Riordan* statements will be given in section 2.

Part A: Sheffer triangles and their diagonals

A *Sheffer* triangle S (an infinite dimensional lower triangular exponential convolution matrix; for practical purpose a $N\times N$ matrix) is denoted by $S\,=\,(g,\,f)$ with *e.g.f.* $g(s)\,=\,\sum_{k=0}^{\infty} g_n\,\frac{s^n}{n!}$, where $g(0)\,=\,g_0\,=\,1$ (*w.l.o.g.*), and $f(s)\,=\,s\,\hat f(s)$ with *e.g.f.* $\hat f(s)\,=\,\sum_{k=0}^{\infty}\hat f_n\,\frac{s^n}{n!}$, where $\hat f(0)\,=\,\hat f_0\,\neq\,0$. The column sequence $SCol(m)\,=\,\{S(n,m)\}_{n=0}^{\infty}$ (with m leading zeros) has *e.g.f.* $ESCol(s,m)\,=\,\sum_{n=m}^{\infty} S(n,m)\,\frac{s^n}{n!}$, for $m\,\in\,\mathbb{N}_0\,:=\,\{0,\,1,\,...\}$, given by

$$ESCol(s,m)\,=\,g(s)\,\frac{f(s)^m}{m!}\,=\,g(s)\,\frac{s^m\,\hat f(s)^m}{m!}\,. \qquad (1)$$

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In this paper formal power series (*f.p.s.*) are considered, and therefore no convergence issues are treated.
The (ordinary, not exponential) row polynomials (called *Sheffer* polynomials) are $PS(n,\,x)\,=\,\sum_{m=0}^{n}\,S(n,\,m)\,x^n$. They have *e.g.f.* $EPS(s,x)\,=\,\sum_{n=0}^{\infty} PS(n,\,x)\,\frac{s^n}{n!}$ given by

$$EPS(s,x)\,=\,g(s)\,e^{x\,f(s)}\,, \tag{2}$$

which is also called the *e.g.f.* of the S triangle.
The important exponential convolution property of *Sheffer* polynomials, implied by eq. (2), is

$$PS(n,x\,+\,y)\,=\,\sum_{k=0}^{n}\binom{n}{k}P(k,x)\,PS(n-k,\,y)\,=\,\sum_{k=0}^{n}\binom{n}{k}PS(k,x)\,P(n-k,\,y)\,, \tag{3}$$

where P are the special *Sheffer* polynomials $P\,=\,(1,\,f)$, called associated polynomials to $S\,=\,(g,\,f)$. (See Roman [11] for *Sheffer* sequences of polynomials. The notation there differs from the present one. See the above mentioned *W. Lang* link for the relation between them.)
The diagonal sequences are labeled by $d\,\in\,\mathbb{N}_0$, with $d\,=\,0$ for the main diagonal. Their entries are

$$DS(d,\,m)\,=\,S(d+m,\,m),\quad \text{for}\quad m\,\in\,\mathbb{N}_0\,. \tag{4}$$

Their *o.g.f.* is

$$GDS(d,\,t)\,=\,\sum_{m=0}^{\infty} DS(d,\,m)\,t^m \tag{5}$$

(the use of t instead of x is motivated by the later appearance of the parameter t), and the *e.g.f.* of $\{GDS(d,\,t)\}_{d=0}^{\infty}$ is taken as

$$EGDS(y,\,t)\,:=\,\sum_{d=0}^{\infty} GDS(d,\,t)\,\frac{y^{d+1}}{(d+1)!}\,. \tag{6}$$

(The unconventional powers for this *e.g.f.* and the use of y instead of s will become clear later).
To derive a formula for this *e.g.f.* $EGDS(y,\,t)$ of *o.g.f.*s of diagonal sequences we need *Lagrange*'s theorem and an application.

Lemma: Lagrange theorem and inversion [4], p. 523, eq. (29), [13], p. 133.

a) For $\widetilde{H}(x) = H(y(x))$ with implicit $y\,=\,y(x)\,=\,a\,+\,x\,\varphi(y)$ (here as *f.p.s.*) one has

$$\widetilde{H}(x)\,=\,H(a)\,+\,\sum_{n=1}^{\infty}\frac{x^n}{n!}\frac{d^{n-1}}{da^{n-1}}\left[\varphi^n(a)\,H'(a)\right]\,. \tag{7}$$

b) With $a\,=\,0$, $y\,=\,y(x)\,=\,x\,\psi(x)$, and the compositional inverse $x\,=\,y^{[-1]}\,=\,x(y)$ it follows that

