PRODUCT FORMULAS ON POSETS, WICK PRODUCTS, AND A CORRECTION FOR THE q-POISSON PROCESS

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ABSTRACT. We give an example showing that the product and linearization formulas for the Wick product versions of the q-Charlier polynomials in [Ans04a] are incorrect. Next, we observe that the relation between monomials and several families of Wick polynomials is governed by "incomplete" versions of familiar posets. We compute Möbius functions for these posets, and prove a general poset product formula. These provide new proofs and new inversion and product formulas for Wick product versions of Hermite, Chebyshev, Charlier, free Charlier, and Laguerre polynomials. By different methods, we prove product and inversion formulas for the Wick product versions of the free Meixner polynomials. As a corollary, we obtain a simple formula for their linearization coefficients.

1. INTRODUCTION

Let $\mathcal{M}$ be a complex star-algebra, and $\langle\cdot\rangle$ a star-linear functional on it. Let $\Gamma(\mathcal{M})$ be the complex unital star-algebra generated by non-commuting symbols $\{X(a) : a \in \mathcal{M}\}$ and 1, subject to the linearity relations

$$X(\alpha a + \beta b) = \alpha X(a) + \beta X(b).$$

Thus $\Gamma(\mathcal{M})$ is naturally isomorphic to the tensor algebra of $\mathcal{M}$, but we prefer to use the polynomial notation rather than this identification. The star-operation on $\Gamma(\mathcal{M})$ is determined by the requirement that $X(a^*) = X(a)^*$.

In this article we will discuss six constructions of *Wick products* (four known and two new), that is, linear maps $W : \mathcal{M}^{\otimes n} \to \Gamma(\mathcal{M})$ whose ranges are linearly independent and (together with the scalars) span $\Gamma(\mathcal{M})$. These have also been called the Kailath-Segall polynomials. As is well-known, there are three reasons to consider such objects.

- Define a star-linear functional φ of $\Gamma(\mathcal{M})$ by $\varphi[1] = 1$, $\varphi[W(a_1 \otimes a_2 \otimes \ldots \otimes a_n)] = 0$ for $n \geqslant 1$. In many examples, φ is positive, the W operators have orthogonal ranges for different n, and we have a Fock representation of $\Gamma(\mathcal{M})$ on (a quotient of) $L^2(\Gamma(\mathcal{M}), \varphi)$. In this case the W operators are indeed Wick products.
- For $\mathcal{M} = (L^1 \cap L^\infty)(\mathbb{R})$, in many examples we have an Itô isometry which allows us to interpret $W(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ as a stochastic integral

$$\int \ldots \int a_1(t_1) \ldots a_n(t_n)\, dX(t_1) \ldots dX(t_n).$$

This isometry may involve unfamiliar inner products on multivariate function spaces, see Remark 43.

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- Suppose all $a_1 = \ldots = a_n = a$. Setting $a = 1$, or more generally a multiple of a projection, $W(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ becomes a polynomial in $X(a)$. In many examples, these polynomials are orthogonal for different n.

Our main interest is in mutual expansions between polynomials $W(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ and monomials, product formulas

$$\prod_{i=1}^{k} W\left(a_{u_i(1)} \otimes a_{u_i(2)} \otimes \ldots \otimes a_{u_i(s(i))}\right) = \sum W,$$

and corresponding linearization coefficients. We show that the product formulas for q-Wick products claimed in [Ans04a] are incorrect. The rest of the article concerns related positive results.

In combinatorics, linearization formulas are often proved using a weight-preserving sign-reversing involution, in other words a version of the inclusion-exclusion principle. We use a different generalization of this principle, namely Möbius inversion. For five out of our six examples, we define posets Π such that

$$X(a_1) \ldots X(a_n) = \sum_{\pi \in \Pi} W(a_1 \otimes \ldots \otimes a_n)^{\pi}.$$

The posets which arise are "incomplete" versions of matchings, non-crossing matchings, set partitions, non-crossing partitions, and permutations. These posets, as posets, may be deserving of further study. We compute their Möbius functions and thus obtain inversion formulas. We also prove a general product formula on posets, and apply it to obtain product formulas for Wick products. For the matchings and partitions the results are known but the proofs are new. For the permutations the results are new. For the non-crossing matchings and partitions, the results are known for the usual Wick products, but the poset method allows us to extend them to operator-valued Wick products with no difficulty. As an application, it was observed by ad hoc methods that in the case of incomplete partitions, inversion formulas involve general open blocks but only singleton closed blocks (see Proposition 7 for terminology). The Möbius function approach provides an explanation for this phenomenon. As expected, in the product formulas, in the partitions cases only inhomogeneous partitions appear, while in the permutation case we encounter "incomplete generalized derangements".

For our sixth and most interesting example, morally corresponding to "non-crossing permutations", we do not know a natural poset structure governing Wick product expansions. So we perform the computations in a more direct way, using induction and generating functions. The combinatorial objects which govern these expansions are pairs $\sigma \ll \pi$ of non-crossing partitions in a relation first observed by Belinschi and Nica [BN08, Ans07, Nic10]. Interestingly, in the product formulas the operators in the sum depend only on π, while the inhomogeneity condition now applies only to σ. The general inversion formula obtained in this approach, with an appropriate choice of coefficients, provides yet another explanation for the phenomenon in the preceding paragraph.

Next, we restrict the results to single-variable polynomials. With some additional work, using the machinery from the sixth example, we obtain the formula for linearization coefficients of the free Meixner polynomials. This complements many known results for the Meixner polynomials [KZ01] and partial results for the q-interpolation between these two families [ISV87, Ans05, KSZ06, KSZ11, IKZ13]. We also list some enumeration results for posets from Section 2.

Finally, we discuss some analytic extensions of the algebraic results from earlier sections. It is natural to ask whether the map $(\mathcal{M}, \langle\cdot\rangle) \mapsto (\Gamma(\mathcal{M}), \varphi)$ can be interpreted as a functor, generalizing the well-known Gaussian/semiciruclar functors. For the case of the incomplete non-crossing partitions, we show that this is the case, although a larger collection of morphisms would be desirable. We also observe that in all six of our examples, if the functionals $\langle\cdot\rangle$ are tracial states, so are the functionals φ. Finally, for the case of incomplete non-crossing partitions, we show that the product formulas hold when one of the factors is in L^2.

The paper is organized as follows. In Section 2, we prove a general linearization result on posets, and compute Möbius functions for five posets. In Section 3, we use these to obtain inversion and product formulas for five types of Wick products. In Section 4, we obtain the corresponding expansions for the free Meixner Wick products. In short Section 5 we give an explicit counterexample to the product formula for q-Wick products claimed in [Ans04a]. In Section 6, we list combinatorial consequences of these results for ordinary polynomials. In particular, in Proposition 30 we compute the linearization coefficients for the free Meixner polynomials. In Section 7, we collect the analytic results.

2. Linearization on posets

Remark 1. Let $(\Pi_n, \leqslant)_{n=1}^{\infty}$ be a family of posets. As usual, we write $\sigma < \pi$ if $\sigma \leqslant \pi$ and $\sigma \neq \pi$. In all our examples, Π_n will be a meet-semilattice, with the meet operation $\wedge$, and the smallest element, denoted by $\hat{0}_n$. Recall that for $\sigma \leqslant \pi$, the Möbius function $\mu(\sigma, \pi)$ on Π is determined by the property that

$$\sum_{\tau : \sigma \leqslant \tau \leqslant \pi} \mu(\sigma, \tau) = \begin{cases} 1, & \sigma = \pi, \\ 0, & \sigma \neq \pi. \end{cases} \tag{1}$$

As a consequence, we have the Möbius inversion formula: if F, G are two functions on Π such that

$$F(\pi) = \sum_{\sigma \geqslant \pi} G(\sigma), \tag{2}$$

then

$$G(\pi) = \sum_{\sigma \geqslant \pi} \mu(\pi, \sigma) F(\sigma).$$

Theorem 2. *Consider a family of posets $(\Pi_i)_{i=1}^{\infty}$. Fix $s(1), s(2), \ldots, s(k) \geqslant 1$, and denote $n = s(1) + \ldots + s(k)$. Suppose we have an order-preserving injection*

$$\alpha : \Pi_{s(1)} \times \ldots \times \Pi_{s(k)} \to \Pi_n$$

with the property that for any $\tau \in \Pi_n$ there exists a $\tau_{s(1),\ldots,s(k)} \in \Pi_n$ with

$$\left\{\sigma \in \alpha(\Pi_{s(1)} \times \ldots \times \Pi_{s(k)}), \sigma \leqslant \tau\right\} = \left\{\sigma \in \Pi_n : \sigma \leqslant \tau_{s(1),\ldots,s(k)}\right\},$$

where $\tau_{s(1),\ldots,s(k)} = \tau$ if $\tau \in \alpha(\Pi_{s(1)} \times \ldots \times \Pi_{s(k)})$. Let G be a function on the disjoint union $\coprod_{n=1}^{\infty} \Pi_n$. By abuse of notation, we extend this function to the direct sum poset $\bigoplus_{i=1}^{\infty} \Pi_i$. Denote, for $\pi \in \Pi_i$,

$$F(\pi) = \sum_{\sigma \geqslant \pi} G(\sigma),$$

and extend it as above. Suppose that for $\pi_i \in \Pi_{s(i)}$,

$$F(\pi_1, \ldots, \pi_k) = F(\alpha(\pi_1, \ldots, \pi_k)). \tag{3}$$

Then

$$G(\hat{0}_{s(1)}) \ldots G(\hat{0}_{s(k)}) = \sum_{\substack{\tau \in \Pi_n \\ \tau_{s(1),\ldots,s(k)} = \hat{0}_n}} G(\tau).$$

Proof. Taking first $\tau = \alpha(\sigma_1, \ldots, \sigma_k)$, we see that since $\tau = \tau_{s(1),\ldots,s(k)}$,

$$\begin{aligned}
[\hat{0}_n, \alpha(\sigma_1, \ldots, \sigma_k)] = [\hat{0}_n, \tau] &= \left\{\sigma \in \Pi_n : \sigma \leqslant \tau_{s(1),\ldots,s(k)}\right\} \\
&= \left\{\sigma \in \alpha(\Pi_{s(1)} \times \ldots \times \Pi_{s(k)}), \sigma \leqslant \alpha(\sigma_1, \ldots, \sigma_k)\right\} \\
&\simeq \left\{\sigma \in \Pi_{s(1)} \times \ldots \times \Pi_{s(k)}, \sigma \leqslant (\sigma_1, \ldots, \sigma_k)\right\} \\
&= [\hat{0}_{s(1)}, \sigma_1] \times \ldots \times [\hat{0}_{s(k)}, \sigma_k]
\end{aligned}$$

Thus, since the Möbius function is multiplicative,

$$\mu(\hat{0}_n, \alpha(\sigma_1, \ldots, \sigma_k)) = \prod_{i=1}^{k} \mu(\hat{0}_{s(i)}, \sigma_i).$$

The rest of the proof is similar to that of Theorem 4 in [RW97]. Using various assumptions,

$$\begin{aligned}
G(\hat{0}_{s(1)}) \ldots G(\hat{0}_{s(k)}) &= \sum_{\sigma_1 \in \Pi_{s(1)}} \ldots \sum_{\sigma_k \in \Pi_{s(k)}} \prod_{i=1}^{k} \mu(\hat{0}_{s(i)}, \sigma_i) \prod_{i=1}^{k} F(\sigma_i) \\
&= \sum_{(\sigma_1,\ldots,\sigma_k) \in \Pi_{s(1)} \times \ldots \times \Pi_{s(k)}} \mu(\hat{0}_n, \alpha(\sigma_1, \ldots, \sigma_k)) F(\alpha(\sigma_1, \ldots, \sigma_k)) \\
&= \sum_{(\sigma_1,\ldots,\sigma_k) \in \Pi_{s(1)} \times \ldots \times \Pi_{s(k)}} \mu(\hat{0}_n, \alpha(\sigma_1, \ldots, \sigma_k)) \sum_{\tau \geqslant \alpha(\sigma_1,\ldots,\sigma_k)} G(\tau) \\
&= \sum_{\tau \in \Pi_n} G(\tau) \sum_{\substack{(\sigma_1,\ldots,\sigma_k) \in \Pi_{s(1)} \times \ldots \times \Pi_{s(k)}, \\ \alpha(\sigma_1,\ldots,\sigma_k) \leqslant \tau}} \mu(\hat{0}_n, \alpha(\sigma_1, \ldots, \sigma_k)) \\
&= \sum_{\tau \in \Pi_n} G(\tau) \sum_{\substack{\sigma \in \Pi_n, \\ \sigma \leqslant \tau_{s(1),\ldots,s(k)}}} \mu(\hat{0}_n, \sigma) \\
&= \sum_{\substack{\tau \in \Pi_n \\ \tau_{s(1),\ldots,s(k)} = \hat{0}_n}} G(\tau).
\end{aligned}$$

□

Remark 3. We will show that for the five posets in the next series of propositions, the conditions above are satisfied, and compute the corresponding $\tau_{s(1),\ldots,s(k)}$. In all the examples, α combines objects defined on each of the subintervals

$$J_1 = [1, \ldots, s(1)], \quad J_2 = [s(1)+1, \ldots, s(1)+s(2)], \quad \ldots, \quad J_k = [n - s(k) + 1, \ldots, n] \tag{4}$$

into a single object on the interval $[1, \ldots, n]$, in a natural way. We will denote by $(\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)})$ the partition on $[n]$ whose blocks are these intervals. Note that Π_i is *not* assumed to have a maximal element.

Notation 4 (Background on partitions)**.** A partition π of an ordered set Λ is *non-crossing* if there are no two blocks $U \neq V$ of π with $i, k \in U$, $j, l \in V$, and $i < j < k < l$. The set of non-crossing partitions is denotes by $\mathcal{NC}(\Lambda)$, or $\mathcal{NC}(n)$ in case $\Lambda = [n] = \{1, 2, \ldots, n\}$. A block $U \in \pi$ is *inner* if there exists a block $V \neq U$ and $i, k \in V$ such that for all $j \in U$, $i < j < k$. Otherwise U is called *outer*. Denote $Out(\pi)$ the outer blocks of π, $Sing(\pi)$ the single-element blocks, and $Pair(\pi)$ the two-element blocks. Denote $\mathcal{NC}_{\geqslant 2}(\Lambda)$ the partitions with no singletons. The *interval partitions* $Int(\Lambda)$ are partitions whose blocks are intervals.

Proposition 5. *Denote by $\mathcal{P}_{1,2}(n)$, the* incomplete matchings*, the partitions of $[n]$ into pairs and singletons. Equip it with the poset structure it inherits from the usual refinement order on partitions. Then the Möbius function on this poset is*

$$\mu(\pi, \sigma) = (-1)^{|Pair(\sigma)| - |Pair(\pi)|}.$$

Also for this poset, $\tau_{s(1), \ldots, s(k)} = \tau \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)})$.

Proof. It suffices to note that, denoting by U the pairs and V the singleton blocks,

$$[(U_1, \ldots, U_u, V'_1, \ldots, V'_v), (U_1, \ldots, U_k, V_1, \ldots, V_\ell)] \simeq [\hat{0}, (U_{u+1}, \ldots, U_k)] \simeq \mathcal{P}_2^{k-u}. \qquad \square$$

Proposition 6. *Denote by $\mathcal{INC}_{1,2}(n)$, the* incomplete non-crossing matchings*, the non-crossing partitions of $[n]$ into pairs and singletons, such that all singletons are outer. Equip it with the poset structure inherited from $\mathcal{P}_{1,2}(n)$. Then the Möbius function on this poset is*

$$\mu((U_1, \ldots, U_u, V'_1, \ldots, V'_v), (U_1, \ldots, U_k, V_1, \ldots, V_\ell)) = \begin{cases} (-1)^{k-u}, & \forall (u+1) \leqslant i \leqslant k : U_i \in Out(\pi), \\ 0, & \textit{otherwise}. \end{cases}$$

For this poset, $\tau_{s(1), \ldots, s(k)}$ is given by the same expression as for $\mathcal{P}_{1,2}$.

