MAXIMAL INDEPENDENT SETS ON A GRID GRAPH

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ABSTRACT. An independent vertex set of a graph is a set of vertices of the graph in which no two vertices are adjacent, and a maximal independent set is one that is not a proper subset of any other independent set. In this paper we count the number of maximal independent sets of vertices on a complete rectangular grid graph. More precisely, we provide a recursive matrix-relation producing the partition function with respect to the number of vertices. The asymptotic behavior of the maximal hard square entropy constant is also provided. We adapt the state matrix recursion algorithm, recently invented by the author to answer various two-dimensional regular lattice model problems in enumerative combinatorics and statistical mechanics.

1. INTRODUCTION

In graph theory, many problems involve subsets of the vertices of a graph that satisfy certain restrictions based on the adjacency relations within the graphs [10, 11]. Among them, counting all maximal independent sets of a given graph is one that has attracted considerable attention. In a graph G, an *independent vertex set* is a subset S of its vertex set $V(G)$ such that there is no edge of G between any two vertices of S. A *maximal independent set* (MIS) is an independent vertex set that is not a proper subset of any other independent vertex set. In other words, it is a set S such that every edge of the graph has at least one endpoint not in S and every vertex not in S has at least one neighbor in S.

Erdös and Moser raised the problem of determining the maximum value of the number of MISs in a general graph with n vertices and those graphs having this maximum value. Moon and Moser [12] presented that a graph can have at most $3^{n/3}$ MISs and that there are graphs achieving this many. Later Griggs, Grinstead and Guichard [5] improved this result for connected graphs. This problem has been extensively studied for various classes of graphs, including trees [18, 22] and graphs with at most r cycles [4, 19].

Recently several significant enumeration problems regarding various combinatorial objects on the $m \times n$ grid graph were solved by means of the *state matrix recursion algorithm*, originated from [15] and later developed by the author [13]. As one of the most interesting applications, this algorithm provides a recursive matrix-relation producing the exact number of independent vertex sets on the $m \times n$ grid graph in the preceding paper [17]. This is well known as the Hard Square Problem or the Merrifield-Simmons index [10, 11]. This index is an important topological index for the study

of the relation between molecular structure and physical/chemical properties of certain hydrocarbon compounds, such as the correlation with boiling points [6]. A good summary of results on the Merrifield-Simmons index of graphs can be found in the survey paper [21].

In this paper, we apply the state matrix recursion algorithm to calculate the number of MISs on the $m \times n$ rectangular grid graph $\mathbb{Z}_{m \times n}$ that is the most interesting two-dimensional regular lattice. A MIS on $\mathbb{Z}_{m \times n}$ is drawn in Figure 1 (a). An independent vertex set is often represented by a hard square lattice gas with nearest-neighbor exclusion. In a MIS, all vertices must be covered by hard squares as in Figure 1 (b). Up to now there are only partial results [2, 3] on counting MISs in $\mathbb{Z}_{m \times n}$.

$$
\begin{array}{ccccc}
0 & 2 & 1 & 0 & 2 \\
3 & 0 & 1 & 3 & 0 \\
0 & 2 & 2 & 0 & 2 \\
2 & 1 & 0 & 2 & 1 \\
0 & 1 & 1 & 1 & 0
\end{array}
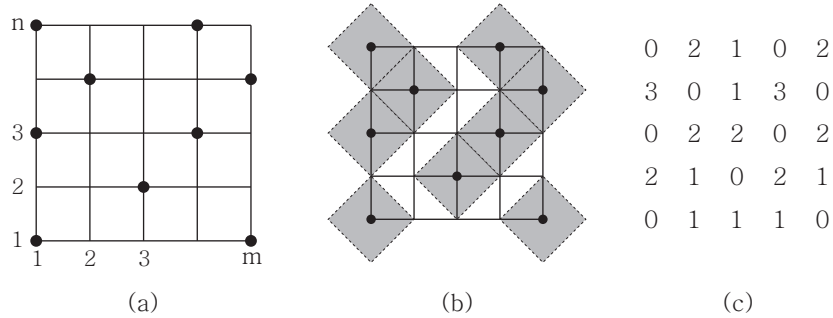$$

(a)   (b)   (c)

FIGURE 1. (a) A MIS on $\mathbb{Z}_{m \times n}$. (b) A maximal hard square lattice gas. (c) Another 0, 1, 2, 3, 4 array form.