$$\begin{aligned}\tilde{H}(y)\,=\,H(x(y))\quad &=\quad H(0)\,+\,\sum_{n=1}^{\infty}\frac{y^n}{n!}\,\frac{d^{n-1}}{da^{n-1}}\left[\left(\frac{1}{\psi(a)}\right)^n\,H'(a)\right]\Bigg|_{a=0}\\ &=\quad H(0)\,+\,\sum_{n=1}^{\infty}\frac{y^n}{n!}\,(n-1)!\,[a^{n-1}]\,\left[\left(\frac{1}{\psi(a)}\right)^n\,H'(a)\right]\end{aligned} \tag{8}$$

where $[a^n]\,h(a)$ picks the coefficient of a^n of a *f.p.s.* $h\,=\,h(a)$. Applying this *Lemma*, part b), introducing a parameter t, to $y\,=\,y(t;x)\,=\,x\,\psi(t;x)\,=\,x\,(1\,-\,t\,\hat{f}(x))\,=\,x\,-\,t\,f(x)$ with the *Sheffer* function f, and taking $H(x)\,=\,\int\,dx\,g(x)$, with the *Sheffer* function g, we obtain with the compositional inverse $x\,=\,x(t;y)$ of $y\,=\,y(t;x)$

Proposition 1:

$$EGDS(y,t)\,=\,H(x(t;y))\,-\,H(0)\,=\,\left[\int\,dx\,g(x)\right]\Bigg|_{x\,=\,x(t;y)}\,-\,\left[\int\,dx\,g(x)\right]\Bigg|_{x\,=\,0}\,. \tag{9}$$

As the last equation shows one has first to compute $x\,=\,x(t;y)$, the compositional (*Lagrange*) inverse of $y\,=\,y(t;x)$. This is the case $H(x) = x$ in the *Lemma*, part b, with the chosen $\psi\,=\,\psi(t;x)\,=\,1-t\,\hat{f}(x)$. This belongs to the associated *Sheffer* case $J = (1, f)$ (the *Jabotinsky* type [6], here called J instead of S). This yields the following corollary which has been treated already by *Bala* [1].

Corollary 1: Jabotinsky case

$$EGDJ(y,t) = x(t;y)\,. \tag{10}$$

This means that for $J = (1, f)$ the *e.g.f.* of the *o.g.f.*s of the diagonal sequences is just the compositional inverse of $y = y(t;x) = x - t\,f(x)$.

Examples

1) ([3], Example 1.10.1, and [1], Example 2) $J = (1, e^s - 1)$, the *Stirling* triangle of the second kind, given in A048993. The *Lagrange* inverse $x = x(t;y)$ of $y = x\left(1 - t\,\frac{e^x - 1}{x}\right)$ turns out to be (for Maple [9] one uses the expansion up to some power to avoid error messages from $x \to 0$)

$$x(t;y) = \frac{1}{1-t}\,y + \frac{t}{(1-t)^3}\,\frac{y^2}{2!} + \frac{t\,(1+2\,t)}{(1-t)^5}\,\frac{y^3}{3!} + \frac{t\,(1+8\,t+6\,t^2)}{(1-t)^7}\,\frac{y^4}{4!} + \frac{t\,(1+22\,t+58\,t^2+24\,t^3)}{(1-t)^9}\,\frac{y^5}{5!} + \dots\,. \tag{11}$$

The coefficients of $\frac{y^{d+1}}{(d+1)!}$, for $d \geq 0$, are the *o.g.f.*s of the diagonal sequences of J. (In [1] $d = n-1$.) *E.g.*, for $d = 2$, $GDJ(2,\,t) = \frac{t\,(1+2\,t)}{(1-t)^5}$ generates the third diagonal sequence $\{0, 1, 7, 25, 65, 140, 266, 462, 750, ...\}$ which is A001296. The coefficients of the numerator polynomials are $[[1], [0, 1], [0, 1, 2],], \dots$. Without the first column and offset 1 this is A008517 (or A201637), the second-order *Eulerian* triangle, call it *Euler*2.

2) P · S2: $S = (e^s, e^s - 1)$. This is the product of the *Sheffer* matrices $P = (e^s, s)$ (of the *Appell* type), the *Pascal* triangle A007318, and $J = (1, e^s - 1)$, *Stirling*2 from the previous example.
Remember that *Sheffer* matrices build a group (for the group law, see, *e.g.*, [7], *Lemma 9*, eq. (139)).
Here $H(x) = \int dx e^x = e^x$, $H(0) = 1$ and the compositional inverse $x(t;\,y)$ is the one from the previous example. Now from eq. (9)

$$\begin{aligned} &EGDS(y,t) = e^{x(t;\,y)} - 1 \\ = \;& \frac{1}{1-t}\,y + \frac{1}{(1-t)^3}\,\frac{y^2}{2!} + \frac{1+2\,t}{(1-t)^5}\,\frac{y^3}{3!} + \frac{1+8\,t+6\,t^2}{(1-t)^7}\,\frac{y^4}{4!} + \frac{1+22\,t+58\,t^2+24\,t^4}{(1-t)^9}\,\frac{y^5}{5!} + \dots \end{aligned} \tag{12}$$

This is similar to the above *e.g.f.* but now the coefficient triangle for the numerator polynomials of the *o.g.f.*s is really A201867 (with the main diagonal $\{1, \text{repeat } 0\}$).
In this way the *Sheffer* triangle **PS · S2** maps to the *Euler*2 triangle A201867 (which is not *Sheffer*).

