Winding of simple walks on the square lattice

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Abstract

A method is described to count simple diagonal walks on $\mathbb{Z}^2$ with a fixed starting point and endpoint on one of the axes and a fixed winding angle around the origin. The method involves the decomposition of such walks into smaller pieces, the generating functions of which are encoded in a commuting set of Hilbert space operators. The general enumeration problem is then solved by obtaining and explicit eigenvalue decomposition of these operators involving elliptic functions. By further restricting the intermediate winding angles of the walks to some open interval, the method can be used to count various classes of walks restricted to cones in $\mathbb{Z}^2$ of opening angles that are integer multiples of $\pi/4$.

We present three applications of this main result. First we find an explicit generating function for the walks in such cones that start and end at the origin. In the particular case of a cone of angle $3\pi/4$ these walks are directly related to Gessel's walks in the quadrant, and we provide a new proof of their enumeration. Next we study the distribution of the winding angle of a simple random walk on $\mathbb{Z}^2$ around a point in the close vicinity of its starting point, and find an intriguing probabilistic interpretation of the Jacobi elliptic functions. Finally we relate the spectrum of one of the Hilbert space operators to the enumeration of closed loops in $\mathbb{Z}^2$ with fixed winding number around the origin.

1 Introduction

Counting of lattice paths has been a major topic in combinatorics (and probability and physics) for many decades. Especially the enumeration of various types of lattice walks confined to convex cones in $\mathbb{Z}^2$, like the positive quadrant, has attracted much attention in recent years, due mainly to the rich algebraic structure of the generating functions involved (see e.g [13, 4] and references therein) and the relations with other combinatorial structures (e.g. [3, 26]). The study of lattice walks in non-convex cones has received much less attention. Notable exception are walks on the slit plane [10, 14] and the three-quarter plane [12]. When describing the plane in polar coordinates, the confinement of walks to cones of different opening angles (with the tip positioned at the origin) may equally be phrased as a restriction on the angular coordinates of the sites visited by the walk. One may generalize this concept by replacing the angular coordinate by a notion of winding angle of the walk around the origin, in which case one can even make sense of cones of angles larger than 2π. It stands to reason that a fine control over the winding angle in the enumeration of lattice walks brings us a long way in the study of walks in (especially non-convex) cones.

Although the winding angle of lattice walks seems to have received little attention in the combinatorics literature, probabilistic aspects of the winding of long random walks have been studied in considerable detail [18, 19, 28, 29]. In particular, it is known that under suitable conditions on the steps of

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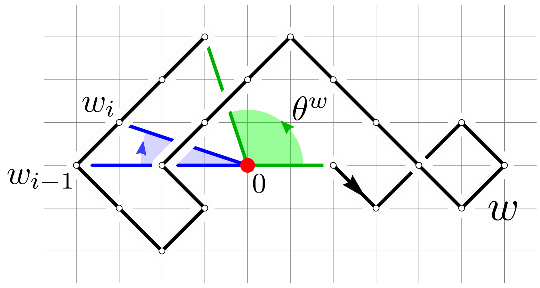


Figure 1: The winding angle of a simple diagonal walk $w \in \mathcal{W}$ of length $|w| = 19$ from $(2,0)$ to $(-1,3)$. The (full) winding angle θ^w is indicated in green, and the winding angle increment $\theta_i^w - \theta_{i-1}^w$ in blue (which is negative in this example).

the random walk the winding angle after n steps is typically of order $\log n$, and that the angle normalized by $2/\log n$ converges in distribution to a standard hyperbolic secant distribution. The methods used all rely on coupling to Brownian motion, for which the winding angle problem is easily studied with the help of its conformal properties. Although quite generally applicable in the asymptotic regime, these techniques tell us little about the underlying combinatorics.

In this paper we initiate the combinatorial study of lattice walks with control on the winding angle, by looking at various classes of simple (rectilinear or diagonal) walks on $\mathbb{Z}^2$. As we will see, the combinatorial tools described in this paper are strong enough to bridge the gap between the combinatorial study of walks in cones and the asymptotic winding of random walks. Before describing the main results of the paper, we should start with some definitions.

We let $\mathcal{W}$ be the set of *simple diagonal walks* w in $\mathbb{Z}^2$ of length $|w| \geq 0$ avoiding the origin, i.e. w is a sequence $(w_i)_{i=0}^{|w|}$ in $\mathbb{Z}^2 \setminus \{(0,0)\}$ with $w_i - w_{i-1} \in \{(1,1),(1,-1),(-1,-1),(-1,1)\}$ for $1 \leq i \leq |w|$. We define the *winding angle* $\theta_i^w \in \mathbb{R}$ *of* w *up to time* i to be the difference in angular coordinates of w_i and w_0 including a contribution 2π (resp. -2π) for each full counterclockwise (resp. clockwise) turn of w around the origin up to time i. Equivalently, $(\theta_i^w)_{i=0}^{|w|}$ is the unique sequence in $\mathbb{R}$ such that $\theta_0^w = 0$ and $\theta_i^w - \theta_{i-1}^w$ is the (counterclockwise) angle between the segments $((0,0), w_{i-1})$ and $((0,0), w_i)$ for $1 \leq i \leq |w|$. The *(full) winding angle* of w is then $\theta^w := \theta_{|w|}^w$. See Figure 1 for an example.

Main result The *Dirichlet space* $\mathcal{D}$ is the Hilbert space of complex analytic functions f on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ that vanish at 0 and have finite norm $\|f\|_{\mathcal{D}}^2$ with respect to the Dirichlet inner product

$$\langle f, g\rangle_{\mathcal{D}} = \int_{\mathbb{D}} \overline{f'(z)}\, g'(z) \mathrm{d}A(z) = \sum_{n=1}^{\infty} n\, \overline{[z^n]f(z)}\; [z^n]g(z), \qquad \mathrm{d}A(x+iy) := \frac{1}{\pi}\mathrm{d}x\mathrm{d}y.$$

See [2] for a review of its properties. We denote by $(e_n)_{n=1}^{\infty}$ the standard orthogonal basis defined by $e_n(x) := x^n$, which is unnormalized since $\|e_n\|_{\mathcal{D}}^2 = n$.

For $k \in (0,1)$ we let $v_k : \mathbb{C} \setminus \{z \in \mathbb{R} : z^2 \geq k\} \to \mathbb{C}$ be the analytic function defined by the elliptic integral

$$v_k(z) := \frac{1}{4K(k)} \int_0^z \frac{\mathrm{d}x}{\sqrt{(k-x^2)(1-kx^2)}} \tag{1}$$

along the simplest path from 0 to z, where

$$K(k) = \int_0^{\sqrt{k}} \frac{\mathrm{d}x}{\sqrt{(k-x^2)(1-kx^2)}} = \int_0^1 \frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-k^2y^2)}}$$

is the complete elliptic integral of the first kind with elliptic modulus k. For fixed k we use the conventional notation

$$k' = \sqrt{1-k^2} \qquad \text{and} \qquad k_1 = \frac{1-k'}{1+k'}$$

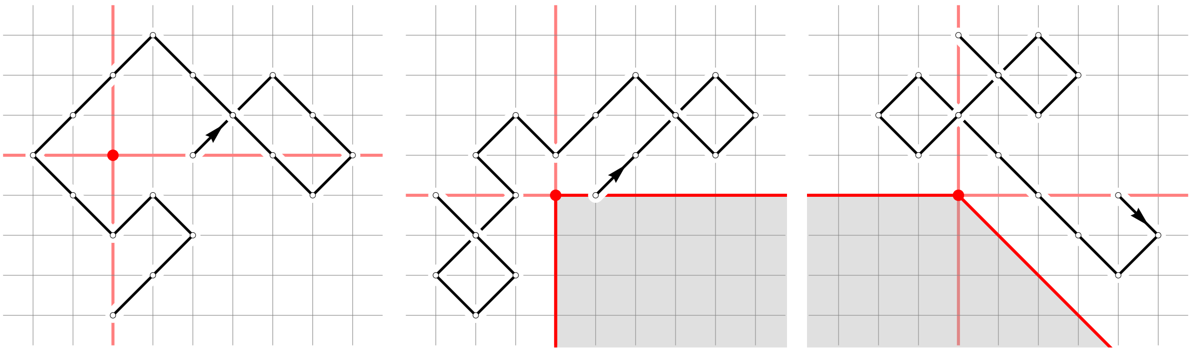
for the *complimentary modulus* k' and the *descending Landen transformation* k_1 of k. Using these we introduce a family $(f_m)_{m=1}^{\infty}$ of analytic functions by setting (notice the k_1 in $v_{k_1}(z)$!)

$$f_m(z) \coloneqq \cos(2\pi m(v_{k_1}(z) + 1/4)) - \cos(\pi m/2). \tag{2}$$

As we will see (in Proposition 2) this family forms yet another orthogonal basis of $\mathcal{D}$, with

$$\|f_m\|^2_{\mathcal{D}} = \frac{m(q_k^{-m} - q_k^{m})}{4}, \qquad \text{where} \qquad q_k \coloneqq e^{-\pi K(k')/K(k)}$$

is the *(elliptic) nome* of modulus k. The main technical result of this paper is the following.



(a) $\mathcal{W}_{4,2}^{(3\pi/2)}$ (b) $\mathcal{W}_{3,1}^{(\pi,(0,3\pi/2))}$ (c) $\mathcal{W}_{4,4}^{(\pi/2,(-\pi/4,\pi))}$

Figure 2: Examples of walks enumerated by Theorem 1.

Theorem 1. *For $l, p \geq 1$ and $\alpha \in \frac{\pi}{2}\mathbb{Z}$, let $\mathcal{W}_{l,p}^{(\alpha)}$ be the set of (possibly empty) simple diagonal walks w on $\mathbb{Z}^2 \setminus \{(0,0)\}$ that start at $(p, 0)$, end on one of the axes at distance l from the origin, and have full winding angle $\theta^w = \alpha$.*

(i) Let $W_{l,p}^{(\alpha)}(t) \coloneqq \sum_{w\in\mathcal{W}_{l,p}^{(\alpha)}} t^{|w|}$ be the generating function of $\mathcal{W}_{l,p}^{(\alpha)}$. For $k = 4t \in (0,1)$ fixed, there exists a compact self-adjoint operator $\mathbf{Y}_k^{(\alpha)}$ on $\mathcal{D}$ with eigenvectors $(f_m)_{m=1}^{\infty}$ such that

$$W_{l,p}^{(\alpha)}(t) = \langle e_l, \mathbf{Y}_k^{(\alpha)} e_p\rangle_{\mathcal{D}}, \qquad \mathbf{Y}_k^{(\alpha)} f_m = \frac{2K(k)}{\pi}\frac{1}{m} q_k^{m|\alpha|/\pi} f_m. \tag{3}$$

(ii) Let $\mathcal{W}_{l,p}^{(\alpha,I)} \subset \mathcal{W}_{l,p}^{(\alpha)}$ be the subset of the aforementioned walks that have intermediate winding angles in $I \subset \mathbb{R}$, i.e. $\theta_i^w \in I$ for $i = 1, 2, \ldots, |w| - 1$, and let $W_{l,p}^{(\alpha,I)}(t)$ be the corresponding generating function. If $I = (\beta_-, \beta_+)$ with $\beta_\pm \in \frac{\pi}{2}\mathbb{Z} \cup \{\pm\infty\}$, $\alpha \in [\beta_-, \beta_+] \cap \frac{\pi}{2}\mathbb{Z}$ and $\alpha \neq 0$ or $\alpha \neq \beta_\pm$, then the generating function $W_{l,p}^{(\alpha,I)}(t)$ is related to a matrix element of a compact self-adjoint operator on $\mathcal{D}$ with the same eigenvectors $(f_m)_{m=1}^{\infty}$, as described in the table below.

