FREE SUBGROUPS OF FREE PRODUCTS AND COMBINATORIAL HYPERMAPS

LAURA CIOBANU & ALEXANDER KOLPAKOV

Abstract. We derive a generating series for the number of free subgroups of finite index in $\Delta^+ = \mathbb{Z}_p * \mathbb{Z}_q$ by using a connection between free subgroups of Δ^+ and certain hypermaps (also known as ribbon graphs or "fat" graphs), and show that this generating series is transcendental. We provide non-linear recurrence relations for the numbers above based on differential equations that are part of the Riccati hierarchy.

We also study the generating series for conjugacy classes of free subgroups of finite index in Δ^+, which correspond to isomorphism classes of hypermaps. Asymptotic formulas are provided for the numbers of free subgroups of given finite index, conjugacy classes of such subgroups, or, equivalently, various types of hypermaps and their isomorphism classes.

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1. Introduction

The main purpose of this paper is to count the number of free subgroups of finite index in the free product of finite cyclic groups $\Delta^+ = \mathbb{Z}_p * \mathbb{Z}_q$, where $p, q \geq 2$, and reveal new aspects of the connection between the number of such subgroups and the number of rooted hypermaps (also known as ribbon graphs or "fat" graphs). We also count the conjugacy classes of free subgroups of finite index, and investigate the link between these and isomorphism classes of hypermaps; the connections between free subgroups (and their conjugacy classes) of finite index in certain Fuchsian triangle groups and hypermaps have been previously exploited by [3, 19, 21, 22, 31]. Our contribution is to give formulas and new information, such as transcendence or non-holonomy, on the growth series of the above objects associated with Δ^+, as well as new recurrence relations and asymptotics, while also creating software that produces these numbers.

The notation Δ^+ is motivated by the fact that the group in question is a (p, q, ∞) Fuchsian triangle group, a group whose relationship with hypermaps has been fruitfully investigated in many papers, of which we mention the groundlaying paper by Jones and Singerman [14], and a recent series of works by Breda-d'Azevedo – Mednykh – Nedela [3], Mednykh [19], and Mednykh – Nedela [21, 22] who solved Tutte's classification problem for maps and hypermaps.

Our methods also provide a solution to Tutte's problem regardless of genus, as described in [3], although do not tackle its genus-specific case, c.f. [21, 22].

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General subgroup growth is the subject of the monograph [18] by Lubotzky and Segal, and further information on counting the number of subgroups in free products of cyclic groups of prime orders can be found in the papers by Müller and Schlage-Puchta [23, 24, 25]. There they enhance the general theory of subgroup structure in free products of (finite and infinite) cyclic groups using representation theory, analytic number theory and probability theory, among other tools.

In our case we use the species theory initiated by Joyal [15] (c.f. the monographs [2, 7]) as our main computational tool, thus generalising and enhancing the results of [28]. This technique allows us to write down the generating series for the number of free subgroups of finite index in Theorem 4.6 (or rooted hypermaps in Theorem 4.1) and the number of their conjugacy classes in Theorem 6.8 (or isomorphism classes of hypermaps in Theorem 6.1) in a relatively simple form suitable for routine calculation and computer experiments.

In Section 5 we use a hierarchy of differential equations in order to obtain certain non-linear recurrence relation for the numbers of finite index subgroups in Δ^+. This hierarchy is known as the Riccati hierarchy c.f. [10], and appears as a simplified version of the general phenomenon described primarily in [9, 27]. Being associated with the classical Riccati equations gives additional information on the series concerned, as it shows that some of the series we consider are non-holonomic (see Corollary 5.5).

Throughout the paper we give concrete formulas for several particular cases of free products of cyclic groups, as well as for the related hypermaps as combinatorial objects, and a SAGE code is provided in the Appendix to support our findings and to provide illustrative examples where necessary.

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2. Preliminaries

2.1. **Hypermaps and (p, q)-hypermaps.** Let $D = [n]$ be a set of n elements, called *darts*. Let $\alpha, \sigma \in \mathfrak{S}_n$ be two permutations such that the group $\langle \alpha, \sigma \rangle$ acts transitively on D. The triple $H = \langle D; \alpha, \sigma \rangle$ is called an *oriented labelled (combinatorial) hypermap* with set of darts D. The orbits of σ are called vertices (σ stands for the French *sommets*), the orbits of α are called hyper-edges (α for *arêtes*). The orbits of $\varphi = \sigma^{-1}\alpha^{-1}$ are called hyper-faces (φ for *faces*). The size of the respective orbit determines the degree of a vertex, hyper-edge or hyper-face. Obviously, $\alpha\,\sigma\,\varphi = \varepsilon$ and H can be equivalently represented as the triple $H = \langle D; \alpha, \varphi \rangle$, where appropriate.

A hypermap naturally appears in the setting of an orientable genus g surface Σ_g and graph Γ embedded in Σ_g as $\iota : \Gamma \to \Sigma$, satisfying

1) the complement $\Sigma_g \setminus \iota(\Gamma)$ is a union of topological discs called faces,
2) the faces are properly two-colourable (e.g. into black and white), i.e. faces of same colour intersect only at vertices of Γ, and
3) the corners of the white faces are labelled with the numbers $1, 2, 3, \dots$ in some fashion (we may think that this information is carried by the embedding map ι), and a black face corner label is equal to the adjacent white face corner label, when moving clockwise around their common vertex.

Then the triple $H = \langle \Sigma_g; \Gamma, \iota \rangle$ is an oriented labelled *topological* hypermap.

By removing a sub-disc in the interior of each face of a hypermap we obtain a *ribbon graph* or *"fat" graph*, which is a graph together with a cyclic ordering on the set of half-edges incident to each vertex; the edges of the ribbon graph can be seen as small rectangles or ribbons attached in a given cyclic order to discs glued at the vertices.

The correspondence between the topological and combinatorial definitions above is as follows:

1) each disjoint cycle of α is obtained from recording the corner labels of a black face in an anticlockwise direction,
2) each disjoint cycle of σ is obtained from recording the labels around a vertex in an anticlockwise direction,
3) each disjoint cycle of φ is obtained from recording the corner labels of a white face in an anticlockwise direction.

Consequently, the set of face labels becomes the set of darts of H, the white faces become hyper-faces of H and the black faces become hyper-edges of H. Thus the combinatorial and topological descriptions of H agree. Indeed, each topological hypermap produces a unique combinatorial hypermap, and given the above combinatorial information one may assemble an oriented connected surface from a number of topological discs, which are represented by polygons with labelled corners.

A topological hypermap H is *rooted* if we mark the first edge encountered while moving clockwise around the white corner of H labelled 1, and a combinatorial hypermap is rooted if one of its darts is marked as a root. We shall always assume that the root dart is 1.

Example 2.1. A partial picture of a rooted hypermap is shown in Figure 1. The root corner is marked with an arrow and numbered 1.

If we ignore the labelling (or marked root) of a hypermap, that means we consider its isomorphism class. Two (combinatorial) hypermaps $H_1 = \langle D_1; \alpha_1, \sigma_1 \rangle$ and $H_2 = \langle D_2; \alpha_2, \sigma_2 \rangle$, assumed to be neither labelled nor rooted, are *isomorphic* if there exists $\psi \in \mathfrak{S}_n$ such that $\psi\, \alpha_1 \psi^{-1} = \alpha_2$ and $\psi\, \sigma_1 \psi^{-1} = \sigma_2$. Their topological counterparts are isomorphic if there exists a homeomorphism between the respective surfaces which preserves the graph embedding. Two (combinatorial) rooted hypermaps are isomorphic if their roots correspond to each other under some hypermap isomorphism. An analogous definition holds in the topological case. Below we shall use the combinatorial and topological descriptions of hypermaps interchangeably.

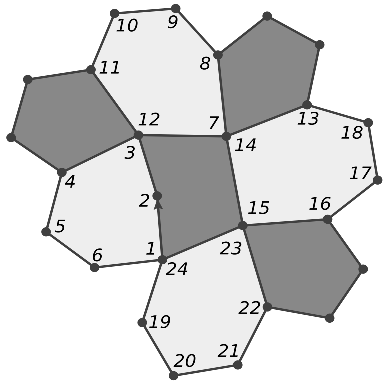

FIGURE 1. A partial drawing of a rooted hypermap (its root is labelled 1)

A (p, q)-hypermap H is one in which α has cycles of length p only and φ has cycles of length q only. In other words, all hyper-edges of H are p-gons and all hyper-faces of H are q-gons. Given this definition, it is more convenient to represent H as $H = \langle D; \alpha, \varphi \rangle$.