3) P · |S1|: $S = (e^s, -\log(1-s))$. This is the product of the *Sheffer* matrices $P = (e^s, s)$ (of the *Appell* type), the *Pascal* triangle A007318, and $J = (1, -\log(1-s)) = |Stirling1|$ given in A132393 = |A048994|. This forms the *Sheffer* triangle A094816 (coefficients of the *Charlier* polynomials, see *e.g.*, [2]).
Here $H(x) = \int dx\,e^x = e^x$, $H(0) = 1$, like in the previous example, and the compositional inverse $x(t;\,y)$ of $y = t(t;x) = x\left(1 - t\left(\frac{-\log(1-x)}{x}\right)\right)$ is (for Maple the expansion up to a certain power is taken)

$$x(t;\,y) = \frac{1}{1-t}\,y + \frac{t}{(1-t)^3}\,\frac{y^2}{2!} + \frac{t\,(2+t)}{(1+t)^5}\,\frac{y^3}{3!} + \frac{t\,(6+8\,t+t^2)}{(1-t)^7}\,\frac{y^4}{4!} + \frac{t\,(24+58\,t+22\,t^2+t^3)}{(1-t)^9}\,\frac{y^5}{5!} + \dots\,. \tag{13}$$

Compare this with the different eq. (11). Now from eq. (9),

$$\begin{aligned} EGDS(y,t) &= e^{x(t;\,y)} - 1 = \frac{1}{1-t}\,y + \frac{1}{(1-t)^3}\,\frac{y^2}{2!} + \frac{1+3\,t-t^2}{(1-t)^5}\,\frac{y^3}{3!} + \\ &+ \frac{t+17\,t-2\,t^2-t^3}{(1-t)^7}\,\frac{y^4}{4!} + \frac{1+80\,t+49\,t^2-27\,t^3+2\,t^4}{(1-t)^9}\,\frac{y^5}{5!} + \dots\,. \end{aligned} \tag{14}$$

The coefficients of the row polynomials are given as signed triangle A290311. Like **P · S2** produced the *Euler*2 triangle in example 2, here **P · |S1|** produces triangle A290311.

4) $\mathbf{S2[d,a]}$. $S = (e^{a\,s},\, e^{d\,s} - 1)$, generalized *Stirling*2 number triangles ([8], also with references). Here $d \in \mathbb{N}_0$, $a \in \mathbb{N}_0$ and $\gcd(d,\,a) = 1$, and for $d = 1$ one puts $a = 0$. Example 1 is the instance $[d,a] = [1,0]$, and we consider here only $d \geq 2$ (*i.e.*, $a \neq 0$). Example 2 would appear as $d = a = 1$.
$y(d;t;x) = x\left(1 - t\,\frac{e^{d\,x} - 1}{x}\right)$ with the compositional inverse $x(d;t;y)$. $H(a;x) = \int dx\, e^{a\,x} = \frac{1}{a}\, e^{a\,x}$, $H(a;0) = \frac{1}{a}$. From eq. (9)

$$EGDS2(d,a;y,t) = \frac{1}{a}\left(e^{a\,x(d;t;y)} - 1\right). \tag{15}$$

We consider two instances.
$\boldsymbol{\alpha}$) $S = S2[2,1] =$ A154537.

$$EGDS2(2,1;y,t) = e^{x(2;t;y)} - 1 = \frac{1}{1 - 2\,t}\,y + \frac{1 + 2\,t}{(1 - 2\,t)^3}\,\frac{y^2}{2!} + \frac{1 + 16\,t + 12\,t^2}{(1 - 2\,t)^5}\,\frac{y^3}{3!}$$
$$+ \frac{1 + 66\,t + 284\,t^2 + 120\,t^3}{(1 - 2\,t)^7}\,\frac{y^4}{4!} + \frac{1 + 224\,t + 2872\,t^2 + 5952\,t^3 + 1680\,t^4}{(1 - 2\,t)^9}\,\frac{y^5}{5!} + \dots. \tag{16}$$

The coefficients of the numerator polynomials are found in A290315.
$\boldsymbol{\beta}$) $S = S2[3,1] =$ A282629.