α	β_-	β_+	$W_{l,p}^{(\alpha,I)}(t)$	*Eigenvalues*
> 0	0	α	$\frac{1}{lp}\langle e_l, \mathbf{A}_k^{(\alpha)} e_p\rangle$	$\mathbf{A}_k^{(\alpha)} f_m = \frac{\pi}{2K(k)} \frac{m}{q_k^{-m\alpha/\pi} - q_k^{m\alpha/\pi}} f_m$
> 0	< 0	α	$\frac{1}{l}\langle e_l, \mathbf{J}_k^{(\alpha,\beta_-)} e_p\rangle$	$\mathbf{J}_k^{(\alpha,\beta_-)} f_m = \frac{q_k^{2m\beta_-/\pi} - 1}{q_k^{2m\beta_-/\pi} - q_k^{2m\alpha/\pi}} q_k^{m\alpha/\pi} f_m$
≥ 0	< 0	$> \alpha$	$\langle e_l, \mathbf{B}_k^{(\alpha,\beta_-,\beta_+)} e_p\rangle$	$\mathbf{B}_k^{(\alpha,\beta_-,\beta_+)} f_m = \frac{2K(k)}{\pi} \frac{q_k^{2m\beta_-/\pi} - 1}{m\, q_k^{m\alpha/\pi}} \frac{q_k^{2m\alpha/\pi} - q_k^{2m\beta_+/\pi}}{q_k^{2m\beta_-/\pi} - q_k^{2m\beta_+/\pi}} f_m$

The remaining cases follow from the symmetries $(\alpha, \beta_-, \beta_+) \to (-\alpha, -\beta_+, -\beta_-)$ and $(\alpha, \beta_-, \beta_+) \to (\alpha, \alpha - \beta_+, \alpha - \beta_-)$, and the cases $\beta_\pm = \pm\infty$ agree with the corresponding limits $\beta\pm \to \pm\infty$ (using that $q_k \in (0, 1)$).

(iii) The statement of (ii) remains valid for $\beta_\pm \in \frac{\pi}{4}\mathbb{Z} \cup \{\pm\infty\}$ and $\alpha \in [\beta_-, \beta_+] \cap \frac{\pi}{2}\mathbb{Z}$ as long as l and p are even.

For example, this theorem states that the set of simple diagonal walks from $(3, 0)$ to $(-3, 0)$ that have winding angle π around the origin has generating function

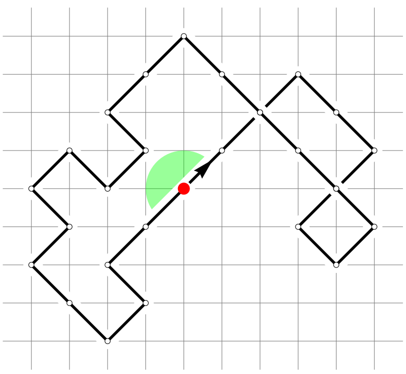
$$W^{(\pi)}_{3,3}(t) = \left\langle e_3, \mathrm{Y}^{(\pi)}_k e_3 \right\rangle_{\mathcal{D}} = \frac{2K(k)}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} q_k^m \frac{\langle e_3, f_m\rangle^2_{\mathcal{D}}}{\|f_m\|^2_{\mathcal{D}}}$$
$$= \frac{2K(k)}{\pi} \sum_{m=1}^{\infty} \frac{4}{m^2} \frac{q_k^m}{q_k^{-m} - q_k^m} \left(3[z^3]f_m(z)\right)^2 = 10\, t^6 + 280\, t^8 + 5661\, t^{10} + \cdots$$

Application: Excursions Theorem 1 can be used to count many specialized classes of walks involving winding angles. The first quite natural counting problem we address is that of the *(diagonal) excursions* $\mathcal{E}$ from the origin, i.e. $\mathcal{E}$ is the set of (non-empty) simple diagonal walks starting and ending at the origin with no intermediate returns (Figure 3a). Actually, in this case we may equally well consider simple rectilinear walks on $\mathbb{Z}^2$, thanks to the obvious linear mapping between the two types of walks (Figure 3b). Even though walks $w \in \mathcal{E}$ do not completely avoid the origin, we may still naturally assign a winding angle sequence to them by imposing that the first and last step do not contribute to the winding angle, i.e. $\theta^w_1 = \theta^w_0 = 0$ and $\theta^w = \theta^w_{|w|} = \theta^w_{|w|-1}$. In Proposition 4 we prove that the generating function for excursions with winding angle $\alpha \in \frac{\pi}{2}\mathbb{Z}$ is given (for $k = 4t \in (0, 1)$ fixed) by

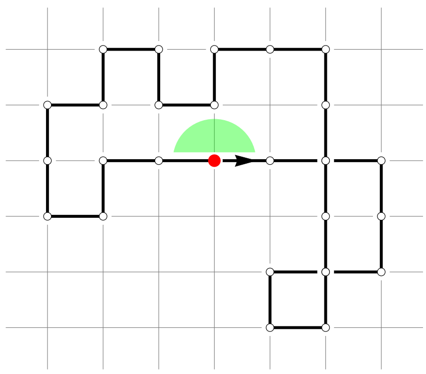
$$F^{(\alpha)}(t) := \sum_{w\in\mathcal{E}} t^{|w|} \mathbf{1}_{\{\theta^w=\alpha\}} = \frac{2\pi}{K(k)} \sum_{n=1}^{\infty} \frac{(1-q_k^n)^2}{1-q_k^{4n}} q_k^{n(\frac{2}{\pi}|\alpha|+1)}.$$

Similarly to Theorem 1(ii) one may further restrict the full winding angle sequence of w to lie in an open interval $I = (\beta_-, \beta_+)$ with $\beta_\pm \in \frac{\pi}{4}\mathbb{Z}$ such that $0 \in I$ and $\alpha \in I \cap \frac{\pi}{2}\mathbb{Z}$. In this case it is more natural to also fix the starting direction, say $w_1 = (1, 1)$, and we introduce the corresponding generating function $F^{(\alpha,I)}(t)$, such that $F^{(\alpha,\mathbb{R})}(t) = F^{(\alpha)}(t)/4$. We prove in Theorem 2 that the generating function $F^{(\alpha,I)}(t)$ is given by the finite sum

$$F^{(\alpha,I)}(t) = \frac{\pi}{8\delta} \sum_{\sigma\in(0,\delta)\cap\frac{\pi}{2}\mathbb{Z}} \left(\cos\left(\frac{4\sigma\alpha}{\delta}\right) - \cos\left(\frac{4\sigma(2\beta_+ - \alpha)}{\delta}\right)\right) F\left(t, \frac{4\sigma}{\delta}\right), \qquad \delta := 2(\beta_+ - \beta_-), \quad (4)$$



(a) Diagonal excursion.



(b) Rectilinear excursion.

Figure 3: Example of an excursion in diagonal and rectilinear form together with its winding angle.

where $F(t,b) := \sum_{\alpha\in\frac{\pi}{2}\mathbb{Z}} F^{(\alpha)}(t)\, e^{ib\alpha}$ is the "characteristic function" associated to $F^{(\alpha)}(t)$. For non-integer values of b (see Proposition 4 for the full expression) the latter can be expressed in closed form as

$$F(t,b) = \frac{1}{\cos\left(\frac{\pi b}{2}\right)}\left[1 - \frac{\pi \tan\left(\frac{\pi b}{4}\right)}{2K(k)} \frac{\theta_1'\left(\frac{\pi b}{4}, \sqrt{q_k}\right)}{\theta_1\left(\frac{\pi b}{4}, \sqrt{q_k}\right)}\right], \qquad (b \in \mathbb{R}\setminus\mathbb{Z})$$

where $\theta_1(z,q)$ is the first Jacobi theta function (see (21) for a definition). In Proposition 1 we prove that $F(t,b)$ is an algebraic power series in t for any $b \in \mathbb{Q}\setminus\mathbb{Z}$, but not for $b\in\mathbb{Z}$. By looking at the terms appearing in (4) we may thus deduce whether $F^{(\alpha,I)}(t)$ is algebraic too.

As a special case we look at the excursions that stay in the angular interval $(-\pi/4,\pi/2)$. Around 2000 Ira Gessel conjectured that the generating function for such excursions is given (in our notation) by

$$F^{(0,(-\pi/4,\pi/2))}(t) = \sum_{n=0}^{\infty} t^{2n+2}\, 16^n \frac{(5/6)_n(1/2)_n}{(2)_n(5/3)_n} = t^2 + 2t^4 + 11t^6 + 85t^8 + \cdots,$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the descending Pochhammer symbol (see also Sloane's *Online Encyclopedia of Integer sequences* (OEIS) sequence A135404). The first computer-aided proof of this conjecture appeared in [25], and it was followed by several "human" proofs in [9, 11, 4]. Here we provide an alternative proof using Theorem 2. Indeed, we have explicitly

$$F^{(0,(-\pi/4,\pi/2))}(t) = \frac{1}{4}F\left(t,\frac{4}{3}\right) = \frac{1}{2}\left[\frac{\sqrt{3}\pi}{2K(4t)}\frac{\theta_1'\left(\frac{\pi}{3},\sqrt{q_k}\right)}{\theta_1\left(\frac{\pi}{3},\sqrt{q_k}\right)} - 1\right].$$

According to our discussion above this is an algebraic power series in t, a fact about $F^{(0,(-\pi/4,\pi/2))}(t)$ that was first observed in [8]. In Corollary 3 we deduce an explicit algebraic equation for $F^{(0,(-\pi/4,\pi/2))}(t)$, and check that it agrees with a known equation for $\sum_{n=0}^{\infty} t^{2n+2}\, 16^n \frac{(5/6)_n(1/2)_n}{(2)_n(5/3)_n}$.

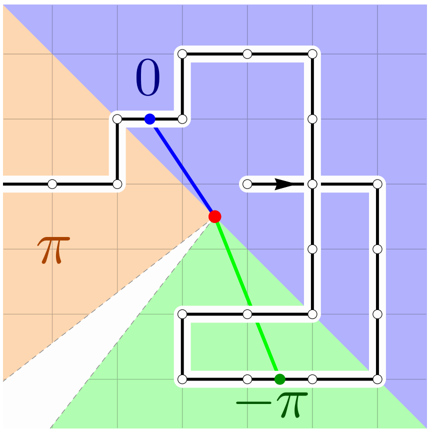
Application: unconstrained random walks Let $(W_i)_{i\geq 0}$ be a simple random walk on $\mathbb{Z}^2$ started at the origin. A natural question is to ask for the (approximate) distribution of the winding angle θ_j of the random walk around some point $(x,y)\in\mathbb{R}^2$ up to time j. As mentioned before, this question has been addressed successfully in the literature in the limit $j\to\infty$ using coupling to Brownian motion. With the help of Theorem 1 we may complement these results by deriving various exact statistics at finite j.