The partition function of MISs at activity z on $\mathbb{Z}_{m \times n}$ is defined by

$$
P_{m \times n}(z) = \sum k(t) \, z^t,
$$

where $k(t)$ is the number of MISs consisting of t vertices. Then, the number of MISs is

$$
\sigma(\mathbb{Z}_{m \times n}) = P_{m \times n}(1).
$$

It is of interest to note that $\sigma(\mathbb{Z}_{m \times n})$ is indeed the number of $m \times n$ arrays of five digits 0, 1, 2, 3, 4 where each entry equals the number of its horizontal and vertical zero neighbors, where 0's are located at the place of a MIS as in Figure 1 (c). The numbers of $\sigma(\mathbb{Z}_{m \times n})$ are registered in Sloane's On-Line Encyclopedia of Integer Sequences [20], namely A197054.

By virtue of the state matrix recursion algorithm, we present a recursive formula for this partition function. Hereafter $\mathbb{O}_k$ denotes the $3^k \times 3^k$ zero-matrix, and $\overset{m}{\otimes} A$ denotes the m-fold tensor product[1] of a matrix A.

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[1] The matrix describing the tensor product $A \otimes B$ is the Kronecker product of the two matrices. For example,

$$
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} & 2 \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} \\ 3 \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} & 4 \cdot \begin{bmatrix} 5 \\ 6 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 & 2 \cdot 5 \\ 1 \cdot 6 & 2 \cdot 6 \\ 3 \cdot 5 & 4 \cdot 5 \\ 3 \cdot 6 & 4 \cdot 6 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 6 & 12 \\ 15 & 20 \\ 18 & 24 \end{bmatrix}.
$$

Theorem 1. *The partition function $P_{m \times n}(z)$ is the unique entry of the 1×1 matrix*

$$D_m \cdot (A_m + B_m)^n \cdot E_m,$$

where A_m and B_m are $3^m \times 3^m$ matrices recursively defined by

$$A_{k+1} = \begin{bmatrix} \mathbb{O}_k & \mathbb{O}_k & z\,C_k \\ \mathbb{O}_k & \mathbb{O}_k & \mathbb{O}_k \\ \mathbb{O}_k & \mathbb{O}_k & \mathbb{O}_k \end{bmatrix}, \quad B_{k+1} = \begin{bmatrix} \mathbb{O}_k & \mathbb{O}_k & \mathbb{O}_k \\ A_k + B_k & A_k & \mathbb{O}_k \\ A_k + B_k & A_k + B_k & \mathbb{O}_k \end{bmatrix}$$

$$and \quad C_{k+1} = \begin{bmatrix} \mathbb{O}_k & \mathbb{O}_k & \mathbb{O}_k \\ A_k + B_k & A_k + B_k & \mathbb{O}_k \\ A_k + B_k & A_k + B_k & \mathbb{O}_k \end{bmatrix},$$

for $k = 0, \ldots, m-1$, starting with $A_0 = \begin{bmatrix} 0 \end{bmatrix}$ and $B_0 = C_0 = \begin{bmatrix} 1 \end{bmatrix}$, and D_m and E_m are respectively 1×3^m and $3^m \times 1$ matrices defined by

$$D_m = \overset{m}{\otimes} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \quad and \quad E_m = \overset{m}{\otimes} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

This partition function gives the following significant consequences: The lowest degree of $P_{m \times n}(z)$ indicates the minimum number of vertices to produce a MIS, and its coefficient is the number of MISs with fewest set.

Note that the square matrices A_m and B_m are exponentially large; this is exactly what one would expect just from applying the standard transfer matrix method used for other lattice models. But these matrices are significantly smaller than what would be obtained from naive application of the transfer matrix method (See Table 1).