Proof. Clearly

$$[(U_1, \ldots, U_u, V'_1, \ldots, V'_v), (U_1, \ldots, U_k, V_1, \ldots, V_\ell)] \simeq [\hat{0}, (U_{u+1}, \ldots, U_k)].$$

Moreover this interval is the product of intervals from $\hat{0}$ to a partition with a single outer block. If that partition is simply $\{U\}$, the Möbius function $\mu(\hat{0}_2, \{U\}) = (-1)$. On the other hand, if it is a larger partition with the single outer block $\{i, j\}$, recalling that all singletons in $\mathcal{INC}_{1,2}$ are outer, we see that

$$\sigma < (U'_1, U'_2, \ldots, \{i, j\}) \quad \Leftrightarrow \sigma \leqslant (U'_1, \ldots, U'_{s-1}, \{i\}, \{j\}).$$

and so from property (1),

$$\mu(\hat{0}, (U'_1, U'_2, \ldots, \{i, j\})) = \sum_{\sigma \leqslant (U'_1, U'_2, \ldots, \{i, j\})} \mu(\hat{0}, \sigma) - \sum_{\sigma < (U'_1, U'_2, \ldots, \{i, j\})} \mu(\hat{0}, \sigma) = 0. \qquad \square$$

Proposition 7. *Denote by $\mathcal{IP}(n)$, the* incomplete partitions*, the collection*

$$\{(\pi, S) : \pi \in \mathcal{P}(n), S \subset \pi\}.$$

Here, and in the subsequent examples, the elements of S will be called open blocks*, those of $\pi \backslash S$* closed blocks*. Denote $s(S) = \bigcup_{V \in S} V$ the union of all the open blocks. Equip $\mathcal{IP}(n)$ with the poset structure*

$$(\pi, S) \leqslant (\sigma, T) \textit{ if } U \in \pi \backslash S \Rightarrow U \in \sigma \backslash T \textit{ and } \pi|_{s(S)} \leqslant \sigma|_{s(S)}.$$

Then the Möbius function on this poset is

$$\mu((\hat{0}, \hat{0}), (\pi, S)) = \begin{cases} (-1)^{n-|S|} \prod_{V \in S} (|V| - 1)!, & \forall U \in \pi \backslash S : |U| = 1, \\ 0, & \textit{otherwise}. \end{cases}$$

Also for this poset, $(\tau, S)_{s(1),\ldots,s(k)} = (\tau \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)}), T)$*, where* $U \in (\tau \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)}))\backslash T$ *if and only if* $U \in \tau \backslash S$*. In particular,* $(\tau, S)_{s(1),\ldots,s(k)} = (\hat{0}_n, \hat{0}_n)$ *if and only if* $\tau \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)}) = \hat{0}_n$ *and all singletons of* τ *are open.*

Proof. Clearly

$$[(\hat{0}, \hat{0}), (\pi, S)] \simeq \prod_{U \in \pi \backslash S} [(\hat{0}, \hat{0}), (\{U\}, \varnothing)] \times \prod_{V \in S} [(\hat{0}, \hat{0}), (\{V\}, \{V\})]$$

and $[(\hat{0}, \hat{0}), (\{V\}, \{V\})] \simeq [\hat{0}, \{V\}] \simeq \mathcal{P}(|V|)$, so $\mu((\hat{0}, \hat{0}), (\{V\}, \{V\})) = (-1)^{|V|-1}(|V| - 1)!$. Also

$$\{(\sigma, T) < (\{U\}, \varnothing)\} = \{(\sigma, T) \leqslant (\{U\}, \{U\})\}$$

so using property (1), $\mu((\hat{0}, \hat{0}), (\{U\}, \varnothing)) = 0$ unless $|U| = 1$. The formula for the Möbius function follows. The final formula follows from the definition of the order. □

Proposition 8. *Denote by* $\mathcal{INC}(n)$*, the* incomplete non-crossing partitions *(called the linear non-crossing half-permutations in* [KMS07]*), the collection*

$$\{(\pi, S) : \pi \in \mathcal{NC}(n), S \subset Out(\pi)\}.$$

Equip it with the poset structure inherited from $\mathcal{IP}(n)$*. Then the Möbius function on this poset is*

$$\mu((\hat{0}, \hat{0}), (\pi, S)) = \begin{cases} (-1)^{n-|S|}, & \pi \in Int(n), \forall U \in \pi \backslash S : |U| = 1, \\ 0, & \textit{otherwise}. \end{cases} \tag{5}$$

For this poset, $\tau_{s(1),\ldots,s(k)}$ *is given by the same expression as for* $\mathcal{IP}$*.*

Proof. The interval $[(\hat{0}, \hat{0}), (\pi, S)]$ is the product of intervals from $(\hat{0}, \hat{0})$ to a partition with a single outer block, so it suffices to prove the result for such partitions. So let (π, S) be a partition with a single outer block B. If B is a closed singleton, $\mu((\hat{0}, \hat{0}), (\{B\}, \varnothing)) = -1$. If B is closed and not a singleton, then $S = \varnothing$ and

$$\{(\sigma, T) < (\pi, \varnothing)\} = \{(\sigma, T) \leqslant (\pi, \{B\})\},$$

so $\mu((\hat{0}, \hat{0}), (\pi, \varnothing)) = 0$. Next, suppose that B is open, π is not an interval partition, and formula (5) holds for all $(\sigma, T) < (\pi, \{B\})$. There exists the largest interval partition $\tau \leqslant \pi$, namely for $i < j$,

$$i \overset{\tau}{\sim} j \quad \Leftrightarrow \quad \forall i \leqslant k \leqslant j, i \overset{\pi}{\sim} k.$$

Let $U \in \tau \backslash T$ if $U \in \tau \cap (\pi \backslash S)$. Then the incomplete interval partitions smaller than (π, S) are exactly those smaller than or equal to (τ, T). Since according to formula (5), the Möbius function is only non-zero on interval partitions, it follows that $\mu((\hat{0}, \hat{0}), (\pi, S)) = 0$. Finally, if B is open and π is an interval partition, the interval in the poset is simply $[(\hat{0}, \hat{0}), (\{B\}, \{B\})]$. Recalling that all open blocks of elements in $\mathcal{INC}$ are outer, we see that this is isomorphic to $Int(|B|)$, with the Möbius function $(-1)^{|B|-1}$. The final formula follows from the definition of the order. □

Proposition 9. *Denote by $\mathcal{IPRM}(n)$, the* incomplete permutations *(sometimes called partial permutations* [BRR89]*, although this term has also been used for different objects), the collection of maps*

$$\mathcal{IPRM}(n) = \{(\Lambda, f) : \Lambda \subset [n], a : \Lambda \to [n] \text{ injective}\}.$$

These may also be identified with pairs (π, S), where π is a partition of $[n]$ with an order on each block of the partition, S is a collection of some blocks of this partition, and the order on the blocks in $\pi \backslash S$ is defined only up to a cyclic permutation. Equivalently, these are collections of words in $[n]$, where each letter appears exactly once, and some of the words are defined only up to cyclic order. Equip $\mathcal{IPRM}(n)$ with the following poset structure:

$$(\Lambda, f) \leqslant (\Omega, g) \text{ if } \Lambda \subset \Omega \text{ and } g|_\Lambda = f.$$

Equivalently, $(\pi, S) \leqslant (\sigma, T)$ if $U \in \pi\backslash S \Rightarrow U \in \sigma\backslash T$, the restriction of partitions $\pi|_{s(S)} \leqslant \sigma|_{s(S)}$, and the words corresponding to blocks of σ combined out of blocks of π are obtained by concatenating the words corresponding to these blocks of π, in some order (and the combined word possibly cyclically rotated if the block of σ is closed). Then the Möbius function on this poset is

$$\mu((\hat{0}, \hat{0}), (\pi, S)) = (-1)^{n-|S|}.$$

$(\Lambda, f)_{s(1),\ldots,s(k)} = (\Omega, g)$, where

$$\Omega = \bigcup_{i=1}^{k} \{x \in \Lambda \cap J_i : f(x) \in J_i\}$$

and $g = f|_\Omega$. In particular, $(\Lambda, f)_{s(1),\ldots,s(k)}$ equals the minimal element of $\mathcal{IPRM}(n)$ if for each i and $x \in \Lambda \cap J_i$, $f(x) \notin J_i$. We will call this final family incomplete derangements *and denote it by $\mathcal{ID}(s(1), \ldots, s(k))$.*

Proof. The blocks of π are simply the orbits of f, with elements of $[n]\backslash\Lambda$ included as open singletons. To compute the Möbius function, it suffices to assume that f has a single orbit. Elements smaller than (Λ, f) are in an ordered bijection with subsets of Λ, and the Möbius function $(-1)^{|\Lambda|-1}$. It remains to note that $\Lambda = [n]$ if the corresponding block is closed, and $|\Lambda| = n - 1$ if the corresponding block is open. Formulas for $(\Lambda, f)_{s(1),\ldots,s(k)}$ follow from the definition of the order. □

3. MULTIPLICATION OF WICK PRODUCTS

Proposition 10. *Let $\mathcal{M}, \Gamma(\mathcal{M})$ be as in the beginning of the introduction. Order the blocks of a partition according to the order of the largest elements of the blocks. For an ordered index set Λ, denote $a_\Lambda = \prod_{i \in \Lambda} a_i$. Finally, write $J_i = \{u_i(1), \ldots, u_i(s(i))\}$, so that*

$$\{1, 2, \ldots, n\} = (u_1(1), \ldots, u_1(s(1)), u_2(1), \ldots, u_2(s(2)), \ldots, u_k(1), \ldots, u_k(s(k))).$$

(a) *Define $W_{\mathcal{P}_{1,2}}(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ recursively by*

$$W_{\mathcal{P}_{1,2}}(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) = W_{\mathcal{P}_{1,2}}(a_1 \otimes \ldots \otimes a_n) X(a_{n+1}) - \sum_{i=1}^{n} W_{\mathcal{P}_{1,2}}(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes a_n) \langle a_i a_{n+1} \rangle.$$

Then

$$X(a_1)\dots X(a_n) = \sum_{\pi \in \mathcal{P}_{1,2}(n)} \prod_{U \in \pi : |U|=2} \langle a_U \rangle \, W_{\mathcal{P}_{1,2}}\left(\bigotimes_{V \in \pi : |V|=1} a_V \right), \tag{6}$$

and

$$W_{\mathcal{P}_{1,2}}(a_1 \otimes \dots \otimes a_n) = \sum_{\pi \in \mathcal{P}_{1,2}(n)} (-1)^{|\pi| - |Sing(\pi)|} \prod_{U \in \pi : |U|=2} \langle a_U \rangle \prod_{V \in \pi : |V|=1} X(a_V),$$

and

$$\prod_{i=1}^{k} W_{\mathcal{P}_{1,2}}\left(a_{u_i(1)} \otimes a_{u_i(2)} \otimes \dots \otimes a_{u_i(s(i))}\right) = \sum_{\substack{\pi \in \mathcal{P}_{1,2}(n) \\ \pi \wedge (\hat{1}_{s(1)}, \dots, \hat{1}_{s(k)}) = \hat{0}_n}} \prod_{U \in \pi : |U|=2} \langle a_U \rangle \, W_{\mathcal{P}_{1,2}}\left(\bigotimes_{V \in \pi : |V|=1} a_V \right).$$

(b) *Define* $W_{\mathcal{IP}}(a_1 \otimes a_2 \otimes \dots \otimes a_n)$ *recursively by*

$$\begin{aligned} W_{\mathcal{IP}}(a_1 \otimes \dots \otimes a_n \otimes a_{n+1}) = {} & W_{\mathcal{IP}}(a_1 \otimes \dots \otimes a_n) X(a_{n+1}) - W_{\mathcal{IP}}(a_1 \otimes \dots \otimes a_n) \langle a_{n+1} \rangle \\ & - \sum_{i=1}^{n} W_{\mathcal{IP}}(a_1 \otimes \dots \otimes \hat{a}_i \otimes \dots \otimes a_n \otimes a_i a_{n+1}) \\ & - \sum_{i=1}^{n} W_{\mathcal{IP}}(a_1 \otimes \dots \otimes \hat{a}_i \otimes \dots \otimes a_n) \langle a_i a_{n+1} \rangle. \end{aligned}$$

Then

$$X(a_1)\dots X(a_n) = \sum_{(\pi, S) \in \mathcal{IP}(n)} \prod_{U \in \pi \setminus S} \langle a_U \rangle \, W_{\mathcal{IP}}\left(\bigotimes_{V \in S} a_V \right). \tag{7}$$

If $\mathcal{M}$ *is commutative, then*

$$W_{\mathcal{IP}}(a_1 \otimes \dots \otimes a_n) = \sum_{\substack{(\pi, S) \in \mathcal{IP}(n) \\ U \in \pi \setminus S \Rightarrow |U|=1}} (-1)^{n - |S|} \prod_{U \in \pi \setminus S} \langle a_U \rangle \prod_{V \in S} (|V| - 1)! X(a_V),$$

and

$$\prod_{i=1}^{k} W_{\mathcal{IP}}\left(a_{u_i(1)} \otimes a_{u_i(2)} \otimes \dots \otimes a_{u_i(s(i))}\right) = \sum_{\substack{(\pi, S) \in \mathcal{IP}(n) \\ \pi \wedge (\hat{1}_{s(1)}, \dots, \hat{1}_{s(k)}) = \hat{0}_n \\ Sing(\pi) \subset S}} \prod_{U \in \pi \setminus S} \langle a_U \rangle \, W_{\mathcal{IP}}\left(\bigotimes_{V \in S} a_V \right).$$

Proof. For part (a), equation (6) is well known, see for example Theorem 2.1 in [EP03] for $q = 1$. It implies that for $\pi \in \mathcal{P}_{1,2}(n)$,

$$\prod_{U \in \pi : |U|=2} \langle a_U \rangle \prod_{V \in \pi : |V|=1} X(a_V) = \sum_{\substack{\sigma \in \mathcal{P}_{1,2}(n) \\ \sigma \geqslant \pi}} \prod_{U \in \sigma : |U|=2} \langle a_U \rangle \, W_{\mathcal{P}_{1,2}}\left(\bigotimes_{V \in \sigma : |V|=1} a_V \right).$$

Denoting the left-hand-side of this equation by $F(\pi)$ and each term in the sum on the right-hand-side by $G(\sigma)$, we see that these functions satisfy the relation (2), and F satisfies the multiplicative property (3). So Theorem 2 and Proposition 5 imply the results.

For part (b), equation (7) is known, see for example Proposition 2.7(a) in [Ans04a]. For commutative $\mathcal{M}$, it implies that for $(\pi, S) \in \mathcal{IP}(n)$,

$$\prod_{U \in \pi \setminus S} \langle a_U \rangle \prod_{V \in S} X(a_V) = \sum_{\substack{(\sigma,T) \in \mathcal{IP}(n) \\ (\sigma,T) \geqslant (\pi,S)}} \prod_{U \in \sigma \setminus T} \langle a_U \rangle \, W_{\mathcal{IP}} \left(\bigotimes_{V \in T} a_V \right).$$

So Theorem 2 and Proposition 7 imply the results. □

The following construction is closely related to Section 4 in [Śni00].

