TORSION TABLE FOR THE LIE ALGEBRA $\mathfrak{nil}_n$

LEON LAMPRET AND ALEŠ VAVPETIČ

Abstract. We study the Lie ring $\mathfrak{nil}_n$ of all strictly upper-triangular $n\times n$ matrices with entries in $\mathbb{Z}$. Its complete homology for $n\leq 8$ is computed.

We prove that every p^m-torsion appears in $H_*(\mathfrak{nil}_n;\mathbb{Z})$ for $p^m\leq n-2$. For $m=1$, Dwyer [1] proved that the bound is sharp, i.e. there is no p-torsion in $H_*(\mathfrak{nil}_n;\mathbb{Z})$ when prime $p>n-2$. In general, for $m>1$ the bound is not sharp, as we show that there is 8-torsion in $H_*(\mathfrak{nil}_8;\mathbb{Z})$.

As a sideproduct, we derive the known result, that the ranks of the free part of $H_*(\mathfrak{nil}_n;\mathbb{Z})$ are the Mahonian numbers (=number of permutations of $[n]$ with k inversions), using a different approach than in [4].

1. Introduction

Let $\mathfrak{nil}_n$ be the Lie algebra of integral $n\times n$ strictly upper-triangular matrices. The complete homology $H_k(\mathfrak{nil}_n;\mathbb{Z})$ is known only for $n\leq 6$ [3]:

$k\backslash n$	2	3	4	5	6
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^5$
2		$\mathbb{Z}^2$	$\mathbb{Z}^5\oplus\mathbb{Z}_2$	$\mathbb{Z}^9\oplus\mathbb{Z}_2^2$	$\mathbb{Z}^{14}\oplus\mathbb{Z}_2^3$
3		$\mathbb{Z}$	$\mathbb{Z}^6\oplus\mathbb{Z}_2$	$\mathbb{Z}^{15}\oplus\mathbb{Z}_2^8\oplus\mathbb{Z}_3^2$	$\mathbb{Z}^{29}\oplus\mathbb{Z}_2^{20}\oplus\mathbb{Z}_3^4$
4			$\mathbb{Z}^5$	$\mathbb{Z}^{20}\oplus\mathbb{Z}_2^{10}\oplus\mathbb{Z}_3^3$	$\mathbb{Z}^{49}\oplus\mathbb{Z}_2^{47}\oplus\mathbb{Z}_3^{13}\oplus\mathbb{Z}_4^3$
5			$\mathbb{Z}^3$	$\mathbb{Z}^{22}\oplus\mathbb{Z}_2^{10}\oplus\mathbb{Z}_3^3$	$\mathbb{Z}^{71}\oplus\mathbb{Z}_2^{79}\oplus\mathbb{Z}_3^{26}\oplus\mathbb{Z}_4^9$
6			$\mathbb{Z}$	$\mathbb{Z}^{20}\oplus\mathbb{Z}_2^8\oplus\mathbb{Z}_3^2$	$\mathbb{Z}^{90}\oplus\mathbb{Z}_2^{118}\oplus\mathbb{Z}_3^{35}\oplus\mathbb{Z}_4^{12}$
7				$\mathbb{Z}^{15}\oplus\mathbb{Z}_2^2$	$\mathbb{Z}^{101}\oplus\mathbb{Z}_2^{138}\oplus\mathbb{Z}_3^{36}\oplus\mathbb{Z}_4^{12}$
8				$\mathbb{Z}^9$	$\mathbb{Z}^{101}\oplus\mathbb{Z}_2^{118}\oplus\mathbb{Z}_3^{35}\oplus\mathbb{Z}_4^{12}$
9				$\mathbb{Z}^4$	$\mathbb{Z}^{90}\oplus\mathbb{Z}_2^{79}\oplus\mathbb{Z}_3^{26}\oplus\mathbb{Z}_4^9$
10				$\mathbb{Z}$	$\mathbb{Z}^{71}\oplus\mathbb{Z}_2^{47}\oplus\mathbb{Z}_3^{13}\oplus\mathbb{Z}_4^3$
11					$\mathbb{Z}^{49}\oplus\mathbb{Z}_2^{20}\oplus\mathbb{Z}_3^4$
12					$\mathbb{Z}^{29}\oplus\mathbb{Z}_2^3$
13					$\mathbb{Z}^{14}$
14					$\mathbb{Z}^5$
15					$\mathbb{Z}$

The main reason why computations for larger n are exceedingly difficult is that the chain complex $C_*=\Lambda^*\mathfrak{nil}_n$ is immense. It has $2^{\binom{n}{2}}$ generators, which is more than 2 million for $n=7$. In the paper, we divide C_* in numerous direct summands $[\![w]\!]$ (corresponding to sequences $w\in\{1,\ldots,n\}^n$ with $w_1+\ldots+w_n=\binom{n+1}{2}$) and show how many of them are isomorphic (up to dimension shift), many are contractible

Date: August 9, 2017.

2010 *Mathematics Subject Classification.* 17B56, 13P20, 13D02, 55-04, 18G35, 58E05.

Key words and phrases. algebraic/discrete Morse theory, homological algebra, chain complex, acyclic matching, nilpotent Lie algebra, triangular matrices, torsion table, algebraic combinatorics.

and many are obtained from smaller ones as the cone of a chain map $[\![w']\!] \xrightarrow{\cdot t} [\![w']\!]$. The direct summands corresponding to permutations of $(1,\ldots,n)$ are generated by just one element, hence those contribute only to free part. We show that any other direct summand contributes only to torsion, so we get the free part of homology.

Complex. Let e_{ij} be the matrix with all entries 0 except 1 in the position (i,j). The chain complex $C_* = \Lambda^*\mathfrak{nil}_n$, due to Chevalley (1948), is generated by wedges $e_{a_1b_1}\wedge\ldots\wedge e_{a_kb_k}$, where $1\le a_i<b_i\le n$ for all i. From now on, for the sake of brevity and clarity, we shall omit the $\wedge$ symbols. The boundary is defined by

$$\partial(e_{a_1b_1}\ldots e_{a_kb_k}) = \textstyle\sum_{i<j}(-1)^{i+j}[e_{a_ib_i},e_{a_jb_j}]e_{a_1b_1}\ldots\widehat{e_{a_ib_i}}\ldots\widehat{e_{a_jb_j}}\ldots e_{a_kb_k},$$

where $[e_{ab},e_{cd}]$ equals e_{ad} if $b=c$, equals $-e_{cb}$ if $a=d$, and equals 0 otherwise.

AMT. For some computations later on, we shall use algebraic Morse theory, so we include a short review of it. To a chain complex of free modules (C_*,∂_*) we associate a weighted digraph Γ_{C_*} (vertices are basis elements of C_*, weights of edges are nonzero entries of matrices ∂_*). Then we carefully select a matching $\mathcal{M}$ in this digraph, so that its edges have invertible weights and if we reverse the direction of every $e\in\mathcal{M}$ in Γ_{B_*}, the obtained digraph $\Gamma_{C_*}^{\mathcal{M}}$ contains no directed cycles and no infinite paths in two adjacent degrees. Under these conditions (i.e. if $\mathcal{M}$ is a *Morse matching*), the AMT theorem ([7], [3], [5]) provides a homotopy equivalent complex $(\mathring{C}_*,\mathring{\partial}_*)$, spanned by the unmatched vertices in $\Gamma_{C_*}^{\mathcal{M}}$, and with the boundary $\mathring{\partial}_*$ of $v\in\mathring{C}_k$ given by the sum of weights of directed paths in $\Gamma_{C_*}^{\mathcal{M}}$ to all critical $v'\in\mathring{C}_{k-1}$. For more details, we refer the reader to the three articles above (which specify the homotopy equivalence), or [6] for a quick introduction and formulation.

2. Subcomplexes

For a set $M=\{(a_1,b_1),\ldots,(a_k,b_k)\}\subseteq\{(i,j);1\le i<j\le n\}$ we denote $e_M = e_{a_1b_1}\ldots e_{a_kb_k}$. For $M_i:=\{x;(i,x)\in M\}$ we have $e_M=\wedge_{i=1}^{n-1}e_{\{i\}\times M_i}$. We define the *weight* vector $\widetilde{w}(e_M)=(\widetilde{w}_1,\ldots,\widetilde{w}_n)$ by $\widetilde{w}_i=|\{x;(x,i)\in M\}|-|\{y;(i,y)\in M\}|$, i.e. the number of times i appears on the right in e_M minus the number of times i appears on the left in e_M. Then $\sum_{i=1}^n\widetilde{w}_i=0$. Every summand in $\partial(e_M)$ has the same weight as e_M. Therefore a submodule $[\widetilde{w}]$ of $\Lambda^*\mathfrak{nil}_n$, spanned by the basis elements with weight $\widetilde{w}$, forms a chain subcomplex which is a direct summand.

Most equalities will be described more conveniently using the *modified weight* $w(e_M)=(1,\ldots,n)-\widetilde{w}(e_M)=(1-\widetilde{w}_1,\ldots,n-\widetilde{w}_n)$. Then $\sum_{i=1}^n w_i=\binom{n+1}{2}$ and $i-n\le\widetilde{w}_i\le i-1$ implies $1\le w_i\le n$ for all i. We denote $[\![w]\!]=[(1,\ldots,n)-w]$ and let $[\![w]\!]_k$ be the complex $[\![w]\!]$ dimensionally shifted by k. Let $\mathcal{S}_n:=\big\{(w_1,\ldots,w_n)\in\{1,\ldots,n\}^n;\ w_1+\ldots+w_n=\binom{n+1}{2}\big\}$, so that $\Lambda^*\mathfrak{nil}_n=\bigoplus_{w\in\mathcal{S}_n}[\![w]\!]$. Notice that $[\![w_1,\ldots,w_{n-1},n]\!]=[\![w_1,\ldots,w_{n-1}]\!]$ and $[\![1,w_2,\ldots,w_n]\!]=[\![w_2-1,\ldots,w_n-1]\!]$.