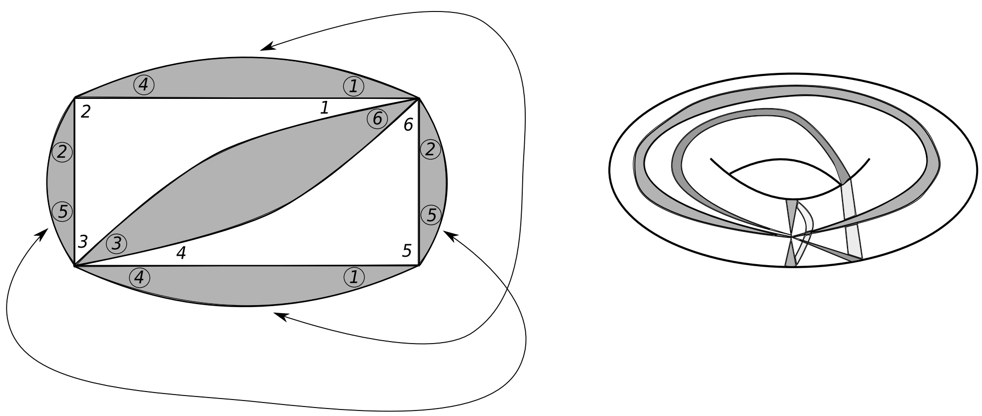

FIGURE 2. A $(2, 3)$-hypermap on a torus: the parallel sides of the rectangular shape on the left are identified in accordance with the circled labels and arrows in order to produce the torus with a hypermap on it depicted on the right.

Example 2.2. The partial picture in Figure 1 features a $(5, 6)$-hypermap. A $(5, 6)$-hypermap whose underlying topological surface is the sphere $\mathbb{S}^2$ is known as *a fullerene*.

Example 2.3. A triangulated surface carries a $(2, 3)$-hypermap all of whose bigonal hyper-edges are collapsed into ordinary edges. We shall refer to a $(2, 3)$-hypermap as a *triangulation* (of an orientable surface), thus allowing identification of two sides of the same triangle. Figure 2 shows a triangulation of a torus with $\alpha = (1, 4)(2, 5)(3, 6)$, $\sigma = (5, 1, 6, 2, 4, 3)$, $\varphi = (1, 2, 3)(5, 6, 4)$.

Another important class of hypermaps is the class of $(2, 4)$-hypermaps, or quadrangulations. In general, every $(2, q)$-hypermap is equivalent to a *map* as described in [14, §2 - §3], and (once labelled) its corner labels of hyper-edges become exactly the dart labels of the resulting map after all bigonal hyper-edges are collapsed into usual edges.

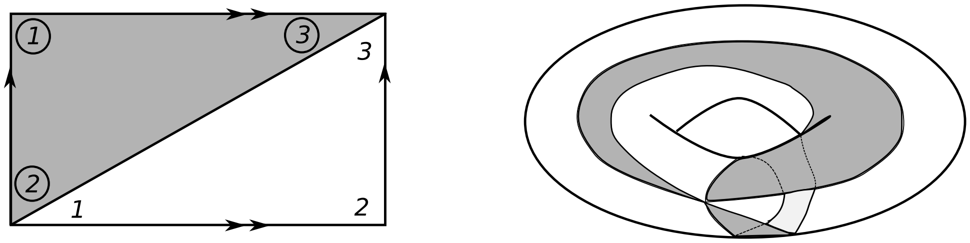

FIGURE 3. A $(3,3)$-hypermap on a torus: the parallel sides of the rectangle on the left are identified in accordance with arrows marking them in order to produce the torus with a hypermap on it depicted on the right.

Example 2.4. The $(3,3)$-hypemaps are triangulations that admit a colouring which is chequerboard around the vertices. We shall call $(3,3)$-hypermaps *bi-coloured triangulations* (whose dual map is a bipartite cubic graph). Figure 3 features a bicoloured triangulation of a torus with $\alpha = \sigma = \varphi = (1,2,3)$.

2.2. **Formal series.** A *hypergeometric sequence* $(c_k)_{k\geq 0}$ is one for which $c_0 = 1$ and the ratio of consecutive terms is a rational function in k, i.e. there exist monic polynomials $P(k)$ and $Q(k)$ such that

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)}.$$

If P and Q can be factored as

$$\frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\dots(k+a_p)}{(k+b_1)(k+b_2)\dots(k+b_q)(k+1)},$$

then we use the notation

$${}_pF_q\left[\begin{array}{c} a_1\dots a_p \\ b_1\dots b_q \end{array};z\right]$$

for the formal series $F(z) = \sum_{k\geq 0} c_k z^k$, c.f. [29, §3.2]. The factor $(k+1)$ belongs to the denominator for historical reasons. Such a hypergeometric series satisfies the differential equation

$$\Big(\vartheta(\vartheta+b_1-1)\cdots(\vartheta+b_q-1) - z(\vartheta+a_1)\cdots(\vartheta+a_p)\Big)\ {}_pF_q(z) = 0, \tag{1}$$

where $\vartheta = z\frac{d}{dz}$, c.f. [5, §16.8(ii)]. Among numerous differential equations related to (1) is the *classical Riccati equation*, which will play an important role in this paper. It is a first order non-linear equation with variable coefficients $f_i(x)$, of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f_1(x) + f_2(x)y + f_3(x)y^2. \tag{2}$$

The *Pocchammer symbol* is connected to hypergeometric series and defined as

$$(a)_n = a(a+1)\dots(a+n-1).$$

It has asymptotic expansion

$$(a)_n \propto \frac{\sqrt{2\pi}}{\Gamma(a)}\, e^{-n}\, n^{a+n-\frac{1}{2}}, \tag{3}$$

where $\Gamma(a)$ is the Gamma function of a, defined as $\Gamma(a) = (a-1)!$ if a is a positive integer, and $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$ for all the non-integer real positive numbers.

A formal power series $y = f(x)$ is said to be *D-finite*, or *differentiably finite*, or *holonomic*, if there exist polynomials $p_0, \ldots, p_m$ (not all zero) such that $p_m(x)y^{(m)} + \cdots + p_0(x)y = 0$, where $y^{(m)}$ denotes the m-th derivative of y with respect to x. All algebraic power series are holonomic, but not vice versa, c.f. [7, Appendix B.4].

Finally recall that the *Hadamard product* of two formal single-variable series $A(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}$ and $B(z) = \sum_{n\geq 0} b_n \frac{z^n}{n!}$ is denoted $(A \odot B)(z)$ and given by $(A \odot B)(z) := \sum_{n\geq 0} a_n b_n \frac{z^n}{n!}$.

Let $\lambda = (n_1, \ldots, n_m)$ be a partition of a natural number $n \geq 0$, i.e. $n = \sum_{i\geq 1} in_i$. We write $\lambda \vdash n$ and define $\lambda! := 1^{n_1} n_1! 2^{n_2} n_2! \ldots m^{n_m} n_m!$. Let $\mathbf{z}^\lambda := z_1^{n_1} z_2^{n_2} \ldots z_m^{n_m}$ for some collection of variables $z_1, z_2, \ldots$. Then for two multi-variable series $A(\mathbf{z}) = \sum_{n\geq 0} \sum_{\lambda\vdash n} a_\lambda \frac{\mathbf{z}^\lambda}{\lambda!}$ and $B(\mathbf{z}) = \sum_{n\geq 0} \sum_{\lambda\vdash n} b_\lambda \frac{\mathbf{z}^\lambda}{\lambda!}$ we have $(A \odot B)(z) := \sum_{n\geq 0} \sum_{\lambda\vdash n} a_\lambda b_\lambda \frac{\mathbf{z}^\lambda}{\lambda!}$.

2.3. **Species theory.** Species theory (théorie des espèces) is initially due to A. Joyal [15] and is a powerful way to describe and count labelled discrete structures. Since it requires a lengthy and formal setup, we give here only the basic ideas and refer the reader to [2, 7] for further details.

A *species of structures* is a rule (or functor) F which produces

i) for each finite set U (of labels), a finite set $F[U]$ of structures on U,
ii) for each bijection $\sigma : U \to V$, a function $F[\sigma] : F[U] \to F[V]$.

The functions $F[\sigma]$ should further satisfy the following functorial properties:

i) for all bijections $\sigma : U \to V$ and $\tau : V \to W$, $F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$,
ii) for the identity map $Id_U : U \to U$, $F[Id_U] = Id_{F[U]}$.

Let $[n] = \{1, 2, \ldots, n\}$ be an n-element set, and assume that $[0] = \emptyset$. A species F of *labelled structures* has a generating function $F(z) = \sum_{n\geq 0} \operatorname{card} F[n] \frac{z^n}{n!}$. For a species of *unlabelled structures* (i.e. structures up to isomorphism) we write $\widetilde{F}$, and its generating function is a specialisation of the cycle index series, in the sense that $\widetilde{F}(z) = \mathcal{Z}_F(z, z^2, z^3 \ldots)$, where the *cycle index series* (see [2, §1.2.3]) is defined as:

$$\mathcal{Z}_F(z_1, z_2, \ldots) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} \operatorname{card} Fix(F[\sigma])\, \mathbf{z}^\sigma.$$

Here $Fix(F[\sigma])$ is the set of elements of $F[n]$ having $F[\sigma]$ as automorphism, and $\mathbf{z}^\sigma = z_1^{c_1} z_2^{c_2} \ldots z_m^{c_m}$ if the cycle type of σ is $c(\sigma) = (c_1, c_2, \ldots, c_m)$ (i.e. c_k is the number of cycles of length k in the decomposition of σ into disjoint cycles).

Species can often be described by functional equations, as in the following example.