Theorem 11. *In the setting of the preceding proposition, define* $W_{\mathcal{IPRM}}(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ *recursively by*

$$\begin{aligned}
&W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) \\
&\quad = W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes a_n) X(a_{n+1}) - W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes a_n) \langle a_{n+1} \rangle \\
&\quad - \sum_{i=1}^{n} W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes a_n \otimes a_i a_{n+1}) \\
&\quad - \sum_{i=1}^{n} W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes a_n \otimes a_{n+1} a_i) \\
&\quad - \sum_{i=1}^{n} W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes a_n) \langle a_i a_{n+1} \rangle \\
&\quad - \sum_{1 \leqslant i < j \leqslant n} W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes \hat{a}_j \otimes \ldots \otimes a_n \otimes a_i a_{n+1} a_j) \\
&\quad - \sum_{1 \leqslant i < j \leqslant n} W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes \hat{a}_j \otimes \ldots \otimes a_n \otimes a_j a_{n+1} a_i).
\end{aligned}$$

For $(\Lambda, f) \in \mathcal{IPRM}(n)$, *let* $(\pi, S) \in \mathcal{INC}(n)$ *be the corresponding orbit decomposition. Order each block* $\{w_1, w_2, \ldots, w_\ell\}$ *of* π *so that* $w_{i+1} = f(w_i)$ *and, in case the block is closed, so that* w_ℓ *is the numerically largest element in the block. Denote*

$$W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes a_n)^{(\Lambda,f)} = \prod_{U \in \pi \setminus S} \langle a_U \rangle \, W_{\mathcal{IPRM}} \left(\bigotimes_{V \in S} a_V \right)$$

and

$$M(a_1 \otimes \ldots \otimes a_n)^{(\Lambda,f)} = \prod_{U \in \pi \setminus S} \langle a_U \rangle \prod_{V \in S} X(a_V),$$

where on each block of π *we use the order described above. Then*

$$X(a_1) \ldots X(a_n) = \sum_{(\Lambda,f) \in \mathcal{IPRM}(n)} W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes a_n)^{(\Lambda,f)}. \tag{8}$$

Also,

$$W_{\mathcal{IPRM}}\left(a_1 \otimes \ldots \otimes a_n\right) = \sum_{(\Lambda, f) \in \mathcal{IPRM}(n)} (-1)^{n-|S|} M(a_1 \otimes \ldots \otimes a_n)^{(\Lambda, f)},$$

and

$$\prod_{i=1}^{k} W_{\mathcal{IPRM}}\left(a_{u_i(1)} \otimes a_{u_i(2)} \otimes \ldots \otimes a_{u_i(s(i))}\right) = \sum_{(\Lambda, f) \in \mathcal{ID}(s(1), \ldots, s(k))} W_{\mathcal{IPRM}}\left(a_1 \otimes \ldots \otimes a_n\right)^{(\Lambda, f)}.$$

Proof. We prove equation (8) by induction. $X(a) = W_{\mathcal{IPRM}}(a) + \langle a \rangle$. Denoting

$$S = \left\{V_1 < V_2 < \ldots < V_{|S|}\right\}$$

and using the inductive hypothesis,

$$
\begin{aligned}
X(a_1) \ldots X(a_n) X(a_{n+1}) &= \sum_{(\Lambda, f) \in \mathcal{IPRM}(n)} W_{\mathcal{IPRM}}\left(a_1 \otimes \ldots \otimes a_n\right)^{(\Lambda, f)} X(a_{n+1}) \\
&= \sum_{(\Lambda, f) \in \mathcal{IPRM}(n)} \prod_{U \in \pi \setminus S} \langle a_U \rangle \\
&\quad \Bigg(W\left(a_{V_1} \otimes a_{V_2} \otimes \ldots \otimes a_{V_{|S|}} \otimes a_{n+1}\right) \\
&\quad + W\left(a_{V_1} \otimes a_{V_2} \otimes \ldots a_{V_{|S|}}\right) \langle a_{n+1} \rangle \\
&\quad + \mathbf{1}_{|S| \geqslant 1} \sum_{i=1}^{|S|} W\left(a_{V_1} \otimes a_{V_2} \otimes \ldots \otimes \hat{a}_{V_i} \otimes \ldots \otimes a_{V_{|S|}} \otimes a_{V_i} a_{n+1}\right) \\
&\quad + \mathbf{1}_{|S| \geqslant 1} \sum_{i=1}^{|S|} W\left(a_{V_1} \otimes a_{V_2} \otimes \ldots \otimes \hat{a}_{V_i} \otimes \ldots \otimes a_{V_{|S|}} \otimes a_{n+1} a_{V_i}\right) \\
&\quad + \mathbf{1}_{|S| \geqslant 1} \sum_{i=1}^{|S|} W\left(a_{V_1} \otimes a_{V_2} \otimes \ldots \otimes \hat{a}_{V_i} \otimes \ldots \otimes a_{V_{|S|}}\right) \langle a_{V_i} a_{n+1} \rangle \\
&\quad + \mathbf{1}_{|S| \geqslant 2} \sum_{1 \leqslant i < j \leqslant |S|} W\left(a_{V_1} \otimes \ldots \otimes \hat{a}_{V_i} \otimes \ldots \otimes \hat{a}_{V_j} \otimes \ldots \otimes a_{V_{|S|}} \otimes a_{V_i} a_{n+1} a_{V_j}\right) \\
&\quad + \mathbf{1}_{|S| \geqslant 2} \sum_{1 \leqslant i < j \leqslant |S|} W\left(a_{V_1} \otimes \ldots \otimes \hat{a}_{V_i} \otimes \ldots \otimes \hat{a}_{V_j} \otimes \ldots \otimes a_{V_{|S|}} \otimes a_{V_j} a_{n+1} a_{V_i}\right) \Bigg).
\end{aligned}
$$

The first term produces all the partitions in $\mathcal{IPRM}(n+1)$ in which $n+1$ is an open singleton. The second term produces all the partitions in which $n+1$ is a closed singleton. The third term produces all the partitions in which $n+1$ is a final letter in an open word of length at least 2. The fourth term produces all the partitions in which $n+1$ is the initial letter in an open word of length at least 2. The fifth term produces all the partitions in which $n+1$ is contained in a closed word of length at least 2. The sixth term produces all the partitions in which $n+1$ is contained in an open word of length at least 3, is neither the initial nor the final letter in it, and the largest letter preceding it is smaller than the largest letter following it. The seventh term produces all the partitions in which

$n+1$ is contained in an open word of length at least 3, is neither the initial nor the final letter in it, and the largest letter preceding it is larger than the largest letter following it. These seven classes are disjoint and exhaust $\mathcal{IPRM}(n+1)$.

It follows that for $(\Lambda, f) \in \mathcal{IPRM}(n)$,

$$M(a_1 \otimes \ldots \otimes a_n)^{(\Lambda,f)} = \sum_{\substack{(\Omega,g)\in\mathcal{IPRM}(n)\\(\Omega,g)\geqslant(\Lambda,f)}} W_{\mathcal{IPRM}}\left(a_1 \otimes \ldots \otimes a_n\right)^{(\Omega,g)}.$$

So Theorem 2 and Proposition 9 imply the results. □

Remark 12. Assume additionally that $\langle\cdot\rangle$ is a trace. By applying the functional $\varphi_{\mathcal{IPRM}}$ to (8), we obtain the moment formula for $\{X(a_i)\}$, which implies that the cumulants of $\varphi_{\mathcal{IPRM}}$ are

$$K^{\varphi_{\mathcal{IPRM}}}[X(a_1), \ldots, X(a_n)] = \frac{1}{n} \sum_{\alpha\in\mathrm{Sym}(n)} \left\langle a_{\alpha(1)} \ldots a_{\alpha(n)} \right\rangle.$$

Remark 13. Note that in Proposition 10, we do not assume that W is symmetric in its arguments, and in Theorem 11, we do not assume that $\mathcal{M}$ is commutative. The results in Proposition 10 are known by direct methods, see Theorems 3.1 and 3.3 in [EP03]. The results in part (b) of that proposition are stated in Proposition 2.7 of [Ans04a].

If $\mathcal{M}$ is commutative, $W_{\mathcal{IPRM}}\left(a_1 \otimes \ldots \otimes a_n\right)$ depends only on the underlying incomplete partition, so we may re-write the expansions in Theorem 11 as

$$X(a_1) \ldots X(a_n) = \sum_{(\pi,S)\in\mathcal{IP}(n)} \prod_{U\in\pi\backslash S} (|U|-1)! \langle a_U\rangle \prod_{V\in S} (|V|)!\, W_{\mathcal{IPRM}}\left(\bigotimes_{V\in S} a_V\right),$$

and

$$W_{\mathcal{IPRM}}\left(a_1 \otimes \ldots \otimes a_n\right) = \sum_{(\pi,S)\in\mathcal{IP}(n)} (-1)^{n-|S|} \prod_{U\in\pi\backslash S} (|U|-1)! \langle a_U\rangle \prod_{V\in S} (|V|)! X(a_V),$$

Note that this additional assumption does not imply that $\Gamma(\mathcal{M})$ is commutative; it is however natural to assume such commutativity to have a non-degenerate representation, see Section 7.

Remark 14. Let $(\Lambda, f) \in \mathcal{IPRM}(n)$. For $w \in [n]$, we say that it is

- A valley if $w \notin \Lambda \cup f(\Lambda)$, or $w \in \Lambda \backslash f(\Lambda)$ and $w < f(w)$, or $w \in f(\Lambda)\backslash\Lambda$ and $f^{-1}(w) > w$, or $w \in \Lambda \cup f(\Lambda)$ and $f^{-1}(w) > w < f(w)$.
- A closed singleton if $f(w) = w$.
- A double rise if $f^{-1}(w) > w$ and either $w > f(w)$ or $w \notin \Lambda$.
- A double fall if $w < f(w)$ and either $f^{-1}(w) < w$ or $w \notin f(\Lambda)$.
- A cycle max if w_i is the (numerically) largest element in a closed word of length at least 2.
- A peak if $f^{-1}(w) < w > f(w)$ and it is not a cycle max.

Clearly each letter in $[n]$ belongs to one of these six types. Then a slight extension of the argument in the previous proposition shows that if we define

$$\begin{aligned} W(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) = {} & W(a_1 \otimes \ldots \otimes a_n) X(a_{n+1}) - \alpha W(a_1 \otimes \ldots \otimes a_n) \langle a_{n+1} \rangle \\ & - \sum_{i=1}^{n} \beta_1 W(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes a_n \otimes a_i a_{n+1}) - \sum_{i=1}^{n} \beta_2 W(a_1, \ldots, \hat{a}_i, \ldots, a_{n+1} a_i) \\ & - \sum_{i=1}^{n} t W(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes a_n) \langle a_i a_{n+1} \rangle \\ & - \sum_{1 \leqslant i < j \leqslant n} \gamma W(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes \hat{a}_j \otimes \ldots \otimes a_n \otimes a_i a_{n+1} a_j) \\ & - \sum_{1 \leqslant i < j \leqslant n} \gamma W(a_1 \otimes \ldots \otimes \hat{a}_i \otimes \ldots \otimes \hat{a}_j \otimes \ldots \otimes a_n \otimes a_j a_{n+1} a_i), \end{aligned}$$

then

$$\begin{aligned} X(a_1) \ldots X(a_n) = & \sum_{(\Lambda, f) \in \mathcal{IPRM}(n)} W_{\mathcal{IPRM}}(a_1 \otimes \ldots \otimes a_n)^{(\Lambda, f)} \\ & \times \alpha^{\#\text{closed singletons}} \beta_1^{\#\text{double rises}} \beta_2^{\#\text{double falls}} t^{\#\text{cycle max}} \gamma^{\#\text{peaks}}. \end{aligned}$$

See [Bia93, SS94, CSZ97, KZ01] for related results. As in most of these references, there is a natural way of including a q parameter in this expansion, based on the values of the i, j indices from the Wick product recursion. However, our technique for obtaining inversion and product formulas does not apply to this extension, and based on the results in Section 5, it is unclear what these formulas should be.

Remark 15. Let $\mathcal{D}$ be a unital star-subalgebra, $\mathcal{M}$ be a complex star-algebra which is also a $\mathcal{D}$-bimodule such that

$$a_1(d a_2) = (a_1 d) a_2, \tag{9}$$

and $\langle \cdot \rangle : \mathcal{M} \to \mathcal{D}$ a star-linear $\mathcal{D}$-bimodule map. (We do not assume that $\mathcal{D} \subset \mathcal{M}$ since $\mathcal{M}$ may not be unital.) Let $\Gamma(\mathcal{M})$ be the complex unital star-algebra generated by non-commuting symbols $\{X(a) : a \in \mathcal{M}\}$ and $\mathcal{D}$, subject to the linearity relations

$$X(\alpha f \beta + \gamma g \delta) = \alpha X(a) \beta + \gamma X(b) \delta, \quad \alpha, \beta, \gamma, \delta \in \mathcal{D}.$$

The star-operation on it is determined by the requirement that all $X(a^*) = X(a)^*$. Thus

$$\Gamma(\mathcal{M}) \simeq \bigoplus_{n=0}^{\infty} \mathcal{M}^{\otimes_{\mathcal{D}} n}.$$

We denote $M(a_1 \otimes \ldots \otimes a_n) = X(a_1) \ldots X(a_n)$, and note that M may be extended to a $\mathcal{D}$-bimodule map on $\mathcal{M}^{\otimes_{\mathcal{D}} n}$.

Let $\pi \in \mathcal{NC}(n)$. We will define a bimodule map $\langle \cdot \rangle^{\pi}$ on $\mathcal{M}^{\otimes_{\mathcal{D}} n}$ recursively as follows. First,

$$\langle d_0 a_1 d_1 \otimes \ldots \otimes a_n d_n \rangle^{\hat{1}_n} = d_0 \langle a_1 d_1 \ldots a_n \rangle d_n.$$

Next let

$$Out(\pi) = \{V_1 < V_2 < \ldots < V_\ell\}, \quad V_i = \{v(i, 1) < \ldots < v(i, t(i))\}.$$

Denote $I_{ij} = [v(i,j)+1, \ldots, v(i,j+1)-1]$ for $1 \leqslant i \leqslant \ell$, $1 \leqslant j \leqslant t(i)-1$, and $\pi_{i,j} = \pi|_{I_{ij}}$. Note that an interval may be empty. Then we recursively define

$$\langle d_0 a_1 d_1 \otimes \ldots \otimes a_n d_n \rangle^{\pi} = d_0 \prod_{i=1}^{\ell} \left\langle \prod_{j=1}^{t(i)-1} \left(a_{v(i,j)} d_{v(i,j)} \langle a_v d_v : v \in I_{i,j} \rangle^{\pi_{ij}} \right) a_n \right\rangle d_n.$$

Note that this is the not the same definition as that in [Spe98] or Section 3 in [ABFN13], although it is related to them and may be expressed in terms of them as long as π is appropriately transformed.

Next, let F be a $\mathcal{D}$-bimodule map on $\mathcal{M}^{\otimes_{\mathcal{D}} n}$ (in our examples, either M or W). Let $(\pi, S) \in \mathcal{INC}(n)$, and this time denote

$$S = \{V_1 < V_2 < \ldots < V_\ell\}, \quad V_i = \{v(i,1) < \ldots < v(i,t(i))\}.$$

Let I_{ij}, π_{ij} be as before, and define additionally $v(\ell+1,1) = n+1$,

$$I_{i,t(i)} = [v(i,t(i))+1, \ldots, v(i+1,1)-1], \quad I_0 = [1, \ldots, v(1,1)-1],$$

and the corresponding π_{ij}, π_0. Denote

$$F(d_0 a_1 d_1 \otimes \ldots \otimes a_n d_n)^{(\pi,S)} = d_0 F\left(\langle a_v d_v : v \in I_0 \rangle^{\pi_0} \bigotimes_{i=1}^{\ell} \prod_{j=1}^{t(i)} \left(a_{v(i,j)} d_{v(i,j)} \langle a_v d_v : v \in I_{i,j} \rangle^{\pi_{ij}} \right) \right).$$

Proposition 16. *We use the notation from the preceding remark.*

(a) *Define* $W_{\mathcal{INC}_{1,2}}(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ *recursively by*

$$\begin{aligned} W_{\mathcal{INC}_{1,2}}(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) &= W_{\mathcal{INC}_{1,2}}(a_1 \otimes \ldots \otimes a_n) X(a_{n+1}) \\ &\quad - W_{\mathcal{INC}_{1,2}}(a_1 \otimes \ldots \otimes a_{n-1}) \langle a_n a_{n+1} \rangle. \end{aligned}$$

Note that $W_{\mathcal{INC}_{1,2}}$ *extends to a* $\mathcal{D}$*-bimodule map on each* $\mathcal{M}^{\otimes_{\mathcal{D}} n}$*. Then*

$$M(a_1 \otimes \ldots \otimes a_n) = \sum_{\pi \in \mathcal{INC}_{1,2}(n)} W_{\mathcal{INC}_{1,2}}(a_1 \otimes \ldots \otimes a_n)^{(\pi, Sing(\pi))}, \tag{10}$$

and

$$W_{\mathcal{INC}_{1,2}}(a_1 \otimes \ldots \otimes a_n) = \sum_{\pi \in Int_{1,2}(n)} (-1)^{|\pi| - |Sing(\pi)|} M(a_1 \otimes \ldots \otimes a_n)^{(\pi, Sing(\pi))},$$

and

$$\prod_{i=1}^{k} W_{\mathcal{INC}_{1,2}}\left(a_{u_i(1)} \otimes a_{u_i(2)} \otimes \ldots \otimes a_{u_i(s(i))}\right) = \sum_{\substack{\pi \in \mathcal{INC}_{1,2}(n) \\ \pi \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)}) = \hat{0}_n}} W_{\mathcal{INC}_{1,2}}(a_1 \otimes \ldots \otimes a_n)^{(\pi, Sing(\pi))}.$$

(b) *Define* $W_{\mathcal{INC}}(a_1 \otimes a_2 \otimes \ldots \otimes a_n)$ *recursively by*

$$\begin{aligned} & W_{\mathcal{INC}}(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) \\ &\quad = W_{\mathcal{INC}}(a_1 \otimes \ldots \otimes a_n) X(a_{n+1}) - W_{\mathcal{INC}}(a_1 \otimes \ldots \otimes a_n) \langle a_{n+1} \rangle \\ &\qquad - W_{\mathcal{INC}}(a_1 \otimes \ldots \otimes a_n a_{n+1}) - W_{\mathcal{INC}}(a_1 \otimes \ldots \otimes a_{n-1}) \langle a_n a_{n+1} \rangle. \end{aligned}$$

Then

$$X(a_1) \dots X(a_n) = \sum_{(\pi,S) \in \mathcal{INC}(n)} W_{\mathcal{INC}}\left(a_1 \otimes \dots \otimes a_n\right)^{(\pi,S)}, \tag{11}$$

and

$$W_{\mathcal{INC}}\left(a_1 \otimes \dots \otimes a_n\right) = \sum_{\substack{(\pi,S) \in \mathcal{INC}(n) \\ \pi \in Int(n), U \in \pi \setminus S \Rightarrow |U|=1}} (-1)^{n-|S|} M(a_1 \otimes \dots \otimes a_n)^{(\pi,S)},$$

and

$$\prod_{i=1}^{k} W_{\mathcal{INC}}\left(a_{u_i(1)} \otimes a_{u_i(2)} \otimes \dots \otimes a_{u_i(s(i))}\right) = \sum_{\substack{(\pi,S) \in \mathcal{INC}(n) \\ \pi \wedge (\hat{1}_{s(1)}, \dots, \hat{1}_{s(k)}) = \hat{0}_n \\ Sing(\pi) \subset S}} W_{\mathcal{INC}}\left(a_1 \otimes \dots \otimes a_n\right)^{(\pi,S)}.$$

In the scalar-valued case, these results are known, see Theorem 3.3 in [EP03] and Proposition 29 in [Ans04b] for $q = 0$.