A particularly appealing statistic is obtained by formulating the problem in the following precise way. We take $(x,y) = (-1/2,-1/2)$, and we lower the resolution at which we measure the winding angle by rounding it to the nearest integer or half-integer multiple of π. To prevent conflicts from occurring in the rounding process it is natural to not look at the winding angle θ_j after the j'th step, but at the winding angle $\theta_{j-1/2}$ half-way the j'th step (or equivalently we may set $\theta_{j-1/2} := \frac{1}{2}(\theta_{j-1}+\theta_j)$). Finally, we replace the fixed time j by a random geometric time ζ_k with parameter $k\in(0,1)$, distributed as $j\mapsto k^j(1-k)$ on $\mathbb{Z}_{\geq 0}$. To be precise, we consider the well-defined random variables $\{\theta_{\zeta_k+1/2}\}_{\pi\mathbb{Z}+\frac{\pi}{2}}$ (see Figure 4) and $\{\theta_{\zeta_k-1/2}\}_{\pi\mathbb{Z}}$, where $\{\cdot\}_A$ means rounding to the nearest element of $A\subset\mathbb{R}$ and by convention we set $\theta_{-1/2} = 0$.

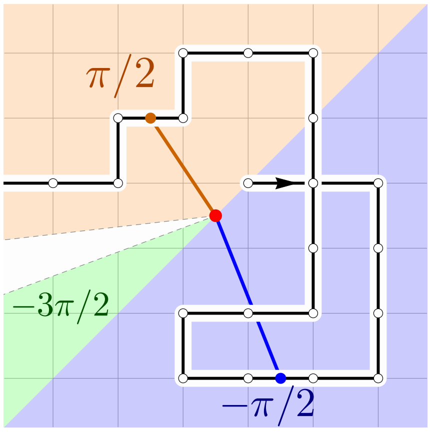
We prove that the characteristic functions of these variables are exactly given by the Jacobi elliptic functions $\mathrm{cn}(\cdot,k)$ and $\mathrm{dn}(\cdot,k)$ of modulus k (with argument normalized for correct periodicity),

$$\mathbb{E}\exp\left(ib\{\theta_{\zeta_k+1/2}\}_{\pi\mathbb{Z}+\frac{\pi}{2}}\right) = \mathrm{cn}(K(k)b,k), \qquad \mathbb{E}\exp\left(ib\{\theta_{\zeta_k-1/2}\}_{\pi\mathbb{Z}}\right) = \mathrm{dn}(K(k)b,k).$$

Since $\mathrm{cn}(y,1) = \mathrm{dn}(y,1) = \mathrm{sech}(y) = \mathbb{E}e^{iy\eta}$ is the characteristic function of a random variable η with the standard hyperbolic secant distribution with density $\frac{1}{2}\,\mathrm{sech}(\pi x/2)\mathrm{d}x$, we may directly conclude the



(a) $\{\theta_{j-1/2}\}_{\pi\mathbb{Z}}$



(b) $\{\theta_{j-1/2}\}_{\pi\mathbb{Z}+\frac{\pi}{2}}$

Figure 4: Example of an unconstrained random walk. The value of the rounded winding angle $\{\theta_{j-1/2}\}_{\pi\mathbb{Z}}$ or $\{\theta_{j-1/2}\}_{\pi\mathbb{Z}+\frac{\pi}{2}}$ around $(-\frac{1}{2},-\frac{1}{2})$ depends on which half plane of the universal cover of $\mathbb{R}^2 \setminus \{(-\frac{1}{2},-\frac{1}{2})\}$ harbors the j'th step of the walk, as illustrated by the color shading.

convergence in distribution as $k \to 1$ of the winding angle at geometric time,

$$\frac{\theta_{\zeta_k+1/2}}{K(k)} \xrightarrow[k\to 1]{(d)} \eta, \qquad\qquad (K(k) \sim -\log\sqrt{1-k^2} \quad \text{as} \quad k \to 1).$$

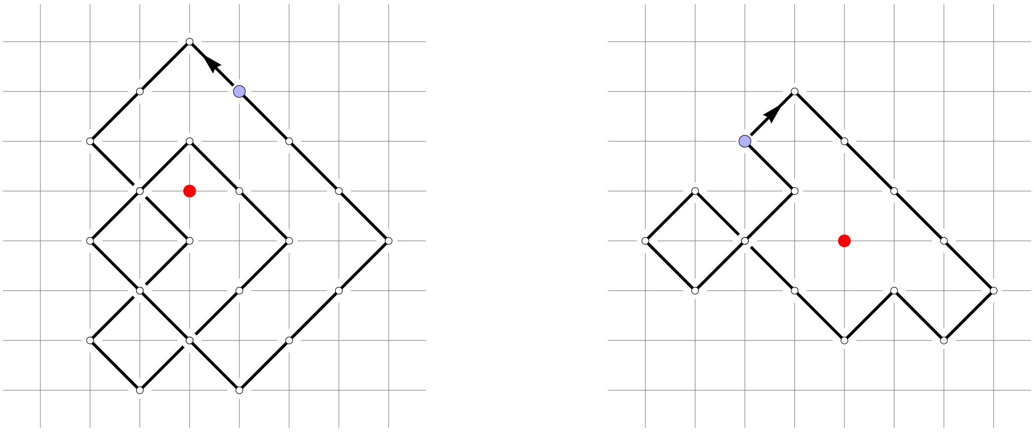
A more delicate singularity analysis yields the same distributional limit for $2\theta_{j+1/2}/\log(j)$ as $j \to \infty$ (in accordance with the probabilistic results of [18, 19, 28, 29]), but this is beyond the scope of this paper.

Application: loops The last application we discuss utilizes the fact that the eigenvalues of the operators in Theorem 1 have much simpler expressions than the components $\langle e_p, f_m\rangle_{\mathcal{D}}$ of the eigenvectors. It is therefore worthwhile to seek combinatorial interpretations of traces of (combinations of) operators, the values of which only depend on the eigenvalues. When writing out the trace in terms of the basis $(e_p)_{p=1}^{\infty}$ it is clear that such an interpretation must involve walks that start and end at arbitrary but equal distance from the origin. If the full winding angle is taken to be a multiple of 2π then such a walk forms a *loop*, i.e. it starts and ends at the same point.

A natural combinatorial set-up is described in Section 5. There we consider the set $\mathcal{L}_n$ of *rooted loops of index n*, $n \in \mathbb{Z}$, which are simple diagonal walks avoiding the origin that start and end at an arbitrary but equal point in $\mathbb{Z}^2$ and have winding angle $2\pi n$ around the origin. The set $\mathcal{L}_n = \mathcal{L}_n^{\text{even}} \cup \mathcal{L}_n^{\text{odd}}$ naturally partitions into loops that visit only sites of even respectively odd parity ($(x,y) \in \mathbb{Z}^2$ with $x+y$ even respectively odd). Theorem 4 states that the corresponding generating functions for $n > 0$ are given by

$$L_n(t) := \sum_{w\in\mathcal{L}_n} \frac{t^{|w|}}{|w|} = \frac{1}{n}\operatorname{tr}\mathbf{J}_k^{(2\pi n,-\infty)} = \frac{1}{n}\frac{q_k^{2n}}{1-q_k^{2n}}, \qquad L_n^{\text{odd}}(t) = \frac{1}{n}\frac{q_k^{2n}}{1-q_k^{4n}}, \qquad L_n^{\text{even}}(t) = \frac{1}{n}\frac{q_k^{4n}}{1-q_k^{4n}}.$$

A simple probabilistic consequence is the following. Let $(W_i)_{i=0}^{2l}$ be a simple random walk on $\mathbb{Z}^2$ started at the origin and conditioned to return after $2l$ steps. For each point $z \in \mathbb{R}^2$ we let the *index* I_z be the signed number of times $(W_i)_{i=0}^{2l}$ winds around z in counterclockwise direction, i.e. $2\pi I_z$ is the winding angle of $(W_i)_{i=0}^{2l}$ around z. If z lies on the trajectory of $(W_i)_{i=0}^{2l}$, then we set $I_z = \infty$. We let *the clusters* C_n *of index n* be the set of connected components of $\{z \in \mathbb{R}^2 : I_z = n\}$, and for $c \in C_n$ we let $|c|$



(a) $\mathcal{L}_2^{\mathrm{odd}}$ (b) $\mathcal{L}_{-1}^{\mathrm{even}}$

Figure 5: Two examples of rooted loops with different index and parity.

and $|\partial c|$ respectively be the area and boundary length of component c. Then for $n > 0$,

$$\mathbb{E}\left[\sum_{c\in C_n} |c|\right] = \frac{4^{2l}}{\binom{2l}{l}^2}\frac{2l}{n}[k^{2l}]\frac{q_k^{2n}}{1-q_k^{4n}} \sim \frac{l}{2\pi n^2},$$

$$\mathbb{E}\left[\sum_{c\in C_n} (|\partial c|-2)\right] = \frac{4^{2l}}{\binom{2l}{l}^2}\frac{4l}{n}[k^{2l}]\frac{q_k^{2n}}{1+q_k^{2n}} \sim \frac{2\pi^3 l}{\log^2 l}.$$

The first result should be compared to the analogous result for Brownian motion: Garban and Ferreras proved in [24] using Yor's work [30] that the expected area of the set of points with index n with respect to a unit time Brownian bridge in $\mathbb{R}^2$ is equal to $1/(2\pi n^2)$. Perhaps surprisingly, we find that the expected boundary length all the components of index n (minus twice the expected number of such components) grows asymptotically at a rate that is independent of n, contrary to the total area.

Open question 1. *Does* $\mathbb{E}\left[\sum_{c\in C_0 \text{ finite}} |c|\right]$, *i.e. the total area of the finite clusters of index* 0, *have a similarly explicit expression? Based on the results of [24] we expect it to be asymptotic to* $\pi l/30$ *as* $l \longrightarrow \infty$.

Finally we mention one more potential application of the enumeration of loops in Theorem 4 in the context of *random walk loop soups* [27], which are certain Poisson processes of loops on $\mathbb{Z}^2$. A natural

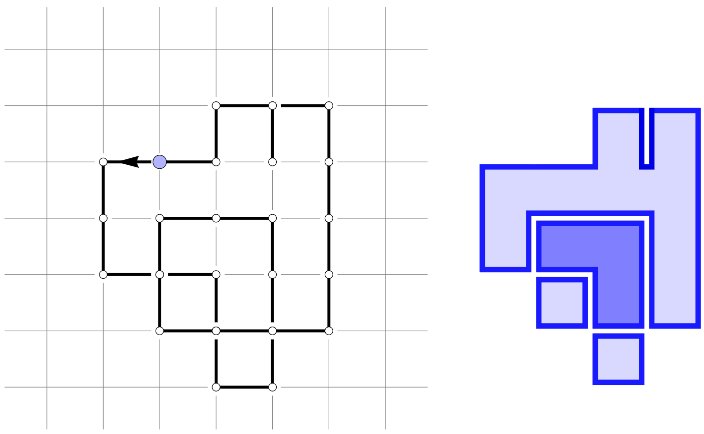


Figure 6: A simple walk on $\mathbb{Z}^2$ together with its clusters of index 1 (light blue) and 2 (dark blue). The largest cluster c has area $|c| = 9$ and boundary length $|\partial c| = 20$ (notice that both sides of the "slit" contribute to the length).

quantity to consider in such a system is the *winding field* which roughly assigns to any point $z \in \mathbb{R}^2$ the total index of all the loops in the process [21, 20]. Theorem 4 may be used to compute explicit expectation values (one-point functions of the corresponding vertex operators to be precise) in the massive version of the loop soups. We will pursue this direction elsewhere.