$$EGDS2(3,1;y,t) = e^{x(3;t;y)} - 1 = \frac{1}{1 - 3\,t}\,y + \frac{1 + 3\,t}{(1 - 3\,t)^3}\,\frac{y^2}{2!} + \frac{1 + 16\,t + 12\,t^2}{(1 - 3\,t)^5}\,\frac{y^3}{3!}$$
$$+ \frac{1 + 66\,t + 284\,t^2 + 120\,t^3}{(1 - 3\,t)^7}\,\frac{y^4}{4!} + \frac{1 + 224\,t + 2872\,t^2 + 5952\,t^3 + 1680\,t^4}{(1 - 3\,t)^9}\,\frac{y^5}{5!} + \dots. \tag{17}$$

The coefficients of the numerator polynomials are found in A290316.

5) $\mathbf{\widehat{S1p}[d,a]}$. $S = ((1 - d\,s)^{-\frac{a}{d}},\, -\frac{1}{d}\log(1 - d\,s))$, generalized signless *Stirling*1 number triangles (see [8], also with references). Here $d \in \mathbb{N}_0$, $a \in \mathbb{N}_0$ and $\gcd(d,\,a) = 1$, and for $d = 1$ one puts $a = 0$. The $[d,a] = [1,0]$ case has been given for the signed *Stirling*1 numbers in the Bala article [1], and we consider here only $d \geq 2$ (*i.e.*, $a \neq 0$).
$y(d;t;x) = x\left(1 - t\left(-\frac{\log(1 - d\,x)}{d\,x}\right)\right)$ with the compositional inverse $x(d;t;y)$. No confusion with above y and x quantities with the same name should arise.
$H(d,a;x) = \int dx\,(1 - d\,x)^{-\frac{a}{d}} = -\frac{1}{d - a}\,(1 - d\,x)^{\frac{d-a}{d}}$, $H(d,a;0) = -\frac{1}{d - a}$. From eq. (9)

$$EGD\widehat{S1p}(d,a;y,t) = \frac{1}{d - a}\left[1 - (1 - d\,x(d;t;y))^{\frac{d-a}{a}}\right]. \tag{18}$$

We consider two instances.
$\boldsymbol{\alpha}$) $S = \widehat{S1p}[2,1] =$ A028338.

$$EGD\widehat{S1p}(2,1;y,t) = 1 - (1 - 2\,x(2;t;y))^{1/2} = \frac{1}{1 - t}\,y + \frac{1 + t}{(1 - t)^3}\,\frac{y^2}{2!} + \frac{3 + 8\,t + t^2}{(1 - t)^5}\,\frac{y^3}{3!}$$
$$+ \frac{15 + 71\,t + 33\,t^2 + t^3}{(1 - t)^7}\,\frac{y^4}{4!} + \frac{105 + 744\,t + 718\,t^2 + 112\,t^3 + t^4}{(1 - t)^9}\,\frac{y^5}{5!} + \dots. \tag{19}$$

The coefficients of the numerator polynomials are found in A288875. The first diagonal sequences of A028338 are A000012, A000290$(n+1)$, A024196$(n+1)$, A024197$(n+1)$, A024198$(n+1)$.

$\boldsymbol{\beta}$) $S = \widehat{S1p}[3,1] =$ A286718.

$$EGD\widehat{S1p}(3,1;y,t) = (1 - (1 - 3\,x(3;t;y))^{2/3})/2 = \frac{1}{1 - t}\,y + \frac{1 + 2\,t}{(1 - t)^3}\,\frac{y^2}{2!} + \frac{4 + 19\,t + 4\,t^2}{(1 - t)^5}\,\frac{y^3}{3!}$$
$$+ \frac{28 + 222\,t + 147\,t^2 + 8\,t^3}{(1 - t)^7}\,\frac{y^4}{4!} + \frac{280 + 3194\,t + 4128\,t^2 + 887\,t^3 + 16\,t^4}{(1 - t)^9}\,\frac{y^5}{5!} + \dots. \tag{20}$$

The coefficients of the numerator polynomials are found in A290318. The first diagonal sequences of A286718 are A000012, A000326$(n+1)$, A024212$(n+1)$, A024213$(n+1)$.

