Studies on the Pea Pattern Sequence

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1 Abstract

The paper will formally define the family of sequences know as "Pea Pattern". It will then anaylize its growth and conditions for fixed points. The paper ends with a list of fixed points.

2 Introduction

This paper was inspired by John Conway's works on the [A005150] sequence, also known as the "Look and Say sequence" (see [2]). The Pea Pattern sequence, tough similarly defined, holds very different properties. The existence of fixed points and cycles in this sequence is the main object of study in this paper, which also compiles every fixed point for the sequences in bases 2 to 9.

3 Initial Definitions and Notations

Note that the existing bijection between the numerical base k of $\mathbb{N}$ and the set of words Σ_k^* over the alphabet $\Sigma_k = \{0, 1, ..., k-1\}$. wil be exploited throught the paper, as no confusion should arise from it. The concatenation operator shall be denoted by the symbol $||$, although it might be omitted when its existence is implicit.

Given $x_0 \in \Sigma_k^*$, define the sequence $(x_n)_n \in \Sigma^*$ recursively by the expression:

$$x_{n+1} = \mathcal{P}_k(x_n) = a_{(n_1,1)}a_{(n_1-1,1)}...a_{(0,1)}b_1a_{(n_2,2)}...a_{(0,2)}b_2...a_{(n_r,r)}...a_{(0,r)}b_r$$

Such that $(a_{(n_j,j)}...a_{(0,j)})_k = |x_n|_{b_j},\ 1 \leq j \leq r$ with $|x_n|_{b_j} \neq 0,\ 1 \leq j \leq r$ and $b_1 > b_2 > ... > b_r$.
Note that $(a_{(n_j,j)}...a_{(0,j)})_k$ is the uniquely defined natural number by base k representation.
For example, given $x_0 = 123 \in \Sigma_{10}^*$, yields: $x_1 = 131211$, $x_2 = 131241, ...$

Theorem 1. *For all $x_0 \in \Sigma_k$, the sequence defined by $x_{n+1} = \mathcal{P}_k(x_n)$ converges to a fixed point or a cycle.*

Proof. Note that there exists a finite number of words of a given length $l \in \mathbb{N}$,so it suffices to show that the sequence terms are bounded in length. Also, note that the numbers letter of $x_{n+1} \in \Sigma_k$ is given by:

$$|x_n| = \lfloor \log_k x_n \rfloor + 1$$

But by definition:

$$\begin{aligned} |x_{n+1}| &= |(|x_n|_{k-1})_k \,| + |(k-1)\,| + |\,...\,| + |\,(|x_n|_0)_k\,| + |\,0| \\ &\le |(|x_n|_{k-1})_k\,| + |\,...\,| + |\,(|x_n|_0)_k| + k \\ &\le k \left| \left(\frac{|x_n|}{k} \right)_k \right| + k = k \left\lfloor \log_k \frac{|x|}{k} \right\rfloor + 2k \end{aligned}$$

Therefore:

$$|x_{n+1}| \le k \lfloor \log_k |x_n| \rfloor + k \tag{1}$$

Since

$$\lfloor \log_k r \rfloor \le r/k^2 + 1 \quad \forall r \in \mathbb{N}$$

Meaning (1) implies in:

$$|x_{n+1}| \le \frac{|x_n|}{k} + 2k \tag{2}$$

Therefore, for $|x_n| > \frac{2k^2}{k-1}$, we have that:

$$|x_{n+1}| < |x_n|$$

Note that, $|x_n| \le \frac{2k^2}{k-1}$ implies that

$$|x_{n+1}| \le \frac{|x_n|}{k} + 2k \le \frac{2k}{k-1} + 2k = \frac{2k^2}{k-1}$$

Therefore, let $l = \max\{|x_0|, \frac{2k^2}{k-1}\}$. Then $|x_n| \le l \quad \forall n \in \mathbb{N}$ ☐

This result also tells us $|x_n| \le \frac{2k^2}{k-1}$ for all $n > N \in \mathbb{N}$ sufficiently large.

Thus one would only need to check for words of length less than $\frac{2k^2}{k-1}$ for fixed points in Σ_k^*. Since there are k letters in Σ_k, that results in a total of

$$\frac{k^{\frac{2k^2}{k-1}} - 1}{k-1}$$

different words, meaning the number of words to be tested grows exponetially with k. In light of that, Theorem 2 reduces the number of words to be tested, as follows.