Discussion The connection between the enumeration of walks and the explicitly diagonalizable operators on Dirichlet space may seem a bit magical to the reader. So perhaps some comments are in order on how we arrived at this result, which originates in the combinatorial study of planar maps.

A *planar map* is a multigraph (a graph with multiple edges and loops allowed) that is properly embedded in the 2-sphere (edges are only allowed to meet at their extremities), viewed up to orientation-preserving homeomorphisms of the sphere. The connected components of the complement of a planar map are called the *faces*, which have a *degree* equal to the number of bounding edges. There exists a relatively simple multivariate generating function for *bipartite* planar maps, i.e. maps with all faces of even degree, that have two distinguished faces of degree p and l and a fixed number of faces of each degree (see e.g. [22]). The surprising fact, for which we will give a combinatorial explanation elsewhere using a peeling exploration [16, 17], is that this generating function has a form that is very similar to that of the generating function $W_{l,p}^{(\pi,(-\pi,\pi))}(t)$ of diagonal walks from $(p,0)$ to $(-l,0)$ that avoid the slit $\{(x,0) : x \le 0\}$ until the end.

If one further decorates the planar maps by a *rigid $O(n)$ loop model* [7], then the combinatorial relation extends to one between walks of fixed winding angle $\mathcal{W}_{l,p}^{(\alpha)}$ with $\alpha \in \pi\mathbb{Z}$ and planar maps with two distinguished faces and a certain collections of non-intersection loops separating the two faces. The combinatorics of the latter has been studied in considerable detail in [7, 6, 5], which has inspired our treatment of the simple walks on $\mathbb{Z}^2$ in this paper. Further details on the connection and an extension to more general lattice walks with *small steps* (i.e. steps in $\{-1,0,1\}^2$) will be provided in forthcoming work.

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2 Winding angle of walks starting and ending on an axis

Our strategy towards proving Theorem 1 will be to first prove part (ii) for three special cases (see Figure 7),

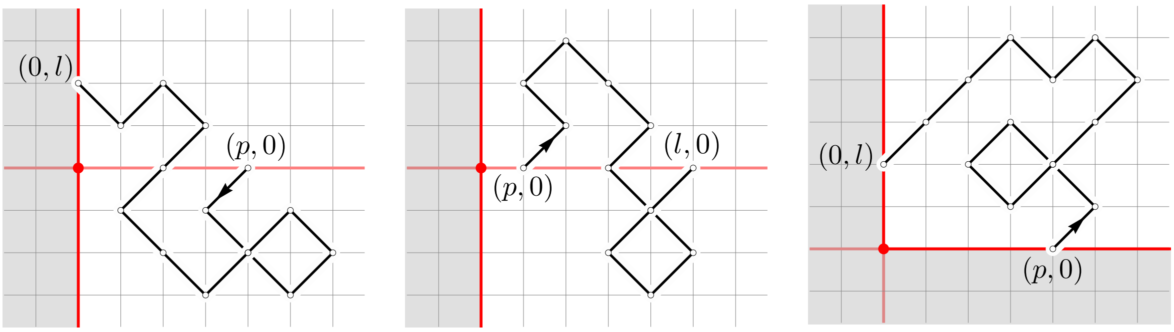
$$\mathcal{J}_{l,p} \coloneqq \mathcal{W}_{l,p}^{(\frac{\pi}{2},(-\frac{\pi}{2},\frac{\pi}{2}))}, \qquad \mathcal{B}_{l,p} \coloneqq \mathcal{W}_{l,p}^{(0,(-\frac{\pi}{2},\frac{\pi}{2}))}, \qquad \mathcal{A}_{l,p} \coloneqq \mathcal{W}_{l,p}^{(\frac{\pi}{2},(0,\frac{\pi}{2}))}.$$

We define three linear operators $\mathbf{J}_k$, $\mathbf{B}_k$ and $\mathbf{A}_k$ on $\mathcal{D}$ by specifying their matrix elements with respect to the standard basis $(e_i)_{i\ge1}$ in terms of the corresponding generating functions $J_{l,p}(t)$, $B_{l,p}(t)$ and $A_{l,p}(t)$ (with $k = 4t$) as

$$\frac{1}{l}\left\langle e_l, \mathbf{J}_k e_p\right\rangle_{\mathcal{D}} = J_{l,p}(t), \qquad \left\langle e_l, \mathbf{B}_k e_p\right\rangle_{\mathcal{D}} = B_{l,p}(t), \qquad \frac{1}{lp}\left\langle e_l, \mathbf{A}_k e_p\right\rangle_{\mathcal{D}} = A_{l,p}(t).$$

The main reason we define the operators in this way is the following.

Proposition 1. *The linear operators $\mathbf{J}_k$, $\mathbf{B}_k$, $\mathbf{A}_k$ on $\mathcal{D}$ are bounded and satisfy $\mathbf{J}_k = \mathbf{A}_k\mathbf{B}_k$.*



(a) $\mathcal{J}_{l,p}$, $J_{l,p}(t) = \frac{1}{l}\left\langle e_l, \mathbf{J}_k e_p \right\rangle_{\mathcal{D}}$ (b) $\mathcal{B}_{l,p}$, $B_{l,p}(t) = \left\langle e_l, \mathbf{B}_k e_p \right\rangle_{\mathcal{D}}$ (c) $\mathcal{A}_{l,p}$, $A_{l,p}(t) = \frac{1}{lp}\left\langle e_l, \mathbf{A}_k e_p \right\rangle_{\mathcal{D}}$

Figure 7: Examples of walks contributing to the three types of building blocks.

Proof. It is not hard to see that (for fixed $k = 4t \in (0,1)$) $A_{l,p}(t)$ falls off exponentially in $l+p$, and therefore $\mathbf{A}_k$ has finite Hilbert-Schmidt norm

$$\sum_{p=1}^{\infty} \frac{\|\mathbf{A}_k e_p\|_{\mathcal{D}}^2}{\|e_p\|_{\mathcal{D}}^2} = \sum_{p,l=1}^{\infty} p\, l\, A_{l,p}^2(t) < \infty.$$

In particular, $\mathbf{A}_k$ is bounded (and even compact).

On the other hand, the matrix elements of $\mathbf{B}_k$ with respect to the orthonormal basis $(e_p/\sqrt{p})_{p\geq 1}$ are given by $B_{l,p}(t)/\sqrt{lp} \leq B_{l,p}(t) \leq b_{|l-p|}$ for some sequence $(b_n)_{n\geq 0}$ that falls off exponentially. Hence, the operator norm of $\mathbf{B}_k$ is bounded by that of the symmetric Toeplitz operator associated to $(b_n)_{n\geq 0}$, which by classical results is finite since $(b_n)_{n\geq 0}$ are the Fourier coefficients of a continuous function.

For $l, p, m \geq 1$, composition of walks determines a bijection between pairs of walks in $\mathcal{A}_{l,m} \times \mathcal{B}_{m,p}$ and walks in $\mathcal{J}_{l,p}$ for which the last intersection with the horizontal axis occurs at $(m, 0)$. Hence

$$\frac{1}{l}\left\langle e_l, \mathbf{J}_k e_p \right\rangle_{\mathcal{D}} = J_{l,p}(t) = \sum_{m=1}^{\infty} A_{l,m}(t) B_{m,p}(t) = \sum_{m=1}^{\infty} \frac{1}{lm} \langle e_l, \mathbf{A}_k e_m \rangle_{\mathcal{D}} \left\langle e_m \mathbf{B}_k, e_p \right\rangle_{\mathcal{D}} = \frac{1}{l}\left\langle e_l, \mathbf{A}_k \mathbf{B}_k e_p \right\rangle_{\mathcal{D}},$$

implying that $\mathbf{J}_k = \mathbf{A}_k \mathbf{B}_k$, which is therefore also bounded. □

It is clear from the definitions that $\mathbf{B}_k$ and $\mathbf{J}_k$ are self-adjoint, and we will soon see (in Lemma 1) that the same is true for $\mathbf{J}_k$. The relation $\mathbf{J}_k = \mathbf{A}_k \mathbf{B}_k$ then implies that all three operators commute. Provided the operators are compact (and we have already seen this to be the case for $\mathbf{A}_k$), they can be simultaneously (unitarily) diagonalized.

In Section 2.1 we will diagonalize $\mathbf{J}_k$, and then we will check in Section 2.2 that $\mathbf{B}_k$ and $\mathbf{A}_k$ possess the same set of eigenvectors, as expected. Finally, in Section 2.3 we will prove Theorem 1 by taking suitable compositions of the operators $\mathbf{J}_k$, $\mathbf{B}_k$. and $\mathbf{A}_k$.

2.1 The operator $\mathbf{J}_k$

We wish to enumerate the walks $w \in \mathcal{J}_{l,p}$, $p, l \geq 1$, that start at $(p, 0)$ and end at $(0, l)$ while maintaining strictly positive first coordinate until the end. By looking at both coordinates of the walks separately, we easily see that these walks are in bijection with pairs of simple walks of equal length on $\mathbb{Z}$, the first of which starts at p and ends at 0 while staying positive until the end, while the second starts at 0 and ends at l without further restrictions. For fixed length n, such walks only exist if both $n-p$ and $n-l$ are non-negative even integers, in which case the Ballot theorem tells us that the former walks are counted by $\frac{p}{n}\binom{n}{(n+p)/2}$ and the latter by $\binom{n}{(n+l)/2}$. Therefore the generating function $J_{l,p}(t)$ is given explicitly by

$$J_{l,p}(t) = \sum_{n=1}^{\infty} \frac{p}{n} \binom{n}{(n+l)/2} \binom{n}{(n+p)/2} \mathbf{1}_{\{n-p \text{ and } n-l \text{ non-negative and even}\}} t^n. \tag{5}$$

It is non-trivial only when $p+l$ is even, in which case it has radius of convergence equal to $1/4$.

For fixed $k = 4t \in (0,1)$ we denote by $\psi_k : \mathbb{D} \to \mathbb{C}$ the analytic function given by

$$\psi_k(x) = \frac{1 - \sqrt{1 - kx^2}}{\sqrt{k}\, x},$$

which maps the unit disk $\mathbb{D}$ biholomorphically onto a strict subset $\psi_k(\mathbb{D}) \subset \mathbb{D}$. It induces a linear operator $\mathbf{R}_k$ on the Dirichlet space $\mathcal{D}$ of analytic functions f on $\mathbb{D}$ that vanish at the origin by setting $\mathbf{R}_k f := f \circ \psi_k$.

Lemma 1. *The linear operator* $\mathbf{R}_k$ *is bounded and* $\mathbf{J}_k = \mathbf{R}_k^\dagger \mathbf{R}_k$. *In particular,* $\mathbf{J}_k$ *is self-adjoint.*

Proof. Since the Dirichlet norm is preserved under conformal mapping, we have

$$\|\mathbf{R}_k f\|_{\mathcal{D}}^2 = \|f \circ \psi_k\|_{\mathcal{D}}^2 = \int_{\psi_k(\mathbb{D})} |f'(z)|^2 \mathrm{d}A(z) \le \|f\|_{\mathcal{D}}^2,$$

implying that $\mathbf{R}_k$ is bounded.