Part B: Riordan triangles and their diagonals multiplied with Pascal diagonals

A *Riordan* triangle R (an infinite dimensional lower triangular (ordinary) convolution matrix; for practical purpose a $N \times N$ matrix) is denoted by $R = (G, F)$ with *o.g.f.* $G(x) = \sum_{k=0}^{\infty} G_n\, x^n$, where $G(0) = G_0 = 1$ (*w.l.o.g.*), and $F(x) = x\,\widehat{F}(x)$ with *o.g.f.* $\widehat{F}(x) = \sum_{k=0}^{\infty} \widehat{F}_n\, x^n$, where $\widehat{F}(0) = \widehat{F}_0 \neq 0$. The column sequence $RCol(m) = \{R(n,m)\}_{n=0}^{\infty}$ (with m leading zeros) has *o.g.f.* $GRCol(x,m) = \sum_{n=m}^{\infty} R(n,m)\,x^n$, for $m \in \mathbb{N}_0$, given by

$$GRCol(x,m) \,=\, G(x)\,F(x)^m \,=\, G(x)\,x^m\,\widehat{F}(x)^m\,. \tag{21}$$

The row polynomials (called *Riordan* polynomials) are $PR(n,\,x) = \sum_{m=0}^{n} R(n,\,m)\,x^n$. They have *o.g.f.*s $GPR(x,z) = \sum_{n=0}^{\infty} PR(n,\,x)\,z^n$ given by

$$GPS(x,z) \,=\, G(z)\,\frac{1}{1 - \,x\,F(z)} \tag{22}$$

which is also called the *o.g.f.* of the R triangle.
The *Riordan* group has been introduced, in analogy to the *Sheffer* group [11] by *Shapiro et al.* [12]
There is no (ordinary) convolution property for *Riordan* polynomials similar to eq. (3). But $P = (1,\,F)$ is also called associated to $R = (g,\,f)$. Such matrices form a subgroup of the *Riordan* group.
The diagonal sequences are labeled by $d \in \mathbb{N}_0$, with $d = 0$ for the main diagonal. Their entries are

$$DR(d,\,m) \,=\, R(d+m,\,m), \quad \text{for} \quad m \in \mathbb{N}_0\,. \tag{23}$$

Their *o.g.f.* is

$$GDR(d,x) \,=\, \sum_{m=0}^{\infty} DR(d,\,m)\,x^m\,, \tag{24}$$

Application of *Lagrange*'s theorem, like in the *Lemma*, part b) does not lead to the *o.g.f.*s of these diagonal sequences directly. Instead one is led to consider the product of the diagonal entries with the corresponding ones of *Pascal*'s *Riordan* triangle $P = \left(\frac{1}{1-\,x},\,\frac{x}{1-\,x}\right)$, A007318. This belongs to the so called *Bell* subgroup of the *Riordan* group of the type $B = (G(x),\,x\,G(x))$. Define

$$\widehat{D}(d,\,m) \,:=\, P(d+m,m)\,R(d+m,\,m) \,=\, \binom{d+m}{m}\,D(d,\,m)\,. \tag{25}$$

The corresponding *o.g.f.* is $G\widehat{D}(d,\,t) = \sum_{m=0}^{\infty} \widehat{D}(d,\,m)\,t^m$ Their logarithmic generating function (*l.g.f.*) $LG\widehat{D}R(y,t)$ is taken as

$$LG\widehat{D}R(y,\,t) \,=\, \sum_{d=0}^{\infty} G\widehat{D}(d,\,t)\,\frac{y^{d+1}}{d+1}\,. \tag{26}$$

(The unconventional powers for this *l.g.f.* and the use of y instead of z will become clear later).
Applying now *Lemma*, part b) to $y = y(t;x) = x\,\psi(t;x) = x\,(1 - \,t\,\widehat{F}(x)) = x - \,t\,F(x)$ with the *Riordan* function F, introducing a parameter t, and taking $H(x) = \int dx\,G(x)$, with the *Riordan* function G, we obtain, with the compositional inverse $x = x(t;y)$ of $y = y(t;x)$, the following proposition.

Proposition 2:

$$LG\widehat{D}R(y,t) \,=\, H(x(t;y)) \,-\, H(0) \,=\, \left[\int dx\,G(x)\right]\Big|_{x\,=\,x(t;y)} \,-\, \left[\int dx\,G(x)\right]\Big|_{x\,=\,0}\,. \tag{27}$$

As in the *Sheffer* section one has first to compute the $x = x(t;y)$, the *Lagrange* inversion of $y = y(t;x)$. This is the case $H(x) = x$ in the *Lemma*, part b, with the chosen $\psi = \psi(t;x) = 1 - t\,\widehat{F}(x)$. It belongs to the associated *Riordan* case $A = (1, F)$ (A for the associated triangle to R). This yields the following corollary.

Corollary 2: Associated Riordan case

$$LG\widehat{D}A(y,t) \;=\; x(t;y)\,. \tag{28}$$

This means that in the $A = (1, F)$ case the *l.g.f.* of the *o.g.f.*s of the sequences of the product of the entries of the diagonals of A and the *Pascal* triangle P is just the compositional inverse of $y = y(t;x) = x - t\,F(x)$.