Example 2.5. Let $\mathcal{A}$ denote the species of rooted trees (i.e. trees with a distinguished vertex, or *arborescences* [15]), and E the species of sets (from the French *ensembles* [15]). Let Z be the singleton species with generating function $Z(z) = z$. Then the functional equation $\mathcal{A} = ZE(\mathcal{A})$ expresses the fact that any rooted tree with vertex labels from a finite set U can be naturally described as a root (a vertex $z \in U$) to which is attached a set of disjoint rooted trees (on $U \setminus \{z\}$) which translate into equalities for generating functions; in this case we have $\mathcal{A}(z) = z\exp(\mathcal{A}(z))$, where $\mathcal{A}(z)$ is the generating function for finite rooted labelled trees.

By using the Lagrange-Brünner inversion formula we get $\mathcal{A}(z) = \sum_{n\geq 2} \frac{n^{n-2}}{(n-1)!} z^n$. This leads to Cayley's formula of n^{n-2} for the number of labelled trees on n vertices via the fact that the number of rooted trees on n vertices is the n-th coefficient of $\mathcal{A}(z)$ and each tree with n vertices rooted at 1 corresponds to $(n-1)!$ labelled trees.

3. Subgroups of free products of cyclic groups

In this section we prove the following two lemmas.

Lemma 3.1. *Let p, q be two natural numbers, $pq \geq 6$. There is a one-to-one correspondence between the set of connected oriented rooted (p,q)-hypermaps $\mathfrak{H}^r_{p,q}(n)$ on n darts and the set of free subgroups of index n in the group $\Delta^+ = \mathbb{Z}_p * \mathbb{Z}_q$.*

Proof. Let $H = \langle D; \alpha, \varphi\rangle$ be a rooted hypermap (with root 1) from $\mathfrak{H}^r_{p,q}(n)$. Then there is an epimorphism from $\Delta^+ = \mathbb{Z}_p * \mathbb{Z}_q \cong \langle \varrho | \varrho^p = \varepsilon\rangle * \langle \delta | \delta^q = \varepsilon\rangle$ to the group $G(H) = \langle \alpha, \varphi\rangle$ given by $\varrho \mapsto \alpha$, $\delta \mapsto \varphi$, and Δ^+ acts transitively on D via this epimorphism. Let $\Gamma := Stab(1)$. Then $[\Delta^+ : \Gamma] = n$ and the action of Δ^+ on D is equivalent to the action of Δ^+ on the cosets of Γ.

Moreover, Γ cannot contain any conjugate of a nontrivial power of ϱ or δ because α and φ have no fixed points, and cycle structure is preserved under conjugation. Thus Γ has no torsion elements, as any torsion element in Δ^+ is conjugate to some power of either ϱ or δ. By the Kurosh theorem on subgroups of free products, Γ is free.

On the other hand, a torsion-free finite index subgroup $\Gamma < \Delta^+$ gives rise to a combinatorial hypermap $H = \langle D_\Gamma; \alpha_\Gamma, \varphi_\Gamma\rangle$, with $D_\Gamma = \{g\,\Gamma | g \in \Delta^+\}$, $\alpha_\Gamma(g\Gamma) = (\varrho g)\Gamma$, $\varphi_\Gamma(g\Gamma) = (\delta g)\Gamma$. The root of H corresponds to the coset $\varepsilon\Gamma$.

Since Γ is torsion-free, it does not contain any conjugates of ϱ, δ, or their powers. Thus, $\alpha^p_\Gamma = \varepsilon$ and $\alpha^k_\Gamma \neq \varepsilon$ for $1 \leq k < p$, which implies that all disjoint cycles of α_Γ have length p. Indeed, a cycle in α_Γ has length d, $d \nmid p$, and once $d < p$, then α^d_Γ has fixed points. Thus Γ contains a conjugate of ϱ^d, which is a contradiction to Γ being torsion-free. Analogously, all disjoint cycles of φ_Γ have length q. Finally, it follows that $H \in \mathfrak{H}^r_{p,q}(n)$, with $n = [\Delta^+ : \Gamma]$. Again, a torsion-free subgroup $\Gamma < \Delta^+$ is actually free. □

By recalling the definition of hypermap isomorphism, we easily arrive at the following lemma.

Lemma 3.2. *Let p, q be two natural numbers, $pq \geq 6$. There is a one-to-one correspondence between the set of isomorphism classes of connected oriented (p,q)-hypermaps $\mathfrak{H}_{p,q}(n)$ on n darts and the set of conjugacy classes of free subgroups of index n in the group $\Delta^+ = \mathbb{Z}_p * \mathbb{Z}_q$.*

Proof. Analogous to the proof of Lemma 3.1. □

4. Counting free subgroups and hypermaps

We proceed by computing the numbers card $\mathfrak{H}^{r}_{p,q}(n)$ of connected oriented rooted (p, q)-hypermaps on n darts. In order to do so we shall use species theory (see Section 2.3) and generalise the results of [28].

Let E be the species of sets, C_i be the species of cyclic permutations of length $i \geq 2$, S_i be the species of permutations with cycles of length exactly i without fixed points (we assume that $S_i[\emptyset] = \{\emptyset\}$), H^* be the species of labelled (p, q)-hypermaps (not necessarily connected) on n darts (we also assume $H^*[\emptyset] = \{\emptyset\}$), H be the species of connected labelled (p, q)-hypermaps on n darts (in contrast, here $H[\emptyset] = \emptyset$), and H° be the species of connected rooted (p, q)-hypermaps on n darts.

The following combinatorial equations describe the relations between the species:

$$S_p = E(C_p),\ S_q = E(C_q),\ H^* = S_p \times S_q. \tag{4}$$

Intuitively, this means that each permutation of S_p has a unique decomposition into cycles of length p, and each hypermap is uniquely determined by a pair of permutations from S_p and S_q. Furthermore

$$H^* = E(H),\ H^\circ = Z \cdot H', \tag{5}$$

where Z is the singleton species with generating function $Z(z) = z$, and H' means species differentiation.

The respective generating functions will be

$$C_p(z) = \frac{z^p}{p},\ C_q(z) = \frac{z^q}{q}, \tag{6}$$

and thus

$$S_p(z) = \exp\left(\frac{z^p}{p}\right) = \sum_{k=0}^{\infty} \frac{z^{pk}}{p^k}\,\frac{1}{k!}, \tag{7}$$

$$S_q(z) = \exp\left(\frac{z^q}{q}\right) = \sum_{k=0}^{\infty} \frac{z^{qk}}{q^k}\,\frac{1}{k!}. \tag{8}$$

We shall use the notation

$$\langle p, q\rangle := lcm(p, q) \text{ and } (p, q) := gcd(p, q),$$

where *lcm* and *gcd* denote as usual the least common multiple and greatest common divisor, respectively.

Identities (4), (7) and (8) imply that

$$H^*(z) = S_p \odot S_q(z) = \sum_{k=0}^{\infty} z^{\langle p,q\rangle k}\ \frac{(\langle p, q\rangle k)!}{p^{k\langle p,q\rangle/p}\left(\frac{\langle p,q\rangle k}{p}\right)!\ q^{k\langle p,q\rangle/q}\left(\frac{\langle p,q\rangle k}{q}\right)!}, \tag{9}$$

where $\odot$ denotes the Hadamard product of $S_p(z)$ and $S_q(z)$. For convenience, we express $H^*(z)$ as $H^*(z) = f(cz^{\langle p,q\rangle})$ with a suitable constant $c > 0$, and let $f(z) = \sum_{k=0}^{\infty} f_k \frac{z^k}{k!}$. First we prove that $f(z)$ is a divergent (i.e. with convergence radius 0) hypergeometric series, and then focus on the properties of its logarithmic derivative $\frac{f'(z)}{f(z)}$ through which the series for $H^\circ(z)$ can be expressed.

Let us choose the constant c so that

$$c^{(p,q)} = \langle p, q\rangle^{(p-1)(q-1)-1}. \tag{10}$$

Then for $k \geq 0$

$$\frac{f_{k+1}}{f_k} = \frac{\prod_{i=1}^{\frac{pq}{(p,q)}-1}\left(k+\frac{i(p,q)}{pq}\right)}{\prod_{i=1}^{\frac{p}{(p,q)}-1}\left(k+\frac{i(p,q)}{p}\right)\prod_{i=1}^{\frac{q}{(p,q)}-1}\left(k+\frac{i(p,q)}{q}\right)}, \tag{11}$$

which, since $f(0) = 1$, implies that the series $f(z)$ is hypergeometric; we obtain

$$f(z) = {}_{\frac{pq}{(p,q)}-1}F_{\frac{p+q}{(p,q)}-2}\left[\begin{matrix} \frac{i(p,q)}{pq}, & i=1,\ldots,\frac{pq}{(p,q)}-1 \\ \frac{j(p,q)}{p}, j=1,\ldots,\frac{p}{(p,q)}-1; & \frac{j(p,q)}{q}, j=1,\ldots,\frac{q}{(p,q)}-1 \end{matrix}; z\right]. \tag{12}$$

However, every time p or q divides i, (11) can be simplified and $\left\lfloor \frac{q}{(p,q)} - \frac{1}{p}\right\rfloor + \left\lfloor \frac{p}{(p,q)} - \frac{1}{q}\right\rfloor = \frac{p+q}{(p,q)} - 2$ terms from both numerator and denominator can be removed. Thus (12) turns into

$$f(z) = {}_{1+\frac{(p-1)(q-1)-1}{(p,q)}}F_0\left[\begin{matrix} \frac{i(p,q)}{pq}, i=1,\ldots,\frac{pq}{(p,q)}-1, p\nmid i, q\nmid i \\ \ldots \end{matrix}; z\right]. \tag{13}$$

Since $f(z)$ is hypergeometric and thus holonomic, the growth function $H^*(z) = f(\xi)$, with $\xi = c\, z^{\langle p,q\rangle}$, is also holonomic. It is also clear that $f(\xi)$ has convergence radius 0. Thus, by [12, Corollary 2], $\frac{1}{f(\xi)}$ is non-holonomic, and $\frac{f'(\xi)}{f(\xi)}$ is transcendental (c.f. also [13, Proposition 3.1.5]).