By the Lagrange inversion theorem one easily finds that for $n-p$ non-negative and even one has

$$[x^n]\psi_k(x)^p = \left(\frac{k}{4}\right)^{n/2} [z^n] \left(\frac{1-\sqrt{1-4z^2}}{2z}\right)^p = \left(\frac{k}{4}\right)^{n/2} \frac{p}{n} [u^{n-p}] \left(1+u^2\right)^n = \left(\frac{k}{4}\right)^{n/2} \frac{p}{n} \binom{n}{(n+p)/2}. \tag{6}$$

Therefore

$$\begin{aligned}
\left\langle e_l, \mathbf{R}_k^\dagger \mathbf{R}_k e_p \right\rangle_{\mathcal{D}} &= \left\langle \mathbf{R}_k e_l, \mathbf{R}_k e_p \right\rangle_{\mathcal{D}} = \sum_{n=1}^{\infty} n\, \overline{[x^n]\psi_k(x)^l}\, [x^n]\psi_k(x)^p \\
&= \sum_{n=1}^{\infty} \left(\frac{k}{4}\right)^n n \frac{l}{n} \binom{n}{(n+l)/2} \frac{p}{n} \binom{n}{(n+p)/2} \mathbf{1}_{\{n-p \text{ and } n-l \text{ non-negative and even}\}} \\
&= l\, J_{l,p}(t) = \left\langle e_l, \mathbf{J}_k e_p \right\rangle_{\mathcal{D}},
\end{aligned}$$

which finishes the proof. □

In order to diagonalize $\mathbf{J}_k$ it suffices to find an orthogonal basis of $\mathcal{D}$ consisting of analytic functions that are also orthogonal with respect to the Dirichlet inner-product on $\psi_k(\mathbb{D})$,

$$\langle f, g \rangle_{\mathcal{D}(\psi_k(\mathbb{D}))} := \int_{\psi_k(\mathbb{D})} \overline{f'(z)}\, g'(z)\, \mathrm{d}A(z).$$

To this end we seek an injective holomorphic mapping that takes both $\mathbb{D}$ and $\psi_k(\mathbb{D})$ to sufficiently simple domains. As we will see the elliptic integral $v_{k_1}(z)$ introduced in (1) does this job. First we notice that $v_k(z)$ can be expressed in terms of the inverse function $\operatorname{arcsn}(\cdot, k)$ (in a suitable neighbourhood of the origin) of the Jacobi elliptic function $\operatorname{sn}(\cdot, k)$ with modulus k,

$$v_k(z) := \frac{1}{4K(k)} \int_0^z \frac{\mathrm{d}x}{\sqrt{(k-x^2)(1-kx^2)}} = \frac{\operatorname{arcsn}\left(\frac{z}{\sqrt{k}}, k\right)}{4K(k)}. \tag{7}$$

As depicted in Figure 8 and proved in the next lemma, after the removal of two slits v_{k_1} maps both $\mathbb{D}$ and $\psi_k(\mathbb{D})$ to rectangles, with the same width but different height.

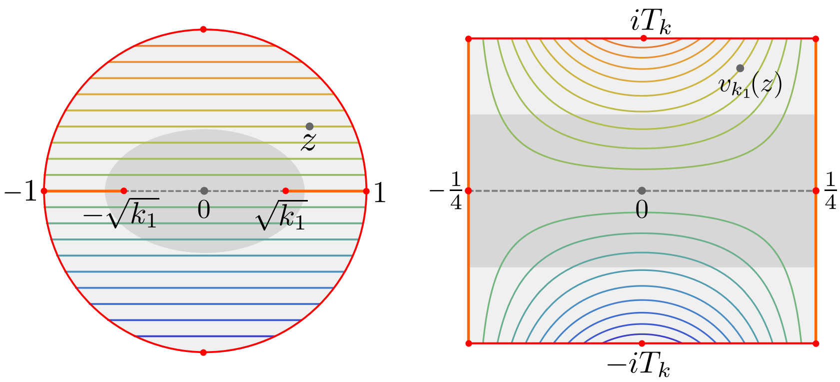


Figure 8: The analytic function v_{k_1} maps the disk with two slits onto a rectangle of width $\frac{1}{2}$ and height $2T_k = K(k')/(2K(k))$. The shaded domains represent $\psi_k(\mathbb{D})$ and its image under v_{k_1}.

Lemma 2. *v_{k_1} maps $\mathbb{D} \setminus \{z \in \mathbb{R} : z^2 \geq k_1\}$, respectively $\psi_k(\mathbb{D}) \setminus \{z \in \mathbb{R} : z^2 \geq k_1\}$, biholomorphically onto the open rectangle $H_{k_1} := (-1/4, 1/4) + i(-T_{k_1}/2, T_{k_1}/2)$, respectively $H_k := (-1/4, 1/4) + i(-T_k/2, T_k/2)$, where $T_k := K(k')/(4K(k))$ satisfies $T_{k_1} = 2T_k$.*

Proof. It follows from elementary properties of the Jacobi elliptic function that $\operatorname{sn}(\cdot, k_1)$ maps $(-K(k_1), K(k_1)) + i(-K(k_1')/2, K(k_1)/2)$ biholomorphically onto the disc of radius $1/\sqrt{k_1}$ with double slits at $\{z \in \mathbb{R} : z^2 > 1\}$. This shows that v_{k_1} indeed maps $\mathbb{D} \setminus \{z \in \mathbb{R} : z^2 \geq k_1\}$ to H_{k_1}.

The descending Landen transformation relates the Jacobi elliptic functions $\operatorname{sn}(\cdot, k)$ and $\operatorname{sn}(\cdot, k_1)$ through (see e.g. [1, 16.12.2])

$$\operatorname{sn}(u, k) = \frac{(1 + k_1)\operatorname{sn}(u/(1 + k_1), k_1)}{1 + k_1 \operatorname{sn}^2(u/(1 + k_1), k_1)}.$$

Inverting the relation yields

$$\sqrt{k_1}\operatorname{sn}(u/(1 + k_1), k_1) = \frac{1 - \sqrt{1 - k^2 \operatorname{sn}^2(u, k)}}{k \operatorname{sn}(u, k)} = \psi_k(\sqrt{k}\operatorname{sn}(u, k)).$$

Setting $u = 4K(k)v_k(x)$ and using that $K(k) = (1 + k_1)K(k_1)$ we observe that $v_{k_1}(\psi_k(x)) = v_k(x)$ for $x \in \mathbb{D} \setminus \{z \in \mathbb{R} : z^2 \geq k\}$. Hence, the image of $\psi_k(\mathbb{D}) \setminus \{z \in \mathbb{R} : z^2 \geq k_1\}$ agrees with that of $\mathbb{D} \setminus \{z \in \mathbb{R} : z^2 \geq k_1\}$ after substituting $k_1 \to k$. □

The presence of slits in the domain of v_{k_1} indicates that for any analytic function f on $\mathbb{D}$, $f \circ v_{k_1}^{-1}$ can be continued to an analytic function g on the infinite strip $\mathbb{R} + i[-T_k, T_k]$, which is 1-periodic and satisfies $g(1/2 - v) = g(v)$. In fact, v_{k_1} induces an isomorphism between $\mathcal{D}$ and the Hilbert space $\mathcal{H}_k$ of such analytic functions g that vanish at 0 with norm

$$\|g\|^2_{\mathcal{H}_k} = \int_{H_{k_1}} |g'(v)|^2 \mathrm{d}A(v).$$

A simple Fourier series expansion now suffices to diagonalize $\mathbf{J}_k$.

Proposition 2. *The operator $\mathbf{J}_k$ on $\mathcal{D}$ is compact and the family $(f_m)_{m=1}^{\infty}$ defined in (2) forms an orthogonal basis of $\mathcal{D}$ satisfying*

$$\mathbf{J}_k f_m = \frac{1}{q_k^{m/2} + q_k^{-m/2}} f_m, \qquad \|f_m\|^2_{\mathcal{D}} = \frac{m}{4}\left(q_k^{-m} - q_k^{m}\right).$$

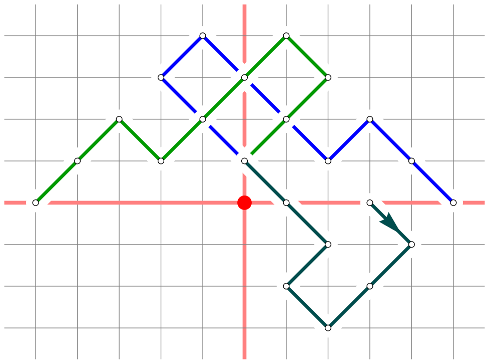


Figure 9: The reflection principle in the vertical axis yields a bijection between walks from $(p,0)$ to $(-l,0)$ and walks $(p,0)$ to $(l,0)$ that visit the vertical axis at least once.

Proof. Clearly $(\cos(2\pi m(\cdot+1/4))-\cos(\pi m/2))_{m=1}^{\infty}$ forms a basis of $\mathcal{H}_k$, which is orthogonal since an explicitly computation shows that

$$\langle \cos(2\pi m(\cdot+1/4))-\cos(\pi m/2), \cos(2\pi n(\cdot+1/4))-\cos(\pi n/2)\rangle_{\mathcal{H}_k} = \frac{1}{2} m \sinh(2m\pi T_{k_1})\mathbf{1}_{\{m=n\}}.$$

But then we also have

$$\langle f_m, f_n\rangle_{\mathcal{D}} = \frac{1}{2} m \sinh(2m\pi T_{k_1})\mathbf{1}_{\{m=n\}} = \frac{m(q_k^{-m}-q_k^{m})}{4}\mathbf{1}_{\{m=n\}}, \tag{8}$$

meaning that $(f_m)_{m=1}^{\infty}$ forms an orthogonal basis of $\mathcal{D}$ with norm $\|f_m\|_{\mathcal{D}}$ as claimed. By Lemma 2 $\langle f_m, f_n\rangle_{\mathcal{D}(\psi_k(\mathbb{D}))}$ takes the same form except for the replacement $k_1 \to k$, i.e.

$$\langle f_m, \mathbf{J}_k f_n\rangle_{\mathcal{D}} = \langle f_m, f_n\rangle_{\mathcal{D}(\psi_k(\mathbb{D}))} = \frac{1}{2} m \sinh(2m\pi T_k)\mathbf{1}_{\{m=n\}}.$$

It follows that f_m is an eigenvector of $\mathbf{J}_k$ with eigenvalue

$$\frac{\langle f_m, \mathbf{J}_k f_n\rangle_{\mathcal{D}}}{\langle f_m, f_n\rangle_{\mathcal{D}}} = \frac{\sinh(2m\pi T_k)}{\sinh(4m\pi T_k)} = \frac{1}{2\cosh(2m\pi T_k)} = \frac{1}{q_k^{m/2}+q_k^{-m/2}},$$

where we used that $T_{k_1} = 2T_k$. In particular, $\mathbf{J}_k$ is a compact operator, since $1/(q_k^{m/2}+q_k^{m/2}) \to 0$ as $m \to \infty$. □

Notice that this verifies Theorem 1(ii) for $\alpha = \pi/2$ and $I = (-\pi/2, \pi/2)$.