Proof. The proof of part (a) is similar to and simpler than that of part (b), so we omit it.

For part (b), the proof of equation (11) is similar to the argument in Theorem 11 above or Theorem 21 below, so we only outline it. It is based on the observation that the four terms in the recursion relation for $W_{\mathcal{INC}}$ correspond to the decomposition of $\mathcal{INC}(n)$ as a disjoint union of four sets: those where $n+1$ is an open singleton, a closed singleton, those where it belongs to larger open block, and those where it belongs to a larger closed block. Equation (11) implies that for $(\pi, S) \in \mathcal{INC}(n)$,

$$M(a_1 \otimes \dots \otimes a_n)^{(\pi,S)} = \sum_{\substack{(\sigma,T) \in \mathcal{INC}(n) \\ (\sigma,T) \geqslant (\pi,S)}} W_{\mathcal{INC}}\left(a_1 \otimes \dots \otimes a_n\right)^{(\pi,S)}.$$

So Theorem 2 and Proposition 8 imply the results. □

4. Product formulas for free Meixner Wick products

Notation 17. A *covered partition* is a partition $\pi \in \mathcal{NC}(\Lambda)$ with a single outer block, or equivalently such that $\min(\Lambda) \overset{\pi}{\sim} \max(\Lambda)$; their set is denoted by $\mathcal{NC}'(\Lambda)$. We define an additional order on $\mathcal{NC}(\Lambda)$: $\pi \ll \sigma$ if $\pi \leqslant \sigma$ and in addition, for each block $U \in \pi$, $\sigma|_U \in \mathcal{NC}'(U)$. See[BN08, Nic10] for more details.

For $(\pi, S) \in \mathcal{INC}(n)$ and $\sigma \ll \pi$, we say that a block of σ is open if it contains the smallest element of an open block of π; their collection is denoted $S'(\sigma, S)$. In particular, each block of π contains at most one open singleton of σ.

Definition 18. For $\mathcal{D}$, $\mathcal{M}$, $\Gamma(\mathcal{M})$, and $X(a)$ as in Proposition 16, define the free Meixner-Kailath-Segall polynomials by the recursion

$$\begin{aligned} W\left(a_1 \otimes \dots \otimes a_n \otimes a_{n+1}\right) = {} & W\left(a_1 \otimes \dots \otimes a_n\right) X(a_{n+1}) - \alpha\, W\left(a_1 \otimes \dots \otimes a_n\right) \langle a_{n+1} \rangle \\ & - \beta\, W\left(a_1 \otimes \dots \otimes a_{n-1} \otimes a_n a_{n+1}\right) - t\, W\left(a_1 \otimes \dots \otimes a_{n-1}\right) \langle a_n a_{n+1} \rangle \\ & - \gamma\, W\left(a_1 \otimes \dots \otimes a_{n-2} \otimes a_{n-1} a_n a_{n+1}\right) \end{aligned}$$

and in particular

$$W(a_1 \otimes a_2) = W(a_1)X(a_2) - \alpha W(a_1)\langle a_2\rangle - \beta W(a_1 a_2) - t\langle a_1 a_2\rangle$$

and $W(a_1) = X(a_1) - \alpha\langle a_1\rangle$. Compare with Section 7 in [Śni00].

Notation 19. For $(\pi, S) \in \mathcal{INC}(n)$, let

$$C^{(\pi,S)}_{\alpha,\beta,t,\gamma} = \sum_{\substack{\sigma\leqslant\pi,\\ U\in\pi\backslash S\Rightarrow\sigma|_U\in\mathcal{NC}'(U),\\ Sing(\sigma)\subset Sing(\pi)\cup S'(\sigma,S)}} \alpha^{|Sing(\pi\backslash S)|}\beta^{n-2|\sigma|+|S|+|Sing(\pi\backslash S)|}t^{|\pi\backslash S|-|Sing(\pi\backslash S)|}\gamma^{|\sigma|-|\pi|}.$$

In particular,

$$C^{\pi}_{\alpha,\beta,t,\gamma} = C^{(\pi,\varnothing)}_{\alpha,\beta,t,\gamma} = \sum_{\substack{\sigma\ll\pi,\\ Sing(\sigma)=Sing(\pi)}} \alpha^{|Sing(\pi)|}\beta^{n-2|\sigma|+|Sing(\pi)|}t^{|\pi|-|Sing(\pi)|}\gamma^{|\sigma|-|\pi|}.$$

We also denote

$$C^{(\pi,S)}_{\beta,t,\gamma}(s(1), s(2), \dots, s(k)) = \sum_{\substack{\sigma\leqslant\pi,\\ U\in\pi\backslash S\Rightarrow\sigma|_U\in\mathcal{NC}'(U),\\ Sing(\sigma)\subset S'(\sigma,S),\\ \sigma\wedge(\hat{1}_{s(1)},\dots,\hat{1}_{s(k)})=\hat{0}_n}} \beta^{n-2|\sigma|+|S|}t^{|\pi\backslash S|}\gamma^{|\sigma|-|\pi|},$$

and in particular

$$C^{\pi}_{\beta,t,\gamma}(s(1), s(2), \dots, s(k)) = C^{(\pi,\varnothing)}_{\beta,t,\gamma}(s(1), s(2), \dots, s(k)) = \sum_{\substack{\sigma\ll\pi,\\ Sing(\sigma)=\varnothing,\\ \sigma\wedge(\hat{1}_{s(1)},\dots,\hat{1}_{s(k)})=\hat{0}_n}} \beta^{n-2|\sigma|}t^{|\pi|}\gamma^{|\sigma|-|\pi|}.$$

Lemma 20. *Denote $M_n(\beta,\gamma)$ a particular case of the Jacobi-Rogers polynomials, the sum over Motzkin paths of length n with flat steps given weight β and down steps given weight γ. Then*

$$C^{(\pi,S)}_{\alpha,\beta,t,\gamma} = \prod_{U\in\pi\backslash S}\kappa^{|U|}_{\alpha,\beta,t,\gamma}\prod_{V\in S}\omega^{|V|}_{\alpha,\beta,t,\gamma},$$

where

$$\omega^n_{\alpha,\beta,t,\gamma} = M_{n-1}(\beta,\gamma), \quad \kappa^1_{\alpha,\beta,t,\gamma} = \alpha, \quad \kappa^n_{\alpha,\beta,t,\gamma} = tM_{n-2}(\beta,\gamma).$$

Similarly,

$$C^{(\pi,S)}_{\beta,t,\gamma}(s(1), s(2), \dots, s(k)) = \prod_{U\in\pi\backslash S}\kappa^{|U|}_{\beta,t,\gamma}((\hat{1}_{s(1)},\dots,\hat{1}_{s(k)})|_U)\prod_{V\in S}\omega^{|V|}_{\beta,t,\gamma}((\hat{1}_{s(1)},\dots,\hat{1}_{s(k)})|_V),$$

where

$$\omega^1_{\beta,t,\gamma}(s(1), s(2), \dots, s(k)) = \kappa^1_{\beta,t,\gamma}(s(1), s(2), \dots, s(k)) = 0, \tag{12}$$

$$\kappa^{n+1}_{\beta,t,\gamma}(s(1), s(2), \dots, s(k), 1) = t\omega^n_{\beta,t,\gamma}(s(1), s(2), \dots, s(k)),$$

and

$$\omega^n_{\beta,t,\gamma}(s(1), s(2), \dots, s(k)) = \sum_{\substack{\tau\in\mathcal{NC}(n),\\ Sing(\tau)\subset\{1\}\\ \tau\wedge(\hat{1}_{s(1)},\dots,\hat{1}_{s(k)})=\hat{0}_n}} \beta^{n-2|\tau|+1}\gamma^{|\tau|-1}. \tag{13}$$

Here the arguments of $\omega^{|U|}_{\beta,t,\gamma}((\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)})|_U)$ *are the sizes of the blocks in the restriction*

$$(\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)})|_U,$$

omitting the empty blocks.

Proof. We note that

$$\kappa^1_{\alpha,\beta,t,\gamma} = \alpha, \quad \kappa^n_{\alpha,\beta,t,\gamma} = t \sum_{\tau \in \mathcal{NC}'_{\geqslant 2}(n)} \beta^{n-2|\tau|} \gamma^{|\tau|-1} = t M_{n-2}(\beta, \gamma),$$

and

$$\omega^1_{\alpha,\beta,t,\gamma} = 1, \quad \omega^n_{\alpha,\beta,t,\gamma} = \sum_{\substack{\tau \in \mathcal{NC}(n), \\ Sing(\tau) \subset \{1\}}} \beta^{n-2|\tau|+1} \gamma^{|\tau|-1} = \sum_{\tau \in \mathcal{NC}'_{\geqslant 2}(n+1)} \beta^{n-2|\tau|+1} \gamma^{|\tau|-1} = M_{n-1}(\beta, \gamma).$$

Also, equations (12) and (13) hold while

$$\kappa^n_{\beta,t,\gamma}(s(1), s(2), \ldots, s(k)) = t \sum_{\substack{\tau \in \mathcal{NC}'_{\geqslant 2}(n) \\ \tau \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)}) = \hat{0}_n}} \beta^{n-2|\tau|} \gamma^{|\tau|-1}. \qquad \square$$

We formulate and prove the following theorem for the case $\mathcal{D} = \mathbb{C}$ to simplify notation, but the result carries over verbatim for general $\mathcal{D}$.

Theorem 21. *We have expansions of monomials*

$$X(a_1)X(a_2)\ldots X(a_n) = \sum_{(\pi,S) \in \mathcal{INC}(n)} C^{(\pi,S)}_{\alpha,\beta,t,\gamma} \prod_{U \in \pi \setminus S} \langle a_U \rangle \, W\left(\bigotimes_{V \in S} a_V \right) \tag{14}$$

and the product formulas

$$\prod_{i=1}^{k} W\left(a_{u_i(1)} \otimes a_{u_i(2)} \otimes \ldots \otimes a_{u_i(s(i))}\right) = \sum_{\substack{(\pi,S) \in \mathcal{INC}(n), \\ Sing(\pi) \subset S}} C^{(\pi,S)}_{\beta,t,\gamma}(s(1), s(2), \ldots, s(k)) \prod_{U \in \pi \setminus S} \langle a_U \rangle \, W\left(\bigotimes_{V \in S} a_V \right). \tag{15}$$

Note that unlike in the earlier examples, in the product formula (15), π itself need not be inhomogeneous with respect to $(\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)})$.

Proof. We will prove the expansion (15); the proof of (14) is similar. It suffices to show that if (15) holds for $(s(1), s(2), \ldots, s(k))$, then it holds for

$$(s(1), s(2), \ldots, s(k), 1) \text{ and } (s(1), s(2), \ldots, s(k) + 1).$$

To verify the expansion for $(s(1), s(2), \ldots, s(k), 1)$, we compute.

$$\prod_{i=1}^{k} W\left(a_{u_i(1)} \otimes a_{u_i(2)} \otimes \ldots \otimes a_{u_i(s(i))}\right) W(a_{n+1})$$

$$= \sum_{\substack{(\pi,S)\in\mathcal{INC}(n),\\ Sing(\pi)\subset S}} C^{(\pi,S)}_{\beta,t,\gamma}(s(1), s(2), \ldots, s(k)) \prod_{U\in\pi\setminus S} \langle a_U\rangle\, W\left(\bigotimes_{V\in S} a_V\right) W(a_{n+1})$$

$$= \sum_{\substack{(\pi,S)\in\mathcal{INC}(n),\\ Sing(\pi)\subset S}} \sum_{\substack{\sigma\leqslant\pi,\\ V\in\pi\setminus S\Rightarrow\sigma|_V\in\mathcal{NC}'(V),\\ Sing(\sigma)\subset S'(\sigma,S),\\ \sigma\wedge(\hat{1}_{s(1)},\ldots,\hat{1}_{s(k)})=\hat{0}_n}} \beta^{n-2|\sigma|+|S|} t^{|\pi\setminus S|} \gamma^{|\sigma|-|\pi|} \prod_{U\in\pi\setminus S}\langle a_U\rangle$$

$$\Bigg(W\left(a_{V_1}\otimes a_{V_2}\otimes\ldots\otimes a_{V_j}\otimes a_{n+1} : \{V_1 < V_2 < \ldots < V_j\} = S\right)$$

$$+ \beta\mathbf{1}_{|S|\geqslant 1} W\left(a_{V_1}\otimes a_{V_2}\otimes\ldots\otimes a_{V_j}a_{n+1} : \{V_1 < V_2 < \ldots < V_j\} = S\right)$$

$$+ t\mathbf{1}_{|S|\geqslant 1} W\left(a_{V_1}\otimes a_{V_2}\otimes\ldots\otimes a_{V_{j-1}} : \{V_1 < V_2 < \ldots < V_j\} = S\right)\left\langle a_{V_j}a_{n+1}\right\rangle$$

$$+ \gamma\mathbf{1}_{|S|\geqslant 2} W\left(a_{V_1}\otimes a_{V_2}\otimes\ldots\otimes a_{V_{j-1}}a_{V_j}a_{n+1} : \{V_1 < V_2 < \ldots < V_j\} = S\right)\Bigg).$$

For fixed (π, S, σ), the first term produces all the triples (π', S', σ') with

$$(\pi', S') \in \mathcal{INC}(n+1), Sing(\pi') \subset S',$$

$$\sigma' \leqslant \pi', V \in \pi'\setminus S' \Rightarrow \sigma'|_V \in \mathcal{NC}'(V), Sing(\sigma') \subset S'(\sigma', S'), \sigma' \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)}, \hat{1}_1) = \hat{0}_{n+1}$$

such that $n+1$ is an open singleton in both π' and σ'. Since n, $|\pi|$, $|S|$, and $|\sigma|$ are all incremented by 1, $C^{(\pi,S)}_{\beta,t,\gamma}(s(1), s(2), \ldots, s(k))$ does not change. The second term produces all triples where $n+1$ belongs to an open block in both σ and π, each of size at least 2, by adjoining it to the largest open block of π and the corresponding open block of σ. Since only n is incremented, $C^{(\pi,S)}$ is multiplied by β. The third term produces all triples in which $n+1$ belongs to a closed block of π (and so also of σ), by adjoining it to the largest open block of π and the corresponding open block of σ, and closing them both. Since n is incremented by 1 and $|S|$ decreased by 1, $C^{(\pi,S)}$ is multiplied by t. The fourth term produces all triples in which $n+1$ belongs to an open block of π but a closed block of σ, by adjoining $n+1$ to the second largest open block of π and the corresponding open block of σ, combining the two largest open blocks of π, and closing the open block of σ which belonged to the largest open block of π. Since n is incremented by 1 while $|\pi|$ and $|S|$ are decreased by 1, $C^{(\pi,S)}$ is multiplied by γ. These four classes are disjoint and exhaust the triples (π, S', σ') above.

