ON A LUSCHNY QUESTION

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Abstract. Let $E_n(x)$ be Euler polynomial, $\nu_2(n)$ be 2−adic order of n, $\{g(n)\}$ be the characteristic sequence for $\{2^n-1\}_{n\geq 1}$. Recently Peter Luschny asked (cf. [5], sequence A135517): is A135517(n) the denominator of $E_n(x)-E_n(1)$? According to a formula in A091090, this question is equivalent to the following one: is the denominator of $E_n(x)-E_n(1)$ equal to $2^{\nu_2(n+1)-g(n)}$? In this note we answer this question in the affirmative.

1. Introduction

Let $E_n(x)$ be Euler polynomial, $\nu_2(n)$ be 2−adic order of n, $\{g(n)\}$ be the characteristic sequence for $\{2^n-1\}_{n\geq 1}$. Recently Peter Luschny asked (cf. [5], sequence A135517): is A135517(n) the denominator of $E_n(x)-E_n(1)$? According to a formula in A091090, this question is equivalent to the following one: is the denominator of $E_n(x)-E_n(1)$ equal to $2^{\nu_2(n+1)-g(n)}$? In this note we answer this question in the affirmative. Our proof is based on finding a simple explicit expression for the coefficients of Euler polynomial.

Remark 1. *Note that Peter Luschny published in OEIS sequence* A290646 *in which he for the first time asked his question: "Is A290646 = A135517?" and in the version 2 of this note we referred to* A290646. *But after publication of version 2, the sequence* A290646 *became a replacement sequence with no relation to this topic. Therefore, in order not to cause any inconvenience to the readers, we are forced to give this third version.*

2. Several classic formulas and theorems

Euler polynomials $E_n(x)$ are defined by generating function

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \tag{1}$$

Below we use several known relations [1]

$$(-1)^nE_n(-x)=2x^n-E_n(x); \tag{2}$$

$$E_n(0)=-E_n(1)=-\frac{2}{n+1}(2^{n+1}-1)B_{n+1},\ \ n=1,2,..., \tag{3}$$

1991 *Mathematics Subject Classification.* 11B68; keywords and phrases: Euler polywhere $\{B_n\}$ are Bernoulli numbers;

$$E'_n(x) = nE_{n-1}(x). \tag{4}$$

We use also the formula which is obtained by combining formulas (14) and (18) in [2] (see also [6]):

$$B_n = \frac{n}{2(2^n-1)}\sum_{j=0}^{n-1}(-1)^j S(n, j+1)\frac{j!}{2^j}, \tag{5}$$

where $\{S(n,j)\}$ are the Stirling numbers of the second kind.

Further recall that, according to Von Staudt-Clausen theorem [4], we have

$$B_{2n} = I_n - \sum \frac{1}{p}, \tag{6}$$

where I_n is an integer, $\{p\}$ are primes for which $p-1$ divides $2n$.
Finally, denote by $t(n,k)$ the number of carries which appear in addition k and $n-k$ in base 2, or, the same, in subtracting k from n. Then, by Kummer's known theorem (cf.[3]),

$$2^{t(n,k)} || \binom{n}{k}, \tag{7}$$

i.e., $t(n,k)$ is 2-adic order of $\binom{n}{k}$.

3. Explicit formula for coefficients of $E_n(x)$

Let

$$E_n(x) = e_0(n)x^n + e_1(n)x^{n-1} + e_2(n)x^{n-2} + ... + e_{n-1}(n)x + e_n(n).$$

Using (2), we immediately find

$$e_0(n) = 1, e_2(n) = e_4(n) = ... = 0.$$

So, we have

$$E_n(x) = x^n + \sum_{odd\ \ k=1,\dots,n} e_k(n)x^{n-k}. \tag{8}$$

Further, by (4), $e_k(n)$ satisfies the difference equation

$$e_k(n) = \frac{n}{n-k}e_k(n-1). \tag{9}$$

It is easy to see that the solution of (9) is

$$e_k(n) = C_k\binom{n}{k}. \tag{10}$$

Firstly, let us find $e_k(n)$ for odd n. Then by (3)

$$e_n(n) = C_n = E_n(0) = -\frac{2}{n+1}(2^{n+1}-1)B_{n+1}.$$

So, by (10) for odd n we find

$$e_k(n) = -\frac{2}{k+1}(2^{k+1}-1)B_{k+1}\binom{n}{k}. \tag{11}$$

Now let n be even. Let us show that the formula for $e_k(n)$ does not change.