From (5) we have the first equality below, and from $H^*(z) = f(\xi)$ the second:

$$H^\circ(z) = z\frac{d}{dz}\log H^*(z) = \langle p, q\rangle\, \xi\, \frac{f'(\xi)}{f(\xi)}. \tag{14}$$

Thus $H^\circ(z)$ is transcendental, and we have shown

Theorem 4.1. *The growth series $H^\circ(z) = \sum_{n=0}^{\infty} \operatorname{card} \mathfrak{H}^r_{p,q}(n)\cdot z^n$ is transcendental and the following asymptotic formula holds for its non-zero coefficients:*

$$[z^{\langle p,q\rangle k}]H^\circ(z) \propto \frac{(2\pi)^{\frac{N-1}{2}}\langle p,q\rangle}{\prod_i \Gamma(a_i)}\; k^{(N-1)k-\frac{N-1}{2}+\sum_i a_i}\; e^{(1+\log c-N)k},\ \text{as } k\to\infty,$$

where $N = 1 + \frac{(p-1)(q-1)-1}{(p,q)}$, and the constants c and a_i satisfy $c^{(p,q)} = \langle p, q\rangle^{(p-1)(q-1)-1}$, $a_i = \frac{i(p,q)}{pq}$, with $i = 1, \ldots, \frac{pq}{(p,q)} - 1$, $p \nmid i$, $q \nmid i$.

Proof. The proof of the formula and transcendence is given above; it only remains to justify the asymptotics of $[z^{\langle p,q\rangle k}]H^\circ(z)$. The series $H^\circ(z)$ is rapidly divergent, and thus we may apply the techinique provided in [6, Theorem 4.1], see also [1] and [26, Theorem 7.2]. Then the coefficients of $H^*(z)$ are given by

$$[z^{\langle p,q\rangle k}]H^*(z) = \frac{c^k}{k!}\prod_{i=1}^{N}(a_i)_k \tag{15}$$

and using the asymptotic expansion for the Pocchammer symbol in (3) we immediately obtain an asymptotic expansion for the coefficients of $\log H^*(z)$, and subsequently for the coefficients of $H^\circ(z) = z\,\frac{d}{dz}\log H^*(z)$. □

Corollary 4.2. *The following asymptotic formula holds for the non-zero coefficients of $H^\circ(z)$:*

$$[z^{\langle p,q\rangle k}]H^\circ(z) \propto Ck^{(N-1)k-c_0}\; e^{c_1 k},\;\; \text{as } k\to\infty,$$

where N, C, c_0 and c_1 are constants that depend only on p and q.

Example 4.3. For the number of rooted triangulations on n darts we have $H^\circ(z) = \sum_{n=0}^{\infty} \mathrm{card}\,\mathfrak{H}^r_{2,3}(n)\cdot z^n = 5z^6 + 60z^{12} + 1105z^{18} + 27120z^{24} + 828250z^{30} + 30220800z^{36} + 1282031525z^{42} + 61999046400z^{48} + 3366961243750z^{54} + 202903221120000z^{60} + \dots$. The coefficient sequence of $H^\circ(z)$ has number A062980 in the OEIS [30].

Example 4.4. For the number of rooted quadrangulations on n darts we have $H^\circ(z) = \sum_{n=0}^{\infty} \mathrm{card}\,\mathfrak{H}^r_{2,4}(n)\cdot z^n = 3z^4 + 24z^8 + 297z^{12} + 4896z^{16} + 100278z^{20} + 2450304z^{24} + 69533397z^{28} + 2247492096z^{32} + 81528066378z^{36} + +3280382613504z^{40} + \dots$.

Example 4.5. For the number of rooted bi-coloured triangulations on n darts we have $H^\circ(z) = \sum_{n=0}^{\infty} \mathrm{card}\,\mathfrak{H}^r_{3,3}(n)\cdot z^n = 2z^3 + 12z^6 + 112z^9 + 1392z^{12} + 21472z^{15} + 394752z^{18} + 8421632z^{21} + 204525312z^{24} + 5572091392z^{27} + 168331164672z^{30} + \dots$.

By reformulating the above results in group-theoretic terms we obtain

Theorem 4.6. *Let $\Delta^+ = \mathbb{Z}_p * \mathbb{Z}_q$ be a free product of cyclic groups with $pq \geq 6$. Then the subgroup growth series $S_f(z) = \sum_{n=0}^{\infty} s_f(n)z^n$, for the number $s_f(n)$ of index n free subgroups of Δ^+, coincides with $H^\circ(z)$.*

Proof. By Lemma 3.1, we have that $S_f(z) = H^\circ(z)$, since the elements of $\mathfrak{H}^r_{p,q}(n)$ are in bijection with free subgroups of index n in Δ^+. □

Example 4.7. The growth series counting the finite index free subgroups in $\mathbb{Z}_2 * \mathbb{Z}_3$, $\mathbb{Z}_2 * \mathbb{Z}_4$ and $\mathbb{Z}_3 * \mathbb{Z}_3$ are given in Examples 4.3, 4.4 and 4.5, respectively.

5. The Riccati hierarchy and recurrence relations

As we mentioned before, the generating series $H^\circ(z)$ cannot be algebraic. Nevertheless, it does satisfy a non-linear *differential* equation; and although this fact does not make $H^\circ(z)$ holonomic, a certain recurrence relation holds for its coefficients $h_{p,q}(n) = \mathrm{card}\,\mathfrak{H}^r_{p,q}(n)$.

Recall that we expressed $H^*(z)$ as $H^*(z) = f(\xi)$, where $\xi = c\,z^{\langle p,q\rangle}$. By (10) – (13), the function $f(\xi)$ is hypergeometric and by (1) it satisfies the hypergeometric differential equation

$$\vartheta f(\xi) = \xi \prod_{i=1}^{N} (\vartheta + a_i) f(\xi), \tag{16}$$

where $\vartheta = \xi \frac{d}{d\xi}$, and a_i, $i = 1, \ldots, N$, are the parameters of the hypergeometric function in equality (13) with $N = 1 + \frac{(p-1)(q-1)-1}{(p,q)}$.

Let $w(\xi) := \xi \frac{f'(\xi)}{f(\xi)}$. Then $w(\xi)$ determines the growth series $H^\circ(z) = \sum_{n\geq 0} h_{p,q}(n)\, z^n$ since (14) leads immediately to

$$H^\circ(z) = \langle p, q\rangle\, \xi\, \frac{f'(\xi)}{f(\xi)} = \langle p, q\rangle\; w(\xi). \tag{17}$$

Lemma 5.1. *The function $w(\xi)$ satisfies*

$$w(\xi) = \xi \sum_{i=0}^{N} \sigma_{N-i}(a_1, \ldots, a_N)\, w_i(\xi), \tag{18}$$

where σ_i is the i-th symmetric polynomial, and the functions $w_i(\xi)$ are defined as

$$w_0(\xi) = 1,\ w_1(\xi) = w(\xi),\ w_i(\xi) = \xi\, \frac{d}{d\xi} w_{i-1}(\xi) + w(\xi) w_{i-1}(\xi),\ \textit{for } i \geq 2. \tag{19}$$

Proof. By (16)

$$\vartheta f(\xi) = \xi \prod_{i=1}^{N} (\vartheta + a_i) f(\xi) = \xi \sum_{i=0}^{N} \sigma_{N-i}(a_1, \ldots, a_N)\, \vartheta^i f(\xi),$$

and from the definition of $w(\xi)$ it follows that $\vartheta f(\xi) = w(\xi)\, f(\xi) = w_1(\xi)\, f(\xi)$; we will prove by induction that $\vartheta^i f(\xi) = w_i(\xi) \cdot f(\xi)$, assuming it holds for all values up to $i-1$:

$$\vartheta^i f(\xi) = \vartheta(w_{i-1}(\xi) \cdot f(\xi)) = \vartheta w_{i-1}(\xi) \cdot f(\xi) + w_{i-1}(\xi) \cdot \vartheta f(\xi) =$$
$$= \left(\xi\, \frac{d}{d\xi} w_{i-1}(\xi) + w(\xi) w_{i-1}(\xi)\right)\; f(\xi) = w_i(\xi) \cdot f(\xi).$$

Now (18) – (19) follow immediately. □

The family of equations (18) – (19) is indexed by the integers p, q, $pq \geq 6$, and the first instance has $p = 2$, $q = 3$; this turns out to be the classical Riccati equation

$$w(\xi) = \xi^2 w'(\xi) + \xi w(\xi) + \xi w^2(\xi) + \frac{5}{36}\xi. \tag{20}$$

All the identities (18) – (19) constitute a part of the Riccati hierarchy (c.f. [10, Equation (7.2)]), and for fixed p, q the corresponding equation gives rise to a non-linear recurrence relation, with polynomials in n as coefficients, for the numbers $h_{p,q}(n)$. For relatively small values of p and q such a recurrence relation can be easily computed.