To verify the expansion for $(s(1), s(2), \ldots, s(k)+1)$, we compute

$$\begin{aligned}
&\prod_{i=1}^{k-1} W\left(a_{u_i(1)} \otimes a_{u_i(2)} \otimes \ldots \otimes a_{u_i(s(i))}\right) \cdot W\left(a_{u_k(1)} \otimes a_{u_k(2)} \otimes \ldots \otimes a_n \otimes a_{n+1}\right) \\
&= \prod_{i=1}^{k-1} W\left(a_{u_i(1)} \otimes a_{u_i(2)} \otimes \ldots \otimes a_{u_i(s(i))}\right) \cdot \Bigg(W\left(a_{u_k(1)} \otimes a_{u_k(2)} \otimes \ldots \otimes a_n\right) W\left(a_{n+1}\right) \\
&\qquad - \beta W\left(a_{u_k(1)} \otimes a_{u_k(2)} \otimes \ldots \otimes a_n a_{n+1}\right) - t W\left(a_{u_k(1)} \otimes a_{u_k(2)} \otimes \ldots \otimes a_{n-1}\right) \langle a_n a_{n+1} \rangle \\
&\qquad - \gamma \mathbf{1}_{s(k) \geqslant 2} W\left(a_{u_k(1)} \otimes a_{u_k(2)} \otimes \ldots \otimes a_{n-1} a_n a_{n+1}\right) \Bigg) \\
&= \sum_{\substack{(\pi, S) \\ \pi \in \mathcal{NC}(n+1), \\ Sing(\pi) \subset S \subset Out(\pi)}} \sum_{\substack{\sigma \leqslant \pi, \\ V \in \pi \backslash S \Rightarrow \sigma|_V \in \mathcal{NC}'(V), \\ Sing(\sigma) \subset S'(\sigma, S), \\ \sigma \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)}, \hat{1}_1) = \hat{0}_{n+1}}} - \sum_{\substack{(\pi, S) \\ \pi \in \mathcal{NC}([n-1] \cup \{\{n, n+1\}\}), \\ Sing(\pi) \subset S \subset Out(\pi)}} \beta \sum_{\substack{\sigma \leqslant \pi, \\ V \in \pi \backslash S \Rightarrow \sigma|_V \in \mathcal{NC}'(V), \\ Sing(\sigma) \subset S'(\sigma, S), \\ \sigma \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)}) = \hat{0}_n}} \\
&\qquad - \sum_{\substack{(\pi, S) \\ \pi \in \mathcal{NC}(n-1), \\ Sing(\pi) \subset S \subset Out(\pi)}} t \sum_{\substack{\sigma \leqslant \pi, \\ V \in \pi \backslash S \Rightarrow \sigma|_V \in \mathcal{NC}'(V), \\ Sing(\sigma) \subset S'(\sigma, S), \\ \sigma \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)-1}) = \hat{0}_{n-1}}} \langle a_n a_{n+1} \rangle \\
&\qquad - \mathbf{1}_{s(k) \geqslant 2} \sum_{\substack{(\pi, S) \\ \pi \in \mathcal{NC}([n-2] \cup \{\{n-1, n, n+1\}\}), \\ Sing(\pi) \subset S \subset Out(\pi)}} \gamma \sum_{\substack{\sigma \leqslant \pi, \\ V \in \pi \backslash S \Rightarrow \sigma|_V \in \mathcal{NC}'(V), \\ Sing(\sigma) \subset S'(\sigma, S), \\ \sigma \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)-1}) = \hat{0}_{n-1}}} ,
\end{aligned}$$

where in each of the four terms we sum the quantity

$$\beta^{n-2|\sigma|+|S|} t^{|\pi \backslash S|} \gamma^{|\sigma|-|\pi|} W\left(a_1 \otimes \ldots \otimes a_n\right)^{(\pi, S)}.$$

We want to conclude that the first sum minus the remaining three sums equals

$$\sum_{\substack{(\pi, S) \\ \pi \in \mathcal{NC}(n+1), \\ Sing(\pi) \subset S \subset Out(\pi)}} \sum_{\substack{\sigma \leqslant \pi, \\ V \in \pi \backslash S \Rightarrow \sigma|_V \in \mathcal{NC}'(V), \\ Sing(\sigma) \subset S'(\sigma, S), \\ \sigma \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)+1}) = \hat{0}_{n+1}}}$$

So consider any term in the first sum in which $(n+1)$ is in the same σ-block as one of the preceding $s(k)$ elements. Since σ is non-crossing and inhomogeneous, $(n+1)$ then has to be in the same block as n. If this block of σ is closed and contains two elements, it is its own block of π. Since the partition contains exactly one extra closed block, while n gets incremented by 2, the term gets cancelled with a term from the third sum. So suppose the block containing $n+1$ is either open or contains at least three elements. If its block of π contains no other elements from the preceding $s(k)$, the number of blocks does not change while n is incremented by 1, and so the term gets cancelled with a term from the second sum. Finally, if its block of π containing n and $n+1$ also contains some other element from the preceding $s(k)$, for the same reason as above it has to contain $(n-1)$, and gets cancelled with the fourth term. □

Lemma 22. *For $\pi \in \mathcal{NC}_{\geqslant 2}(n)$, the set*

$$\left\{\sigma \ll \pi : \sigma \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)}) = \hat{0}_n\right\}$$

is non-empty if and only if in the restriction of $(\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)})$ *to each block of* π*, the size of each block in the restriction is at most the sum of the sizes of the remaining blocks in this restriction.*

Proof. It suffices to prove the result for each block of π individually, so without loss of generality we assume that $\pi = \hat{1}_n$. Clearly if some $s(i)$ is larger than the sum of the rest, there is no σ with $Sing(\sigma) = \emptyset, \sigma \wedge (\hat{1}_{s(1)}, \ldots, \hat{1}_{s(k)}) = \hat{0}_n$. Thus suppose that for each i, $s(i)$ is at most the sum of the rest. We will now construct a σ satisfying the conditions above. Choose a largest $s(i)$, write $J_i = [a, b]$, and suppose without loss of generality that $i < k$. Include in σ non-crossing pairs $\{b, b+1\}$, $b-1, b+2$, etc., until either j_{i+1} is exhausted or the number of unmatched elements in J_i is equal to the second largest $s(j)$. If at any point we include one of the endpoints of $[n]$ in the matching, include the other endpoint in the same block. Remove the matched elements, and continue the same procedure (starting with the largest of the remaining intervals) until all points are included in non-singleton blocks, or all remaining intervals have equal size. If the number of remaining intervals is even, divide them into consecutive pairs and match elements of these pairs as above. If the number of remaining intervals is odd and they all have equal even length, we may match elements of the right half of each interval with the elements of the left half of the (cyclically) following interval. Finally, if we are left with an odd number of intervals of odd equal length, form a block of their middle elements, then match the remaining points half-by-half in a chain as above. This procedure will produce an inhomogeneous non-crossing partition with no singletons. □

Corollary 23. *For the state corresponding to the Wick products from Definition 18, the joint moments are*

$$\varphi\left[X(a_1)X(a_2)\ldots X(a_n)\right] = \sum_{\pi \in \mathcal{NC}(n)} C^{\pi}_{\alpha,\beta,t,\gamma} \left\langle a_1 \otimes \ldots \otimes a_n \right\rangle^{\pi}, \tag{16}$$

and the linearization coefficients are

$$\varphi\left[\prod_{i=1}^{k} W\left(a_{u_i(1)} \otimes a_{u_i(2)} \otimes \ldots \otimes a_{u_i(s(i))}\right)\right] = \sum_{\pi \in \mathcal{NC}_{\geqslant 2}(n)} C^{\pi}_{\beta,t,\gamma}(s(1), s(2), \ldots, s(k)) \left\langle a_1 \otimes \ldots \otimes a_n \right\rangle^{\pi}. \tag{17}$$

Remark 24. Using Lemma 20, we may re-write formula (16) as

$$\varphi\left[X(a_1)X(a_2)\ldots X(a_n)\right] = \sum_{\pi \in \mathcal{NC}(n)} \left(\prod_{U \in \pi} \kappa^{|U|}_{\alpha,\beta,t,\gamma}\right) \left\langle a_1 \otimes \ldots \otimes a_n \right\rangle^{\pi}.$$

Therefore by definition, the joint free cumulants of $\{X(a_i)\}$ are

$$R[X(a_1)] = \kappa^1_{\alpha,\beta,t,\gamma} \left\langle a_1 \right\rangle = \alpha \left\langle a_1 \right\rangle,$$

$$R[X(a_1), \ldots, X(a_n)] = \kappa^n_{\alpha,\beta,t,\gamma} \left\langle a_1 \ldots a_n \right\rangle = t M_{n-2}(\beta, \gamma) \left\langle a_1 \ldots a_n \right\rangle.$$

Compare with Theorem 8 in [Śni00]. Note that $M_{n-2}(1,1) = M_{n-2}$, the Motzkin number. On the other hand,

$$M_{n-2}(2,1) = \sum_{\tau \in \mathcal{NC}'_{\geqslant 2}(n)} 2^{n-2|\tau|} = |\mathcal{NC}'(n)| = c_{n-1},$$

the Catalan number. In general $M_n(\beta, \gamma)$ are the moments of a semicircular distribution with mean β and variance γ. Cf. Theorem 2 in [Ans07].

Theorem 25. *We may expand*

$$W(a_1 \otimes \ldots \otimes a_n) = \sum_{\substack{(\pi,S) \\ \pi \in Int(n) \\ S \subset \pi}} (-1)^{n-|S|} \prod_{U \in \pi \setminus S} c_{|U|} \langle a_U \rangle \prod_{V \in S} o_{|V|} X(a_V), \tag{18}$$

where

$$c_k = \alpha o_k - t o_{k-1}.$$

Case I: $\gamma = 0$. Then

$$o_k = \beta^{k-1}, \qquad c_k = (\alpha\beta - t)\beta^{k-2}$$

For $\gamma \neq 0$, factor $1 - \beta z + \gamma z^2 = (1-uz)(1-vz)$.

Case II: $\gamma \neq 0$, $\beta^2 \neq 4\gamma$, so that $u \neq v$. Then

$$o_k = \frac{1}{u-v}(u^k - v^k), \qquad c_k = \frac{1}{u-v}\left(\alpha(u^k - v^k) - t(u^{k-1} - v^{k-1})\right).$$

Case II′: if in addition, $\alpha^2 - \alpha\beta t + \gamma t^2 = 0$, so that $v = t/\alpha$, then

$$o_k = \frac{1}{\beta - 2t/\alpha}((\beta - t/\alpha)^k - (t/\alpha)^k), \qquad c_k = \alpha(\beta - t/\alpha)^{k-1}.$$

Case III: $\gamma \neq 0$, $\beta^2 = 4\gamma$, so that $u = v = \beta/2$. Then

$$o_k = k(\beta/2)^{k-1}, \qquad c_k = (\alpha k(\beta/2) - t(k-1))(\beta/2)^{k-2}.$$

Case III′: if in addition, $\alpha\beta = 2t$, so that $u = v = t/\alpha$, then

$$o_k = k(\beta/2)^{k-1}, \qquad c_k = \alpha(\beta/2)^{k-1}.$$

Proof. Write $W(a_1 \otimes \ldots \otimes a_n)$ in the form (18); we will show that this is possible by exhibiting coefficients in this expansion. Plugging in this expansion into the recursion in Definition 18, we obtain

$$\begin{aligned}
\sum_{\substack{(\pi,S) \\ \pi \in Int(n+1) \\ S \subset \pi}} (-1)^{n+1-|S|} = & \sum_{\substack{(\pi,S) \\ \pi \in Int(n) \\ S \subset \pi}} (-1)^{n-|S|} X(a_{n+1}) - \alpha \sum_{\substack{(\pi,S) \\ \pi \in Int(n) \\ S \subset \pi}} (-1)^{n-|S|} \langle a_{n+1} \rangle \\
& - \beta \sum_{\substack{(\pi,S) \\ \pi \in Int([n-1] \cup \{\{n,n+1\}\}) \\ S \subset \pi}} (-1)^{n-|S|} \\
& - t \sum_{\substack{(\pi,S) \\ \pi \in Int(n-1) \\ S \subset \pi}} (-1)^{n-1-|S|} \langle a_n a_{n+1} \rangle - \gamma \sum_{\substack{(\pi,S) \\ \pi \in Int([n-2] \cup \{\{n-1,n,n+1\}\}) \\ S \subset \pi}} (-1)^{n-1-|S|},
\end{aligned}$$

where in each term we sum the expression

$$\prod_{U \in \pi \setminus S} c_{|U|} \langle a_U \rangle \prod_{V \in S} o_{|V|} X(a_V).$$

Now compare the factors corresponding to the block B containing $n+1$ on the left-hand-side. If B is an open singleton, it matches with a term in the first sum, with the same coefficient (since the number of open blocks on the left is one more than on the right). Thus $o_1 = 1$. For the remaining terms, the size of S does not change, so we omit it from the coefficients. If B is a closed singleton, it matches with a term from the second sum, and the coefficients are $(-1)^{n+1} c_1 = (-1)^n(-\alpha)$, so $c_1 = \alpha$. If B is an open pair, it matches with a term in the third sum, and the coefficients are $(-1)^{n+1} o_2 = (-1)^n (-\beta) o_1$, so $o_2 = \beta o_1$. If B is a closed pair, it matches with terms in the third and the fourth sums, and the coefficients are $(-1)^{n+1} c_2 = (-1)^n (-\beta) c_1 + (-1)^{n-1}(-t)$, so $c_2 = \beta c_1 - t$. If B is a larger block, it matches with terms in the third and the fifth sums, and the coefficients are $(-1)^{n+1} o_k = (-1)^n (-\beta) o_{k-1} + (-1)^{n-1}(-\gamma) o_{k-2}$ (and the corresponding expression for c_k), so that

$$o_k = \beta o_{k-1} - \gamma o_{k-2}, \qquad c_k = \beta c_{k-1} - \gamma c_{k-2}$$

for $k \geqslant 3$. Let

$$O(z) = \sum_{k=1}^{\infty} o_k z^{k-1}, \qquad C(z) = \sum_{k=1}^{\infty} c_k z^{k-1}.$$

Then

$$O(z) = 1 + \beta z O(z) - \gamma z^2 O(z), \qquad C(z) = \alpha - tz + \beta z C(z) - \gamma z^2 C(z),$$

so

$$O(z) = \frac{1}{1 - \beta z + \gamma z^2}, \qquad C(z) = \frac{\alpha - tz}{1 - \beta z + \gamma z^2} = \alpha O(z) - tz O(z).$$

The specific cases follow. □

Remark 26. An alternative combinatorial structure we could have used are linked partitions. According to [Dyk07, Nic10], the pairs $\{\sigma \ll \pi : \sigma, \pi \in \mathcal{NC}(n)\}$ are in a natural bijection with the set of non-crossing linked partitions $\mathcal{NCL}(n)$, and doubling the value of β gives a bijection between such pairs with $Sing(\sigma) = Sing(\pi)$ and all such pairs. Moveover, according to [CWY08, CLW13], permutations are in a natural bijection with the set of all linked partitions $\mathcal{LP}(n)$. The results of this section and Theorem 11 can be phrased in terms of these objects, see [YY09] for related moment computations. This approach has not led us to any clarification in the inversion or product formulas.

In place of partitions, we could also (of course) have used colored Motzkin paths. From the point of view of Definition 18, the most natural family are those with a single color for rising steps and flat and falling steps at height zero, two colors for the other falling steps, and three colors for the rest of flat steps. It is not hard to see using the continued fraction form of the generating functions that the number of such paths of length $n+1$ is equal to the number of large $(3,2)$-Motzkin paths of length n in the sense of [CW12] (similar to the above, except their falling and flat steps at height zero are allowed two colors). This number in turn is known to be the (large) Schröder number, see Remark 32.

Remark 27. Unlike in the expansions in the five examples in Section 3, the terms on the right hand side of (14) have multiplicities. One can modify Definition 18 to obtain bijective representations. For example, we may define instead

$$\begin{aligned} W(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) = {} & W(a_1 \otimes \ldots \otimes a_n) X(a_{n+1}) - \alpha W(a_1 \otimes \ldots \otimes a_n) \langle a_{n+1} \rangle \\ & - \beta W(a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n a_{n+1}) - t W(a_1 \otimes \ldots \otimes a_{n-1}) \langle a_n a_{n+1} \rangle \\ & - \gamma W(a_1 \otimes \ldots \otimes a_{n-2} \otimes a_n a_{n+1} a_{n-1}). \end{aligned}$$

Note that this definition works only in the scalar-valued and not in the operator-valued case. The corresponding terms are in a bijection with the following collection of incomplete permutations. First, they have no double descents. Second, arrange each closed block so that it ends in its largest element. Then the descent-ascents in each block appear in decreasing order. Finally, split each block into sub-words, ending with the final letter or a descent-ascent, and beginning with the initial letter or right after the preceding descent-ascent. Then the partition into these sub-words is non-crossing.