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
$m=1$	1							
$m=2$	2	2						
$m=3$	2	4	10					
$m=4$	3	6	18	42				
$m=5$	4	10	38	108	358			
$m=6$	5	16	78	274	1132	4468		
$m=7$	7	26	156	692	3580	17742	88056	
$m=8$	9	42	320	1754	11382	70616	439338	2745186
$m=9$	12	68	654	4442	36270	281202	2192602	17155374
$m=10$	16	110	1326	11248	114992	1117442	10912392	106972582

TABLE 1. List of the exact numbers of $\sigma(\mathbb{Z}_{m \times n})$

We are turning now to the growth rate per vertex of the number of MISs $\sigma(\mathbb{Z}_{m \times n})$ as defined by

$$\lim_{m,\,n \to \infty} (\sigma(\mathbb{Z}_{m \times n}))^{\frac{1}{mn}} = \kappa.$$

We call this limit κ the *maximal hard square entropy constant*. A two-dimensional application of Fekete's lemma gives a mathematical proof of the existence of the limit.

Theorem 2. *The maximal hard square entropy constant κ exists. More precisely,*

$$\kappa = \sup_{m,\,n \geq 1} \left(\sigma(\mathbb{Z}_{m \times n}) \right)^{\frac{1}{(m+1)(n+1)}}.$$

We approximate the maximal hard square entropy constant as little greater than $1.225084\cdots$ from $\sigma(\mathbb{Z}_{8 \times 380}) = 2.0932\cdots \times 10^{302}$.

2. State matrix recursion algorithm

In this section we prove Theorem 1 by means of the state matrix recursion algorithm. This algorithm is divided into three stages. The first stage is devoted to the installation of the mosaic system for MISs on the grid graph. Note that the original construction of the mosaic system for quantum knots was invented by Lomonaco and Kauffman to represent an actual physical quantum system [9]. Recently, the author *et al.* have developed a state matrix argument for knot mosaic enumeration in a series of papers [7, 8, 14, 15, 16]. The state matrix recursion algorithm is a well-formalized version of this argument to answer various two-dimensional square lattice model problems in enumerative combinatorics and statistical mechanics [13, 17]. In the second stage, we find recursive matrix-relations producing the exact enumeration. Our proofs of two lemmas in this stage parallel those of Lemmas 3 and 4 in [17], with slight modification to MISs. In the third stage, we analyze the state matrix obtained in the second stage to complete the proof.

2.1. **Stage 1: Conversion to the MIS mosaic system.**

In this paper, we consider the sixteen *mosaic tiles* illustrated in Figure 2. Their side edges are labeled with three letters a, b and c. Note that the dot in the first mosaic tile T_1 indicates a vertex in a MIS. In detail, T_1 has four side edges labeled with one letter a only. All the other fifteen mosaic tiles have side edges labeled with letters b and c, except that four b's are not allowed.
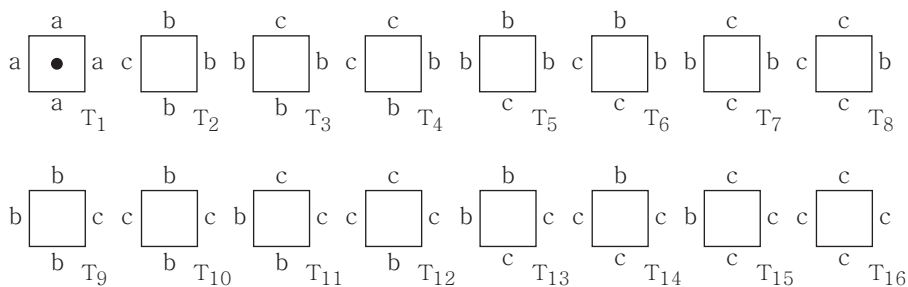

FIGURE 2. Sixteen mosaic tiles labeled with three letters

For positive integers m and n, an $m \times n$–*mosaic* is an $m \times n$ rectangular array $M = (M_{ij})$ of those tiles, where M_{ij} denotes the mosaic tile placed at the ith column from 'left' to 'right' and the jth row from 'bottom' to 'top'. We are exclusively interested in mosaics whose tiles match each other properly to represent MISs. For this purpose we consider the following rules.

Adjacency rule (abc–cba type): Adjacent edges of adjacent mosaic tiles in a mosaic must be labeled with any of the following pairs of letters: a/c, b/b.

Boundary state requirement: All boundary edges in a mosaic are labeled with letters a and b (but, not c).