We would like to remark that $w(\xi)$ is a solution to *one* of the possible equations (18) – (19), and not the *whole* Riccati hierarchy.

Example 5.2. If we consider the formal series expansion $w(\xi) = \sum_{n=0}^{\infty} w_n \, \xi^n$, then by the Riccati equation (20) we obtain

$$w_{n+1} = (n+1)\, w_n + \sum_{i=0}^{n} w_i w_{n-i}, \text{ for } n \geq 2,$$

with initial conditions $w_0 = 0$ and $w_1 = \frac{5}{36}$ (c.f. [28]). Thus

$$h_{2,3}(6n+6) = 6(n+1)\, h_{2,3}(6n) + \sum_{i=0}^{n} h_{2,3}(6i)\, h_{2,3}(6n-6i), \text{ for } n \geq 2, \tag{21}$$

with $h_{2,3}(0) = 0$, $h_{2,3}(6) = 5$, and $h_{2,3}(d) = 0$ for any non-zero $d \neq 0 \mod 6$.

This is a recurrence relation for the number of rooted triangulations with n darts (equivalently, $n/3$ triangles).

Example 5.3. In the case $p = 2, q = 4$ we arrive at the following equation for $w(\xi)$:

$$w(\xi) = \xi^2 w'(\xi) + \xi w(\xi) + \xi w^2(\xi) + \frac{3}{16}\, \xi, \tag{22}$$

which is also a classical Riccati equation. It gives rise to the following relation between the coefficients of the series $w(\xi) = \sum_{n=0}^{\infty} w_n \, \xi^n$:

$$w_{n+1} = (n+1)\, w_n + \sum_{i=0}^{n} w_i w_{n-i}, \text{ for } n \geq 2,$$

with initial conditions $w_0 = 0$ and $w_1 = \frac{3}{16}$. Thus

$$h_{2,4}(4n+4) = 4(n+1)\, h_{2,4}(4n) + \sum_{i=0}^{n} h_{2,4}(4i)\, h_{2,4}(4n-4i), \text{ for } n \geq 2, \tag{23}$$

with $h_{2,4}(0) = 0$, $h_{2,4}(4) = 3$, and $h_{2,4}(d) = 0$ for any non-zero $d \neq 0 \mod 4$, and we obtained a recurrence for the number of rooted quadrangulations with n darts (equivalently, $n/4$ squares).

Example 5.4. In the case $p = q = 3$ we arrive at yet another Riccati equation for $w(\xi)$:

$$w(\xi) = \xi^2 w'(\xi) + \xi w(\xi) + \xi w^2(\xi) + \frac{2}{9}\, \xi, \tag{24}$$

which gives rise to the recurrence relation for the number of rooted bi-coloured triangulations with n darts (equivalently, with $n/3$ triangles):

$$h_{3,3}(3n+3) = 3(n+1)\, h_{3,3}(3n) + \sum_{i=0}^{n} h_{3,3}(3i)\, h_{3,3}(3n-3i), \text{ for } n \geq 2, \tag{25}$$

with $h_{3,3}(0) = 0$, $h_{3,3}(3) = 2$, and $h_{3,3}(d) = 0$ for any non-zero $d \neq 0 \mod 3$.

Now consider the generating function $H^\circ(z)$ for any family of (p,q)-hypermaps associated with the classical Riccati equation, that is, the case when $N = 2$ in (18) – (19), or equivalently, when p and q satisfy the identity

$$(p-1)\ (q-1) = (p,q) + 1.$$

An easy calculation shows that the only possible values are (i) $p = 2, q = 3$, (ii) $p = 2, q = 4$ and (iii) $p = 3, q = 3$, which correspond to Examples 5.2, 5.3 and 5.4, respectively. In each case the Riccati equation has the form

$$w'(\xi) = \frac{w(\xi) - \xi w(\xi) - \xi w^2(\xi) - k}{\xi^2}, \tag{26}$$

where k is a constant. This is the form of the equation in [16, Theorem 5.2] of Klazar, so one can easily deduce the following:

Corollary 5.5. *The generating function $H^\circ(z)$ for any family of (p, q)-hypermaps associated with the classical Riccati equation is non-holonomic.*

Question 5.6. Are the generating functions $H^\circ(z)$ associated with higher-order equations in the Riccati hierarchy also non-holonomic?

All the above results and questions easily translate into the language of subgroup generating functions for the number of free subgroups of finite index in $\mathbb{Z}_p * \mathbb{Z}_q$.

6. Subgroup conjugacy and hypermap isomorphism

In order to count the conjugacy classes of index n free subgroups in Δ^+, we shall use Lemma 3.2 and first compute the number $\operatorname{card} \mathfrak{H}_{p,q}(n)$ of isomorphism classes of (p, q)-hypermaps with n darts.

We first recall the equations that hold for the species H^* of labelled (not necessarily connected) (p, q)-hypermaps and the species H of labelled connected (p, q)-hypermaps. Let S_p and S_q be the species of permutations (acting on the same set of darts) with only p and q cycles, respectively. Then

$$H^* = S_p \times S_q, \ \ H^* = E(H). \tag{27}$$

Let C_p and C_q be the species of p- and q-cycles, respectively, with their corresponding cycle indices

$$\mathcal{Z}_{C_p}(z_1, z_2, \dots, z_p) = \frac{1}{p} \sum_{d|p} \phi(d)\, z_d^{p/d}, \ \ \mathcal{Z}_{C_q}(z_1, z_2, \dots, z_q) = \frac{1}{q} \sum_{d|q} \phi(d)\, z_d^{q/d}, \tag{28}$$

where $\phi(d)$ is the Euler totient function.

The species E of sets has cycle index

$$\mathcal{Z}_E(z_1, z_2, \dots) = \exp\left(\sum_{k\geq 1} \frac{z_k}{k}\right). \tag{29}$$

Then, the cycle index of S_p becomes

$$\mathcal{Z}_{S_p}(z_1, z_2, \dots) = \mathcal{Z}_E\left(\frac{1}{p} \sum_{d|p} \phi(d)\, z_d^{p/d}, \frac{1}{p} \sum_{d|p} \phi(d)\, z_{2d}^{p/d}, \frac{1}{p} \sum_{d|p} \phi(d)\, z_{3d}^{p/d}, \dots\right) = \tag{30}$$

$$= \exp\left(\sum_{k\geq 1} \frac{1}{pk} \sum_{d|p} \phi(d)\, z_{kd}^{p/d}\right) = \exp\left(\sum_{k\geq 1} \sum_{d|p} \frac{1}{pk} \phi(d)\, z_{kd}^{p/d}\right) = \tag{31}$$

$$= \exp\left(\sum_{n=1}^{\infty} \sum_{d|(n,p)} \frac{d}{np} \phi(d)\, z_n^{p/d}\right) = \prod_{n=1}^{\infty} \exp\left(\frac{1}{np} \sum_{d|(n,p)} d\,\phi(d)\, z_n^{p/d}\right), \tag{32}$$

where the last identity is obtained by setting $n = kd$, and (n,p) denotes the greatest common divisor of n and p. An analogous expression holds for the cycle index $\mathcal{Z}_{S_q}(z_1, z_2, \dots)$ of the species S_q (up to the substitution of q in place of p).

Let us set

$$P_n(z_n) = \exp\left(\frac{1}{np} \sum_{d|(n,p)} d\,\phi(d)\, z_n^{p/d}\right),\ Q_n(z_n) = \exp\left(\frac{1}{nq} \sum_{d|(n,q)} d\,\phi(d)\, z_n^{q/d}\right). \tag{33}$$

Then

$$\mathcal{Z}_{S_p}(z_1, \dots, z_p) = \prod_{n=1}^{\infty} P_n(z_n),\ \mathcal{Z}_{S_q}(z_1, \dots, z_q) = \prod_{n=1}^{\infty} Q_n(z_n). \tag{34}$$

The cycle indices $\mathcal{Z}_{S_p}$ and $\mathcal{Z}_{S_q}$ are separable as defined in [28, Section A.3.3]. Thus, the cycle index $\mathcal{Z}_{H^*}$ can be expressed as

$$\mathcal{Z}_{H^*}(z_1, z_2, \dots) = \prod_{n=1}^{\infty} P_n(z_n) \odot Q_n(z_n). \tag{35}$$

Anytime two species A and B satisfy $A = E(B)$, by [2, §1.4, Exercice 9] we can obtain a formula for $\mathcal{Z}_B$ from $\mathcal{Z}_A$. In particular, the cycle index $\mathcal{Z}_H$ is

$$\mathcal{Z}_H(z_1, z_2, \dots) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \mathcal{Z}_{H^*}(z_n, z_{2n}, z_{3n}, \dots), \tag{36}$$

where $\mu(n)$ is the Möbius function. The generating function $\widetilde{H}(z) = \sum_{n\geq 0} \mathrm{card}\, \mathfrak{H}_{p,q}(n) z^n$ satisfies

$$\widetilde{H}(z) = \mathcal{Z}_H(z, z^2, z^3, \dots) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \mathcal{Z}_{H^*}(z^n, z^{2n}, z^{3n}, \dots) = \tag{37}$$

$$= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} \log(P_n \odot Q_n)(z_n)|_{z_n = z^{nk}}. \tag{38}$$