We may also define

$$\begin{aligned} W(a_1 \otimes \ldots \otimes a_n \otimes a_{n+1}) = & \; W(a_1 \otimes \ldots \otimes a_n) X(a_{n+1}) - \alpha W(a_1 \otimes \ldots \otimes a_n) \langle a_{n+1} \rangle \\ & - \beta_1 W(a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n a_{n+1}) \\ & - \beta_2 W(a_1 \otimes \ldots \otimes a_{n-1} \otimes a_{n+1} a_n) \\ & - t W(a_1 \otimes \ldots \otimes a_{n-1}) \langle a_n a_{n+1} \rangle \\ & - \gamma W(a_1 \otimes \ldots \otimes a_{n-2} \otimes a_n a_{n+1} a_{n-1}), \end{aligned}$$

The corresponding terms are in a bijection with the following collection of incomplete permutations. Arrange each closed block so that it ends in its largest element. Then the descent-ascents in each block appear in decreasing order. Split each block as above. Then the partition into these sub-words is non-crossing, and on each sub-block, the letters are decreasing and then increasing, with the sub-block maximum at the end.

This description appears related to the work of West [Wes95], who studied permutations avoiding the patters $(3142, 2413)$ (sometimes called separable permutations). He proved that the cardinality of this set is the Schröder number (see Remark 32), and the argument uses trees reminiscent of the construction above.

5. COUNTEREXAMPLE

The following is Definition 4.9 from [Ans04a]. Here $\mathcal{M}, \Gamma(\mathcal{M})$ are as in the introduction.

Definition 28. For $a_i \in \mathcal{M}^{sa}$, define the q-Kailath-Segall polynomials by $W_q(a) = X(a) - \langle a \rangle$ and

$$\begin{aligned} W_q(a, a_1, a_2, \ldots, a_n) = & \; X(a) W_q(a_1, a_2, \ldots, a_n) - \sum_{i=1}^{n} q^{i-1} \langle a a_i \rangle W_q(a_1, \ldots, \hat{a}_i, \ldots, a_n) \\ & - \sum_{i=1}^{n} q^{i-1} W_q(a a_i, \ldots, \hat{a}_i, \ldots, a_n) - \langle a \rangle W_q(a_1, a_2, \ldots, a_n). \end{aligned} \tag{19}$$

This map has a $\mathbb{C}$-linear extension, so that each W is really a multi-linear map from $\mathcal{M}$ to $\Gamma(\mathcal{M})$.

Example 29. According to Corollary 4.13 from [Ans04a],

$$\varphi_q [W_q(a_0) \, W_q(a_1, a_2, a_3) \, W_q(a_4)] = 0.$$

However a direct calculation shows that in fact

$$\varphi_q [W_q(a_0) \, W_q(a_1, a_2, a_3) \, W_q(a_4)] = (q - q^2)(\langle a_0 a_2 \rangle \langle a_1 a_3 a_4 \rangle - \langle a_0 a_2 a_4 \rangle \langle a_1 a_3 \rangle).$$

To be completely explicit, we consider the case where $a_0 = a_2 = \mathbf{1}_I$, $a_1 = a_3 = a_4 = \mathbf{1}_J$, $I \cap J = \emptyset$, and the state is the Lebesgue measure. Then we get

$$\varphi_q [W_q(a_0) \, W_q(a_1, a_2, a_3) \, W_q(a_4)] = (q - q^2) \, |I| \cdot |J| \, .$$

Thus Corollary 4.13, and so also Theorem 4.11 part (c) in [Ans04a], are false.

The formula in Theorem 4.11(c) is true if the arguments of each W are orthogonal; however this does not imply the general result since φ_q is not tracial. See Remark 43. There are many particular cases when 4.11(c) is true. For the case $q = 1$ (classical), and $q = 0$ (free), the proof provided in [Ans04a] still works. For the q-Gaussian case, this is Theorem 3.3 in [EP03]. Finally, for univariate polynomials obtained for equal idempotent a and general q, the linearization formulas in Corollary 4.13 also hold [KSZ06, IKZ13].

6. Combinatorial corollaries

Proposition 30. *Define the free Meixner polynomials by the recursion*

$$\begin{aligned} xP_0(x) &= P_1(x) + \alpha P_0(x), \\ xP_1(x) &= P_2(x) + (\alpha+\beta)P_1(x) + tP_0(x), \\ xP_n(x) &= P_{n+1}(x) + (\alpha+\beta)P_n(x) + (t+\gamma)P_{n-1}(x), \end{aligned}$$

and denote by $\mu_{\alpha,\beta,\gamma,t}$ their measure of orthogonality. Then their linearization coefficients are

$$\int P_{s(1)}(x) \dots P_{s(k)}(x)\, d\mu_{\alpha,\beta,\gamma,t}(x) = \sum_{\substack{\sigma \in \mathcal{NC}_{\geqslant 2}(n), \\ \sigma \wedge (\hat{1}_{s(1)}, \dots, \hat{1}_{s(k)}) = \hat{0}_n}} \beta^{n-2|\sigma|} t^{|Out(\sigma)|} (t+\gamma)^{|\sigma| - |Out(\sigma)|}.$$

Proof. Setting all $a = 1$ in Definition 18 and denoting $x = X(1)$, we see that $P_n(x) = W(1^{\otimes n})$. So using Theorem 21, the linearization coefficients are

$$\sum_{\substack{(\pi,\sigma): \pi \in \mathcal{NC}(n), \\ \sigma \ll \pi, \\ Sing(\sigma) = \emptyset, \\ \sigma \wedge (\hat{1}_{s(1)}, \dots, \hat{1}_{s(k)}) = \hat{0}_n}} \beta^{n-2|\sigma|} t^{|\pi|} \gamma^{|\sigma|-|\pi|} = \sum_{\substack{\sigma \in \mathcal{NC}_{\geqslant 2}(n), \\ \sigma \wedge (\hat{1}_{s(1)}, \dots, \hat{1}_{s(k)}) = \hat{0}_n}} \beta^{n-2|\sigma|} \sum_{\pi \gg \sigma} t^{|\pi|} \gamma^{|\sigma|-|\pi|}.$$

Using Lemma 31 below, it follows that the linearization coefficients are

$$\begin{aligned} &\sum_{\substack{\sigma \in \mathcal{NC}_{\geqslant 2}(n), \\ \sigma \wedge (\hat{1}_{s(1)}, \dots, \hat{1}_{s(k)}) = \hat{0}_n}} \beta^{n-2|\sigma|} \sum_{Out(\sigma) \subset A \subset \sigma} t^{|A|} \gamma^{|\sigma|-|A|} \\ &\qquad = \sum_{\substack{\sigma \in \mathcal{NC}_{\geqslant 2}(n), \\ \sigma \wedge (\hat{1}_{s(1)}, \dots, \hat{1}_{s(k)}) = \hat{0}_n}} \beta^{n-2|\sigma|} t^{|Out(\sigma)|} (t+\gamma)^{|\sigma| - |Out(\sigma)|}. \quad \square \end{aligned}$$

Lemma 31 (Proposition 2.13 and Remark 2.14 in [BN08], Proposition 2.5 in [Nic10], cf. Lemma 3 in [Ans07])**.** *For a fixed $\sigma \in \mathcal{NC}(n)$, there is a bijection*

$$f : \{\pi : \pi \gg \sigma\} \to \{A : Out(\sigma) \subset A \subset \sigma\}$$

such that $|f(\pi)| = |\pi|$.

Remark 32. A calculation similar to Proposition 30 can be done for moments, but the result can also be obtained more directly. From the recursion for its orthogonal polynomials, the Jacobi-Szegő parameters of the measure $\mu_{\alpha,\beta,\gamma,t}$ are

$$\begin{pmatrix} \alpha, & \alpha+\beta, & \alpha+\beta, & \dots \\ t, & t+\gamma, & t+\gamma, & \dots \end{pmatrix}.$$

Then from the Viennot-Flajolet theorem, the n'th moment of this measure is

$$(20)\quad \sum_{\tau \in \mathcal{NC}_{1,2}(n)} \prod_{\substack{V \in Out(\tau),\\ |V|=1}} \alpha \prod_{\substack{V \in \tau \setminus Out(\tau),\\ |V|=1}} (\alpha+\beta) \prod_{\substack{U \in Out(\tau),\\ |U|=2}} t \prod_{\substack{U \in \tau \setminus Out(\tau),\\ |U|=2}} (t+\gamma)$$
$$= \sum_{\sigma \in \mathcal{NC}(n)} \alpha^{|Sing(\sigma)|} \beta^{n-2|\sigma|+|Sing(\sigma)|} t^{|Out(\sigma) \setminus Sing(\sigma)|} (t+\gamma)^{|\sigma|-|Sing(\sigma)|-|Out(\sigma) \setminus Sing(\sigma)|}.$$

We may interpret this as saying that the two-state free cumulants of the pair of free Meixner and free Poisson measures $(\mu_{\alpha,\beta,\gamma,t}, \mu_{\alpha,\beta,0,t})$ are

$$R_1 = r_1 = \alpha, \quad R_j = (t+\gamma)\beta^{j-2}, \quad r_j = t\beta^{j-2}.$$

Cf. Proposition 10 in [Ans09].

Various classical combinatorial sequences appearing as moments of these measures are listed in Section 7.4 of [Aig07]. These include Catalan, Motzkin, and Schröder numbers. Expansions (16) and (20) then give us various combinatorial identities. For example, for $\alpha = t = \gamma = 1$ and $\beta = 2$, the free cumulants are Catalan numbers while the moments are the large Schröder numbers, and we obtain the relations

$$\mathrm{Sch}_{n-1} = \sum_{\pi \in \mathcal{NC}(n)} \prod_{U \in \pi} c_{|U|-1} = \sum_{\sigma \in \mathcal{NC}(n)} 2^{n-|\sigma|-|Out(\sigma) \setminus Sing(\sigma)|}.$$

For the first relation, cf. Corollary 8.4 in [Dyk07]. If $\beta = t = \gamma = 1$, and $\alpha = 0$ the free cumulants are Motzkin numbers, and the moments are

$$\sum_{\pi \in \mathcal{NC}_{\geqslant 2}(n)} \prod_{U \in \pi} M_{|U|-2} = \sum_{\sigma \in \mathcal{NC}_{\geqslant 2}(n)} 2^{|\sigma|-|Out(\sigma)|}.$$

Either for $\alpha = 1$ or $\alpha = 0$ this moment sequence does not appear in [OEIS17].

Example 33. From Theorem 25, we can get a variety of different-looking combinatorial expansions.

For $\alpha = \beta = \gamma = t = 1$, Case II. $u, v = e^{\pm(\pi/3)i}$.

$$o_k = 1, 1, 0, -1, -1, 0, \ldots, \qquad c_k = 1, 0, -1, -1, 0, 1, \ldots.$$

$$W(a_1 \otimes \ldots \otimes a_n) = \sum_{\substack{(\pi,S)\\ \pi \in Int(n)\\ S \subset \pi\\ U \in \pi \setminus S \Rightarrow |U| \neq 2 \mod 3\\ V \in S \Rightarrow |V| \neq 0 \mod 3}} (-1)^{n-|U \in \pi \setminus S : |U| = 3 \text{ or } 4 \bmod 6| - |V \in S : |V| = 1 \text{ or } 2 \bmod 6|} \prod_{U \in \pi \setminus S} \langle a_U \rangle \prod_{V \in S} X(a_V).$$

For $\alpha = 1$, $\gamma = t$, $\beta = t+1$, Case II$'$. $u = 1$, $v = t$.

$$o_k = \frac{1}{1-t}(1-t^k), \qquad c_k = 1.$$

$$W(a_1 \otimes \ldots \otimes a_n) = \sum_{\substack{(\pi,S)\\ \pi \in Int(n)\\ S \subset \pi}} (-1)^{n-|S|} \prod_{U \in \pi \setminus S} \langle a_U \rangle \prod_{V \in S} \left(\frac{1-t^{|V|}}{1-t} \right) X(a_V).$$

For $\alpha = 0$, $\gamma = t = 1$, $\beta = 2$, Case III.

$$o_k = k, \qquad c_k = -(k-1).$$

$$W(a_1 \otimes \ldots \otimes a_n) = \sum_{\substack{(\pi, S) \\ \pi \in Int(n) \\ Sing(\pi) \subset S \subset \pi}} (-1)^{n-|\pi|} \prod_{U \in \pi \backslash S} (|U| - 1) \langle a_U \rangle \prod_{V \in S} |V| \, X(a_V).$$

For $\alpha = \gamma = t = 1$, $\beta = 2$, Case III$'$.

$$o_k = k, \qquad c_k = 1.$$

$$W(a_1 \otimes \ldots \otimes a_n) = \sum_{\substack{(\pi, S) \\ \pi \in Int(n) \\ S \subset \pi}} (-1)^{n-|S|} \prod_{U \in \pi \backslash S} \langle a_U \rangle \prod_{V \in S} |V| \, X(a_V).$$

These in turn give expansions for free Meixner polynomials and may serve as a source of combinatorial identities.

Proposition 34. *All five examples of incomplete posets in Section 2 are graded by the number of open blocks.*

(a) *The numbers of incomplete partitions (analog of Bell numbers) are*

$$|\mathcal{IP}(n)| = \sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i},$$

sequence A001861 in [OEIS17]. *The incomplete Stirling numbers of the second kind are*

$$S_{n,k,\ell} = |\{(\pi, S) \in \mathcal{IP}(n) : |\pi \backslash S| = k, |S| = \ell\}| = \binom{k+\ell}{\ell} S_{n,k+\ell},$$

and the number of elements of rank ℓ is $\sum_k \binom{k+\ell}{\ell} S_{n,k+\ell}$, sequence A049020.

(b) *The numbers of incomplete non-crossing partitions (analog of Catalan numbers) are*

$$|\mathcal{INC}(n)| = \binom{2n}{n},$$

the central binomial coefficients, sequence A000984. Define the incomplete Narayana numbers

$$N_{n,k,\ell} = |\{(\pi, S) \in \mathcal{INC}(n) : |\pi \backslash S| = k, |S| = \ell\}| .$$

Then denoting

$$F(t, x, z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} N_{n,k,\ell} t^k x^\ell z^n$$

their generating function and $\tilde{F}(t, z) = F(t, 0, z)$ the known generating function of the regular Narayana numbers,

$$F(t, x, z) = \frac{1 - z\tilde{F}(t, z)}{1 - z(t + x + \tilde{F}(t, z))}.$$

The rank generating function is $F(1, x, z)$, generating sequence A039599.

(c) *The numbers of partial permutations are* $|\mathcal{IPRM}(n)| = \sum_{\ell=0}^{n} \binom{n}{\ell} \ell!$*, sequence A002720. The incomplete Stirling numbers of the first kind*

$$s_{n,k,\ell} = |\{(\pi, S) \in \mathcal{IPRM}(n) : |\pi \backslash S| = k, |S| = \ell\}|$$

have the generating function

$$\sum_{k=0}^{n} s_{n,k,\ell} t^k = \binom{n}{\ell} (t + \ell) \dots (t + n - 1),$$

and the number of elements of rank ℓ *is* $\binom{n}{\ell}^2 \ell!$.