The meaning of the abc–cba type is that if the labeled state of one side of a pair of adjacent edges is a (b or c), then the other side is c (b or a, respectively). As illustrated in Figure 3, every MIS in $\mathbb{Z}_{m \times n}$ can be converted into an $m \times n$–mosaic which satisfies the two rules. According to the adjacency rule, we avoid putting two mosaic tiles T_1 next to each other (to be an independent vertex set), and guarantee that each of fifteen mosaic tiles $T_2 \sim T_{16}$ has at least one T_1 as neighbors (to be maximal).

FIGURE 3. Conversion of the MIS in Figure 1 to a MIS $m \times n$–mosaic

A mosaic is said to be *suitably adjacent* if any pair of mosaic tiles sharing an edge satisfies the adjacency rule. A suitably adjacent $m \times n$–mosaic is called a *MIS $m \times n$–mosaic* if it additionally satisfies the boundary state requirement. The boundary state requirement guarantees the uniqueness of a MIS $m \times n$–mosaic representing a given MIS. The following one-to-one conversion arises naturally.

One-to-one conversion: There is a one-to-one correspondence between MISs on $\mathbb{Z}_{m \times n}$ and MIS $m \times n$–mosaics. Furthermore, the number of vertices in a MIS is equal to the number of T_1 mosaic tiles in the corresponding MIS $m \times n$–mosaic.

2.2. **Stage 2: State matrix recursion formula.**

Let $p \leq m$ and $q \leq n$ be positive integers. Consider a suitably adjacent $p \times q$–mosaic M, possibly labeled c on boundary edges. We use $v(M)$ to denote the number of appearances of T_1 tiles in M. A *state* is a finite sequence of three letters a, b and c. The *b–state* $s_b(M)$ (*t–state* $s_t(M)$) is the state of length p obtained by reading off letters on the bottom (top, respectively) boundary edges from right to left, and the *l–state* $s_l(M)$ (*r–state* $s_r(M)$) is the state of length q on the left (right, respectively) boundary edges from top to bottom. For example, the MIS 5×5–mosaic drawn in

Figure 3 has four state indications: s_b = abbba, s_t = babba, s_l = ababa, and s_r = babba.

Given a triple $\langle s_r, s_b, s_t \rangle$ of r–, b– and t–states, we associate the *state polynomial*:

$$Q_{\langle s_r, s_b, s_t \rangle}(z) = \sum k(v)\, z^v,$$

where $k(v)$ is the number of all suitably adjacent $p \times q$–mosaics M such that $v(M) = v$, $s_r(M) = s_r$, $s_b(M) = s_b$, $s_t(M) = s_t$ and $s_l(M)$ is any state of length q consisting of only two letters a and b. The last condition for $s_l(M)$ is due to the left boundary state requirement.

Now we focus on mosaics of width 1. Consider a suitably adjacent $p \times 1$–mosaic ($p \leq m$), which is called a *bar mosaic*. Bar mosaics of length p have 3^p kinds of b– and t–states, especially called *bar states*. We arrange all bar states in two ways as follows: for example if $p = 2$, the abc-ordered state set as aa, ab, ac, ba, bb, bc, ca, cb and cc (the lexicographic order), and the cba-ordered state set as cc, cb, ca, bc, bb, ba, ac, ab and aa (the reverse lexicographic order). For $1 \leq i \leq 3^p$, let ϵ_i^p and λ_i^p denote the ith bar states of length p among the abc- and cba-ordered state sets, respectively.

The *bar state matrix* X_p ($X = A, B, C$) for the set of suitably adjacent bar mosaics of length p is a $3^p \times 3^p$ matrix (m_{ij}) given by

$$m_{ij} = Q_{\langle \mathrm{x}, \epsilon_i^p, \lambda_j^p \rangle}(z)$$

where x = a, b, c, respectively. We remark that information on suitably adjacent bar mosaics is completely encoded in the three bar state matrices A_p, B_p and C_p.