Instead of the *l.g.f.* of the *o.g.f.*s of diagonal sequences of the triangle with entries $\widehat{D}(d,\,m)$ one could as well take the *e.g.f.* of the *e.g.f.*s of the diagonal sequences of the triangle with entries $\widetilde{D}(d,\,m) := (d+m)!\,D(d,\,m)$. This leads to

Corollary 3: With the *e.g.f.*

$$E\widetilde{D}(d,\,t) \;:=\; \sum_{m=0}^{\infty} \widetilde{D}(d,\,m)\,\frac{t^m}{m!} \;=\; \sum_{m=0}^{\infty} (d+m)!\,D(d,\,m)\,\frac{t^m}{m!}\,, \tag{29}$$

and the further *e.g.f.*

$$EE\widetilde{D}R(y,\,t) \;=\; \sum_{d=0}^{\infty} E\widetilde{D}(d,\,t)\,\frac{y^{d+1}}{(d+1)!} \tag{30}$$

one has

$$EE\widetilde{D}R(y,\,t) \;=\; H(x(t;y)) \;-\; H(0) \;=\; \left[\int dx\,G(x)\right]\Bigg|_{x\,=\,x(t;y)} \;-\; \left[\int dx\,G(x)\right]\Bigg|_{x\,=\,0}\,. \tag{31}$$

Examples

1) $A \;=\; \left(1,\,\frac{x}{1\,-\,x}\right)$, the **Pascal** triangle variant given in A097805. The *Lagrange* inverse $x = x(t;y)$ of $y \;=\; x\,\left(1\,-\,\frac{t}{1\,-\,x}\right)$ turns out to be

$$x(t;y) \;=\; \frac{1}{1\,-\,t}\,y + \frac{2\,t}{(1\,-\,t)^3}\,\frac{y^2}{2} + \frac{3\,t\,(1\,+\,t)}{(1\,-\,t)^5}\,\frac{y^3}{3} + \frac{4\,t\,(1\,+\,3\,t\,+\,t^2)}{(1\,-\,t)^7}\,\frac{y^4}{4} + \frac{5\,t\,(1\,+\,6\,t\,+\,6\,t^2\,+\,t^3)}{(1\,-\,t)^9}\,\frac{y^5}{5} + \ldots\,. \tag{32}$$

See [3], Example 1.10.8.

This is a *l.g.f.* , therefore the coefficients of $\frac{y^{d+1}}{d+1}$, for $d \geq 0$, are the *o.g.f.* of the diagonal sequences of the triangle $[[1], [0, 1], [0, 2, 1], [0, 3, 6, 1], [0, 4, 18, 12, 1], [0, 5, 40, 60, 20, 1], [0, 6, 75, 200, 150, 30, 1], [0, 7, 126, 525, 700, 315, 42, 1], ...]$ obtained by multiplying the entries of *Pascal*'s triangle and A = A097805. *E.g.*, the fourth diagonal ($d = 3$) $[0,\,4,\,40,\,...]$ has o.g.f. $G(3,x) = \frac{4\,t\,(1\,+\,3\,t\,+\,t^2)}{(1\,-\,t)^7}$. The numerator polynomials divided by $(d+1)\,t$, for $d \geq 1$, are found as row d polynomials of A001263 (*Narayana* triangle).

2) Generalized Pascal triangles.

$R \;=\; \left(G(x),\,\frac{x}{1\,-\,x}\right)$, and the *Lagrange* inverse $x(t;y)$ is given in eq. (32). Now eq. (27) applies with $H(x) \;=\; \int dx\,G(x)$.

Two instances:

$\boldsymbol{\alpha}$) $R \;=\; \left(\frac{1}{1\,-\,x},\,\frac{x}{1\,-\,x}\right)$. This is the *Pascal* triangle A007318. Here $H(x) \;=\; -\log(1\,-\,x)$, $H(0) \;=\; 0$, and one obtains the *l.g.f.*

$$\begin{aligned} LG\widehat{D}R(y,t) \;&=\; -\log(1\,-\,x(y;t)) \;=\; \frac{1}{1\,-\,t}\,y + \frac{1\,+\,t}{(1\,-\,t)^3}\,\frac{y^2}{2} + \frac{1\,+\,4\,t\,+\,t^2}{(1\,-\,t)^5}\,\frac{y^3}{3} + \\ &+\; \frac{1\,+\,9\,t\,+\,9\,t^2\,+\,t^3}{(1\,-\,t)^7}\,\frac{y^4}{4} + \frac{1\,+\,16\,t\,+\,36\,t^2\,+\,16\,t^3\,+\,t^4}{(1\,-\,t)^9}\,\frac{y^5}{5} + \ldots \end{aligned} \tag{33}$$

The numerator polynomials are the row polynomials of A008459, the square entries of *Pascal*'s triangle. The *o.g.f.*s for the diagonal sequences of A008459 are given by $GDR(d,\, x) \;=\; \left[\frac{y^{d+1}}{d+1}\right]\, LG\widehat{D}R(y,t)$ for $d \ge 0$. *E.g.*, the fourth diagonal sequence [1, 16, 100, ...] has *o.g.f.* $GDR(3,\, x) \;=\; \frac{1\,+\,9\,t\,+\,9\,t^2\,+\,t^3}{(1\,-\,x)^7}$.