Example 2.1. Let us take a look at bracket subcomplexes in $\Lambda^*\mathfrak{nil}_n$ for $n\le 4$.

Set $\mathcal{S}_2$ consists of permutations of $(1,2)$. Furthermore, there holds $H_k[\![1,2]\!]=H_k\langle\emptyset\rangle\cong\begin{cases}\mathbb{Z}; & \text{if } k=0\\ 0; & \text{if } k\neq 0\end{cases}$ and $H_k[\![2,1]\!]=H_k\langle e_{12}\rangle\cong\begin{cases}\mathbb{Z}; & \text{if } k=1\\ 0; & \text{if } k\neq 1\end{cases}$.

Set $\mathcal{S}_3$ consists of permutations of $(1,2,3),(2,2,2)$. Furthermore, $H_k[\![1,2,3]\!]=H_k\langle\emptyset\rangle\cong\begin{cases}\mathbb{Z}; & \text{if } k=0\\ 0; & \text{if } k\neq 0\end{cases}$, $H_k[\![1,3,2]\!]=H_k\langle e_{23}\rangle\cong\begin{cases}\mathbb{Z}; & \text{if } k=1\\ 0; & \text{if } k\neq 1\end{cases}$, $H_k[\![2,1,3]\!]=H_k\langle e_{12}\rangle\cong$

$\begin{cases}\mathbb{Z}; & \text{if } k{=}1\\ 0; & \text{if } k{\neq}1\end{cases}$, $H_k[\![2,3,1]\!]=H_k\langle e_{13}e_{23}\rangle\cong\begin{cases}\mathbb{Z}; & \text{if } k{=}2\\ 0; & \text{if } k{\neq}2\end{cases}$, $H_k[\![3,1,2]\!]=H_k\langle e_{12}e_{13}\rangle\cong\begin{cases}\mathbb{Z}; & \text{if } k{=}2\\ 0; & \text{if } k{\neq}2\end{cases}$, $H_k[\![3,2,1]\!]=H_k\langle e_{12}e_{13}e_{23}\rangle\cong\begin{cases}\mathbb{Z}; & \text{if } k{=}3\\ 0; & \text{if } k{\neq}3\end{cases}$, $H_k[\![2,2,2]\!]=H_k\langle e_{13}, e_{12}e_{23}\rangle\simeq 0$.

Set $\mathcal{S}_4$ consists of permutations of (1,1,4,4), (1,2,3,4), (1,3,3,3), (2,2,2,4), (2,2,3,3). The largest complexes are $[\![3{,}2{,}3{,}2]\!]=\langle e_{12}e_{14}e_{24}, e_{13}e_{14}e_{34}, e_{12}e_{14}e_{23}e_{34}, e_{12}e_{13}e_{24}e_{34}\rangle$ and $[\![2{,}3{,}2{,}3]\!]=\langle e_{14}e_{23}, e_{13}e_{24}, e_{12}e_{23}e_{24}, e_{13}e_{23}e_{34}\rangle$, with $[\![3{,}2{,}3{,}2]\!]\cong[\![2{,}3{,}2{,}3]\!]_1$. See the final chapter for the complete computation of $H_*\mathfrak{nil}_4$. $\Diamond$

Lemma 2.2. $[\![w_1,\ldots,w_n]\!]\cong[\![n{+}1{-}w_n,\ldots,n{+}1{-}w_1]\!]$.

Proof. Define $\tau(e_{ab})=e_{n+1-b,n+1-a}$ and $\tau(\wedge_{i=1}^k e_{a_ib_i})=(-1)^{k+1}\wedge_{i=1}^k\tau(e_{a_ib_i})$. Now $e_M\in[\![w_1,\ldots,w_n]\!]$ implies $\tau(e_M)\in[\![n{+}1{-}w_n,\ldots,n{+}1{-}w_1]\!]$, because

$$\begin{aligned}w_{n+1-i}(\tau(e_{a_1b_1}\ldots e_{a_kb_k})) &= w_{n+1-i}(e_{n+1-b_1,n+1-a_1}\ldots e_{n+1-b_k,n+1-a_k})\\ &= n{+}1-i-|\{j;a_j{=}i\}|+|\{j;b_j{=}i\}|\\ &= n{+}1-(i{-}|\{j;b_j{=}i\}|{+}|\{j;a_j{=}i\}|)\\ &= n{+}1-w_i(e_{a_1b_1}\ldots e_{a_kb_k}).\end{aligned}$$

From $[\tau(e_{ab}),\tau(e_{cd})]=-\tau([e_{ab},e_{cd}])$, we obtain

$$\begin{aligned}\partial\tau(e_{a_1b_1}\ldots e_{a_kb_k}) &= (-1)^{k+1}\textstyle\sum_{i<j}(-1)^{i+j}[\tau e_{a_ib_i},\tau e_{a_jb_j}]\ldots\widehat{\tau}(e_{a_ib_i})\ldots\widehat{\tau}(e_{a_jb_j})\ldots\\ &= (-1)^k\textstyle\sum_{i<j}(-1)^{i+j}\tau[e_{a_ib_i},e_{a_jb_j}]\ldots\widehat{\tau}(e_{a_ib_i})\ldots\widehat{\tau}(e_{a_jb_j})\ldots\\ &= \tau\partial(e_{a_1b_1}\ldots e_{a_kb_k}),\end{aligned}$$

so τ is a chain map. Since $\tau\circ\tau=\mathrm{id}$, our τ is an isomorphism of chain complexes. $\square$

Lemma 2.3. $[\![w_1,w_2,\ldots,w_n]\!]\cong[\![w_2,\ldots,w_n,w_1]\!]_{2w_1-n-1}$.

Proof. Define a linear map $\varphi\colon[\![w_1,w_2,\ldots,w_n]\!]\longrightarrow[\![w_2,\ldots,w_n,w_1]\!]_{2w_1-n-1}$ by

$$\varphi\big(\wedge_{i=1}^{n-1}e_{\{i\}\times M_i}\big)=(-1)^{\Sigma M_1}e_{M_1^C\times\{n+1\}}\wedge_{i=2}^{n-1}e_{\{i\}\times M_i},$$

where $M_1^C=\{2,\ldots,n\}\setminus M_1$; it is convenient to have indices in the codomain go from 2 to $n{+}1$ instead of from 1 to n. There holds

$$\begin{aligned}w_i(\varphi(e_M)) &= i-|\{x;e_{x,i+1}\in\varphi(e_M)\}|+|\{y;e_{i+1,y}\in\varphi(e_M)\}|\\ &= i-\big(|\{x;e_{x,i+1}{\in}e_M\}|+\{\begin{smallmatrix}-1; & e_{1,i+1}\in e_M\\ 0; & e_{1,i+1}\notin e_M\end{smallmatrix}\big)+\big(|\{y;e_{i+1,y}{\in}e_M\}|+\{\begin{smallmatrix}1; & e_{1,i+1}\notin e_M\\ 0; & e_{1,i+1}\in e_M\end{smallmatrix}\big)\\ &= i{+}1-|\{x;e_{x,i+1}{\in}e_M\}|+|\{y;e_{i+1,y}{\in}e_M\}|=w_{i+1}(e_M)\quad\text{for } i{<}n \quad\text{and}\\ w_n(\varphi(e_M)) &= n-|\{x;e_{x,n+1}\in\varphi(e_M)\}|+|\{y;e_{n+1,y}\in\varphi(e_M)\}|\\ &= n-(n{-}1{-}|M_1|)+0=1-0+|M_1|=w_1(e_M).\end{aligned}$$

Length difference of e_M and $\varphi(e_M)$ is $(n{-}1{-}|M_1|)-|M_1|=n{-}1{-}2(w_1{-}1)=n{+}1{-}2w_1$. Thus φ is a well-defined bijection. Denoting $M\setminus x\cup y:=(M\setminus\{x\})\cup\{y\}$, we have

$$\begin{aligned}&\varphi\partial(e_{\{1\}\times M_1}e_N)=\varphi\big(\textstyle\sum_{x\in M_1,\,y\in N_x\setminus M_1}\varepsilon_{xy}e_{\{1\}\times(M_1\setminus x\cup y)}e_{N\setminus\{(x,y)\}}+(-1)^{|M_1|}e_{\{1\}\times M_1}\partial e_N\big)\\ &=\sum_{x\in M_1,\,y\in N_x\setminus M_1}(-1)^{y-x+\Sigma M_1}\varepsilon_{xy}e_{(M_1\setminus x\cup y)^C\times\{n+1\}}e_{N\setminus\{(x,y)\}}+(-1)^{|M_1|+\Sigma M_1}e_{M_1^C\times\{n+1\}}\partial e_N,\\ &\partial\varphi(e_{\{1\}\times M_1}e_N)=\partial\big((-1)^{\Sigma M_1}e_{M_1^C\times\{n+1\}}e_N\big)\\ &=\sum_{y\in N_x\cap M_1^C,\,x\notin M_1^C}(-1)^{\Sigma M_1}(-\varepsilon'_{xy})e_{(M_1^C\setminus y\cup x)\times\{n+1\}}e_{N\setminus\{(x,y)\}}+(-1)^{\Sigma M_1+|M_1^C|}e_{M_1^C\times\{n+1\}}\partial e_N\\ &=\sum_{x\in M_1,\,y\in N_x\setminus M_1}(-1)^{1+\Sigma M_1}\varepsilon'_{xy}e_{(M_1\setminus x\cup y)^C\times\{n+1\}}e_{N\setminus\{(x,y)\}}+(-1)^{n-1+|M_1|+\Sigma M_1}e_{M_1^C\times\{n+1\}}\partial e_N.\end{aligned}$$

for $\varepsilon_{xy},\varepsilon'_{xy}\in\{1,-1\}$. Since $[e_{y,n+1},e_{x,y}]=-e_{x,n+1}$, there is a minus before ε'_{xy}. We must show that $(-1)^{y-x}\varepsilon_{xy}=(-1)^n\varepsilon'_{xy}$: if $\alpha=$(position of x in M_1), $\beta=$(position of (x,y) in N), $\gamma=$(position of y in $M_1\setminus x\cup y$), then y in M_1^C has position $y{-}\gamma{-}1$ and x in $M_1^C\setminus y\cup x$ has position $x{-}\alpha$, so $(-1)^{y-x}\varepsilon_{xy}=(-1)^{y-x+\alpha+(|M_1|+\beta)+(\gamma-1)}=$