Lemma 3. *The bar state matrices A_p, B_p and C_p are obtained by the recurrence relations:*

$$A_{k+1} = \begin{bmatrix} \mathbb{O}_k & \mathbb{O}_k & z\,C_k \\ \mathbb{O}_k & \mathbb{O}_k & \mathbb{O}_k \\ \mathbb{O}_k & \mathbb{O}_k & \mathbb{O}_k \end{bmatrix}, \quad B_{k+1} = \begin{bmatrix} \mathbb{O}_k & \mathbb{O}_k & \mathbb{O}_k \\ A_k + B_k & A_k & \mathbb{O}_k \\ A_k + B_k & A_k + B_k & \mathbb{O}_k \end{bmatrix}$$

$$\text{and} \quad C_{k+1} = \begin{bmatrix} \mathbb{O}_k & \mathbb{O}_k & \mathbb{O}_k \\ A_k + B_k & A_k + B_k & \mathbb{O}_k \\ A_k + B_k & A_k + B_k & \mathbb{O}_k \end{bmatrix}$$

with seed matrices $A_1 = \begin{bmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ *and* $C_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

Note that we may start with matrices $A_0 = \begin{bmatrix} 0 \end{bmatrix}$ and $B_0 = C_0 = \begin{bmatrix} 1 \end{bmatrix}$ instead of A_1, B_1 and C_1.

Proof. The following proof parallels the inductive proof of [17, Lemma 3] with slight modification. By observing the sixteen mosaic tiles, we find the first bar state matrices A_1, B_1 and C_1 as in the lemma. For example, $(1, 3)$-entry of A_1 is

$$Q_{\langle \mathrm{a}, \epsilon_1^1, \lambda_3^1 \rangle}(z) = Q_{\langle \mathrm{a}, \mathrm{a}, \mathrm{a} \rangle}(z) = z$$

since exactly one mosaic tile T_1 satisfies this requirement.

Assume that the bar state matrices A_k, B_k and C_k satisfy the statement. Consider the matrix B_{k+1}, which is of size $3^{k+1} \times 3^{k+1}$. Partition this matrix

into nine block submatrices of size $3^k \times 3^k$, and consider the 21-submatrix of B_{k+1} i.e., the $(2,1)$-component in the 3×3 array of the nine blocks. Due to the abc- and cba-orders, the (i,j)-entry of the 21-submatrix is the state polynomial $Q_{\langle \mathrm{b}, \mathrm{b}\epsilon_i^k, \mathrm{c}\lambda_j^k \rangle}(z)$ where $\mathrm{b}\epsilon_i^k$ (similarly $\mathrm{c}\lambda_j^k$) is a bar state of length $k+1$ obtained by concatenating two bar states b and ϵ_i^k. A suitably adjacent $(k+1) \times 1$–mosaic corresponding to this triple $\langle \mathrm{b}, \mathrm{b}\epsilon_i^k, \mathrm{c}\lambda_j^k \rangle$ must have a tile either T_3 or T_4 at the place of the rightmost mosaic tile, and so its second rightmost tile must have r–state b or a, respectively, by the adjacency rule. By considering the contribution of the rightmost tile T_3 or T_4 to the state polynomial, one easily gets

$$Q_{\langle \mathrm{b}, \mathrm{b}\epsilon_i^k, \mathrm{c}\lambda_j^k \rangle}(z) = (i,j)\text{-entry of } (A_k + B_k).$$

Thus the 21-submatrix of B_{k+1} is $A_k + B_k$. Using the same argument, we derive Table 2 presenting all possible twenty seven cases as desired. □

	Submatrix for $\langle s_r, s_b, s_t \rangle$	Rightmost tile	Submatrix
A_{k+1}	13-submatrix $\langle \mathrm{a}, \mathrm{a}\cdots, \mathrm{a}\cdots \rangle$	T_1	$z\,C_k$
B_{k+1}	21-submatrix $\langle \mathrm{b}, \mathrm{b}\cdots, \mathrm{c}\cdots \rangle$	T_3, T_4	$A_k + B_k$
	22-submatrix $\langle \mathrm{b}, \mathrm{b}\cdots, \mathrm{b}\cdots \rangle$	T_2	A_k
	31-submatrix $\langle \mathrm{b}, \mathrm{c}\cdots, \mathrm{c}\cdots \rangle$	T_7, T_8	$A_k + B_k$
	32-submatrix $\langle \mathrm{b}, \mathrm{c}\cdots, \mathrm{b}\cdots \rangle$	T_5, T_6	$A_k + B_k$
C_{k+1}	21-submatrix $\langle \mathrm{c}, \mathrm{b}\cdots, \mathrm{c}\cdots \rangle$	T_{11}, T_{12}	$A_k + B_k$
	22-submatrix $\langle \mathrm{c}, \mathrm{b}\cdots, \mathrm{b}\cdots \rangle$	T_9, T_{10}	$A_k + B_k$
	31-submatrix $\langle \mathrm{c}, \mathrm{c}\cdots, \mathrm{c}\cdots \rangle$	T_{15}, T_{16}	$A_k + B_k$
	32-submatrix $\langle \mathrm{c}, \mathrm{c}\cdots, \mathrm{b}\cdots \rangle$	T_{13}, T_{14}	$A_k + B_k$
	The other 18 cases	None	$\mathbb{O}_k$