β) $R \;=\; \left(\frac{1}{(1\,-\,x)^2},\, \frac{x}{1\,-\,x}\right)$. This is the *Riordan* triangle A135278. Here $H(x) \;=\; \frac{1}{1\,-\,x}$, $H(0) \;=\; 1$, and one obtains the *l.g.f.*

$$
\begin{aligned}
LG\widehat{D}R(y,t) \;&=\; \frac{1}{1\,-\,x(y;t)} \,-\, 1 \;=\; \frac{1}{1\,-\,t}\, y \,+\, \frac{2}{(1\,-\,t)^3}\,\frac{y^2}{2} \,+\, \frac{3\,(1\,+\,t)}{(1\,-\,t)^5}\,\frac{y^3}{3} \,+\\
&+\; \frac{4\,(1\,+\,3\,t\,+\,t^2)}{(1\,-\,t)^7}\,\frac{y^4}{4} \,+\, \frac{5\,(1\,+\,6\,t\,+\,6\,t^2\,+\,t^3)}{(1\,-\,t)^9}\,\frac{y^5}{5} \,+\, \ldots
\end{aligned}
\tag{34}
$$

The numerator polynomials are again the row polynomials of A008459 (*Narayana* triangle) multiplied here by $d+1$. Therefore, the *o.g.f.*s for the diagonal sequences with entries A103371$(n,\, k)$ = A135278$(n,\, k)$ A007318$(n,\, k)$ are given by $GDR(d,\, x) \;=\; (d+1)\, \frac{\sum_{k=1}^{d} N(d,\, k)\, x^{k-1}}{(1\,-\,x)^{2\,d+1}}$ for $d \ge 1$, with $N(d,\, k)$ = A008459$(d,\, k)$, and for $d \;=\; 0$ the *o.g.f.* is $GDR(0,\, x) \;=\; \frac{1}{1\,-\,x}$.

2 Proofs

Part A

1. Proof of the Lemma: Lagrange theorem and inversion [4], p. 523. eq. (29), [13], p. 133.

Part **a)** is the standard theorem of *Lagrange* with the proof given in the references.

Part **b)**: The first two equations of eq. (8) follow from part **a)** for $a \;=\; 0$, interchanging the rôle of x and y, and using $\varphi(x) \;=\; \frac{1}{\psi(x)}$ (See[4], pp. 524-525 for the case $H(x) \;=\; x$). The last eq. is then obvious with the definition of $[a^n]\, h(a)$ given there.

2. Proof of Proposition 1

From the *Lemma*, part b), one has, with $y \;=\; y(t;x) \;=\; x\,\psi(t;x) \;=\; x\,(1\,-\,t\,\hat{f}(x))$, and $H(x) \;=\; \int dx\, g(x)$, where the *Sheffer* triangle is $S \;=\; (g(x),\, x\,\hat{f}(x))$,

$$
H(x(t;y)) \,-\, H(0) \;=\; \sum_{n=1}^{\infty} \frac{y^n}{n!}\,(n-1)!\,[a^{n-1}]\, \left[(1\,-\,t\,\hat{f}(a))^{-n}\, g(a)\right]\,.
\tag{35}
$$

The binomial theorem $(1\,-\,t\,\hat{f}(a))^{-n} \;=\; \sum_{p=0}^{\infty} \binom{-n}{p}\, (-t)^p\, (\hat{f}(a))^p$ is applied. Then the binomial with negative upper entry is transformed in one with non-negative entries, using the identity (see [5], p. 164, eq. (5.14))

$$
\binom{-n}{p} \;=\; (-1)^p\, \binom{p+n-1}{p}\,.
\tag{36}
$$

$$
H(x(t;y)) \,-\, H(0) \;=\; \sum_{n=1}^{\infty} \frac{y^n}{n!}\,(n-1)!\, \sum_{p=0}^{\infty} \binom{p+n-1}{p}\, t^p\, p!\, [a^{n-1}]\, \left[\frac{(\hat{f}(a))^p}{p!}\, g(a)\right]\,.
\tag{37}
$$