$(-1)^{n+(y-\gamma-1)+(n-1-|M_1|+\beta)+(x-\alpha-1)} = (-1)^n\varepsilon'_{xy}$. Therefore $\varphi\partial = (-1)^{n-1}\partial\varphi$, hence $\overline{\varphi}(e_M) := \begin{cases}\varphi(e_M)(-1)^{n-1}; & \text{if } |M|\in 2\mathbb{N}\\ \varphi(e_M); & \text{if } |M|\notin 2\mathbb{N}\end{cases}$ is an isomorphism of chain complexes. □

Lemma 2.4. *$[\![w_1,\dots,w_{k-1},n,w_{k+1},\dots,w_n]\!] \cong [\![w_1,\dots,w_{k-1},w_{k+1},\dots,w_n]\!]_{n-k}$ and $[\![w_1,\dots,w_{k-1},1,w_{k+1},\dots,w_n]\!] \cong [\![w_1-1,\dots,w_{k-1}-1,w_{k+1}-1,\dots,w_n-1]\!]_{k-1}$.*

Proof. We can identify $[\![w_1,\dots,w_{n-1},n]\!]$ with $[\![w_1,\dots,w_{n-1}]\!]$. By Lemma 2.3,

$$\begin{aligned}[\![w_1,\dots,w_{k-1},n,w_{k+1},\dots,w_n]\!] &\cong [\![w_{k+1},\dots,w_n,w_1,\dots,w_{k-1},n]\!]_{-2\sum_{i=k+1}^n w_i+(n-k)(n+1)}\\ &\cong [\![w_{k+1},\dots,w_n,w_1,\dots,w_{k-1}]\!]_{-2\sum_{i=k+1}^n w_i+(n-k)(n+1)}\\ &\cong [\![w_1,\dots,w_{k-1},n,w_{k+1},\dots,w_n]\!]_{(n-k)\cdot 1}.\end{aligned}$$

We can identify $[\![1,w_2,\dots,w_n]\!]$ with $[\![w_2-1,\dots,w_n-1]\!]$. By Lemma 2.3,

$$\begin{aligned}[\![w_1,\dots,w_{k-1},1,w_{k+1},\dots,w_n]\!] &\cong [\![1,w_{k+1},\dots,w_n,w_1,\dots,w_{k-1}]\!]_{2\sum_{i=1}^{k-1} w_i-(k-1)(n+1)}\\ &\cong [\![w_{k+1}-1,\dots,w_n-1,w_1-1,\dots,w_{k-1}-1]\!]_{2\sum_{i=1}^{k-1} w_i-(k-1)(n+1)}\\ &\cong [\![w_1-1,\dots,w_{k-1}-1,w_{k+1}-1,\dots,w_n-1]\!]_{2(k-1)-(k-1)}.\end{aligned}$$

This establishes the first and second part of the claim. □

If all elements in a sequence $w = (w_1,\dots,w_n)$ are distinct (i.e. w is a permutation of $(1,\dots,n)$), then by applying Lemma 2.4 n times, we see that $[\![w]\!]$ has only one generator, namely $e_M = \bigwedge_{i<j,w_i>w_j} e_{ij}$, the inversions of w. Indeed, $w_k(e_M) = k-|\{i; e_{ik}\in e_M\}|+|\{j; e_{kj}\in e_M\}| = k-|\{i; i<k, w_i>w_k\}|+|\{j; k<j, w_k>w_j\}| = 1+|\{i; i<k, w_i<w_k\}|+|\{j; k<j, w_k>w_j\}| = 1+|\{r; r\neq k, w_r<w_k\}| = w_k$.

Let $\mathcal{F}_n = \{(w_1,\dots,w_n)\in\mathcal{S}_n;\ w_i\neq w_j \text{ for } i\neq j\}$. Then $\bigoplus_{w\in\mathcal{F}_n}[\![w]\!]$ is a submodule of the free part $FH_*(\mathfrak{nil}_n)$. In Lemma 4.2, we will show that for every $w\notin\mathcal{F}_n$ the homology of the complex $[\![w]\!]$ has only torsion, so $FH_*(\mathfrak{nil}_n) \cong \bigoplus_{w\in\mathcal{F}_n}[\![w]\!]$. Thus we can calculate $FH_*(\mathfrak{nil}_n)$, which was already known by Kostant [4, Theorem 5.14], who used the Laplacian operator on $\Lambda^*\mathfrak{nil}_n$ to get the following result:

Theorem 2.5. *$FH_k(\mathfrak{nil}_n) \cong \mathbb{Z}^{T(n-1,k)}$, where $T(n-1,k)$ is a Mahonian number.*

Proof. Mahonian numbers (OEIS A008302) are given by the recurrence $T(0,0)=1$, $T(0,k)=0$ for $k\neq 0$, and $T(n,k)=\sum_{i=0}^{n-1}T(n-1,k-i)$ for $n>0$. Since $T(1,k) = 1,1,0,\dots$, the theorem is true for $n=2$. By Lemma 2.4, we have $F_* := \bigoplus_{w\in\mathcal{F}_n}[\![w]\!] = \bigoplus_{i\in[n],w\in\mathcal{F}_{n-1}}[\![w_1,\dots,w_{i-1},n,w_i,\dots,w_{n-1}]\!] \cong \bigoplus_{i\in[n],w\in\mathcal{F}_{n-1}}[\![w]\!]_{n-i}$. By induction, $\operatorname{rank} F_k = \sum_{i=1}^n T(n-2,k-(n-i)) = \sum_{j=0}^{n-1} T(n-2,k-j) = T(n-1,k)$. □

3. Filtrations

Let $w = (2,w_2,\dots,w_n)$, so every wedge in $[\![w]\!]$ contains exactly one e_{1*}. There is a natural filtration of $[\![w]\!]$ by subcomplexes: if F_k^w is spanned by $\{e_{1i}e_M; i\geq k\}$, then $0 = F_{n+1}^w \leq F_n^w \leq \dots \leq F_2^w = [\![w]\!]$. The quotient F_k^w/F_{k+1}^w has generators $\{[e_{1k}e_M];\ e_{1k}e_M\in F_k^w\}$ and boundary $\partial[e_{1k}e_M] = -[e_{1k}\,\partial e_M]$, therefore

$$\begin{aligned}F_k^w/F_{k+1}^w &\cong [\![1,w_2,\dots,w_{k-1},w_k+1,w_{k+1},\dots,w_n]\!]_1\\ &\cong [\![w_2-1,\dots,w_{k-1}-1,w_k,w_{k+1}-1,\dots,w_n-1]\!]_1.\end{aligned} \tag{3.1}$$

Lemma 3.1. *If $w_i = w_j = w_k \in \{2,n-1\}$ for distinct i,j,k, then $H_*[\![w_1,\dots,w_n]\!] \cong 0$.*

Proof. By Lemmas 2.3 and 2.2, we may assume that $i=1$ and $w_i=w_j=w_k=2$. For any $r\notin\{j,k\}$ there holds $F^w_r/F^w_{r+1}\cong[\![w_2{-}1,\ldots,1,\ldots,w_r,\ldots,1,\ldots,w_n{-}1]\!]_1\cong[\![w_2{-}2,\ldots,0,\ldots,w_r{-}1,\ldots,w_n{-}2]\!]_k\cong 0$, by (3.1) and Lemma 2.4. Thus we have $0=F^w_n=\ldots=F^w_{k+1}<F^w_k=\ldots=F^w_{j+1}<F^w_j=\ldots=[\![w]\!]$ and a long exact sequence of a pair $\ldots\to H_{n+1}\frac{[\![w]\!]}{F^w_k}\overset{\chi}{\to}H_nF^w_k\to H_n[\![w]\!]\to H_n\frac{[\![w]\!]}{F^w_k}\overset{\chi}{\to}H_{n-1}F^w_k\to\ldots.$ To prove $H_*[\![w]\!]\cong 0$, it suffices to show that χ is an isomorphism, where $\chi(x+F^w_k)=[\partial(x)]$.

Let $x\in[\![w]\!]/F^w_k=F^w_j/F^w_{j+1}$, so $x=e_{1j}\cdots$. By $w_k=2$, $x=e_{1j}e_{\{2,\ldots,j,\ldots,k-1\}\times\{k\}}\cdots$. By $w_j=2$, $x=[e_{1j}e_{\{2,\ldots,j-1\}\times\{j\}}e_{\{2,\ldots,j,\ldots,k-1\}\times\{k\}}e_M]$ with no indices j and k in M.