TABLE 2. 27 submatrices of A_{k+1}, B_{k+1} and C_{k+1}

Now we extend to mosaics of any width. The *state matrix* $Y_{m \times q}$ for the set of suitably adjacent $m \times q$–mosaics ($q \leq n$) is a $3^m \times 3^m$ matrix (y_{ij}) given by

$$y_{ij} = \sum Q_{\langle s_r, \epsilon_i^m, \lambda_j^m \rangle}(z),$$

where the summation is taken over all r–states s_r of length q consisting of only two letters a and b. The summation condition of s_r is due to the right boundary state requirement.

Lemma 4. *The state matrix $Y_{m \times n}$ is obtained by*

$$Y_{m \times n} = (A_m + B_m)^n.$$

Proof. The following proof parallels the inductive proof of [17, Lemma 4] with slight modification. For $n = 1$, $Y_{m \times 1} = A_m + B_m$ since $Y_{m \times 1}$ counts suitably adjacent $m \times 1$–mosaics with l– and r–states consisting of letters a or b. Assume that $Y_{m \times k} = (A_m + B_m)^k$. Consider a suitably adjacent $m \times (k+1)$–mosaic $M^{m \times (k+1)}$ with r– and l–states consisting of letters a and b. Split it into two suitably adjacent $m \times k$– and $m \times 1$–mosaics $M^{m \times k}$ and $M^{m \times 1}$ by tearing off the topmost bar mosaic. There is a certain relation

between the t–state of $M^{m \times k}$ and the b–state of $M^{m \times 1}$ as shown in Figure 4. To satisfy the adjacency rule, the letters a and c are changed by c and a, respectively, from one state to the other. The key point is that, for some $r = 1, \ldots, 3^m$, $s_t(M^{m \times k}) = \lambda_r^m$ and $s_b(M^{m \times 1}) = \epsilon_r^m$.
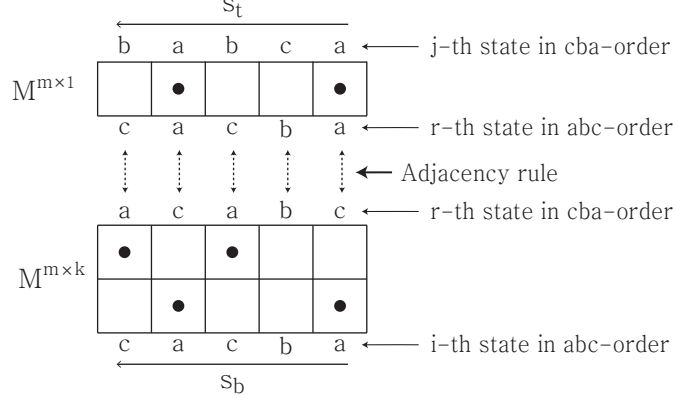

FIGURE 4. Expanding $M^{m \times k}$ to $M^{m \times (k+1)}$

Let $Y_{m \times (k+1)} = (y_{ij})$, $Y_{m \times k} = (y'_{ij})$ and $Y_{m \times 1} = (y''_{ij})$. Note that y_{ij} is the state polynomial for the set of suitably adjacent $m \times (k+1)$–mosaics M which admit splittings into $M^{m \times k}$ and $M^{m \times 1}$ satisfying $s_b(M) = s_b(M^{m \times k}) = \epsilon_i^m$, $s_t(M) = s_t(M^{m \times 1}) = \lambda_j^m$, and for some $r = 1, \ldots, 3^m$, $s_t(M^{m \times k}) = \lambda_r^m$ and $s_b(M^{m \times 1}) = \epsilon_r^m$. Obviously, all their l– and r–states consist of letters a and b. Since all 3^m kinds of bar states arise as states of these m horizontal adjacent edges,