In order to obtain $f(a) \;=\; a\,\hat{f}(a)$ one uses $[a^{n-1}]\,h(a) \;=\; [a^{n-1+p}](a^p\, h(a))$. Then the definition of the *e.g.f.* of the sequence of column p of the *Sheffer* triangle is used: $\sum_{k=p\,(0)}^{\infty} S(k,\, p)\, \frac{a^k}{k!} \;=\; \frac{(f(a))^p}{p!}\, g(a)$ (One can start with $p = 0$

because $S(k,\,p)\;=\;0$ for $0\;\le\;k\;<\;p$.) Thus $[a^m]\left(\frac{(f(a))^p}{p!}\,g(a)\right)\;=\;S(m,\,p)\,\frac{1}{m!}$.

$$
\begin{aligned}
H(x(t;y))\;-\;H(0)\quad&=\quad\sum_{n=1}^{\infty}\frac{y^n}{n!}\,(n-1)!\,\sum_{p=0}^{\infty}\binom{p+n-1}{p}\,t^p\,p!\,\frac{1}{(n-1+p)!}\,S(n-1+p,\,p)\\
&=\quad\sum_{n=1}^{\infty}\frac{y^n}{n!}\left(\sum_{p=0}^{\infty}t^p\,S(n-1+p,\,p)\right)\,. \qquad (38)
\end{aligned}
$$

But the *o.g.f.* of the diagonal sequences of S is $GDS(n-1,\,t)\;=\;\sum_{p=0}^{\infty}\,t^p\,S(n-1+p,\,p)$, for $n\,\ge 1$, and because we take $d\;=\;n-1$ to label the diagonals, we get

$$
H(x(t;y))\;-\;H(0)\;=\;\sum_{d=0}^{\infty}\frac{y^{d+1}}{(d+1)!}\,GDS(d,\,t)\;=:\;EGDS(y,\,t) \qquad (39)
$$

□

Part B 3. Proof of Proposition 2

From the *Lemma*, part b), one has, with $y\;=\;y(t;x)\;=\;x\,\psi(t;x)\;=\;x\,(1\;-\;t\,\widehat{F}(x))$, and $H(x)\;=\;\int dx\,G(x)$, where the *Riordan* triangle is $R\;=\;(G(x),\,x\,\widehat{F}(x))$

$$
H(x(t;y))\;-\;H(0)\;=\;\sum_{n=1}^{\infty}\frac{y^n}{n!}\,(n-1)!\,[a^{n-1}]\,\left[(1\;-\;t\,\widehat{F}(a))^{-n}\,G(a)\right]\,. \qquad (40)
$$

Using the binomial theorem and the binomial identity eq. (36) one finds

$$
H(x(t;y))\;-\;H(0)\;=\;\sum_{n=1}^{\infty}\frac{y^n}{n!}\,(n-1)!\,\sum_{p=0}^{\infty}\binom{p+n-1}{p}\,t^p\,[a^{n-1}]\,\left[(\widehat{F}(a))^p\,G(a)\right]\,. \qquad (41)
$$

In order to obtain $F(a)\;=\;a\,\widehat{F}(a)$ one uses $[a^{n-1}]\,h(a)\;=\;[a^{n-1+p}](a^p\,h(a))$. Then the definition of the *o.g.f.* of the sequence of column labeled p of triangle R is used: $\sum_{k=p\,(0)}^{\infty}R(k,\,p)\,a^k\;=\;(F(a))^p\,G(a)$. Thus $[a^m]\,((F(a))^p\,G(a))\;=\;R(m,\,p)$.

$$
\begin{aligned}
H(x(t;y))\;-\;H(0)\quad&=\quad\sum_{n=1}^{\infty}\frac{y^n}{n!}\,(n-1)!\,\sum_{p=0}^{\infty}\binom{p+n-1}{p}\,t^p\,R(n-1+p,\,p)\\
&=\quad\sum_{n=1}^{\infty}\frac{y^n}{n!}\,\sum_{p=0}^{\infty}\frac{t^p}{p!}\,(p+n-1)!\;R(n-1+p,\,p)\,. \qquad (42)
\end{aligned}
$$

At this stage the *Corollary 3* has been proved, if one uses $n-1\;=\;d$ (and $p\;\to\;m$). But we prefer to use the binomial coefficient to multiply the diagonal R entries, *i.e.*, we use the first equation. With $n-1\;=\;d$ this becomes

$$
H(x(t;y))\;-\;H(0)\;=\;\sum_{d=0}^{\infty}\frac{y^{d+1}}{d+1}\left(\sum_{p=0}^{\infty}\binom{d+p}{p}\,R(d+p,\,p)\,t^p\right)\,. \qquad (43)
$$

This is the the *l.g.f.* eq. (26) of the *o.g.f.*s $G\widehat{D}(d,t)$ of the product of the diagonal entries in *Pascal*'s triangle and the ones of the *Riordan* triangle, called $\widehat{D}(d\,m)$ in eq. (25). □

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