Let $y\in F^w_k=F^w_k/F^w_{k+1}$, so $y=e_{1k}\cdots$. Since $w_j=2$, $y=e_{1k}e_{\{2,\ldots,j-1\}\times\{j\}}\cdots$. Since $w_k=2$, $y=e_{1k}e_{\{2,\ldots,j-1\}\times\{j\}}e_{\{2,\ldots,\hat{j},\ldots,k-1\}\times\{k\}}e_M$ with no indices j and k in M.

Since $H_n\frac{[\![w]\!]}{F^w_k}=\frac{\mathrm{Ker}\partial}{\mathrm{Im}\partial}$, its elements are sent by ∂ to F^w_k, so in x the only multiplication is $[e_{1j},e_{jk}]=e_{1k}$. Thus χ sends $x\mapsto y$ and is bijective. $\square$

Lemma 3.2. $[\![\ldots,2,2,\ldots]\!]\simeq 0$ *and* $[\![\ldots,n{-}1,n{-}1,\ldots]\!]\simeq 0$.

Proof. By Lemmas 2.3 and 2.2, it suffices to show that $[\![w]\!]:=[\![2,2,w_3,\ldots,w_n]\!]\simeq 0$. Now $[\![w]\!]$ consists of $e_{1i}e_M$ with $i\geq 3$ and $e_{12}e_{2i}e_M$, where 2 is not an index in M. Hence $\mathcal{M}=\{e_{12}e_{2i}e_M\to e_{1i}e_M;\ e_{1i}e_M\in[\![w]\!]\}$ is a Morse matching with $\mathring{\mathcal{M}}=\emptyset$. $\square$

Lemma 3.3. *Let* $w=(2,w_2,w_3,\ldots,w_n)$ *and* $w'=(2,w_3,\ldots,w_n,w_2)$*. Then* $F^w_3\cong[\![w']\!]_{2w_2-n-2}/F^{w'}_n$*. If* $H_*[\![w_2,w_3{-}1,\ldots,w_n{-}1]\!]\cong 0$*, then* $H_*[\![w]\!]\cong H_*[\![w']\!]_{2w_2-n-2}$.

Proof. Define a linear map $\varphi\colon F^w_3\to[\![w']\!]_{2w_2-n-2}/F^{w'}_n$ by

$$\varphi\big(e_{1b}\wedge_{i=2}^{n-1}e_{\{i\}\times M_i}\big)=(-1)^{\Sigma M_2}[e_{1b}\,e_{M_2^C\times\{n+1\}}\wedge_{i=3}^{n-1}e_{\{i\}\times M_i}],$$

where $M_2^C=\{3,\ldots,n\}\backslash M_2$ and indices in the codomain are $1,3,\ldots,n{+}1$. Our φ is a bijection and proof that it is a chain map is similar to the one in Lemma 2.3.

Let $H_*[\![w_2,w_3-1,\ldots,w_n-1]\!]\cong 0$, which by (3.1) is $H_*(F^w_2/F^w_3)$. By the long exact sequence and first part, $H_*[\![w]\!]=H_*F^w_2\cong H_*F^w_3\cong H_*([\![w']\!]_{2w_2-n-2}/F^{w'}_n)$. By (3.1), $H_*F^{w'}_n\cong H_*[\![w_3-1,\ldots,w_n-1,w_2]\!]\cong H_*[\![w_2,w_3-1,\ldots,w_n-1]\!]_{n-2w_2}\cong 0$, so by the long exact sequence, $H_*([\![w']\!]/F^{w'}_n)\cong H_*[\![w']\!]$ and the result follows. $\square$

Recall that any chain map $\varphi\colon B_*\to C_*$ induces a chain complex $D_*=\mathrm{Cone}\,\varphi$, where $D_n=B_{n-1}\oplus C_n$ and $\partial(b,c)=\big(\partial(b),\varphi(b)-\partial(c)\big)$. Furthermore, there is an exact sequence $\ldots\to H_{n+1}D_*\to H_nB_*\overset{\varphi_*}{\to}H_nC_*\to H_nD_*\to H_{n-1}B_*\overset{\varphi_*}{\to}\ldots$

Lemma 3.4. *Let* $w=(2,w_2,\ldots,w_k,3,3,w_{k+3},\ldots,w_n)$ *and* $w'=(w_2{-}2,\ldots,w_k{-}2,3,w_{k+3}-2,\ldots,w_n-2)$*. Then* $H_*[\![w]\!]\cong H_*\mathrm{Cone}\big([\![w']\!]_k\xrightarrow{\cdot 2}[\![w']\!]_k\big)$.

Proof. Let $k{=}1$, so $w=(2,3,3,\ldots)$. By (3.1) and Lemma 3.2, $F^w_k/F^w_{k+1}\simeq 0$ for $k\geq 4$, so $H_*F^w_4\cong 0$ and $H_*[\![w]\!]\cong H_*([\![w]\!]/F^w_4)$. There are 4 types of generators in $[\![w]\!]/F^w_4$:

- $A=\{[e_{12}e_{23}e_{2a}e_{3b}e_M];$ all indices in M are $\geq 4\}$,
- $B=\{[e_{13}e_{23}e_{3a}e_{3b}e_M];$ all indices in M are $\geq 4\}$,
- $C=\{[e_{12}e_{2a}e_{2b}e_M];$ all indices in M are $\geq 4\}$,
- $D=\{[e_{13}e_{2a}e_{3b}e_M];$ all indices in M are $\geq 4\}$.

The set $\mathcal{M}=\{A\ni e_{12}e_{23}e_{2a}e_M\to e_{13}e_{2a}e_M\in D\}$ is a Morse matching, with critical elements $\mathring{\mathcal{M}}=B\cup C$. Nontrivial zig-zag paths go from B to C and come in pairs:

$$\begin{array}{ccc} [e_{13}e_{23}e_{3a}e_{3b}e_M] & \xrightarrow{\ 1\ } & [e_{13}e_{2a}e_{3b}e_M] \\ & \overset{-1}{\nearrow\!\!\swarrow} & \\ [e_{12}e_{23}e_{2a}e_{3b}e_M] & \xrightarrow{\ 1\ } & [e_{12}e_{2a}e_{2b}e_M] \end{array} \quad \text{and} \quad \begin{array}{ccc} [e_{13}e_{23}e_{3a}e_{3b}e_M] & \xrightarrow{\ -1\ } & [e_{13}e_{2b}e_{3a}e_M] \\ & \overset{-1}{\nearrow\!\!\swarrow} & \\ [e_{12}e_{23}e_{2b}e_{3a}e_M] & \xrightarrow{\ -1\ } & [e_{12}e_{2a}e_{2b}e_M], \end{array}$$

which add up to $\cdot 2$. We have $\langle \mathring{\mathcal{M}} \rangle / \langle C \rangle \cong [\![w']\!]_2$ (omit $e_{13}e_{23}$ and indices $1,2$) and $\langle C \rangle \cong [\![w']\!]_1$ (omit e_{12} and indices $1,3$), so $H_*[\![w]\!] \cong H_* \langle \mathring{\mathcal{M}} \rangle \cong H_* \mathrm{Cone}\big([\![w']\!]_1 \xrightarrow{\cdot 2} [\![w']\!]_1\big)$.

Finally, if $k \geq 2$, then $H_*[\![w]\!] \cong H_*[\![2,3,3,w_{k+3},\ldots,w_n,w_2,\ldots,w_k]\!]_{\sum_{i=2}^{k}(2w_i-n-2)}$ $\cong H_* \,\mathrm{Cone}\big(\cdot 2 \circlearrowright [\![3,w_{k+3}-2,\ldots,w_n-2,w_2-2,\ldots,w_k-2]\!]_{1+\sum_{i=2}^{k}(2w_i-n-2)}\big) \cong H_* \,\mathrm{Cone}$ $\big(\cdot 2 \circlearrowright [\![w_2-2,\ldots,w_k-2,3,w_{k+3}-2,\ldots,w_n-2]\!]_{1+\sum_{i=2}^{k}(2w_i-n-2)-\sum_{i=2}^{k}(2(w_i-2)-(n-1))}\big) \cong$ $H_* \mathrm{Cone}\big([\![w']\!]_k \xrightarrow{\cdot 2} [\![w']\!]_k\big)$ by Lemmas 3.2, 3.3, 2.3, so the job is done. □

Lemma 3.5. *Let* $w = (2, w_2, \ldots, w_{k-1}, 2, w_{k+1}, \ldots, w_n)$.

(1) Let $A = \{e_{\{1,\ldots,k-1\}\times\{k\}} e_{ka} e_M \in [\![w]\!]\}$ *and* $B = \{e_{1a} e_{\{2,\ldots,k-1\}\times\{k\}} e_M \in [\![w]\!];\ a > k\}$. *There exists a Morse matching* $\mathcal{M}$ *for* $[\![w]\!]$, *such that* $\mathring{\mathcal{M}} = A \cup B$, $\mathring{\partial}|_B = \partial|_B$,

$$\mathring{\partial}|_A\colon e_{\{1,\ldots,k-1\}\times\{k\}} e_{ka} e_M \mapsto (-1)^{k+1}\big(e_{\{1,\ldots,k-1\}\times\{k\}} \partial(e_{ka}e_M) + n_M e_{1a} e_{\{2,\ldots,k-1\}\times\{k\}} e_M\big) + \textstyle\sum_{(b,c)\in X} (-1)^{\epsilon_{bc}+k+1} e_{1c} e_{\{2,\ldots,k-1\}\times\{k\}} e_{ba} e_{M\setminus\{(b,c)\}},$$

where $n_M = |\{b \in \{1,\ldots,k-1\};\ (b,a) \notin M\}|$, $\epsilon_{bc} = ($*position of* (b,c) *in* $M)$, *and* $X = \{(b,c) \in M;\ b < k < c,\ (b,a) \notin M\}$.