$$y_{ij} = \sum_{r=1}^{3^m} y'_{ir} \cdot y''_{rj}.$$

This implies

$$Y_{m \times (k+1)} = Y_{m \times k} \cdot Y_{m \times 1} = (A_m + B_m)^{k+1},$$

and the induction step is finished. □

2.3. Stage 3: State matrix analyzing.

Proof of Theorem 1. The (i, j)-entry of $Y_{m \times n}$ is the state polynomial for the set of suitably adjacent $m \times n$–mosaics M with $s_b(M) = \epsilon_i^m$, $s_t(M) = \lambda_j^m$ and r– and l–states consisting of letters a and b. According to the boundary state requirement, MISs in $\mathbb{Z}_{m \times n}$ are converted into suitably adjacent $m \times n$–mosaics M with b–, t–, r– and l–states consisting of letters a and b (but not c). Thus the partition function $P_{m \times n}(z)$ is the sum of all entries of $Y_{m \times n}$ associated to b– and t–states only consisting of letters a and b.

Now we define the two matrices D_m and E_m as in Theorem 1. Obviously, $D_m \cdot Y_{m \times n}$ is the 1×3^m matrix obtained from $Y_{m \times n}$ by nullifying each ith row where the corresponding ith state in the abc-order has at least one letter of c, followed by summing each column. Again, $D_m \cdot Y_{m \times n} \cdot E_m$ is the 1×1 matrix obtained from $D_m \cdot Y_{m \times n}$ by nullifying each jth column where the corresponding jth state in the cba-order has at least one letter of c,

followed by summing the unique row. Therefore we get the partition function $P_{m \times n}(z)$ from the unique entry of $D_m \cdot Y_{m \times n} \cdot E_m$. This fact combined with Lemmas 3 and 4 completes the proof. □

3. Maximal hard square entropy constant

We will need the following result called Fekete's lemma with slight modification.

Lemma 5. [13, Lemma 7] *If a double sequence $\{a_{m,n}\}_{m,n \in \mathbb{N}}$ with $a_{m,n} \geq 1$ satisfies $a_{m_1,n} \cdot a_{m_2,n} \leq a_{m_1+m_2+1,n}$ and $a_{m,n_1} \cdot a_{m,n_2} \leq a_{m,n_1+n_2+1}$ for all m, m_1, m_2, n, n_1 and n_2, then*

$$\lim_{m,n \to \infty} (a_{m,n})^{\frac{1}{mn}} = \sup_{m,n \geq 1} (a_{m,n})^{\frac{1}{(m+1)(n+1)}},$$

provided that the supremum exists.

Proof of Theorem 2. Obviously, $\sigma(\mathbb{Z}_{m \times n})$ is at least 1 for all m, n. For any two MIS $m_1 \times n-$ and $m_2 \times n$–mosaics, we can always create a new MIS $(m_1+m_2+1) \times n$–mosaic by inserting a proper $1 \times n$–mosaic between them as in Figure 5. More precisely, we put T_1 tiles in all other places in each consecutive shaded region. Note that all tiles in the $1 \times n$–mosaic whose neither left nor right neighbors are T_1 tiles are shaded. Therefore $\sigma(\mathbb{Z}_{m_1 \times n}) \cdot \sigma(\mathbb{Z}_{m_2 \times n}) \leq \sigma(\mathbb{Z}_{(m_1+m_2+1) \times n})$, and similarly for the other index. Since we use total sixteen mosaic tiles at each site, $\sup_{m,n}(\sigma(\mathbb{Z}_{m \times n}))^{\frac{1}{(m+1)(n+1)}} \leq 16$, and now apply Lemma 5. □
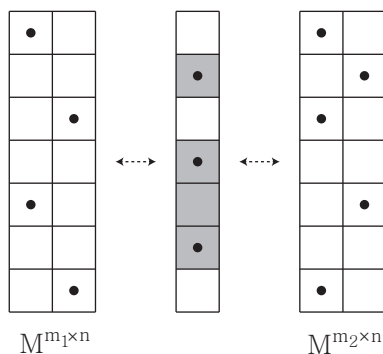

Figure 5. Adjoining two MIS mosaics

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