Theorem 2. *Let $\overline{x} \in \Sigma_k^*$ be a fixed point of the sequence defined by $x_{n+1} = \mathcal{P}_k(x_n)$, such that:*

$$\overline{x} = a_{(n_1,1)}a_{(n_1-1,1)}...a_{(0,1)}b_1a_{(n_2,2)}...a_{(0,2)}b_2...a_{(n_r,r)}...a_{(0,r)}b_r$$

Where $a_{(i,j)} \in \Sigma_k,\ \forall i < n_j$ and $a_{(n_j,j)} \in \Sigma_k - \{0\}$ e $0 \leq n_j;\ 1 \leq j \leq r$.
Then:

$$\sum_{j=1}^{r} n_j \geq \sum_{j=1}^{r} k^{n_j} - 2r \tag{3}$$

Proof. Initially, note that:

$$|\overline{x}| = (n_1+1)+1+(n_2+1)+1+...+(n_r+1)+1 = \sum_{j=1}^{r} n_j + 2r \tag{4}$$

Also, since $\overline{x}$ is a fixed point, $\mathcal{P}_k(\overline{x}) = \overline{x}$, therefore:

$$\begin{gathered}
a_{(n_1,1)} \cdot k^{n_1} + ... + a_{(0,1)} \cdot k^0 = |\overline{x}|_{b_1} \\
a_{(n_2,2)} \cdot k^{n_2} + ... + a_{(0,2)} \cdot k^0 = |\overline{x}|_{b_2} \\
\vdots \\
a_{(n_r,r)} \cdot k^{n_r} + ... + a_{(0,r)} \cdot k^0 = |\overline{x}|_{b_r} \\
\therefore \\
\sum_{j=1}^{r} \left(a_{(n_j,j)} \cdot k^{n_j} + ... + a_{(0,j)} \cdot k^0 \right) = |\overline{x}|
\end{gathered}$$

Since $0 \leq a_{(i,j)}$:

$$\sum_{j=1}^{r} \left(a_{(n_j,j)} \cdot k^{n_j} \right) \leq |\overline{x}|$$

And $1 \leq a_{(n_j,j)}$ implies that:

$$\sum_{j=1}^{r} k^{n_j} \leq |\overline{x}|$$

Equation (4) yields:

$$\sum_{j=1}^{r} k^{n_j} \leq \sum_{j=1}^{r} n_j + 2r$$

Thus,

$$\sum_{j=1}^{r} n_j \geq \sum_{j=1}^{r} k^{n_j} - 2r$$

□

This result can be generalized into the following:
Denote by $\overline{x}_i$ para $i = 1, ..., p$ be the words contained in a p-cycle, such that, for $i = 1, ..., p$:

$$\overline{x} = a_{(n_{(1,i)},1,i)} a_{(n_{(1,i)}-1,1,i)} ... a_{(0,1,i)} b_1 a_{(n_{(2,i)},2,i)} ... a_{(0,2,i)} b_2 ... a_{(n_{(r,i)},r_i,i)} ... a_{(0,r,i)} b_r$$

Theorem 3. *Let $\overline{x}_i$ be the words in a p-cycle in Σ_k^*. Then:*

$$\sum_{i=1}^{p} \sum_{j=1}^{r} n_{(j,i)} \geq \sum_{i=1}^{p} \sum_{j=1}^{r} k^{n_{(j,i)}} - 2pr$$

Proof. Analogous to the previous proof. □

As a consequence from these results, we can easily calculate all the fixed points and cycles in a given alphabet. Therefore, it follows:

Corollary 1. *The sequence defined by $x_{n+1} = \mathcal{P}_2(x_n)$, converges to the fixed point: $\overline{x} = 1001110$, for all $x_0 \in \Sigma_2^*$, except for the fixed point $x_0 = \overline{x}' = 111$.*

Proof. It is easily checked by exasution that that is the case □

4 Conclusion

The paper succed in proving simple, yet fundamental results about the Pea Pattern sequence. The autor believes that the next steps should be in finding better conditions for fixed points, aiding their numerical computation.

Fixed Points for Σ_k^*, with $k =$				
2	3	4	5	6
111	22	22	22	22
1001110	11110	1211110	14233221	14233221
	12111	1311110	14331231	14331231
	101100	1312111	14333110	14333110
	1022120	23322110	23322110	15143331
	2211110	33123110	33123110	15233221
	22101100	132211110	131211110	15331231
			141211110	15333110
			141311110	23322110
			141312111	33123110
			1433223110	1433223110
			14132211110	1514332231
				1533223110
				14131211110
				15131211110
				15141311110
				15141312111
				1514132211110

Table 1: Fixed points for the first few values of k.

References

[1] Conway, J. H. "The Weird and Wonderful Chemistry of Audioactive Decay." Eureka 46, 5-18, 1986.

[2] OEIS Foundation Inc. (2017), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A005150