Proof. The formula for the incomplete Stirling numbers of the second kind is obvious. Then using for example the solved Exercise 1.32 in [Aig07],

$$|\mathcal{IP}(n)| = \sum_{k,\ell=0}^{n} \binom{k+\ell}{\ell} S_{n,k+\ell} = \sum_{k,\ell=0}^{n} \sum_{i=\ell}^{n-k} \binom{n}{i} S_{i,\ell} S_{n-i,k} = \sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i}.$$

The incomplete Narayana numbers satisfy the recursion relation

$$N_{n+1,k,\ell} = N_{n,k-1,\ell} + N_{n,k,\ell-1} + \sum_{i=1}^{n} \sum_{j=0}^{k} N_{i,j,\ell} N_{n-i,k-j,0}.$$

It follows that

$$F(t, x, z) = 1 + ztF(t, x, z) + zxF(t, x, z) + z(F(t, x, z) - 1)\tilde{F}(t, z).$$

The generating function for $|\mathcal{INC}(n)|$ is easily computed to be

$$F(1, 1, z) = \frac{1}{\sqrt{1 - 4z}}.$$

The formula for $|\mathcal{IPRM}(n)|$ is obvious. The formula for the incomplete Stirling numbers of the first kind follows from the recursion relation

$$s_{n+1,k,\ell} = s_{n,k-1,\ell} + s_{n,k,\ell-1} + (n + \ell) s_{n,k,\ell},$$

obtained in the usual way by adjoining $n + 1$ to an incomplete permutation of n; note that in a closed work of length u, $n + 1$ can be inserted in u places, while in an open word it can be inserted in $u + 1$ spaces. □

Remark 35. For completeness, we include combinatorial corollaries of Proposition 10 and Theorem 11. Take a to a projection, so that $a^2 = a$ and $\langle a \rangle = t$. Denote $X(a) = x$. Then

$$x\, W_{\mathcal{IP}_{1,2}}\left(a^{\otimes n}\right) = W_{\mathcal{IP}_{1,2}}\left(a^{\otimes n+1}\right) + tn\, W_{\mathcal{IP}_{1,2}}\left(a^{\otimes n-1}\right),$$

$$x\, W_{\mathcal{IP}}\left(a^{\otimes n}\right) = W_{\mathcal{IP}}\left(a^{\otimes n+1}\right) + (t + n)\, W_{\mathcal{IP}}\left(a^{\otimes n}\right) + tn\, W_{\mathcal{IP}}\left(a^{\otimes n-1}\right),$$

and

$$x\, W_{\mathcal{IPRM}}\left(a^{\otimes n}\right) = W_{\mathcal{IPRM}}\left(a^{\otimes n+1}\right) + (t + 2n)\, W_{\mathcal{IPRM}}\left(a^{\otimes n}\right) + (tn + n(n-1))\, W_{\mathcal{IPRM}}\left(a^{\otimes n-1}\right).$$

Thus $W_{\mathcal{IP}_{1,2}}\left(a^{\otimes n}\right) = H_n(x, t)$, the Hermite polynomial; $W_{\mathcal{IP}}\left(a^{\otimes n}\right) = C_n(x, t)$, the Charlier polynomial; and $W_{\mathcal{IPRM}}\left(a^{\otimes n}\right) = L_n^{(t-1)}(x)$, the Laguerre polynomial. We thus get (mostly known) inversion, moment, product, and linearization formulas for these polynomials. For example, for the Laguerre case

$$x^n = \sum_{(\pi,S)\in\mathcal{IPRM}(n)} t^{|\pi\setminus S|} L_{|S|}^{(t-1)}(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} (t+\ell)\dots(t+n-1) L_{|S|}^{(t-1)}(x),$$

$$L_n^{(t-1)}(x) = \sum_{(\pi,S)\in\mathcal{IPRM}(n)} (-1)^{n-|S|} t^{|\pi\setminus S|} x^{|S|} = \sum_{\ell=0}^{n} (-1)^{n-\ell} \binom{n}{\ell} (t+\ell)\dots(t+n-1) x^{\ell},$$

and

$$\prod_{i=1}^{k} L_{s(i)}^{(t-1)}(x) = \sum_{(\pi,S)\in\mathcal{ID}(s(1),\dots,s(k))} t^{|\pi\setminus S|} L_{|S|}^{(t-1)}(x).$$

extending the moment and linearization formulas [FZ88]

$$\int x^n \, d\mu(x) = \sum_{\pi\in\mathrm{Sym}(n)} t^{\mathrm{cyc}\,\pi} = t(1+1)\dots(t+n-1),$$

$$\int \prod_{i=1}^{k} L_{s(i)}^{(t-1)}(x) \, d\mu(x) = \sum_{\pi\in\mathcal{D}(s(1),\dots,s(k))} t^{\mathrm{cyc}\,\pi}.$$

Similarly, since

$$\sum_{\substack{(\pi,S)\in\mathcal{IP}(n), U\in\pi\setminus S\Rightarrow|U|=1,\\ |\pi\setminus S|=k, |S|=\ell}} \prod_{V\in S} (|V|-1)! = \binom{n}{k} \sum_{\pi\in\mathcal{P}(n-k), |\pi|=\ell} \prod_{V\in\pi} (|V|-1)! = \binom{n}{k} s_{n-k,\ell},$$

we obtain the familiar result that the Charlier polynomials are

$$C_n(x,t) = \sum_{k,\ell=0}^{n} (-1)^{n-\ell} \binom{n}{k} s_{n-k,\ell} t^k x^{\ell} = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} t^k \binom{x}{n-k}.$$

Finally, since

$$|\{(\pi,S)\in\mathcal{INC}(n), \pi\in Int(n), V\in\pi\setminus S\Rightarrow|V|=1, |\pi\setminus S|=k, |S|=\ell\}| = \binom{n-k}{\ell-1}\binom{k+\ell}{\ell},$$

the free Charlier polynomials are

$$P_n(x,t) = \sum_{k,\ell} (-1)^{n-\ell} \binom{n-k}{\ell-1}\binom{k+\ell}{\ell} t^k x^{\ell}.$$

See, for example, Chapter 7 in [Aig07] for many related combinatorial results.

7. Representations and completions

Let $\mathcal{M}$ and $\mathcal{B}$ be $\mathcal{D}$-bimodules with the actions satisfying (9). For a linear $\mathcal{D}$-bimodule map $F : \mathcal{M} \to \mathcal{B}$, define the map $\mathcal{F}(F) : \mathcal{M}^{\otimes_{\mathcal{D}} n} \to \mathcal{B}^{\otimes_{\mathcal{D}} n}$ by

$$\mathcal{F}(F)[d_0 a_1 d_1 \otimes \dots \otimes a_n d_n] = F(d_0 a_1 d_1) \otimes \dots \otimes F(a_n d_n),$$

and the map $\Gamma(F) : \Gamma(\mathcal{M}) \to \Gamma(\mathcal{B})$ by $\Gamma(F)[d] = d$ for $d \in \mathcal{D}$ and

$$\Gamma(F)[W_{\mathcal{INC}}(\mathbf{a})] = W_{\mathcal{INC}}(\mathcal{F}(F)[\mathbf{a}]).$$

Definition 36. Let $\mathcal{M}$ be a star-algebra and $\mathcal{B}$ a star-subalgebra. An *algebraic conditional expectation* is a star-linear $\mathcal{B}$-bimodule map $F : \mathcal{M} \to \mathcal{B}$ such that $F^2 = F$. If $\mathcal{M}$ is a $\mathcal{D}$-bimodule, $\mathcal{D}\mathcal{B}\mathcal{D} \subset \mathcal{B}$, and $\tau : \mathcal{M} \to \mathcal{D}$ is a star-linear functional, we say that F preserves τ if $\tau[F(a)] = \tau[a]$ for $a \in \mathcal{M}$.

Proposition 37. *Let $\mathcal{D}, \mathcal{B}, \mathcal{M}$ be as in the preceding definition, and $F : \mathcal{M} \to \mathcal{B}$ an algebraic conditional expectation preserving $\langle \cdot \rangle$. Then $\Gamma(F) : \Gamma(\mathcal{M}) \to \Gamma(\mathcal{B})$ is an algebraic conditional expectation preserving $\varphi_{\mathcal{INC}}$.*

Proof. Clearly $\Gamma(F)$ is the identity on $\Gamma(\mathcal{B})$. For $\mathbf{a} \in \mathcal{B}^{\otimes_{\mathcal{D}} n}$, $\mathbf{b} \in \mathcal{M}^{\otimes_{\mathcal{D}} k}$, $\mathbf{c} \in \mathcal{B}^{\otimes_{\mathcal{D}} \ell}$,

$$
\begin{aligned}
&\Gamma(F) \left[W_{\mathcal{INC}} (\mathbf{a}) \; W_{\mathcal{INC}} (\mathbf{b}) \; W_{\mathcal{INC}} (\mathbf{c}) \right] \\
&= \sum_{\substack{(\pi, S) \in \mathcal{INC}(n+k+\ell) \\ \pi \wedge (\hat{1}_n, \hat{1}_k, \hat{1}_\ell) = \hat{0}_{n+k+\ell} \\ Sing(\pi) \subset S}} \Gamma(F) \left[W_{\mathcal{INC}} (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c})^{(\pi, S)} \right] \\
&= \sum_{\substack{(\pi, S) \in \mathcal{INC}(n+k+\ell) \\ \pi \wedge (\hat{1}_n, \hat{1}_k, \hat{1}_\ell) = \hat{0}_{n+k+\ell} \\ Sing(\pi) \subset S}} W_{\mathcal{INC}} (\mathbf{a} \otimes \mathcal{F}(F)[\mathbf{b}] \otimes \mathbf{c})^{(\pi, S)} \\
&= W_{\mathcal{INC}} (\mathbf{a}) \, \Gamma(F) \left[W_{\mathcal{INC}} (\mathbf{b}) \right] W_{\mathcal{INC}} (\mathbf{c}) \, ,
\end{aligned}
$$

where we have used the inhomogeneity of the partitions, the bimodule property of F for open blocks, and both properties of F for closed blocks. The final property is clear. □

Proposition 38. *In all six examples above,*

$$
W (a_1 \otimes a_2 \otimes \ldots \otimes a_n)^* = W (a_n^* \otimes \ldots \otimes a_2^* \otimes a_1^*) \, ,
$$

where for $\varphi_{\mathcal{IP}}$ we additionally assume that $\mathcal{M}$ is commutative. If $\mathcal{D} = \mathbb{C}$ and the linear functional $\langle \cdot \rangle$ on $\mathcal{M}$ is tracial, all six linear functionals φ are tracial. If $\mathcal{D}$ is a unital C^*-algebra, and $\langle \cdot \rangle$ is positive, the functionals φ are positive, where for $\varphi_{\mathcal{IP}}$ and $\varphi_{\mathcal{IPRM}}$ we additionally assume that $\mathcal{M}$ is commutative.

Proof. The trace and adjoint properties follow from the moment formulas and expansions of Wick products in terms of monomials, since in all cases the coefficients in the expansions depend only on the size of the blocks. For positivity,

$$
\begin{aligned}
&\varphi_{\mathcal{INC}_{1,2}} \left[W_{\mathcal{INC}_{1,2}} (a_1 \otimes \ldots \otimes a_n)^* \; W_{\mathcal{IP}_{1,2}} (b_1 \otimes \ldots \otimes b_k) \right] \\
&\quad = \varphi_{\mathcal{INC}} \left[W_{\mathcal{INC}} (a_1 \otimes \ldots \otimes a_n)^* \; W_{\mathcal{INC}} (b_1 \otimes \ldots \otimes b_k) \right] \\
&\quad = \delta_{n=k} \left\langle a_n^* \otimes \ldots \otimes a_1^* \otimes b_1 \otimes \ldots \otimes b_n \right\rangle^{\{(1,2n),(2,2n-1),\ldots,(n,n+1)\}} ,
\end{aligned}
$$

The proof of positivity of this inner product on $\mathcal{M}^{\otimes_{\mathcal{D}} n}$ (which we denote $\langle \cdot, \cdot \rangle_n$) is almost verbatim the argument in Theorem 3.5.6 of [Spe98]. Also,

$$
\begin{aligned}
&\varphi \left[W (a_1 \otimes \ldots \otimes a_n)^* \; W (b_1 \otimes \ldots \otimes b_k) \right] \\
&\quad = \delta_{n=k} \sum_{\pi \in Int(n)} t^{|\pi|} \gamma^{n-|\pi|} \left\langle a_n^* \otimes \ldots \otimes a_1^* \otimes b_1 \otimes \ldots \otimes b_n \right\rangle^{\{U \cup (2n+1-U) : U \in \pi\}} \\
&\quad = \delta_{n=k} \sum_{\pi \in Int(n)} t^{|\pi|} \gamma^{n-|\pi|} \left\langle \bigotimes_{U \in \pi} a_U , \bigotimes_{U \in \pi} b_U \right\rangle_{|\pi|} ,
\end{aligned}
\tag{21}
$$

and so this inner product is also positive. For commutative $\mathcal{M}$,

$$\begin{aligned}&\varphi_{\mathcal{IP}_{1,2}}\left[W_{\mathcal{IP}_{1,2}}(a_1\otimes\ldots\otimes a_n)^* W_{\mathcal{IP}_{1,2}}(b_1\otimes\ldots\otimes b_k)\right]\\&\quad=\varphi_{\mathcal{IP}}\left[W_{\mathcal{IP}}(a_1\otimes\ldots\otimes a_n)^* W_{\mathcal{IP}}(b_1\otimes\ldots\otimes b_k)\right]=\delta_{n=k}\sum_{\alpha\in\mathrm{Sym}(n)}\left\langle a^*_{\alpha(1)}b_1\right\rangle\ldots\left\langle a^*_{\alpha(n)}b_n\right\rangle,\end{aligned}$$

This inner product on $\mathcal{M}^{\otimes n}$ is well known to be positive semi-definite. Finally,

$$\begin{aligned}&\varphi_{\mathcal{IPRM}}\left[W_{\mathcal{IPRM}}(a_1\otimes\ldots\otimes a_n)^* W_{\mathcal{IPRM}}(b_1\otimes\ldots\otimes b_k)\right]\\&\quad=\delta_{n=k}\sum_{\alpha,\beta\in\mathrm{Sym}(n)}\prod_{U\in\pi(\beta)}\left\langle\prod_{i\in U}\left(a^*_{\alpha(i)}b_i\right)\right\rangle,\end{aligned}$$

where $\pi(\beta)$ is the orbit decomposition of β and the order in each $U\in pi(\beta)$ is as according to β as in Theorem 11. As observed in Section 4 of [Śni00], this inner product is in general not positive. If $\mathcal{M}$ is commutative, we may re-write

$$\begin{aligned}&\varphi_{\mathcal{IPRM}}\left[W_{\mathcal{IPRM}}(a_1\otimes\ldots\otimes a_n)^* W_{\mathcal{IPRM}}(a_1\otimes\ldots\otimes a_k)\right]\\&\quad=\delta_{n=k}\sum_{\alpha\in\mathrm{Sym}(n)}\sum_{\pi\in\mathcal{P}(n)}\prod_{U\in\pi}(|U|-1)!\left\langle\left(\prod_{i\in U}a_{\alpha(i)}\right)^*\left(\prod_{i\in U}a_i\right)\right\rangle\\&\quad=\delta_{n=k}\frac{1}{n!}\sum_{\pi\in\mathcal{P}(n)}\sum_{\alpha,\beta\in\mathrm{Sym}(n)}\prod_{U\in\pi}(|U|-1)!\left\langle\left(\prod_{i\in U}a_{\alpha(i)}\right)^*\left(\prod_{i\in U}a_{\beta(i)}\right)\right\rangle\\&\quad=\delta_{n=k}\frac{1}{n!}\sum_{\pi\in\mathcal{P}(n)}\prod_{U\in\pi}(|U|-1)!\left\langle\left(\prod_{i\in U}(P_n\mathbf{a})_i\right)^*\left(\prod_{i\in U}(P_n\mathbf{a})_i\right)\right\rangle\geqslant 0,\end{aligned}$$

where in the next-to-last term we replaced $\alpha\circ\beta$ with α, and P_n is the symmetrization operator. ☐

Remark 39. For $\mathcal{D}=\mathbb{C}$, $\langle\cdot,\cdot\rangle_n$ is essentially the induced inner product on a tensor product of Hilbert spaces, and so is non-degenerate if $\langle\cdot\rangle$ is faithful. In general, $\langle\cdot,\cdot\rangle_n$, and so $\varphi_{\mathcal{INC}}$, is rarely faithful. For example, let $\mathcal{D}=\mathcal{M}$, $\langle a\rangle=a$, and $p\in\mathcal{M}$ be a idempotent. Then $\langle(1-p)\otimes p,(1-p)\otimes p\rangle_2=0$.

Notation 40. For $\mathbf{a}\in\mathcal{M}^{\otimes n}$ and $(\pi,S)\in\mathcal{INC}(n)$, define the contraction $\mathcal{C}^{(\pi,S)}(\mathbf{a})$ by a linear extension of

$$\mathcal{C}^{(\pi,S)}(a_1\otimes\ldots\otimes a_n)=\prod_{U\in\pi\setminus S}\langle a_U\rangle\bigotimes_{V\in S}a_V.$$

Note that in all our examples with $\mathcal{D}=\mathbb{C}$,

$$W(\mathbf{a})^{(\pi,S)}=W\left(\mathcal{C}^{(\pi,S)}(\mathbf{a})\right).$$

Proposition 41. *Assume $\langle\cdot\rangle$ is a faithful state such that in its representation on $L^2(\mathcal{M},\langle\cdot\rangle)$, $\mathcal{M}$ is represented by bounded operators. Let $\mathbf{a}\in\mathcal{M}^{\otimes n}$ and $\mathbf{b}\in\mathcal{M}^{\otimes k}$. Denote $\|\mathbf{a}\|_2=\sqrt{\langle\mathbf{a},\mathbf{a}\rangle_n}$ and $\|W(\mathbf{a})\|_\varphi=\sqrt{\varphi\left[W(\mathbf{a})^* W(\mathbf{a})\right]}$. Denote*

$$Z_{n,k}=\left\{(\pi,S)\in\mathcal{INC}(n+k):\pi\wedge(\hat{1}_n,\hat{1}_k)=\hat{0}_{n+k},Sing(\pi)\subset S\right\}.$$

For $(\pi,S)\in Z_{n,k}$, the map

$$\mathbf{b}\mapsto W_{\mathcal{INC}}(\mathbf{a}\otimes\mathbf{b})^{(\pi,S)}$$

is bounded as a map from $L^2(\mathcal{M}, \langle\cdot\rangle)^{\otimes k}$ to $L^2(\mathcal{M}, \langle\cdot\rangle)^{\otimes |S|}$, as is the map

$$\mathbf{b} \mapsto W_{\mathcal{INC}}(\mathbf{a})\, W_{\mathcal{INC}}(\mathbf{b})\,.$$

Therefore the definitions of $\mathcal{C}^{(\pi,S)}(\mathbf{a}\otimes\mathbf{b})$ and $W_{\mathcal{INC}}(\mathbf{a}\otimes\mathbf{b})^{(\pi,S)}$ and the identity

$$W_{\mathcal{INC}}(\mathbf{a})\, W_{\mathcal{INC}}(\mathbf{b}) = \sum_{(\pi,S)\in Z_{n,k}} W_{\mathcal{INC}}\left(\mathcal{C}^{(\pi,S)}(\mathbf{a}\otimes\mathbf{b})\right) = \sum_{(\pi,S)\in Z_{n,k}} W_{\mathcal{INC}}(\mathbf{a}\otimes\mathbf{b})^{(\pi,S)}$$

extend to $\mathbf{a} \in \mathcal{M}^{\otimes n}$ (algebraic tensor product) and $\mathbf{b} \in L^2(\mathcal{M}, \langle\cdot\rangle)^{\otimes k}$ (Hilbert space tensor product). If $\langle\cdot\rangle$ is tracial, we may switch $\mathbf{a}$ and $\mathbf{b}$.