2.2 The operators $\mathbf{B}_k$ and $\mathbf{A}_k$

Recall that $B_{l,p}(t) = \langle e_l, \mathbf{B}_k e_p\rangle_{\mathcal{D}}$ is the generating function for the set $\mathcal{B}_{l,p}$ of simple diagonal walks from $(p,0)$ to $(l,0)$ that maintain strictly positive first coordinate. A simple reflection principle (see Figure 9) teaches us that $B_{l,p}(t)$ is given by

$$B_{l,p}(t) = B_{l-p}(t) - B_{-l-p}(t), \tag{9}$$

where $B_m(t)$, $m \in \mathbb{Z}$, is the generating function of simple diagonal walks from $(0,0)$ to $(m,0)$ without further restrictions.

Lemma 3. *For fixed $t \in (0,1/4)$, $B_m(t)$ can be expressed as a contour integral as*

$$B_m(t) = \frac{1}{2\pi i}\int_{\gamma} \frac{\mathrm{d}z}{z^{m+1}\sqrt{1-4t^2(z+1/z)^2}}, \tag{10}$$

where γ traces the unit circle in counterclockwise direction.

Proof. The contribution of walks from $(0,0)$ to $(m,0)$ of length $2n$ and m even is

$$\binom{2n}{n}\binom{2n}{n+m/2}t^{2n} = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z^{m+1}}\binom{2n}{n}(z+1/z)^{2n}t^{2n}.$$

The result then follows from summing over $n \geq 0$ and relying on absolute convergence to interchange the summation and integral. □

Based on the similarity between the integrand of (10) and the one in the definition (1) of v_{k_1}, we find the following useful representation of $\mathbf{B}_k$.

Lemma 4. *If $f, g \in \mathcal{D}$ are analytic in a neighbourhood of the closed unit disk, then*

$$\langle f, \mathbf{B}_k g\rangle_{\mathcal{D}} = \frac{2K(k)}{\pi}\int_{\gamma'}(f(z)-f(z^{-1}))(g(z)-g(z^{-1}))v'_{k_1}(z)\mathrm{d}z, \tag{11}$$

where γ' traces the upper half of the unit circle starting at 1 and ending at -1.

Proof. From the definition (1) we see that the integrand of (11) is continuous and bounded on the upper-half circle, and therefore the right-hand side of (11) converges absolutely. Hence, it suffices to check the identity for $f = e_l$ and $g = e_p$, $p, l \geq 1$. Combining (9) and Lemma 3 we find

$$\begin{aligned}\langle e_l, \mathbf{B}e_p\rangle_{\mathcal{D}} &= B_{l-p}(t) - B_{-l-p}(t) = \frac{1}{2}\left(B_{l-p}(t) - B_{-l-p}(t) + B_{p-l}(t) - B_{l+p}(t)\right)\\ &= \frac{-1}{4\pi i}\int_\gamma \frac{(z^l - z^{-l})(z^p - z^{-p})\mathrm{d}z}{z\sqrt{1-4t^2(z+1/z)^2}} = \frac{-1}{2\pi i}\int_{\gamma'} \frac{(z^l - z^{-l})(z^p - z^{-p})\mathrm{d}z}{z\sqrt{1-4t^2(z+1/z)^2}},\end{aligned} \tag{12}$$

where in the last equality we used that both sides vanish for $p+l$ odd, while for $p+l$ even the upper and lower half circles contribute equally. Note that

$$(k_1 - z^2)(1-k_1z^2) = -z^2((1+k_1)^2 - k_1(z+1/z)^2) = -z^2(1+k_1)^2\left(1 - \frac{k^2}{4}(z+1/z)^2\right).$$

Hence, for z on the upper-half circle (1) implies that

$$v'_{k_1}(z) = \frac{1}{4K(k_1)}\frac{1}{\sqrt{(k_1-z^2)(1-k_1z^2)}} = \frac{i}{4K(k)}\frac{1}{z\sqrt{1-\frac{k^2}{4}(z+1/z)^2}},$$

where we used that $(1+k_1)K(k_1) = K(k)$. Combining with (12) we indeed reproduce the right-hand side of (11). □

With the contour integral representation in hand it is now straightforward to evaluate $\mathbf{B}_k$ (and $\mathbf{A}_k$ subsequently) with respect to the basis $(f_m)_{m=1}^\infty$.

Proposition 3. *The linear operators $\mathbf{B}_k$ and $\mathbf{A}_k$ are compact and have the same eigenvectors $(f_m)_{m=1}^\infty$ as $\mathbf{J}_k$ satisfying*

$$\begin{aligned}\mathbf{B}_k f_m &= \frac{2K(k)}{\pi}\frac{1}{m}\frac{1-q_k^m}{1+q_k^m}f_m,\\ \mathbf{A}_k f_m &= \frac{\pi}{2K(k)}\frac{m}{q_k^{-m/2}-q_k^{m/2}}f_m.\end{aligned}$$

Proof. The functions $f_m(z)$ are seen to be analytic for $|z| < 1/\sqrt{k_1}$, and therefore we may apply Lemma 4 to obtain

$$\begin{aligned}\langle f_n, \mathbf{B}_k f_m \rangle_{\mathcal{D}} &= \frac{2K(k)}{\pi} \int_{\gamma'} (f_n(z) - f_n(z^{-1}))(f_m(z) - f_m(z^{-1}))v'_{k_1}(z)\mathrm{d}z \\ &= -\frac{2K(k)}{\pi} \int_0^{1/2} (\cos(2\pi n(v + iT_k)) - \cos(2\pi n(v - iT_k))) \\ &\qquad\qquad \times (\cos(2\pi m(v + iT_k)) - \cos(2\pi m(v - iT_k)))\mathrm{d}v \\ &= \frac{8K(k)}{\pi} \sinh(2\pi n T_k) \sinh(2\pi m T_k) \int_0^{1/2} \sin(2\pi n v)\sin(2\pi m v)\mathrm{d}v \\ &= \frac{2K(k)}{\pi} \sinh^2(2\pi m T_k) \mathbf{1}_{\{m=n\}}.\end{aligned}$$

Together with (8) we conclude that f_m is an eigenmode of $\mathbf{B}_k$ with eigenvalue

$$\frac{\langle f_m, \mathbf{B}_k f_m \rangle_{\mathcal{D}}}{\langle f_m, f_m \rangle_{\mathcal{D}}} = \frac{4K(k)}{\pi} \frac{\sinh^2(2m\pi T_k)}{m \sinh(4m\pi T_k)} = \frac{2K(k)}{\pi} \frac{\tanh(2m\pi T_k)}{m} = \frac{2K(k)}{\pi} \frac{1}{m} \frac{1 - q_k^m}{1 + q_k^m}.$$

Since $\mathbf{J}_k = \mathbf{A}_k \mathbf{B}_k$ is injective, we find using Proposition 2 that

$$\mathbf{A}_k f_m = \frac{\pi}{2K(k)} m \frac{1 + q_k^m}{1 - q_k^m} \frac{1}{q_k^{m/2} + q_k^{-m/2}} f_m = \frac{\pi}{2K(k)} \frac{m}{q_k^{-m/2} - q_k^{m/2}} f_m.$$

The eigenvalues of $\mathbf{B}_k$ and $\mathbf{A}_k$ both approach 0 as $m \to \infty$, implying compactness. □

2.3 Proof of Theorem 1

We start with part (i) with $\alpha \in \frac{\pi}{2}\mathbb{Z}$. Given a walk $w \in \mathcal{W}_{l,p}^{(\alpha)}$, let $(s_i)_{i=1}^N$ be the sequence of times at which w *alternates between the axes*, i.e. $s_0 = 0$ and for each $j \geq 0$ we set $s_{j+1} = \inf\{s > s_j : |\theta_s^w - \theta_{s_j}^w| = \pi/2\}$ provided it exists (otherwise $s_j = s_N$ is the last entry in the sequence). Let $(\alpha_j)_{j=0}^N$ and $(l_j)_{j=0}^N$ be the sequences of winding angles and distances to the origin defined by $\alpha_j = \theta_{s_j}^w$ respectively $l_j = |w_{s_j}|$ for $0 \leq j \leq N$. It is now easy to see that for $0 \leq j < N$ the part of the walk between time s_j and s_{j+1} is (up to a unique rotation around the origin and/or reflection in the horizontal axis) of the form of a walk $w^{(j)} \in \mathcal{J}_{l_{j+1}, l_j}$. Similarly, the last part of the walk between time s_N and $|w|$ is (up to rotation) of the form of a walk $w^{(N)} \in \mathcal{B}_{l, l_N}$. See Figure 10 for an example.

In fact, this construction is seen to yield a bijection between $\mathcal{W}_{l,p}^{(\alpha)}$ and the set of tuples

$$\left(N, (l_j)_{j=0}^N, (\alpha_j)_{j=0}^N, (w^{(j)})_{j=0}^N\right)$$

where $N \geq 0$, $l_0 = p$, $l_j \geq 1$, $(\alpha_j)_{j=0}^N$ is a simple walk on $\frac{\pi}{2}\mathbb{Z}$ from $\alpha_0 = 0$ to $\alpha_N = \alpha$, $w^{(j)} \in \mathcal{J}_{l_{j+1}, l_j}$ for $0 \leq j < N$ and $w^{(N)} \in \mathcal{B}_{l, l_N}$. If we denote by

$$a_N^{(\alpha)} = \binom{N}{\frac{N - \frac{2|\alpha|}{\pi}}{2}} \mathbf{1}_{\{N - \frac{2|\alpha|}{\pi} \text{ even and non-negative}\}}$$

the number of simple walks on $\frac{\pi}{2}\mathbb{Z}$ from 0 to α of length $N \geq 0$, then we may identify the generating

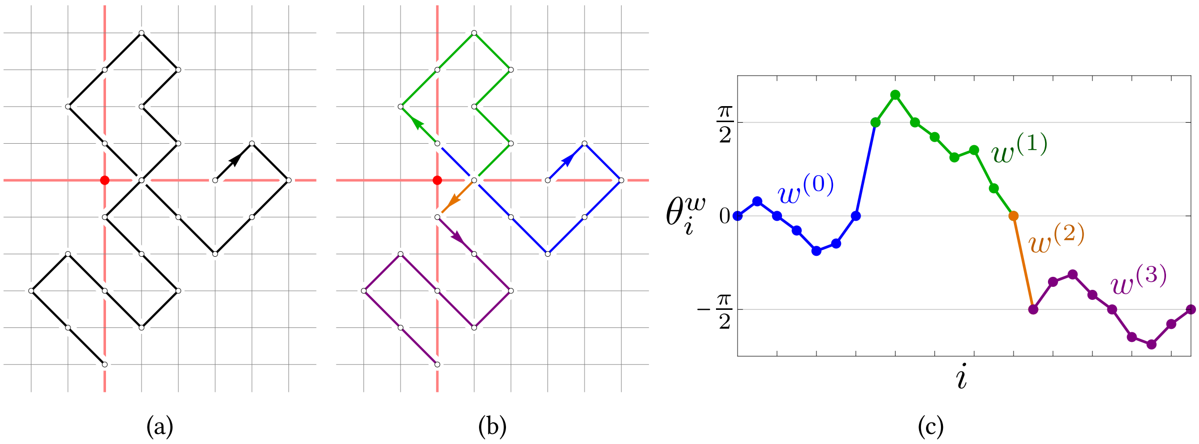


Figure 10: (a) A walk $w \in \mathcal{W}_{5,3}^{(-\pi/2)}$; (b) its decomposition into $w^{(0)} \in \mathcal{J}_{1,3}$, $w^{(1)}, w^{(2)} \in \mathcal{J}_{1,1}$, $w^{(3)} \in \mathcal{B}_{5,1}$; (c) its winding angle sequence (θ_i^w).