Thus we arrive at the following theorem:

Theorem 6.1. *The growth series* $\widetilde{H}(z) = \sum_{n=0}^{\infty} \mathrm{card}\, \mathfrak{H}_{p,q}(n) \cdot z^n$ *is given by the formula*

$$\widetilde{H}(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} \log(P_n \odot Q_n)(z_n)|_{z_n = z^{nk}},$$

with

$$P_n(z_n) = \exp\left(\frac{1}{np}\sum_{d|(n,p)} d\,\phi(d)\,z_n^{p/d}\right),\; Q_n(z_n) = \exp\left(\frac{1}{nq}\sum_{d|(n,q)} d\,\phi(d)\,z_n^{q/d}\right).$$

Its non-zero coefficients have the following asymptotic expansion:

$$[z^{\langle p,q\rangle k}]\widetilde{H}(z) \propto \frac{(2\pi)^{\frac{N-1}{2}}}{\prod_i \Gamma(a_i)}\; k^{(N-1)k-\frac{N+1}{2}+\sum_i a_i}\; e^{(1+\log c-N)k},\; \text{as } k\to\infty,$$

where $N = 1 + \frac{(p-1)(q-1)-1}{(p,q)}$, *and the constants* c *and* a_i *satisfy* $c^{(p,q)} = \langle p,q\rangle^{(p-1)(q-1)-1}$, $a_i = \frac{i(p,q)}{pq}$, *with* $i = 1,\ldots,\frac{pq}{(p,q)}-1$, $p\nmid i$, $q\nmid i$.

Proof. Most of the proof has been already provided above. The remaining part is the asymptotic behaviour of $[z^n]\widetilde{H}(z)$, as $n\to\infty$. Since most of the (p,q)-hypermaps on n darts are asymmetric (one can adapt the argument from [6, Section 7.1] in order to show this), we have that $\operatorname{card}\mathfrak{H}_{p,q}(n) \propto \frac{1}{n}\operatorname{card}\mathfrak{H}^r_{p,q}(n)$, as $n\to\infty$. Then we apply Theorem 4.1 to obtain the required asymptotic formula. □

Corollary 6.2. *The following asymptotic formula holds for the non-zero coefficients of* $\widetilde{H}(z)$*:*

$$[z^{\langle p,q\rangle k}]\widetilde{H}(z) \propto Ck^{(N-1)k-c_0}\; e^{c_1 k},\; \text{as } k\to\infty,$$

where N, C, c_0 *and* c_1 *are constants that depend only on* p *and* q.

Example 6.3. For the number of isomorphism classes of triangulations on n darts we have $\widetilde{H}(z) = \sum_{n=0}^{\infty}\operatorname{card}\mathfrak{H}_{2,3}\cdot z^n = 3z^6+11z^{12}+81z^{18}+1228z^{24}+28174z^{30}+843186z^{36}+30551755z^{42}+1291861997z^{48}+62352938720z^{54}+3381736322813z^{60}+\ldots$. The coefficient sequence of $\widetilde{H}(z)$ has number A129114 in the OEIS [30].

Example 6.4. For the number of isomorphism classes of quadrangulations on n darts we have $\widetilde{H}(z) = \sum_{n=0}^{\infty}\operatorname{card}\mathfrak{H}_{2,4}\cdot z^n = 2z^4+7z^8+36z^{12}+365z^{16}+5250z^{20}+103801z^{24}+2492164z^{28}+70304018z^{32}+2265110191z^{36}+82013270998z^{40}+\ldots$.

Example 6.5. For the number of isomorphism classes of bi-coloured triangulations on n darts we have $\tilde{H}(z) = \sum_{n=0}^{\infty}\operatorname{card}\mathfrak{H}_{3,3}\cdot z^n = 2z^3+3z^6+16z^9+133z^{12}+1440z^{15}+22076z^{18}+401200z^{21}+8523946z^{24}+206375088z^{27}+5611089408z^{30}+\ldots$.

To the best of our knowledge, the following questions remain at the moment unsettled.

Question 6.6. Is the growth series $\widetilde{H}(z)$ given in Theorem 6.1 algebraic or transcendental?

Question 6.7. Is $\widetilde{H}(z)$ a solution to any differential equation or system of equations? Is there a recurrence relation of any sort that the coefficients of $\widetilde{H}(z)$ satisfy?

Again, we can reformulate our result in group-theoretic terms via Lemma 3.2.

Theorem 6.8. *Let* $\Delta^+ = \mathbb{Z}_p * \mathbb{Z}_q$ *be a free product of cyclic groups with* $pq \geq 6$. *Then the conjugacy growth series* $C_f(z) = \sum_{n=0}^{\infty} c_f(n)z^n$, *for the number* $c_f(n)$ *of conjugacy classes of index* n *free subgroups in* Δ^+, *coincides with* $\widetilde{H}(z)$.

Example 6.9. The conjugacy growth series for free subgroups in $\mathbb{Z}_2 * \mathbb{Z}_3$, $\mathbb{Z}_2 * \mathbb{Z}_4$ and $\mathbb{Z}_3 * \mathbb{Z}_3$ are given in Examples 6.3, 6.4 and 6.5, respectively. An independent computation with GAP [8] by issuing `LowIndexSubgroupsFPGroup` command gives matching results.

7. Other free products and their free subgroups

As we mentioned in the introduction, the group $\Delta^+ = \mathbb{Z}_p * \mathbb{Z}_q$ is also known as the (p, q, ∞)-triangle group, a Fuchsian group with rich geometry. With the above technique we can also count the number of free subgroups of finite index and their conjugacy classes in $\Delta^+ = \mathbb{Z}_p * \mathbb{Z}$ and $\Delta^+ = \mathbb{Z} * \mathbb{Z}$, which are the (p, ∞, ∞)- and (∞, ∞, ∞)-triangle groups, respectively. These two groups have been fruitfully used by Breda d'Azevedo – Menykh – Nedela [3] and later on by Mednykh [19], Mednykh – Nedela [21, 22] for counting hypermaps: $\Delta^+ = \mathbb{Z}_2 * \mathbb{Z}$ was used for counting maps without semiedges, and $\Delta^+ = \mathbb{Z} * \mathbb{Z}$ was used for counting general hypermaps (both rooted hypermaps and their isomorphism classes).

Since most of the study has been already done in [3, 19, 21, 22], we briefly formulate the necessary statements below without a proof. Where necessary, we give more details.

7.1. **Free subgroups of $\mathbb{Z}_p * \mathbb{Z}$.** If $\Delta^+ = \mathbb{Z}_p * \mathbb{Z}$, then its free subgroups of index n are in bijection with the elements of $\mathfrak{H}^r_p(n)$, which is the set of rooted connected hypermaps on n darts with p-gonal hyperedges. The conjugacy classes of free index n subgroups in Δ^+ are in bijection with the elements of $\mathfrak{H}_p(n)$, which is the set of isomorphism classes of rooted connected hypermaps on n darts with p-gonal hyperedges. We shall call such hypermaps rooted, resp. unrooted, (p, ∞)-hypermaps.

Let us first define the species H° by the following equations

$$H^\circ = Z \cdot H',\ H^* = E(H),\ \text{and } H^* = S_p \times S, \tag{39}$$

analogous to equations (4) – (5), where S is the species of all permutations, H^* is the species of labelled, not necessarily connected, (p, ∞)-hypermaps, H is the species of labelled connected (p, ∞)-hypermaps, and H° of rooted ones. Thus, we have that $\mathfrak{H}^r_p(n) = H^\circ[n]$.

Using the fact that

$$S_p(z) = \sum_{k=0}^{\infty} \frac{z^{pk}}{p^k k!},\ S(z) = \frac{1}{1-z}, \tag{40}$$

we obtain

$$H^*(z) = (S_p \odot S)(z) = \sum_{k=0}^{\infty} \frac{(pk)!}{p^k k!} z^{pk}. \tag{41}$$

Then, analogous to Section 2.1, $H^*(z) = f(p^{p-1} z^p)$, where $f(\xi)$ is a hypergeometric function:

$$f(\xi) = {}_pF_0 \left[\begin{matrix} \frac{1}{p}, \ldots, \frac{p-1}{p}, 1 \\ \ldots \end{matrix} ; \xi \right]. \tag{42}$$

We also have that $H^\circ(z) = p\xi\frac{f'(\xi)}{f(\xi)} = pw(\xi)$, with $\xi = p^{p-1}z^p$, and $w(\xi)$ is a solution to an equation from the Riccati hierarchy with $N = p$, and $a_i = \frac{i}{p}$, for $i = 1, \ldots, p$, c.f. Section 5.