(2) $H_*[\![w]\!] \cong H_* \,\mathrm{Cone}\,\varphi$ *for some chain map* $\varphi\colon F^w_{k+1} \to F^w_{k+1}$.

(3) $H_*[\![2,w_2,\ldots,w_{n-2},2,w_n]\!] \cong H_* \,\mathrm{Cone}\big(\cdot(w_n-1) \circlearrowright [\![w_2-1,\ldots,w_{n-2}-1,1,w_n]\!]_1\big)$.

Proof. *(1):* There are four types of generators in $[\![w]\!]$: A, B,

$C = \{e_{1a} e_{\{2,\ldots,k-1\}\times\{k\}} e_M;\ a < k,$ there is no index 1 or k in $M\}$,

$D = \{e_{\{1,\ldots,\hat{a},\ldots,k-1\}\times\{k\}} e_M;\ 1 < a < k,$ there is no index 1 or k in $M\}$.

Set $\mathcal{M} = \big\{C \ni e_{1a} e_{\{2,\ldots,k-1\}\times\{k\}} e_M \to e_{\{1,\ldots,\hat{a},\ldots,k-1\}\times\{k\}} e_M \in D\big\}$ is a Morse matching, with $\mathring{\mathcal{M}} = A \cup B$. Zig-zag paths starting in B are arrows and end in B, so $\mathring{\partial}|_B = \partial|_B$. Zig-zag paths starting in A are $e_{\{1,\ldots,k-1\}\times\{k\}} e_{ka} e_M \xrightarrow{(-1)^{k+1}} e_{1a} e_{\{2,\ldots,k-1\}\times\{k\}} e_M$ and

$$\begin{array}{lll} e_{\{1,\ldots,k-1\}\times\{k\}} e_{ka} e_M & \xrightarrow{\qquad (-1)^b \qquad} & e_{\{1,\ldots,\hat{b},\ldots,k-1\}\times\{k\}} e_{ba} e_M \\ & \swarrow (-1)^{b+1} & \\ e_{1b} e_{\{2,\ldots,k-1\}\times\{k\}} e_{ba} e_M & \xrightarrow{\ (-1)^{k+1}\ } & e_{1a} e_{\{2,\ldots,k-1\}\times\{k\}} e_M \quad \text{for } n_M \text{ choices,} \\ & \searrow (-1)^{\epsilon_{bc}+k+1} & \\ & & e_{1c} e_{\{2,\ldots,k-1\}\times\{k\}} e_{ba} e_{M\setminus\{(b,c)\}} \quad \text{for } (b,c) \in X. \end{array}$$

(2): Follows from (1), because $\langle B \rangle = F^w_{k+1}$ and $\langle \mathring{\mathcal{M}} \rangle / \langle B \rangle \cong \langle B \rangle_1$ (we mod out B, so 2nd and 3rd summand in $\mathring{\partial}|_A$ are 0, thus $e_{\{1,\ldots,k-1\}\times\{k\}} e_{ka} e_M \mapsto e_{1a} e_{\{2,\ldots,k-1\}\times\{k\}} e_M$ is a chain isomorphism). Ergo φ is the part of $\mathring{\partial}|_A$ that goes to B.

(3): Follows from (2), since $F^w_{k+1} = F^w_n \cong [\![w']\!]_1$ (omit e_{1n} and index 1) and $X = \emptyset$ and $n_M = |\{b \in \{1,\ldots,n-2\};\ (b,n) \notin M\}| = n-1-(n-w_n) = w_n - 1$. □

Dwyer [1] reports how Kunkel proved that $H_* \mathfrak{nil}_n$ has p-torsion for prime $p < n-1$. Now we can easily see that $H_* \mathfrak{nil}_n$ also has p^m-torsion for every $p^m < n-1$:

Example 3.6. Let $q = p^m = n-2$ and $w = (2,3,\ldots,q+1,2,q+1)$. By Lemma 3.5, $H_*[\![w]\!] \cong H_* \,\mathrm{Cone}([\![w']\!]_1 \xrightarrow{\cdot q} [\![w']\!]_1)$. Since $w' = (2,\ldots,q,1,q+1)$ is a permutation of $(1,\ldots,q+1)$ and $|\{(i,j);\ i<j, w'_i > w'_j\}| = q-1$, we have $H_k[\![w']\!] \cong \begin{cases} \mathbb{Z}; & \text{if } k=q-1 \\ 0; & \text{if } k \neq q-1 \end{cases}$, so $H_k[\![w]\!] \cong \begin{cases} \mathbb{Z}_q; & \text{if } k=q \\ 0\ ; & \text{if } k \neq q \end{cases}$. If $q < n-2$, then $H_*[\![w,q+3,\ldots,n]\!] \cong H_*[\![w]\!]$. ◊

In [1], Dwyer proved that there is no p-torsion in $H_*\mathfrak{nil}_n$ for any prime $p\geq n-1$. The next example shows that $H_*\mathfrak{nil}_n$ can have p^m-torsion for some $p^m\geq n-1$:

Example 3.7. Let $w=(2,4,7,5,4,2,5,7)$. By Lemma 3.5, $H_*[\![w]\!]\cong H_*\mathrm{Cone}(\varphi\circlearrowright F_7^w)$. By (3.1), $F_8^w\cong[\![3,6,4,3,1,4,7]\!]_1\cong[\![3,6,4,3,1,4]\!]_1\cong[\![2,5,3,2,3]\!]_5\cong[\![2,3,2,3]\!]_8$ has $H_*\cong\mathbb{Z}_2$ generated by $[e_{18}e_M]$ with $M=\{(2,7),(4,5)\}\cup\{(i,6);i=2,\ldots,5\}\cup\{(3,i);i=4,5,7\}$.

By Lemmas 2.4, 2.2, 3.5, we get

$$\begin{aligned}H_*F_7^w/F_8^w &\cong H_*[\![3,6,4,3,1,5,6]\!]_1\cong H_*[\![2,5,3,2,4,5]\!]_5\cong\\ &\cong H_*[\![2,3,5,4,2,5]\!]_5\cong H_*\,\mathrm{Cone}\big(\cdot 4\circlearrowright[\![2,4,3,1,5]\!]_6\big).\end{aligned}$$

Because $H_*[\![2,4,3,1,5]\!]\cong\mathbb{Z}$ generated by $[e_{14}e_{23}e_{24}e_{34}]$, we have $H_*[\![2,3,5,4,2,5]\!]\cong\mathbb{Z}_4$ generated by $[e_{16}e_N]$, where $N=\{(2,3),(2,4),(2,5),(3,4)\}$ and

$$\partial\big(\textstyle\sum_{i=2}^5 e_{1i}e_{i6}e_M\big)=-4e_{16}e_M\in[\![2,3,5,4,2,5]\!]. \tag{3.2}$$

Then $[e_{16}e_{N'}]$ generates $H_5[\![2,5,3,2,4,5]\!]\cong H_5[\![2,3,5,4,2,5]\!]$, where $N'=\{(7-y,7-x);(x,y)\in N\}$, and $[e_{17}e_{28}e_P]$ generates $H_{10}F_7^w/F_8^w\cong H_{10}[\![2,5,3,2,4,5]\!]_5$, where $P=\{(3,4),(3,5),(3,7),(4,5)\}\cup\{(i,6);i=2,\ldots,5\}$. From the long exact sequence for $F_8^w\leq F_7^w$ we get that $H_*F_7^w\cong 0$ for $*\neq 10$ and $H_{10}F_7^w$ is an extension of $H_{10}F_7^w/F_8^w\cong\mathbb{Z}_4$ by $H_{10}F_7^w\cong\mathbb{Z}_2$, so it is isomorphic to $\mathbb{Z}_4\times\mathbb{Z}_2$ or $\mathbb{Z}_8$. By (3.2),

$$\partial\big(\textstyle\sum_{i\in\{3,4,5,7\}}[e_{17}e_{2i}e_{i8}e_P]\big)=4[e_{17}e_{28}e_P]\in F_7^w/F_8^w,\quad\text{but}$$

$$\partial\big(\textstyle\sum_{i\in\{3,4,5,7\}} e_{17}e_{2i}e_{i7}e_P\big)=-4e_{17}e_{27}e_P+e_{18}e_M\in F_7^w. \tag{3.3}$$

Thus $[e_{17}e_{27}e_P]$ is of order $\neq 4$, so $H_{10}F_7^w\cong\mathbb{Z}_8$. If $R=\{(2,8),(3,4),(3,5),(3,7),(4,5)\}$, then $e_{16}e_{67}e_{27}e_P=e_{\{1,\ldots,5\}\times\{6\}}e_{67}e_R$. By Lemma 3.5,

$$\mathring{\partial}(e_{\{1,\ldots,5\}\times\{6\}}e_{67}e_R)=\pm n_R e_{17}e_{\{2,\ldots,5\}\times\{6\}}e_R\pm e_{18}e_{\{2,\ldots,5\}\times\{6\}}e_{27}e_{R\setminus\{(2,8)\}},$$

where $n_R=|b\in\{1,\ldots,5\};\,(b,7)\notin R\}|=4$. By (3.3), $\mathring{\partial}(e_{\{1,\ldots,5\}\times\{6\}}e_{67}e_R)$ is trivial or $\pm 8e_{17}e_{2i}e_{i8}e_P$. In both cases the morphism $\varphi\colon F_7^w\to F_7^w$ is homologically trivial. Consequently, we can conclude that $H_k[\![w]\!]\cong\begin{cases}\mathbb{Z}_8; & \text{if } k\in\{10,11\}\\ 0\ ; & \text{if } k\notin\{10,11\}\end{cases}$. $\lozenge$

4. Free part of homology

We can generalise the filtration from the previous section to an arbitrary complex $[\![w_1,\ldots,w_n]\!]=[\![w]\!]$. In every $e_M\in[\![w]\!]$, there are exactly $t:=w_1-1$ occurences of e_{1*}. Thus $F_k^w:=\big\langle e_{1i_1}\ldots e_{1i_t}e_M;\,i_1+\ldots+i_t\geq k,\,1\text{ is not in }M\big\rangle\leq[\![w]\!]$ is a subcomplex.