Proof. Since

$$\left\| W_{\mathcal{INC}}(\mathbf{a}\otimes\mathbf{b})^{(\pi,S)} \right\|_{\varphi_{\mathcal{INC}}} = \left\| \mathcal{C}^{(\pi,S)}(\mathbf{a}\otimes\mathbf{b}) \right\|_2,$$

it suffices to consider the map $\mathbf{b} \mapsto \mathcal{C}^{(\pi,S)}(\mathbf{a}\otimes\mathbf{b})$. Denote

$$\pi_\ell = \{(i) : 1 \leqslant i \leqslant n-\ell, n+\ell+1 \leqslant i \leqslant n+k; (n-j+1, n+j) : 1 \leqslant j \leqslant \ell\},$$

and

$$S_\ell = \{(i) : 1 \leqslant i \leqslant n-\ell, n+\ell+1 \leqslant i \leqslant n+k\}, \quad S'_\ell = S_\ell \cup \{(n-\ell+1, n+\ell)\}.$$

Note $|S_\ell| = n+k-2\ell$, $|S'_\ell| = n+k-2\ell+1$, and $|Z_{n,k}| = 2\min(n,k)$. Then

$$Z_{n,k} = \{(\pi_\ell, S_\ell), (\pi_\ell, S'_\ell) : 0 \leqslant \ell \leqslant \min(n-k)\}.$$

For $\mathbf{a} = a_1 \otimes \ldots \otimes a_n$ and $\mathbf{b} = \sum_i b_{i1} \otimes \ldots \otimes b_{ik}$,

$$\begin{aligned}
\left\| \mathcal{C}^{(\pi_\ell, S'_\ell)}(\mathbf{a}\otimes\mathbf{b}) \right\|_2^2 &= \left\| \sum_i \prod_{r=1}^{\ell-1} \langle a_{n-r+1} b_{i,r} \rangle\, a_1 \otimes \ldots \otimes a_{n-\ell} \otimes a_{n-\ell+1} b_{i\ell} \otimes \ldots \otimes b_{ik} \right\|_2^2 \\
&= \sum_{i,j} \prod_{r=1}^{\ell-1} \overline{\langle a_{n-r+1} b_{i,r} \rangle} \prod_{s=1}^{\ell-1} \langle a_{n-s+1} b_{j,s} \rangle \langle a_1^* a_1 \rangle \ldots \left\langle a_{n-\ell}^* a_{n-\ell} \right\rangle \left\langle b_{i\ell}^* a_{n-\ell+1}^* a_{n-\ell+1} b_{j\ell} \right\rangle \ldots \langle b_{ik}^* b_{jk} \rangle \\
&\leqslant \left\| a_{n-\ell+1} \right\|^2 \left\| \mathcal{C}^{(\pi_{\ell-1}, S_{\ell-1})}(\mathbf{a}\otimes\mathbf{b}) \right\|_2^2
\end{aligned}$$

Also, the statement

$$\left\| \mathcal{C}^{(\pi_\ell, S_\ell)}(\mathbf{a}\otimes\mathbf{b}) \right\|_2 \leqslant \|\mathbf{a}\|_2 \|\mathbf{b}\|_2$$

is about Hilbert spaces and not algebras, and as such is well known, see for example Proposition 5.3.3 in [BS98] (one may identify the Hilbert space with the space of square-integrable functions on a measure space, and apply coordinate-wise Cauchy-Schwarz inequality). The boundedness of the first map follows.

Next, note that $W_{\mathcal{INC}}\left(\mathcal{C}^{(\pi,S)}(\mathbf{a}\otimes\mathbf{b})\right)$ are orthogonal for different $|S|$. Thus

$$\| W_{\mathcal{INC}}(\mathbf{a})\, W_{\mathcal{INC}}(\mathbf{b}) \|_{\varphi_{\mathcal{INC}}}^2 = \sum_{(\pi,S)\in Z_{n,k}} \left\| W_{\mathcal{INC}}\left(\mathcal{C}^{(\pi,S)}(\mathbf{a}\otimes\mathbf{b})\right) \right\|_{\varphi_{\mathcal{INC}}}^2.$$

The results follow. □

Example 42. Let $f(x, y) = \mathbf{1}_{[0,1]}(y)\mathbf{1}_{[0,y^{-1/4}]}(x)$ and $g(y) = y^{-1/4}$. Then $f \in L^1 \cap L^\infty(\mathbb{R}^2)$ (and so in $L^2(\mathbb{R}^2)$) and $g \in L^2(\mathbb{R})$, but

$$\mathcal{C}^{(\{(1),(2,3)\},\{(1),(2,3)\})}(f \otimes g)(x, y) = f(x, y)g(y)$$

is not in $L^2(\mathbb{R}^2)$. Cf. Remark 3.3 in [BP14].

Remark 43. In stochastic analysis, see for example [PT11] or [BS98], it is usual to prove product formulas

$$W (a_1 \otimes \ldots \otimes a_n)\, W (b_1 \otimes \ldots \otimes b_k) = \sum W \tag{22}$$

for all a_i's, and separately all b_j's, orthogonal to each other. One can then conclude using the Itô isometry that the same formula holds for general a_i, b_j. Some, but not all, of the ingredients of this approach generalize to the Wick product setting.

- In the case of $W_{\mathcal{INC}}$, W, and W_q, we have isometries between $\Gamma(\mathcal{M})$ and $\bigoplus_{n=0}^{\infty} \mathcal{M}^{\otimes n}$ with, respectively, the usual inner product induced by $\langle\cdot\rangle$, the inner product (21), and the appropriate q-inner product (equation 4.73 in [Ans04a]). So in all these cases, one may extend the definition of W to the appropriate closure, which however is different in all three cases.
- Instead of starting with general simple tensors, we could have started with the analog of functions supported away from diagonals. As noted in Lemma 44, in the infinite-dimensional setting such elements are still dense with respect to the usual inner product. However they are clearly not dense for the inner product (21). For example, in the natural commutative setting of $(\mathcal{M}, \langle\cdot\rangle) = ((L^1 \cap L^\infty(\mathbb{R}), dx)$, the inner product on functions of n arguments is

$$\sum_{\pi \in Int(n)} \overline{f(\mathbf{x})} g(\mathbf{x})\, d\mu_\pi(\mathbf{x}),$$

  where μ_π is a multiple of the $|\pi|$-dimensional Lebesgue measure on the diagonal set

$$\left\{\mathbf{x} \in \mathbb{R}^n : x_i = x_j \Leftrightarrow i \overset{\pi}{\sim} j\right\}.$$

  This is the reason why the formulas in Theorem 21 take a considerably simpler form if the arguments have orthogonal components.
- Finally, to extend the product relation (22), we need the product map to be continuous, at least when one of the arguments is in the algebraic tensor product and the other is bounded in two-norm. If the state φ is not tracial, this need not be the case. Since φ_q is not tracial, it is natural to expect a counterexample in Section 5.

It is well-known that for non-atomic measures, functions supported away from diagonals are dense in the product space of all square integrable functions. The next lemma (applied to $L^2(\mathcal{M}, \langle\cdot\rangle)$. shows that this results remains true for non-commutative algebras, in fact with no assumptions on the state other than faithfulness. The result is surely known, but we could not find it in the literature.

Lemma 44. *Let H be a Hilbert space. In the Hilbert space tensor product $H \otimes H$, consider the span S of tensors of the form $f \otimes g$ with $\langle f, g\rangle = 0$. This span is dense if and only if H is infinite dimensional.*

Proof. Choose an orthonormal basis $\{e_i\}$ for H. Identify $H \otimes H$ with the space of Hilbert-Schmidt operators $\mathrm{HS}(H)$,

$$f \otimes g \mapsto f \langle \cdot, g \rangle .$$

Then for any f, g, we have

$$\mathrm{tr}(f \otimes g) = \sum_i \langle f, e_i \rangle \langle e_i, g \rangle = \langle f, g \rangle .$$

In particular, if $\langle f, g \rangle = 0$, $\mathrm{tr}(f \otimes g) = 0$. So if $\dim H < \infty$, all operators in S have trace zero, and so S is not dense.

Now suppose that $\dim H = \infty$. Then H is isomorphic to $L^2([0,1], dx)$, in which case the result is well-known (it is also not hard to give a direct argument in terms of Hilbert-Schmidt operators; it is left to the interested reader). □

References

[Aig07] Martin Aigner, *A course in enumeration*, Graduate Texts in Mathematics, vol. 238, Springer, Berlin, 2007. MR 2339282 (2008f:05001)

[Ans04a] Michael Anshelevich, *Appell polynomials and their relatives*, Int. Math. Res. Not. (2004), no. 65, 3469–3531. MR 2101359 (2005k:33012)

[Ans04b] ———, *q-Lévy processes*, J. Reine Angew. Math. **576** (2004), 181–207. MR 2099204 (2006c:46052)

[Ans05] ———, *Linearization coefficients for orthogonal polynomials using stochastic processes*, Ann. Probab. **33** (2005), no. 1, 114–136. MR 2118861 (2005m:05243)

[Ans07] ———, *Free Meixner states*, Comm. Math. Phys. **276** (2007), no. 3, 863–899. MR 2350440 (2009b:81106)

[Ans09] ———, *Appell polynomials and their relatives. III. Conditionally free theory*, Illinois J. Math. **53** (2009), no. 1, 39–66. MR 2584934

[ABFN13] Michael Anshelevich, Serban T. Belinschi, Maxime Fevrier, and Alexandru Nica, *Convolution powers in the operator-valued framework*, Trans. Amer. Math. Soc. **365** (2013), no. 4, 2063–2097. MR 3009653

[BN08] Serban T. Belinschi and Alexandru Nica, *η-series and a Boolean Bercovici-Pata bijection for bounded k-tuples*, Adv. Math. **217** (2008), no. 1, 1–41. MR 2357321

[Bia93] Philippe Biane, *Permutations suivant le type d'excédance et le nombre d'inversions et interprétation combinatoire d'une fraction continue de Heine*, European J. Combin. **14** (1993), no. 4, 277–284. MR 1226575

[BS98] Philippe Biane and Roland Speicher, *Stochastic calculus with respect to free Brownian motion and analysis on Wigner space*, Probab. Theory Related Fields **112** (1998), no. 3, 373–409. MR 99i:60108

[BRR89] David Borwein, Stuart Rankin, and Lex Renner, *Enumeration of injective partial transformations*, Discrete Math. **73** (1989), no. 3, 291–296. MR 983027

[BP14] Solesne Bourguin and Giovanni Peccati, *Semicircular limits on the free Poisson chaos: counterexamples to a transfer principle*, J. Funct. Anal. **267** (2014), no. 4, 963–997. MR 3217054

[CLW13] William Y. C. Chen, Lewis H. Liu, and Carol J. Wang, *Linked partitions and permutation tableaux*, Electron. J. Combin. **20** (2013), no. 3, Paper 53, 10. MR 3118961

[CW12] William Y. C. Chen and Carol J. Wang, *Noncrossing linked partitions and large $(3,2)$-Motzkin paths*, Discrete Math. **312** (2012), no. 11, 1918–1922. MR 2913085

[CWY08] William Y. C. Chen, Susan Y. J. Wu, and Catherine H. Yan, *Linked partitions and linked cycles*, European J. Combin. **29** (2008), no. 6, 1408–1426. MR 2423730

[CSZ97] Robert J. Clarke, Einar Steingrímsson, and Jiang Zeng, *New Euler-Mahonian statistics on permutations and words*, Adv. in Appl. Math. **18** (1997), no. 3, 237–270. MR 1436481

[Dyk07] Kenneth J. Dykema, *Multilinear function series and transforms in free probability theory*, Adv. Math. **208** (2007), no. 1, 351–407. MR 2304321 (2008k:46193)

[EP03] Edward G. Effros and Mihai Popa, *Feynman diagrams and Wick products associated with q-Fock space*, Proc. Natl. Acad. Sci. USA **100** (2003), no. 15, 8629–8633 (electronic). MR 1994546 (2004h:81099)

[FZ88] Dominique Foata and Doron Zeilberger, *Laguerre polynomials, weighted derangements, and positivity*, SIAM J. Discrete Math. **1** (1988), no. 4, 425–433. MR 968850

[IKZ13] Mourad E. H. Ismail, Anisse Kasraoui, and Jiang Zeng, *Separation of variables and combinatorics of linearization coefficients of orthogonal polynomials*, J. Combin. Theory Ser. A **120** (2013), no. 3, 561–599. MR 3007137

[ISV87] Mourad E. H. Ismail, Dennis Stanton, and Gérard Viennot, *The combinatorics of q-Hermite polynomials and the Askey-Wilson integral*, European J. Combin. **8** (1987), no. 4, 379–392. MR 89h:33015

[KSZ11] Anisse Kasraoui, Dennis Stanton, and Jiang Zeng, *The combinatorics of Al-Salam-Chihara q-Laguerre polynomials*, Adv. in Appl. Math. **47** (2011), no. 2, 216–239.

[KSZ06] Dongsu Kim, Dennis Stanton, and Jiang Zeng, *The combinatorics of the Al-Salam-Chihara q-Charlier polynomials*, Sém. Lothar. Combin. **54** (2005/06), Art. B54i, 15 pp. (electronic). MR 2223031 (2007b:05024)

[KZ01] Dongsu Kim and Jiang Zeng, *A combinatorial formula for the linearization coefficients of general Sheffer polynomials*, European J. Combin. **22** (2001), no. 3, 313–332. MR 1822720 (2001k:05023)

[KMS07] Timothy Kusalik, James A. Mingo, and Roland Speicher, *Orthogonal polynomials and fluctuations of random matrices*, J. Reine Angew. Math. **604** (2007), 1–46. MR 2320312

[Nic10] Alexandru Nica, *Non-crossing linked partitions, the partial order $\ll$ on $\mathrm{NC}(n)$, and the S-transform*, Proc. Amer. Math. Soc. **138** (2010), no. 4, 1273–1285. MR 2578521

[OEIS17] OEIS Foundation Inc., *The on-line encyclopedia of integer sequences*, 2017, http://oeis.org.

[PT11] Giovanni Peccati and Murad S. Taqqu, *Wiener chaos: moments, cumulants and diagrams*, Bocconi & Springer Series, vol. 1, Springer, Milan; Bocconi University Press, Milan, 2011, A survey with computer implementation, Supplementary material available online. MR 2791919 (2012d:60157)

[RW97] Gian-Carlo Rota and Timothy C. Wallstrom, *Stochastic integrals: a combinatorial approach*, Ann. Probab. **25** (1997), no. 3, 1257–1283. MR 98m:60081

[SS94] R. Simion and D. Stanton, *Specializations of generalized Laguerre polynomials*, SIAM J. Math. Anal. **25** (1994), no. 2, 712–719. MR 1266585

[Śni00] Piotr Śniady, *Quadratic bosonic and free white noises*, Comm. Math. Phys. **211** (2000), no. 3, 615–628. MR 1773810 (2001i:81148)

[Spe98] Roland Speicher, *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory*, Mem. Amer. Math. Soc. **132** (1998), no. 627, x+88. MR 1407898 (98i:46071)

[Wes95] Julian West, *Generating trees and the Catalan and Schröder numbers*, Discrete Math. **146** (1995), no. 1-3, 247–262. MR 1360119

[YY09] Fujine Yano and Hiroaki Yoshida, *A representation of linked partitions and continued fractions*, unpublished, 2009?

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