Example 7.1. If $p = 2$, then $H^*(z)$ enumerates labelled sensed pre-maps (without semi-edges) on n darts or, equivalently, with $n/2$ edges, $n = 1, 2, \ldots$, so that the odd coefficients of $H^*(z)$ vanish. We also have that $H^\circ(z) = \sum_{n=0}^{\infty} h(n)z^n$ enumerates sensed rooted maps on n darts. All the odd coefficients of $H^\circ(z)$ vanish as well. Finally, $H^\circ(z) = 2w(\xi)$, where $\xi = 2z^2$ and $w(\xi)$ satisfies the following Riccati equation:

$$w(\xi) = \xi^2 w'(\xi) + \frac{3}{2}\xi w(\xi) + \xi w^2(\xi) + \frac{1}{2}\xi.$$

Thus the coefficients of the series $w(\xi) = \sum_{n=0}^{\infty} w_n z^n$ satisfy $w_0 = 0$, $w_1 = \frac{1}{2}$ and

$$w_{n+1} = \left(n + \frac{3}{2}\right) w_n + \sum_{i=0}^{n} w_i w_{n-i},\ n \geq 1.$$

This implies

$$h(2n+2) = (2n+3)h(2n) + \sum_{i=0}^{n} h(2i)h(2n-2i),\ n \geq 1,$$

with $h(0) = 0$, $h(2) = 2$, and $h(2n+1) = 0$ for all natural $n \geq 0$.

By analogy to Theorems 4.1 and 4.6 the following holds.

Theorem 7.2. *The growth series $H^\circ(z) = \sum_{n=0}^{\infty} \operatorname{card} \mathfrak{H}_p^r(n) \cdot z^n$ is transcendental and the following asymptotic formula holds for its non-zero coefficients:*

$$[z^{pk}]H^\circ(z) \propto \frac{(2\pi)^{\frac{p-1}{2}}\, p}{\prod_{i=1}^{p} \Gamma\left(\frac{i}{p}\right)}\ k^{(p-1)k+1}\ e^{(p-1)(\log p - 1)k},\ \textit{as } k \to \infty.$$

Example 7.3. For the case of sensed maps on n darts without semi-edges (i.e. orientable connected $(2, \infty)$-hypermaps), we have that $H^\circ(z) = \sum_{n=0}^{\infty} \operatorname{card} \mathfrak{H}_2^r(n) \cdot z^n = 2z^2 + 10z^4 + 74z^6 + 706z^8 + 8162z^{10} + 110410z^{12} + 1708394z^{14} + 29752066z^{16} + 576037442z^{18} + 12277827850z^{20} + \ldots$. The coefficient sequence of $H^\circ(z)$ has number A000698 in the OEIS [30]. Since $H^\circ(z)$ satisfies a classical Riccati equation, by Corollary 5.5 this sequence is non-holonomic.

Theorem 7.4. *Let $\Delta^+ = \mathbb{Z}_p * \mathbb{Z}$. Then the subgroup growth series $S_f(z) = \sum_{n=0}^{\infty} s_f(n)z^n$, for the number $s_f(n)$ of index n free subgroups of Δ^+, coincides with $H^\circ(z)$.*

Example 7.5. For $\Delta^+ = \mathbb{Z}_2 * \mathbb{Z}$, the free subgroup growth series $S_f(z)$ is given in Example 7.3.

The number of isomorphism classes of hypermaps or, equivalently, the number of conjugacy classes of free subgroups in Δ^+, are given in the two theorems below.

Theorem 7.6. *The growth series* $\widetilde{H}(z) = \sum_{n=0}^{\infty} \operatorname{card} \mathfrak{H}_{p,q}(n) \cdot z^n$ *is given by the formula*

$$\widetilde{H}(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} \log(P_n \odot Q_n)(z_n)|_{z_n = z^{nk}},$$

with

$$P_n(z_n) = \exp\left(\frac{1}{np} \sum_{d|(n,p)} d\, \phi(d)\, z_n^{p/d}\right),\ Q_n(z_n) = \sum_{d=1}^{\infty} z_n^d = \frac{1}{1 - z_n}.$$

Its non-zero coefficients have the following asymptotic expansion:

$$[z^{pk}] H^{\circ}(z) \propto \frac{(2\pi)^{\frac{p-1}{2}}}{\prod_{i=1}^{p} \Gamma\left(\frac{i}{p}\right)}\, k^{(p-1)k}\, e^{(p-1)(\log p - 1)k},\ \textit{as } k \to \infty.$$

Example 7.7. For the case of isomorphism classes of sensed maps on n darts (i.e. isomorphism classes of orientable connected $(2,\infty)$-hypermaps), we have that $\widetilde{H}(z) = \sum_{n=0}^{\infty} \operatorname{card} \mathfrak{H}_2(n) \cdot z^n = 2z^2 + 5z^4 + 20z^6 + 107z^8 + 870z^{10} + 9436z^{12} + 122840z^{14} + 1863359z^{16} + 32019826z^{18} + 613981447z^{20} + \ldots$. The coefficient sequence of $\widetilde{H}(z)$ has number A170946 in the OEIS [30].

Theorem 7.8. *Let* $\Delta^+ = \mathbb{Z}_p * \mathbb{Z}$ *be a free product of cyclic groups with* $pq \geq 6$*. Then the conjugacy growth series* $C_f(z) = \sum_{n=0}^{\infty} c_f(n) z^n$*, for the number* $c_f(n)$ *of conjugacy classes of index* n *free subgroups in* Δ^+*, coincides with* $\widetilde{H}(z)$*.*

Example 7.9. For $\Delta^+ = \mathbb{Z}_2 * \mathbb{Z}$, the conjugacy counting function $C_f(z)$ coincides with $\widetilde{H}(z)$ from Example 7.7. This can be verified by an independent computation using GAP's `LowIndexSubgroupsFPGroup` command [8].

7.2. **Free subgroups of $\mathbb{Z} * \mathbb{Z}$.** If $\Delta^+ = \mathbb{Z} * \mathbb{Z} \cong F_2$, the free group on two generators, then all of its index n subgroups are free, and correspond bijectively to the elements of the set $\mathfrak{H}^r(n)$ of rooted hypermaps on n darts. The conjugacy classes of index n subgroups in Δ^+ are in bijection with the elements of $\mathfrak{H}(n)$, which is the set of isomorphism classes of hypermaps on n darts. In this case, the formulas for the number of index n subgroups in Δ^+ and their conjugacy classes are given by multiple authors including, respectively, Hall [11], for the former, and Liskovec [17], Mednykh [20], for the latter. The advantage of a species theory approach is that the proof turns out to be much simpler and shorter. Initially, we have that

$$H^*(z) = (S \odot S)(z) = \sum_{n=0}^{\infty} n! \cdot z^n, \tag{43}$$

and

$$H^{\circ}(z) = z \frac{d}{dz} \log H^*(z) = \sum_{n=0}^{\infty} s_n \cdot z^n, \tag{44}$$

where s_n is given by the recurrence relation (c.f. [11])

$$s_n = n \cdot n! - \sum_{k=1}^{n-1} k! \cdot s_{n-k},\ \text{and } s_0 = 0,\ s_1 = 1. \tag{45}$$

The sequence (45) is known to be non-holonomic due to Klazar (c.f. discussion after [16, Proposition 5.1]).

Also,

$$\mathcal{Z}_{H^*}(z_1, z_2, \ldots) = \prod_{n=1}^{\infty} \left(\sum_{i=0}^{\infty} i! \cdot n^i \cdot z_n^i \right) \tag{46}$$

and

$$\widetilde{H}(z) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log \mathcal{Z}_{H^*}(z^k, z^{2k}, \ldots) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \sum_{n=1}^{\infty} \log \left(\sum_{i=0}^{\infty} i! \cdot n^i \cdot z^{ikn} \right). \tag{47}$$

By using the fact that $\log\left(\sum_{i=0}^{\infty} i! z^i\right) = \sum_{i=1}^{\infty} \frac{s_i}{i} z^i$ (already used in formulas (43)-(44)), and substitution $z \to n \cdot z^{kn}$ we may simplify formula (47) down to

$$\widetilde{H}(z) = \sum_{n=1}^{\infty} z^n \cdot \frac{1}{n} \sum_{i|n} s_i \sum_{m|\frac{n}{i}} \mu\left(\frac{n}{i\,m}\right) \cdot m^{i+1} = \tag{48}$$

$$= \sum_{n=1}^{\infty} z^n \cdot \frac{1}{n} \sum_{\ell|n} s_{n/\ell}\, \varphi_{n/\ell+1}(\ell), \tag{49}$$

where $\varphi_m(\ell) = \sum_{d|\ell} \mu\left(\frac{\ell}{d}\right) d^m$ is the Jordan totient function. The coefficient $[z^n]\widetilde{H}(z)$ in (49) coincides with the one given in [4, Proposition 2.1], [17] or [20, Theorem 2].