Define $w(i_1,\ldots,i_t)\in\mathcal{S}_{n-1}$ as $(w_2',\ldots,w_n')$, where $w_j'=\begin{cases}w_j & ;\ \text{if } j\in\{i_1,\ldots,i_t\}\\ w_j-1; & \text{if } j\notin\{i_1,\ldots,i_t\}\end{cases}$. Then

$$F_k^w/F_{k+1}^w\cong\textstyle\bigoplus_{i_1+\ldots+i_t=k}[\![w(i_1,\ldots,i_t)]\!]_t. \tag{4.1}$$

Example 4.1. Let us compute $H_*[\![3,3,3,3,3]\!]$. By Lemmas 3.2, 3.5, (4.1), $F_{10}^w=0$, $F_9^w/F_{10}^w\cong[\![2,2,3,3]\!]_2\simeq 0$, $H_*F_8^w/F_9^w\cong H_*[\![2,3,2,3]\!]_2\cong\mathbb{Z}_2$ generated by $[e_{13}e_{15}\,e_{24}e_{35}]$, $F_7^w/F_8^w\cong[\![3,2,2,3]\!]_2\oplus[\![2,3,3,2]\!]_2\simeq 0$, $H_*F_6^w/F_7^w\cong H_*[\![3,2,3,2]\!]_2\cong H_*[\![2,3,2,3]\!]_3\cong\mathbb{Z}_2$ generated by $[e_{12}e_{14}\,e_{23}e_{25}e_{35}]$, $F_5^w/F_6^w\cong[\![3,3,2,2]\!]_2\simeq 0$, $F_5^w=[\![w]\!]$. Thus $H_*[\![w]\!]\cong H_*F_6$ and $H_*F_7^w\cong H_*F_8^w\cong H_*F_8^w/F_9^w\cong\mathbb{Z}_2$ generated by $[e_{13}e_{15}\,e_{24}e_{35}]$. In the exact sequence $\ldots\to H_{k+1}\frac{F_6^w}{F_7^w}\xrightarrow{\chi}H_kF_7^w\to H_kF_6^w\to H_k\frac{F_6^w}{F_7^w}\xrightarrow{\chi}H_{k-1}F_7^w\to\ldots$ our χ sends

$$\begin{aligned}&[e_{12}e_{14}\,e_{23}e_{25}e_{35}]\mapsto[e_{13}e_{14}e_{25}e_{35}+e_{14}e_{15}e_{23}e_{35}]=\\ &\qquad[-e_{13}e_{15}e_{24}e_{35}-\partial(e_{13}e_{15}e_{23}e_{34}e_{35})+\partial(e_{13}e_{14}e_{25}e_{34}e_{45})].\end{aligned}$$

It is an isomorphism, hence by exactness, $H_*[\![3,3,3,3,3]\!]\cong H_*F_6^w\cong 0$. $\lozenge$

Lemma 4.2. *For $w\in\mathcal{S}_n\backslash\mathcal{F}_n=:\mathcal{T}_n$, the homology of $[\![w]\!]$ has only torsion.*

Proof. The proof is by induction on n. The claim is trivial for $n=2$ because $\mathcal{T}_2=\emptyset$. Let $n>2$ and $w\in\mathcal{T}_n$. By Lemma 2.3 we may assume that $w_1\leq w_i$ for all i. If $w_1=1$, then by Lemma 2.4, $[\![w]\!]\cong[\![w_2{-}1,\dots w_n{-}1]\!]$, the claim holds by induction. If $w_1\geq 3$, all elements of the sequence $w(i_1,\dots,i_{w_1-1})$ are at least 2, so $w(i_1,\dots,i_{w_1-1})\in\mathcal{T}_{n-1}$. By (4.1) and induction, $FH_*(F^w_k/F^w_{k+1})\cong 0$ for all k, hence $FH_*[\![w]\!]\cong 0$.

Now let $w_1=2$. If $w(i)\in\mathcal{T}_{n-1}$ for all $i\in\{2,\dots n\}$, then the same argument as for the case $w_1\geq 3$ shows that $FH_*[\![w]\!]\cong 0$. Suppose there exists i such that $w(i)\in\mathcal{F}_{n-1}$. Then $w_i=n-1$, $w_1=2$, and the other elements of the sequence w form a permutation of $(2,\dots,n-1)$, so there are exactly two numbers $j<i$ such that $w(j),w(i)\in\mathcal{S}_{n-1}$. If the second number 2 (the first one is $w_1=2$) appears after the position i, we can use Lemma 3.5 to show that the free part of $H_*[\![w]\!]$ is trivial. Anyway, in the filtrations of $[\![w]\!]$ there are exactly two subquotients with nontrivial free part of homology, and in both cases the rank of free part is 1. So to prove that $FH_*[\![w]\!]\cong 0$, it is enough to show that some nontrivial multiple of the generator of $FH_*F^w_{j+1}$ is in the image of the boundary morphism of the long exact sequence of the pair $F^w_{j+1}\leq F^w_j$. Because $FH_*F^w_{i+1}\cong 0$, the pair $F^w_{j+1}\leq F^w_j$ may be replaced with the pair $F^w_{j+1}/F^w_{i+1}\leq F^w_j/F^w_{i+1}$. Because $F^w_i/F^w_{i+1}\cong[\![w(i)]\!]$ and $w(i)\in\mathcal{F}_{n-1}$, a generator of $FH_*(F^w_i/F^w_{i+1})$ is of a form $[e_{1i}e_M]=:[x]$, such that $\partial e_M=0$ and no multiple of e_M is in $\mathrm{Im}\partial$. For $a\in\{2,\dots,i-1\}$ let $x_a=e_{1a}e_{M_a}$, where $M_a=M\cup\{(a,i)\}$. Let $n_a(b,c)$ be the position of (b,c) in M_a. Then

$$\begin{aligned}
&\partial\big(\textstyle\sum_{a=2}^{i-1}(-1)^{n_a(a,i)}[x_a]\big)\\
&=\textstyle\sum_{a=2}^{i-1}\Big((-1)^{n_a(a,i)+1+n_a(a,i)}[x]+\sum_{(a,b)\in M_a}(-1)^{n_a(a,i)+1+n_a(a,b)}[e_{1b}e_{M_a\backslash\{(a,b)\}}]\\
&+\textstyle\sum_{(c,a)\in M_a}(-1)^{n_a(a,i)+n_a(c,a)+n_a(a,i)+n_c(c,i)-2}[e_{1a}e_{M_c-\{(c,a)\}}]\Big)\\
&=-(i-2)[x]+\textstyle\sum_{a=2}^{i-1}\sum_{(a,b)\in M_a}(-1)^{n_a(a,i)+n_a(a,b)+1}[e_{1b}e_{M_a\backslash\{(a,b)\}}]\\
&+\textstyle\sum_{a=2}^{i-1}\sum_{(c,a)\in M_a}(-1)^{n_c(c,i)+n_a(c,a)-2}[e_{1a}e_{M_c\backslash\{(a,b)\}}].
\end{aligned}$$

Because $[e_{1b}e_M]=0$ for $b>i$, the middle sum runs only over $(a,b)\in M_a$ with $b\leq i$. Because $n_a(a,b)=n_b(a,b)$, the last two sums are the same with the opposite sign, therefore $\partial(\sum_{a=2}^{i-1}(-1)^{n_a(a,i)}[x_a])=-(i-2)[x]$. Since $i\geq 3$, our $[x]$ is of finite order in $H_*(F^w_j/F^w_{i+1})\cong H_*[\![w]\!]$. $\square$

Lemma 4.3. $H_k[\![w_n,\dots,w_1]\!]\cong H_{\binom{n}{2}-k-1}[\![w_1,\dots,w_n]\!]$ *for* $(w_1,\dots,w_n)\in\mathcal{T}_n$.