Indeed, again by (4), we have

$$(n+1)E_n(x) = E'_{n+1}(x).$$

So, by (9) and (11), we have

$$x^n + \sum_{odd\ \ k=1,\dots,n} e_k(n)x^{n-k} =$$

$$\frac{1}{n+1}((n+1)x^n + \sum_{odd\ \ k=1,\dots,n+1} C_k\binom{n+1}{k}(n+1-k)x^{n-k}) =$$

$$x^n + \sum_{odd\ \ k=1,\dots,n-1} C_k\binom{n+1}{k}\frac{n+1-k}{n+1}x^{n-k}.$$

Hence, for even n we find

$$e_k(n) = -\frac{2}{k+1}(2^{k+1}-1)B_{k+1}\binom{n}{k}, \tag{12}$$

that coincides with (11).

Let x be a rational number. Below we denote by $N(x)$ the numerator of x and by $D(x)$ the denominator of x, such that $N(x)$ and $D(x)$ are relatively prime.

Now note, that by (7), $D(B_{k+1})$ (k is odd) is an even square-free number, while $N(B_{k+1})$ is odd. Hence, $(2^{k+1}-1)N(2B_{k+1})$ is odd number. Finally, by (11)-(12) and (5) we have (for odd k):

$$e_k(n) = -\binom{n}{k}\sum_{j=0}^{k}(-1)^j S(k+1,j+1)\frac{j!}{2^j}. \tag{13}$$

This yields that $D(e_k(n))$ could be only a power of 2. This means for (11)-(12), that $D(e_k(n))$ is really

$$D(e_k(n)) = 2^{\nu_2(k+1)-\nu_2(\binom{n}{k})}, \tag{14}$$

where $\nu_2(n)$ is 2−adic order of n.

Add that, since $sign(B_{2n}) = (-1)^{n-1}$, then, by (11)-(12), $sign(e_k(n)) = (-1)^{\frac{k+1}{2}}$.

4. Answer in the affirmative on the Peter Luschny question

Let, according to the question,

$$E_n^*(x) = E_n(x) - E_n(1).$$

By (3),

$$E_n^*(x) = E_n(x) + E_n(0). \tag{15}$$

In case of odd n, when $E_n(0) \neq 0$, $E_n^*(0) = 2E_n(0)$. So, the formula (14) for the corresponding coefficients $e_k^*(n)$ takes the form

$$D(e_k^*(n)) = 2^{\nu_2(k+1) - \nu_2(\binom{n}{k}) - \delta_{n,k}}, \tag{16}$$

where $\delta_{n,k} = 1$, if $k = n$, and $\delta_{n,k} = 0$, otherwise. Further, by (16), we have

$$D(e_k^*(n)) = 2^{\nu_2((k+1)/\binom{n}{k}) - \delta_{n,k}} = 2^{\nu_2((n+1)/\binom{n+1}{k+1}) - \delta_{n,k}} =$$
$$2^{\nu_2(n+1) - \nu_2(\binom{n+1}{k+1}) - \delta_{n,k}}. \tag{17}$$

Now, by (7), we have

$$D(E_n^*(x)) = 2^{\max(odd\ \ k=1,\dots,n)(\nu_2(n+1) - t(n+1,k+1) - \delta_{n,k})} =$$
$$2^{\nu_2(n+1) - \min(odd\ \ k=1,\dots,n)(t(n+1,k+1) + \delta_{n,k})}. \tag{18}$$

Let firstly $n = 2^m - 1,\ \ m \geq 1$. Then in (18) we obtain the minimum in case $k = n$ when $t(n+1, k+1) = 0$. So, since $delta(n, n) = 1$, the minimum is $1 = g(n)$. So, by (18),

$$D(E_n^*(x)) = 2^{\nu_2(n+1) - g(n)}.$$

Let now a positive n have not a form $2^m - 1$. Let us show that in this case the minimum in (18) is 0. Indeed, take $k = 2^{u(n)-1} - 1$, where $u(n)$ is the number of $(0,1)-$digits in the binary expansion of n. Then $k < n$ and evidently $t(n+1, k+1) = 0$. Since also $\delta(n, k) = 0$, then the minimum in (18) is 0. So,

$$D(E_n^*(x)) = 2^{\nu_2(n+1)}$$

and since here $g(n) = 0$, we complete the proof.

Acknowledgement. The author thanks Jean-Paul Allouche for sending article [2] and for useful discussions.

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