function of $\mathcal{W}_{l,p}^{(\alpha)}$ as

$$\begin{aligned} W_{l,p}^{(\alpha)}(t) &= \sum_{N=0}^{\infty} a_N^{(\alpha)} \sum_{l_1,\ldots,l_N=1}^{\infty} B_{l,l_N}(t) \prod_{i=0}^{N-1} J_{l_{i+1},l_i}(t) \\ &= \sum_{N=0}^{\infty} a_N^{(\alpha)} \sum_{l_1,\ldots,l_N=1}^{\infty} \langle e_l, \mathbf{B}_k e_{l_N} \rangle_{\mathfrak{D}} \prod_{i=0}^{N-1} \frac{1}{l_{i+1}} \langle e_{l_{i+1}}, \mathbf{J}_k e_{l_i} \rangle_{\mathfrak{D}} \\ &= \sum_{N=0}^{\infty} a_N^{(\alpha)} \langle e_l, \mathbf{B}_k \mathbf{J}_k^N e_p \rangle_{\mathfrak{D}} . \end{aligned}$$

Since the eigenvalues of $\mathbf{J}_k$ are all strictly smaller than $1/2$ and $a_N^{(\alpha)} \le 2^N$, $\sum_{N=0}^{\infty} a_N^{(\alpha)} \mathbf{B}_k \mathbf{J}_k^N$ converges to a compact self-adjoint operator $\mathbf{Y}_k^{(\alpha)}$ satisfying

$$W_{l,p}^{(\alpha)}(t) = \left\langle e_l, \mathbf{Y}_k^{(\alpha)} e_p \right\rangle_{\mathfrak{D}} . \tag{13}$$

With a little help of (6), we find the (formal) generating function

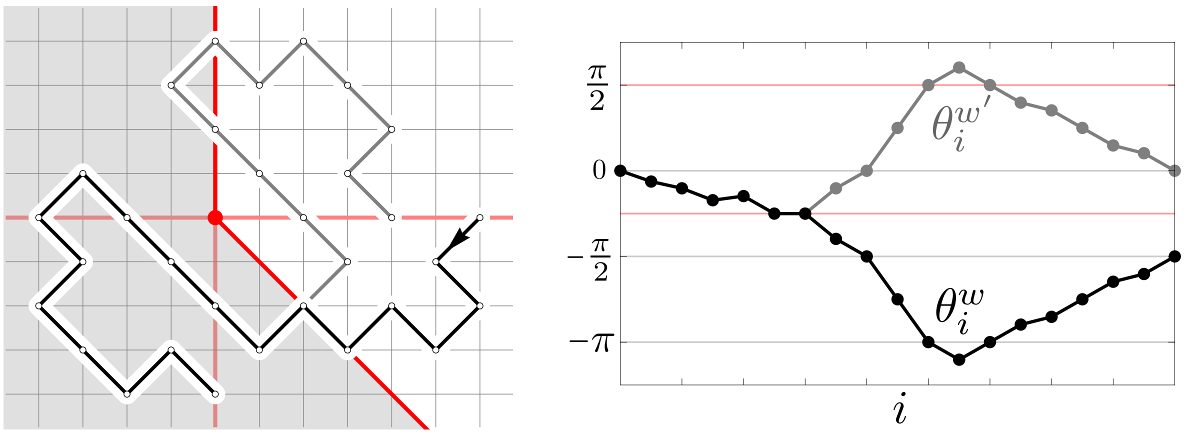
$$a^{(\alpha)}(x) = \sum_{N=0}^{\infty} a_N^{(\alpha)} x^N = \frac{1}{\sqrt{1-4x^2}} \left(\frac{1-\sqrt{1-4x^2}}{2x} \right)^{2|\alpha|/\pi} .$$

Then one may deduce after some simplification that

$$\begin{aligned} \mathbf{Y}_k^{(\alpha)} f_m = \mathbf{B}_k\, a^{(\alpha)}(\mathbf{J}_k) f_m &= \frac{2K(k)}{\pi} \frac{1}{m} \frac{1-q_k^m}{1+q_k^m} a^{(\alpha)} \left(\frac{1}{q_k^{m/2} + q_k^{-m/2}} \right) f_m \\ &= \frac{2K(k)}{\pi} \frac{1}{m} q_k^{|\alpha| m/\pi} f_m, \end{aligned}$$

in agreement with part (i) of Theorem 1.

We could easily extend this result to the case $I = (\beta_-, \beta_+)$ with $\beta_\pm \in \frac{\pi}{2}\mathbb{Z} \cup \{\pm\infty\}$ such that $0 \in I$ and $\alpha \in I \cap \frac{\pi}{2}\mathbb{Z}$, by replacing $a^{(\alpha)}(x)$ by the generating function of simple walks on $\frac{\pi}{2}\mathbb{Z}$ confined to an interval. Instead, we choose to discuss a reflection principle at the level of the simple diagonal walks, which allows us to directly generalize to the case of $\beta_\pm \in \frac{\pi}{4}\mathbb{Z} \cup \{\pm\infty\}$ of part (iii).



(a) (b)

Figure 11: Example with $\alpha = 0$, $I = (-\pi/4, \pi/2)$. (a) A walk $w \in \mathcal{W}_{4,6}^{(-\pi/2)}$ is depicted in black, and its reflection $w' \in \mathcal{W}_{4,6}^{(0)}$ in gray. (b) The corresponding winding angle sequences (θ_i^w) and $(\theta_i^{w'})$.

Lemma 5. *Suppose l and p are both odd and $\beta_\pm \in \frac{\pi}{2}\mathbb{Z}$, or l and p are both even and $\beta_\pm \in \frac{\pi}{4}\mathbb{Z}$, such that $0 \in I$. If in addition $\alpha \in \frac{\pi}{2}\mathbb{Z} \cap I$, then the generating function for $\mathcal{W}_{l,p}^{(\alpha,I)}$ is given by*

$$W_{l,p}^{(\alpha,I)}(t) = \sum_{n=-\infty}^{\infty} \left(W_{l,p}^{(\alpha+n\delta)}(t) - W_{l,p}^{(2\beta_+-\alpha+n\delta)}(t) \right), \qquad \delta := 2(\beta_+ - \beta_-).$$

Proof. Consider any walk $w \in \bigcup_{n=-\infty}^{\infty} \mathcal{W}_{l,p}^{(2\beta_+-\alpha+n\delta)}$ and let $s = \inf\{j \ge 0 : \theta_j^w \notin I\}$ be the first time w leaves I, which is well-defined since $\theta^w \notin I$. It is not hard to see that under the stated conditions on $l, p, \beta_\pm$ the winding angle sequence $(\theta_i^w)_{i=0}^{|w|}$ cannot cross $\beta_\pm$ without visiting $\beta_\pm$, and therefore $\theta_s^w = \beta_\pm$ and w_s lies on an axis or a diagonal of $\mathbb{Z}^2$. Then we let w' be the walk obtained from w by reflecting the portion of w after time s in this axis or diagonal (see Figure 11). Then $\theta^{w'} = 2\beta_+ - \theta^w$ or $\theta^{w'} = 2\beta_- - \theta^w$. Hence $w' \in \bigcup_{n=-\infty}^{\infty} \mathcal{W}_{l,p}^{(\alpha+n\delta)}$. It is not hard to see that this mapping $w \mapsto w'$ is injective (the inverse $w' \mapsto w$ is given by the exact same reflection operation). Moreover, any walk $w' \in \bigcup_{n=-\infty}^{\infty} \mathcal{W}_{l,p}^{(\alpha+n\delta)}$ is obtained in such way provided $(\theta_i^{w'})_{i=0}^{|w'|}$ visits $\beta_\pm$ at least once. Clearly, the only walks w' not satisfying the latter condition are the ones in $\mathcal{W}_{l,p}^{(\alpha,I)}$. The claimed result for the generating function $W_{l,p}^{(\alpha,I)}(t)$ readily follows (absolute convergence is granted because $\sum_{\alpha' \in \frac{\pi}{2}\mathbb{Z}} W_{l,p}^{(\alpha')}(t) < \infty$). □

Inspired by this result let us introduce for $\beta_\pm \in \frac{\pi}{4}\mathbb{Z}$ such that $0 \in I$ and $\alpha \in \frac{\pi}{2}\mathbb{Z} \cap I$, the operator $\mathbf{B}_k^{(\alpha,\beta_-,\beta_+)}$ on $\mathcal{D}$ defined by

$$\mathbf{B}_k^{(\alpha,\beta_-,\beta_+)} := \sum_{n=-\infty}^{\infty} \left(\mathbf{Y}_k^{(|\alpha+n\delta|)} - \mathbf{Y}_k^{(|2\beta_+-\alpha+n\delta|)} \right), \qquad \delta := 2(\beta_+ - \beta_-). \tag{14}$$

By Theorem 1(i) it is well-defined, compact and self-adjoint and has eigenvalues

$$\frac{2K(k)}{\pi}\frac{1}{m}\sum_{n=-\infty}^{\infty}\left(q_k^{|\alpha+n\delta|m/\pi} - q_k^{|2\beta_+-\alpha+n\delta|m/\pi}\right) = \frac{2K(k)}{\pi}\,\frac{q_k^{2m\beta_-/\pi}-1}{m\,q_k^{m\alpha/\pi}}\,\frac{q_k^{2m\alpha/\pi}-q_k^{2m\beta_+/\pi}}{q_k^{2m\beta_-/\pi}-q_k^{2m\beta_+/\pi}}. \tag{15}$$

Lemma 5 then tells us that

$$W_{l,p}^{(\alpha,I)}(t) = \left\langle e_l, \mathbf{B}_k^{(\alpha,\beta_-,\beta_+)} e_p \right\rangle_{\mathcal{D}} \tag{16}$$

holds under the conditions stated in the lemma, which exactly verifies part (ii) and (iii) for $0, \alpha \in I$ and $\beta_\pm$ finite.

Similarly when $\beta_- = -\infty$ or $\beta_+ = \infty$ one may introduce the operators $\mathbf{B}_k^{(\alpha,-\infty,\beta_+)} = \mathbf{Y}_k^{(\alpha)} - \mathbf{Y}_k^{(2\beta_+-\alpha)}$ and $\mathbf{B}_k^{(\alpha,\beta_-,\infty)} = \mathbf{Y}_k^{(\alpha)} - \mathbf{Y}_k^{(-2\beta_--\alpha)}$. It is straightforward to check that then (16) still holds, and that the eigenvalues are given by (15) in the appropriate limit $\beta_- \to -\infty$ or $\beta_+ \to \infty$.