Proof. Let (C^*,δ^*) be the dual of the complex (C_*,∂_*) of $\mathfrak{nil}_n$, let f_M be the dual of a basis element e_M, and $N=\binom{n}{2}$. Define $\tau_*\colon C_*\to C^{N-*}$ by $\tau(e_M)=\varepsilon_M f_{M^C}$, where ε_M is the sign of the permutation (M,M^C) of $\{(i,j);\,1\leq i<j\leq n\}$. By [2, p.640], τ_* is a chain isomorphism, i.e. $\tau_{k-1}\partial_k=\delta^{N-k}\tau_k$. For $e_M\in[\![w_1,\dots,w_n]\!]$ we have

$$\begin{aligned}
w_i\big(\tau(e_M)\big)&=i-|\{x;(x,i)\in M^C\}|+|\{x;(i,x)\in M^C\}|\\
&=i-\big(i-1-|\{x;(x,i)\in M\}|\big)+\big(n-i-|\{x;(i,x)\in M\}|\big)=n+1-w_i,
\end{aligned}$$

hence $\tau([\![w_1,\dots,w_n]\!])=[\![n{+}1{-}w_1,\dots,n{+}1{-}w_n]\!]^*\cong[\![w_n,\dots,w_1]\!]^*$ by Lemma 2.2. Now the result follows from Lemma 4.2 and the universal coefficient theorem. $\square$

5. Computations

We have $H_*\mathfrak{nil}_n \cong \big(\bigoplus_{w\in\mathcal{F}_n} H_*[\![w]\!]\big) \oplus \big(\bigoplus_{w\in\mathcal{T}_n} H_*[\![w]\!]\big)$. Free part $\bigoplus_{w\in\mathcal{F}_n} H_*[\![w]\!]$ is known from 2.5. For the torsion part, we use Lemma 2.4:

$$\begin{aligned} TH_*(\mathfrak{nil}_n) &\cong \big(\oplus_{1,n\notin w\in\mathcal{T}_n} H_*[\![w]\!]\big) \oplus \frac{\big(\oplus_{1\in w\in\mathcal{T}_n} H_*[\![w]\!]\big)\oplus\big(\oplus_{n\in w\in\mathcal{T}_n} H_*[\![w]\!]\big)}{\oplus_{1,n\in w\in\mathcal{T}_n} H_*[\![w]\!]} \\ &\cong \big(\oplus_{1,n\notin w\in\mathcal{T}_n} H_*[\![w]\!]\big) \oplus \frac{\oplus_{w\in\mathcal{T}_{n-1}}\oplus_{k=1}^{n} H_*[\![w]\!]_{k-1}\oplus H_*[\![w]\!]_{n-k}}{\oplus_{w\in\mathcal{T}_{n-2}}\oplus_{i\in[n-1],j\in[n]} H_*[\![w]\!]_{i-1+n-j}} \\ &\cong \big(\oplus_{1,n\notin w\in\mathcal{T}_n} H_*[\![w]\!]\big) \oplus \frac{\oplus_{k=0}^{n-1} TH_{*+k}(\mathfrak{nil}_{n-1})^2}{\oplus_{i=0}^{2n-3} TH_{*+i}(\mathfrak{nil}_{n-2})^{\min\{i+1,2n-2-i\}}}. \end{aligned}$$

By induction and Lemmas 3.1, 3.2, it suffices to calculate only $H_*[\![w]\!]$ coming from $\widetilde{\mathcal{T}}_n := \big\{w\in\mathcal{T}_n;\, 1,n\notin w,\, \nexists i\colon w_i=w_{i+1}\in\{2,n{-}1\},\, \nexists i<j<k\colon w_i=w_j=w_k\in\{2,n{-}1\}\big\}$. Define maps $\alpha,\beta,\gamma\colon \widetilde{\mathcal{T}}_n\to\widetilde{\mathcal{T}}_n$ by $\alpha(w_1,\ldots,w_n)=(w_2,\ldots,w_n,w_1)$, $\beta(w_1,\ldots,w_n)=(n+1-w_n,\ldots,n+1-w_1)$, $\gamma(w_1,\ldots,w_n)=(w_n,\ldots,w_1)$. Let $\sim$ be the smallest equivalence relation on $\widetilde{\mathcal{T}}_n$ with $w\sim\xi(w)$ for $\xi\in\{\alpha,\beta,\gamma\}$. By Lemmas 2.3, 2.2, 4.3, we need to compute H_* only for one complex in each equivalence class.

Case $n=4$: The set $\mathcal{S}_4$ consists of all permutations of (1,1,4,4), (1,2,3,4), (1,3,3,3), (2,2,2,4), (2,2,3,3), whilst $\widetilde{\mathcal{T}}_4/_\sim = \{(2,3,2,3)\}$. Now $[\![3,2,3,2]\!] \cong [\![2,3,2,3]\!]_1$ has only 4 generators, so H_* can be computed directly, but let us use Lemma 3.5: $H_*[\![2,3,2,3]\!] \cong H_*\,\mathrm{Cone}([\![2,1,3]\!]_1 \xrightarrow{\cdot 2} [\![2,1,3]\!]_1)$. Since $[\![2,1,3]\!]=\langle e_{12}\rangle$, we conclude that $H_k[\![2,3,2,3]\!]\cong\begin{cases}\mathbb{Z}_2; & \text{if } k=2\\ 0\ ; & \text{if } k\neq 2\end{cases}$. Hence the torsion part is $TH_k(\mathfrak{nil}_4)\cong\begin{cases}\mathbb{Z}_2; & \text{if } k\in\{2,3\}\\ 0\ ; & \text{if } k\notin\{2,3\}\end{cases}$.

Case $n=5$: Set $\widetilde{\mathcal{T}}_5/_\sim$ consists of $a=(2,3,4,2,4)$, $b=(2,3,3,3,4)$, $c=(2,3,3,4,3)$, $d=(3,3,3,3,3)$. By Lemma 3.5, $H_*[\![a]\!] \cong H_*\,\mathrm{Cone}([\![2,3,1,4]\!]_1 \xrightarrow{\cdot 3} [\![2,3,1,4]\!]_1) \cong \begin{cases}\mathbb{Z}_3; & \text{if } k=3\\ 0\ ; & \text{if } k\neq 3\end{cases}$. By Lemma 3.4, $H_*[\![b]\!] \cong H_*\,\mathrm{Cone}([\![1,3,2]\!]_2 \xrightarrow{\cdot 2} [\![1,3,2]\!]_2) \cong \begin{cases}\mathbb{Z}_2; & \text{if } k=3\\ 0\ ; & \text{if } k\neq 3\end{cases}$, and $H_*[\![c]\!] \cong H_*\,\mathrm{Cone}([\![3,2,1]\!]_1 \xrightarrow{\cdot 2} [\![3,2,1]\!]_1) \cong \begin{cases}\mathbb{Z}_2; & \text{if } k=4\\ 0\ ; & \text{if } k\neq 4\end{cases}$. By Example 4.1, $H_*[\![d]\!]\cong 0$. Because $\beta(a)=(2,4,2,3,4)=\alpha^3(a)$, $\beta(b)=(2,3,3,3,4)=b$, $\beta(c)=(3,2,3,3,4)=\alpha^4(c)$, and $\gamma(x)\neq\alpha^i(x)$ for all $x\in\{a,b,c\}$ and all i, we conclude that

$$\bigoplus_{w\in\widetilde{\mathcal{T}}_n} H_k[\![w]\!] = \bigoplus_{x\in\{a,b,c\}}\ \bigoplus_{i\in\{0,\ldots,4\}} \big(H_k[\![\alpha^i(x)]\!]\oplus H_k[\![\gamma\alpha^i(x)]\!]\big) = \begin{cases}\mathbb{Z}_2^4\oplus\mathbb{Z}_3^2; & k=3,\\ \mathbb{Z}_2^6\oplus\mathbb{Z}_3^3; & k=4,\\ \mathbb{Z}_2^6\oplus\mathbb{Z}_3^3; & k=5,\\ \mathbb{Z}_2^4\oplus\mathbb{Z}_3^2; & k=6.\end{cases}$$

Case $n=6$ is still doable by hand. Set $\widetilde{\mathcal{T}}_6/_\sim$ has 28 elements: 9 cases are done by Lemma 3.4, 6 by Lemma 3.5, and the rest by examining their filtration. There are only 3 classes containing no 2 or $n{-}1$: $(3,3,3,4,4,4)$, $(3,3,4,3,4,4)$, $(3,4,3,4,3,4)$.

Cases $n=7,8$ require a computer. The set $\widetilde{\mathcal{T}}_7/_\sim$ has 250 elements, and $\widetilde{\mathcal{T}}_8/_\sim$ has 3485 elements. See the table below for the homology of $\mathfrak{nil}_7$ and $\mathfrak{nil}_8$.

Cases $n\geq 9$: The set $\widetilde{\mathcal{T}}_9/_\sim$ has 59 102 elements. We have not been able to compute, among other things, the homology of the complex $[\![5,5,5,5,5,5,5,5,5]\!]$.

6. Afterword

6.1. **Conclusion.** We have seen that methods, designed for a specific family of Lie algebras, where we partition the problem into smaller pieces and solve only the nontrivial nonequivalent parts, can enable us to compute more than twice as much data compared with the usual approach.

6.2. **Acknowledgment.** This research was supported by the Slovenian Research Agency (research core funding no. P1-0292, J1-7025, J1-8131).