Appendix

I. Counting free subgroups in $\mathbb{Z}_p * \mathbb{Z}_q$. First we define the necessary parameters: p, q, and n, the index up to which we shall count the number of free subgroups in $\mathbb{Z}_p * \mathbb{Z}_q$.

```
#computing the number of free subgroups in Z_p*Z_q of index < n
#or: computing the number of rooted (p,q)-hypermaps on < n darts

#defining p, q
p = 2; q = 3;

#defining n
n = 36;

#defining power series ring over \mathbb{Q}
R.<z> = PowerSeriesRing(QQ, default_prec = 2*n);
```

Then we define the auxiliary function $f(\xi)$.

```
#defining f(z), such that H^\ast(z) = f(c*z^lcm(p,q))
f = 1; coeff = 1;
for k in range(1,n):
    mult = 1;
    for i in range(1, lcm(p,q)):
```

```
        if (not(i%p==0) and not(i%q==0)):
            mult = mult*(k-1+i/lcm(p,q));
    coeff = coeff*mult;
    f = f + coeff*power(z,k)/factorial(k);
```

We shall also need the constant c for the substitution $\xi = cz^{\langle p,q \rangle}$.

```
#defining the constant c
c = power(lcm(p,q), (p*q - p - q)/gcd(p,q));
```

Finally, $H^\circ(z)$ can be computed.

```
#computing H^\circ
Hcirc = lcm(p,q)*z*derivative(f)/f;
Hcirc.substitute(z = c*power(z,lcm(p,q))).truncate(n);
```

```
>828250*z^30 + 27120*z^24 + 1105*z^18 + 60*z^12 + 5*z^6
```

II. Finding a generalised Riccati equation for $w(\xi)$. First we define p and q, and then compute the number of terms in equation (18).

```
#defining p, q
p = 2;
q = 3;
```

```
#computing N
N = 1 + (p*q - p - q)/gcd(p,q);
N;
>2
```

Next we prepare all the ingredients for formula (18), starting with the symmetric functions on the a_i's, then computing the $w_i(\xi)$'s, and finally the equation itself. Below we use Z instead of ξ in the SAGE code throughout.

```
#defining symmetric functions
e = SymmetricFunctions(QQ).elementary();
```

```
#creating the list of a_i's
a_val = [];
for i in range(1,lcm(p,q)):
    if (not(i%p==0) and not(i%q==0)):
        a_val.append(i/lcm(p,q));
```

```
#defining the symmetric coefficient \sigma_k(a_1,\dots,a_N)
def sym_coeff(k):
    coeff = e([k]).expand(int(N), alphabet='a');
    return coeff(*a_val);
```

```
#defining \xi and w as a function of \xi
var('Z'); w = function('w')(Z); w;
>Z
>w(Z)
```

```
#defining w_i(\xi) for i = 1,\dots,N
w_lst = [1,w];
for i in range(1,N):
     w_lst.append(Z*w_lst[i].derivative(Z) + w*w_lst[i]);

#defining the list of symmetric coefficients of the equation
coeff_lst = [sym_coeff(i) for i in range(N+1)];
```

Now we compose the equation itself, and view the output.

```
#composing the equation
eqn = expand(w - Z*sum([a*b for a,b in zip(reversed(coeff_lst), w_lst)]));
#and viewing it
view(eqn);

> - Z*w(Z)^2 - Z^2*D[0](w)(Z) - Z*w(Z) - 5/36*Z + w(Z)
```

III. Counting free conjugacy classes in $\mathbb{Z}_p * \mathbb{Z}_q$. We start by defining the parameters: p, q, and n, and the polynomial ring that we shall use. Although cycle indices are multivariate polynomials, we need only one variable z to present the final result of computation as $\widetilde{H}(z)$.

```
#computing the number of conjugacy classes of
# free subgroups in Z_p*Z_q of index < n
#or: computing the number of isomorphism classes
# of (p,q)-hypermaps on < n darts

#defining p, q
p = 2;
q = 3;

#defining n
n = 36;

#defining power series ring over \mathbb{Q}
R.<z> = PowerSeriesRing(QQ, default_prec=2*n);
```

Next we define the auxiliary functions $P_m(z_m)$ and $Q_m(z_m)$. Note that we use z as a variable instead of z_m (computing series in a single variable is usually faster in SAGE).

```
def P(m):
    sum = 0;
    for d in divisors(gcd(m,p)):
        sum = sum + d*euler_phi(d)*power(z, p//d);
    sum = sum/(m*p);
    return sum.exp(2*n);

def Q(m):
    sum = 0;
    for d in divisors(gcd(m,q)):
```

```
        sum = sum + d*euler_phi(d)*power(z, q//d);
    sum = sum/(m*q);
    return sum.exp(2*n);
```

Next we define the Hadamard product of $P_m(z_m)$ and $Q_m(z_m)$

```
def h_prod_PQ(m):
    prod = 0;
    P_coeff = P(m).dict();
    Q_coeff = Q(m).dict();
    for k in Set(P_coeff.keys()).intersection(Set(Q_coeff.keys())):
            prod = prod \\
            + power(z,k)*P_coeff[k]*Q_coeff[k]*factorial(k)*power(m,k);
    return prod;
```

and its logarithm $\log(P_m \odot Q_m)(z_m)$ upon the substitution $z_m = z^{km}$

```
def log_h_prod_PQ(m,k):
    return log(h_prod_PQ(m)).substitute(z=power(z,m*k));
```

Finally, we define the general term of the double sum in Theorem 6.1

```
@parallel
def term(m,k):
    return moebius(k)/k*log_h_prod_PQ(m,k);
```

in a way that allows parallel computing in order to speed up the computation.

The function $\widetilde{H}(z)$ is the double sum of the terms `term(m,k)` above. We define it, and compute its output.

```
#defining H_tilde(n):
def H_tilde(n):
    return sum([t[1] for t in list(term([(m,k) for m in range(1,n) \\
    for k in range(1,n)]))]).truncate(n);
#and computing it:
H_tilde(n);
```

```
>28174*z^30 + 1228*z^24 + 81*z^18 + 11*z^12 + 3*z^6
```

IV. Free products $\mathbb{Z}_p * \mathbb{Z}$ and $\mathbb{Z} * \mathbb{Z}$. In order to compute the subgroup growth series for the number of free subgroups having index $< n$ in $\mathbb{Z}_p * \mathbb{Z}$ we use essentially the same code as in Appendix I.

```
#computing the number of free subgroups in Z_p*Z of index < n
#or: computing the number of rooted (p,\infty)-hypermaps on < n darts
```

```
#defining p
p = 2;
#defining n
n = 22;
```

```
#defining the power series ring
R.<z> = PowerSeriesRing(QQ, default_prec=2*n);
```

The auxiliary function $f(\xi)$ is defined, where $\xi = p^{p-1}\, z^p$. Analogous to Appendix I, we have that $H^*(z) = f(p^{p-1}\, z^p)$ and $H^\circ(z) = p^{p-1} z^p \frac{f'(z)}{f(z)}$.

```
#defining the auxiliary function f(\xi)
f = sum( [ factorial(p*k)/power(p, p*k)*power(z, k)/factorial(k) \\
for k in range(n) ] );
```

```
#computing H^\circ
Hcirc = p*z*derivative(f)/f;
Hcirc.substitute(z=power(p,p-1)*power(z,p)).truncate(n);
```

```
>12277827850*z^20 + 576037442*z^18 + 29752066*z^16 + 1708394*z^14 +
110410*z^12 + 8162*z^10 + 706*z^8 + 74*z^6 + 10*z^4 + 2*z^2
```

Similarly we use our SAGE code from Appendix III in order to compute the number of conjugacy classes of free subgroups in $\mathbb{Z}_p * \mathbb{Z}$. The only change that we need to perform is setting

```
def P(m):
    sum = 0;
    for d in divisors(gcd(m,p)):
        sum = sum + d*euler_phi(d)*power(z, p//d);
    sum = sum/(m*p);
    return sum.exp(2*n);
```

```
def Q(m):
    return sum([power(z,k) for k in range(2*n)]);
```

For instance, computing the number conjugacy classes of free subgroups in $\mathbb{Z}_p * \mathbb{Z}$ of index $< n$ with $p = 2$ and $n = 22$ produces the following output which is consistent with [3, Table 1].

```
H_tilde(n);
>613981447*z^20 + 32019826*z^18 + 1863359*z^16 + 122840*z^14 + 9436*z^12
+ 870*z^10 + 107*z^8 + 20*z^6 + 5*z^4 + 2*z^2
```

Computing the number of subgroups in $\mathbb{Z} * \mathbb{Z} \cong F_2$ can be achieved by using Hall's recursive formula [11].

In order to compute the number of conjugacy classes of subgroups in $\mathbb{Z} * \mathbb{Z} \cong F_2$, we again use the SAGE code from Appendix III while setting $P_m(z_m)$ and $Q_m(z_m)$ to be

```
def P(m):
    return sum([power(z,k) for k in range(2*n)]);
```

and

```
def Q(m):
    return sum([power(z,k) for k in range(2*n)]);
```

This produces the following output (as before we keep $n = 22$):

```
> 48509766592893402121*z^21 +
2303332664693034476*z^20 + 114794574818830735*z^19 +
6019770408287089*z^18 + 333041104402877*z^17 + 19496955286194*z^16 +
1211781910755*z^15 + 80257406982*z^14 + 5687955737*z^13 + 433460014*z^12
+ 35704007*z^11 + 3202839*z^10 + 314493*z^9 + 34470*z^8 + 4163*z^7 +
624*z^6 + 97*z^5 + 26*z^4 + 7*z^3 + 3*z^2 + z
```

The coefficient sequence of the above series has number A057005 in the OEIS [30].

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