$k\backslash n$	7	8
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}^6$	$\mathbb{Z}^7$
2	$\mathbb{Z}^{20}\oplus\mathbb{Z}_2^4$	$\mathbb{Z}^{27}\oplus\mathbb{Z}_2^5$
3	$\mathbb{Z}^{49}\oplus\mathbb{Z}_2^{35}\oplus\mathbb{Z}_3^6$	$\mathbb{Z}^{76}\oplus\mathbb{Z}_2^{57}\oplus\mathbb{Z}_3^8$
4	$\mathbb{Z}^{98}\oplus\mathbb{Z}_2^{124}\oplus\mathbb{Z}_3^{27}\oplus\mathbb{Z}_4^6$	$\mathbb{Z}^{174}\oplus\mathbb{Z}_2^{253}\oplus\mathbb{Z}_3^{45}\oplus\mathbb{Z}_4^9$
5	$\mathbb{Z}^{169}\oplus\mathbb{Z}_2^{303}\oplus\mathbb{Z}_3^{78}\oplus\mathbb{Z}_4^{28}\oplus\mathbb{Z}_5^4$	$\mathbb{Z}^{343}\oplus\mathbb{Z}_2^{793}\oplus\mathbb{Z}_3^{168}\oplus\mathbb{Z}_4^{53}\oplus\mathbb{Z}_5^8$
6	$\mathbb{Z}^{259}\oplus\mathbb{Z}_2^{635}\oplus\mathbb{Z}_3^{168}\oplus\mathbb{Z}_4^{65}\oplus\mathbb{Z}_5^{17}$	$\mathbb{Z}^{602}\oplus\mathbb{Z}_2^{2132}\oplus\mathbb{Z}_3^{479}\oplus\mathbb{Z}_4^{164}\oplus\mathbb{Z}_5^{47}$
7	$\mathbb{Z}^{359}\oplus\mathbb{Z}_2^{1122}\oplus\mathbb{Z}_3^{275}\oplus\mathbb{Z}_4^{112}\oplus\mathbb{Z}_5^{38}$	$\mathbb{Z}^{961}\oplus\mathbb{Z}_2^{4880}\oplus\mathbb{Z}_3^{1050}\oplus\mathbb{Z}_4^{380}\oplus\mathbb{Z}_5^{145}$
8	$\mathbb{Z}^{455}\oplus\mathbb{Z}_2^{1674}\oplus\mathbb{Z}_3^{384}\oplus\mathbb{Z}_4^{160}\oplus\mathbb{Z}_5^{56}$	$\mathbb{Z}^{1415}\oplus\mathbb{Z}_2^{9882}\oplus\mathbb{Z}_3^{1927}\oplus\mathbb{Z}_4^{730}\oplus\mathbb{Z}_5^{309}\oplus\mathbb{Z}_8$
9	$\mathbb{Z}^{531}\oplus\mathbb{Z}_2^{2096}\oplus\mathbb{Z}_3^{481}\oplus\mathbb{Z}_4^{196}\oplus\mathbb{Z}_5^{63}$	$\mathbb{Z}^{1940}\oplus\mathbb{Z}_2^{17721}\oplus\mathbb{Z}_3^{3178}\oplus\mathbb{Z}_4^{1200}\oplus\mathbb{Z}_5^{524}\oplus\mathbb{Z}_8^5$
10	$\mathbb{Z}^{573}\oplus\mathbb{Z}_2^{2238}\oplus\mathbb{Z}_3^{522}\oplus\mathbb{Z}_4^{210}\oplus\mathbb{Z}_5^{64}$	$\mathbb{Z}^{2493}\oplus\mathbb{Z}_2^{27826}\oplus\mathbb{Z}_3^{4781}\oplus\mathbb{Z}_4^{1728}\oplus\mathbb{Z}_5^{766}\oplus\mathbb{Z}_8^{12}$
11	$\mathbb{Z}^{573}\oplus\mathbb{Z}_2^{2096}\oplus\mathbb{Z}_3^{481}\oplus\mathbb{Z}_4^{196}\oplus\mathbb{Z}_5^{63}$	$\mathbb{Z}^{3017}\oplus\mathbb{Z}_2^{38810}\oplus\mathbb{Z}_3^{6504}\oplus\mathbb{Z}_4^{2253}\oplus\mathbb{Z}_5^{1007}\oplus\mathbb{Z}_8^{18}$
12	$\mathbb{Z}^{531}\oplus\mathbb{Z}_2^{1674}\oplus\mathbb{Z}_3^{384}\oplus\mathbb{Z}_4^{160}\oplus\mathbb{Z}_5^{56}$	$\mathbb{Z}^{3450}\oplus\mathbb{Z}_2^{48576}\oplus\mathbb{Z}_3^{7902}\oplus\mathbb{Z}_4^{2720}\oplus\mathbb{Z}_5^{1219}\oplus\mathbb{Z}_8^{17}$
13	$\mathbb{Z}^{455}\oplus\mathbb{Z}_2^{1122}\oplus\mathbb{Z}_3^{275}\oplus\mathbb{Z}_4^{112}\oplus\mathbb{Z}_5^{38}$	$\mathbb{Z}^{3736}\oplus\mathbb{Z}_2^{54457}\oplus\mathbb{Z}_3^{8614}\oplus\mathbb{Z}_4^{3011}\oplus\mathbb{Z}_5^{1351}\oplus\mathbb{Z}_8^{11}$
14	$\mathbb{Z}^{359}\oplus\mathbb{Z}_2^{635}\oplus\mathbb{Z}_3^{168}\oplus\mathbb{Z}_4^{65}\oplus\mathbb{Z}_5^{17}$	$\mathbb{Z}^{3836}\oplus\mathbb{Z}_2^{54457}\oplus\mathbb{Z}_3^{8614}\oplus\mathbb{Z}_4^{3011}\oplus\mathbb{Z}_5^{1351}\oplus\mathbb{Z}_8^{11}$
15	$\mathbb{Z}^{259}\oplus\mathbb{Z}_2^{303}\oplus\mathbb{Z}_3^{78}\oplus\mathbb{Z}_4^{28}\oplus\mathbb{Z}_5^4$	$\mathbb{Z}^{3736}\oplus\mathbb{Z}_2^{48576}\oplus\mathbb{Z}_3^{7902}\oplus\mathbb{Z}_4^{2720}\oplus\mathbb{Z}_5^{1219}\oplus\mathbb{Z}_8^{17}$
16	$\mathbb{Z}^{169}\oplus\mathbb{Z}_2^{124}\oplus\mathbb{Z}_3^{27}\oplus\mathbb{Z}_4^6$	$\mathbb{Z}^{3450}\oplus\mathbb{Z}_2^{38810}\oplus\mathbb{Z}_3^{6504}\oplus\mathbb{Z}_4^{2253}\oplus\mathbb{Z}_5^{1007}\oplus\mathbb{Z}_8^{18}$
17	$\mathbb{Z}^{98}\oplus\mathbb{Z}_2^{35}\oplus\mathbb{Z}_3^6$	$\mathbb{Z}^{3017}\oplus\mathbb{Z}_2^{27826}\oplus\mathbb{Z}_3^{4781}\oplus\mathbb{Z}_4^{1728}\oplus\mathbb{Z}_5^{766}\oplus\mathbb{Z}_8^{12}$
18	$\mathbb{Z}^{49}\oplus\mathbb{Z}_2^4$	$\mathbb{Z}^{2493}\oplus\mathbb{Z}_2^{17721}\oplus\mathbb{Z}_3^{3178}\oplus\mathbb{Z}_4^{1200}\oplus\mathbb{Z}_5^{524}\oplus\mathbb{Z}_8^5$
19	$\mathbb{Z}^{20}$	$\mathbb{Z}^{1940}\oplus\mathbb{Z}_2^{9882}\oplus\mathbb{Z}_3^{1927}\oplus\mathbb{Z}_4^{730}\oplus\mathbb{Z}_5^{309}\oplus\mathbb{Z}_8$
20	$\mathbb{Z}^6$	$\mathbb{Z}^{1415}\oplus\mathbb{Z}_2^{4880}\oplus\mathbb{Z}_3^{1050}\oplus\mathbb{Z}_4^{380}\oplus\mathbb{Z}_5^{145}$
21	$\mathbb{Z}$	$\mathbb{Z}^{961}\oplus\mathbb{Z}_2^{2132}\oplus\mathbb{Z}_3^{479}\oplus\mathbb{Z}_4^{164}\oplus\mathbb{Z}_5^{47}$
22		$\mathbb{Z}^{602}\oplus\mathbb{Z}_2^{793}\oplus\mathbb{Z}_3^{168}\oplus\mathbb{Z}_4^{53}\oplus\mathbb{Z}_5^8$
23		$\mathbb{Z}^{343}\oplus\mathbb{Z}_2^{253}\oplus\mathbb{Z}_3^{45}\oplus\mathbb{Z}_4^9$
24		$\mathbb{Z}^{174}\oplus\mathbb{Z}_2^{57}\oplus\mathbb{Z}_3^8$
25		$\mathbb{Z}^{76}\oplus\mathbb{Z}_2^5$
26		$\mathbb{Z}^{27}$
27		$\mathbb{Z}^7$
28		$\mathbb{Z}$

References

[1] W.G. Dwyer, *Homology of integral upper-triangular Matrices*, Proc. Am. Math. Soc. **94** (1985), 523–528.

[2] M. Hazewinkel, *A duality theorem for cohomology of Lie algebras*, (Russian) Mat. Sb. (N.S.) **83 (125)** 1970 639–644.

[3] M. Jöllenbeck, *Algebraic Discrete Morse Theory and Applications to Commutative Algebra*, Thesis, (2005).

[4] B. Kostant, *Lie Algebra Homology and the Generalized Borel-Weil Theorem*, Ann. of Math. **74** (1961), 329–387.

[5] D.N. Kozlov, *Discrete Morse theory for free chain complexes*, C. R. Acad. Sci. Paris **340** (2005), 867–872.

[6] L. Lampret, A. Vavpetič, *(Co)homology of Lie algebras via algebraic Morse theory*, J. Algebra **463** (2016), 254–277.

[7] E. Sköldberg, *Morse theory from an algebraic viewpoint*, Trans. Amer. Math. Soc. **358** (2006), no. 1, 115–129.

Institute for Mathematics, Physics and Mechanics, Ljubljana, Slovenia
Faculty of Computer and Information Science, University of Ljubljana, Slovenia
E-mail address: leon.lampret@fri.uni-lj.si

Department of Mathematics, University of Ljubljana, Slovenia
E-mail address: ales.vavpetic@fmf.uni-lj.si