Next, let us consider the case $0 < \alpha \in \frac{\pi}{2}\mathbb{Z}$ and $I = (0, \alpha)$. The case $\alpha = \pi/2$ with the corresponding operator $\mathbf{A}_k^{(\pi/2)} = \mathbf{A}_k$ has already been settled above, so let us assume $\alpha \geq \pi$. Any such walk $w \in \mathcal{W}_{l,p}^{(\alpha,(0,\alpha))}$ is naturally encoded in a triple $w^{(1)}, w^{(2)}, w^{(3)}$ of walks with $w^{(1)} \in \mathcal{A}_{l_1,p}$, $w^{(2)} \in \mathcal{W}_{l_2,l_1}^{(\alpha-\pi,(-\pi/2,\alpha-\pi/2))}$ and $w^{(3)} \in \mathcal{A}_{l,l_2}$ for some $l_1, l_2 \geq 1$. Hence

$$W_{l,p}^{(\alpha,(0,\alpha))}(t) = \sum_{l_1,l_2=1}^{\infty} \frac{1}{l l_2} \left\langle e_l, \mathbf{A}_k e_{l_2} \right\rangle_{\mathcal{D}} \left\langle e_{l_2}, \mathbf{B}_k^{(\alpha,-\pi/2,\alpha-\pi/2)} e_{l_1} \right\rangle_{\mathcal{D}} \frac{1}{l_1 p} \left\langle e_{l_1}, \mathbf{A}_k e_p \right\rangle_{\mathcal{D}}$$
$$= \frac{1}{lp} \left\langle e_l, \mathbf{A}_k \mathbf{B}_k^{(\alpha,-\pi/2,\alpha-\pi/2)} \mathbf{A}_k e_p \right\rangle_{\mathcal{D}}.$$

One may easily verify the claimed eigenvalues of $\mathbf{A}^{(\alpha)} := \mathbf{A}_k \mathbf{B}_k^{(\alpha,-\pi/2,\alpha-\pi/2)} \mathbf{A}_k$ and its compactness.

Finally, a similar argument shows that for $0 < \alpha \in \frac{\pi}{2}\mathbb{Z}$, $0 > \beta_- \in \frac{\pi}{4}\mathbb{Z}$,

$$W_{l,p}^{(\alpha,(\beta_-,\alpha))}(t) = \sum_{l_1=1}^{\infty} \frac{1}{l l_1} \left\langle e_l, \mathbf{A}_k e_{l_1} \right\rangle_{\mathcal{D}} \left\langle e_{l_1}, \mathbf{B}_k^{(\alpha-\pi/2,\beta_-,\alpha)} e_p \right\rangle_{\mathcal{D}} = \frac{1}{l} \left\langle e_l, \mathbf{A}_k \mathbf{B}_k^{(\alpha-\pi/2,\beta_-,\alpha)} e_p \right\rangle_{\mathcal{D}},$$

and once again one may directly verify the eigenvalues of $\mathbf{A}_k \mathbf{B}_k^{(\alpha-\pi/2,\beta_-,\alpha)}$. This finishes the proof of Theorem 1.

3 Excursions

Recall from the introduction the set of excursions $\mathcal{E}$ consisting of (non-empty) simple diagonal walks starting and ending at the origin with no intermediate returns. For such an excursion $w \in \mathcal{E}$ we have a well-defined winding angle sequence $(\theta_i^w)_{i=0}^{|w|}$ with $\theta_1^w = \theta_0^w = 0$ and $\theta^w = \theta_{|w|}^w = \theta_{|w|-1}^w$. Our first goal is to compute the generating function of excursions with winding angle equal to $\alpha \in \frac{\pi}{2}\mathbb{Z}$,

$$F^{(\alpha)}(t) := \sum_{w \in \mathcal{E}} t^{|w|} \mathbf{1}_{\{\theta_w = \alpha\}}. \tag{17}$$

To this end we cannot directly apply Theorem 1(i) because the excursions start and end at the origin. Nevertheless, a combinatorial trick allows us to relate $F^{(\alpha)}(t)$ to the generating functions $W_{l,p}^{(\alpha')}$ with $\alpha' > |\alpha|$.

Lemma 6. *For $\alpha \in \frac{\pi}{2}\mathbb{Z}$, $F^{(\alpha)}(t)$ may be expressed as the absolutely convergent sum*

$$F^{(\alpha)}(t) = 4 \sum_{m,l,p=1}^{\infty} (-1)^{l+p+m+1} m \, W_{2l,2p}^{(|\alpha|+m\pi/2)}(t).$$

Proof. For the absolute convergence it suffices to note that $\sum_{l,p=1}^{\infty} W_{l,p}^{(\pi/2)}(t) \leq 1/(1-4t)$ and for $n > 0$

$$\sum_{l,p=1}^{\infty} W_{l,p}^{(n\pi/2)}(t) \leq \left(\sum_{l,p=1}^{\infty} W_{l,p}^{(\pi/2)}(t) \right)^n \leq (1-4t)^{-n}.$$

Next we use that for $\alpha' \in \frac{\pi}{2}\mathbb{Z}_{>0}$, the sets $\bigcup_{p \in 4\mathbb{Z}_{>0}} \mathcal{W}_{l,p}^{(\alpha')}$ and $\bigcup_{p \in 4\mathbb{Z}_{>0}-2} \mathcal{W}_{l,p}^{(\alpha')}$ are nearly in bijection. Indeed, a walk in the former is mapped to a unique walk in the latter by moving its starting point by

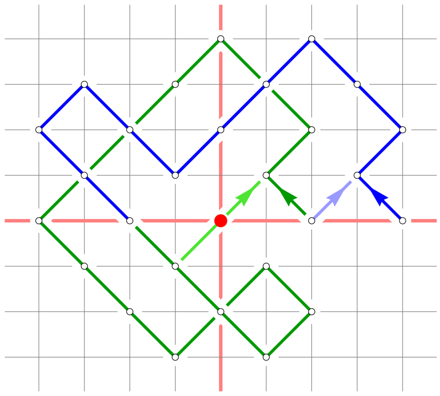


Figure 12: By changing the starting point the dark blue walk in $\mathcal{W}^{(\pi)}_{2,4}$ is mapped to a walk in $\mathcal{W}^{(\pi)}_{2,2}$. The dark green walk $w \in \mathcal{W}^{(\pi)}_{2,2}$ satisfies $|w_1| = |w_{|w|-1}| = \sqrt{2}$, such that moving its starting point and endpoint to the origin gives an excursion.

$(\pm 2, 0)$ depending on the direction of the first step (see Figure 12), while keeping the other sites fixed. It is not hard to see that any walk $\bigcup_{p\in 4\mathbb{Z}_{>0}-2} \mathcal{W}^{(\alpha')}_{l,p}$ is obtained in such way except for those walks in $\mathcal{W}^{(\alpha')}_{l,2}$ that have $w_1 = (1, \pm 1)$. The generating function of such walks is therefore given by

$$\sum_{w\in\mathcal{W}^{(\alpha')}_{l,2}} t^{|w|}\mathbf{1}_{\{|w_1|=\sqrt{2}\}} = \sum_{p=1}^{\infty}(-1)^{p+1}W^{(\alpha')}_{l,2p}(t).$$

An analogous argument for the endpoint then yields

$$S^{(\alpha')}(t) := \sum_{w\in\mathcal{W}^{(\alpha')}_{2,2}} t^{|w|}\mathbf{1}_{\{|w_1|=|w_{|w|-1}|=\sqrt{2}\}} = \sum_{p,l=1}^{\infty}(-1)^{p+l}W^{(\alpha')}_{2l,2p}(t). \tag{18}$$

Let X be the set of four possible tuples $(w_1, w_{|w|-1}, \theta^w_{|w|-1} - \theta^w_1)$, then we may write

$$S^{(\alpha')}(t) = \sum_{(x,y,\alpha)\in X}\sum_{w\in\mathcal{W}^{(\alpha')}_{2,2}} t^{|w|}\mathbf{1}_{\{w_1=x,\, w_{|w|-1}=y\}} = \sum_{(x,y,\alpha)\in X}\sum_{w\in\mathcal{E}} t^{|w|}\mathbf{1}_{\{w_1=x,\, w_{|w|-1}=y\}}\mathbf{1}_{\{\theta^w=\alpha\}},$$

where we used the obvious mapping to excursions by merely moving the starting point and endpoint to the origin (see Figure 12). By the rotational symmetry of the set of excursions we may replace the $\mathbf{1}_{\{w_1=x,\, w_{|w|-1}=y\}}$ by $1/4$ in the last sum. By observing that α takes exactly the four values $\alpha' - \pi/2$, α', α', $\alpha' + \pi/2$, one finds that

$$S^{(\alpha')}(t) = \frac{1}{4}\sum_{(x,y,\alpha)\in X}\sum_{w\in\mathcal{E}} t^{|w|}\mathbf{1}_{\{\theta^w=\alpha\}} = \frac{1}{4}F^{(\alpha'-\pi/2)}(t) + \frac{1}{2}F^{(\alpha')}(t) + \frac{1}{4}F^{(\alpha'+\pi/2)}(t).$$

From here it is an easy check by substitution that for $\alpha \in \frac{\pi}{2}\mathbb{Z}_{>0}$,

$$4\sum_{m=1}^{\infty}(-1)^{m+1}\, m\, S^{(\alpha+m\pi/2)}(t) = F^{(\alpha)}(t) = F^{(-\alpha)}(t),$$

which together with (18) yields the claimed identity. □

We are now in the position to apply Theorem 1(i) and explicitly evaluate $F^{(\alpha)}(t)$, as well as its "characteristic function"

$$F(t,b) := \sum_{w\in\mathcal{E}} t^{|w|}e^{ib\theta^w} = \sum_{\alpha\in\frac{\pi}{2}\mathbb{Z}} F^{(\alpha)}(t)\, e^{ib\alpha}.$$

Proposition 4. *The excursion generating functions are given by*

$$F^{(\alpha)}(t) = \frac{2\pi}{K(k)} \sum_{n=1}^{\infty} \frac{(1-q_k^n)^2}{1-q_k^{4n}} q_k^{n(\frac{2}{\pi}|\alpha|+1)}, \tag{19}$$

$$F(t,b) = \begin{cases} \frac{1}{\cos\left(\frac{\pi b}{2}\right)} \left[1 - \frac{\pi \tan\left(\frac{\pi b}{4}\right)}{2K(k)} \frac{\theta_1'\left(\frac{\pi b}{4}, \sqrt{q_k}\right)}{\theta_1\left(\frac{\pi b}{4}, \sqrt{q_k}\right)}\right] & \text{for } b \in \mathbb{R}\setminus\mathbb{Z}, \\ 1 - \frac{\pi}{2K(k)} & \text{for } b \in 4\mathbb{Z}, \\ 1 - \frac{2E(k)}{\pi} & \text{for } b \in 2\mathbb{Z}+1, \\ -1 + \frac{4E(k)}{\pi} - (1-k^2)\frac{2K(k)}{\pi} & \text{for } b \in 4\mathbb{Z}+2, \end{cases} \tag{20}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, and $\theta_1(z,q)$ is the Jacobi theta function

$$\theta_1(z,q) := 2\sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin((2n+1)z). \tag{21}$$

Proof. Combining Lemma 6 with Theorem 1(i) we find

$$\begin{aligned} F^{(\alpha)}(t) &= 4 \sum_{m,l,p=1}^{\infty} (-1)^{l+p+m+1} m \left\langle e_{2l}, \mathrm{Y}_k^{(|\alpha|+m\pi/2)} e_{2p} \right\rangle_{\mathfrak{D}} \\ &= \frac{4K(k)}{\pi} \sum_{m,n=1}^{\infty} (-1)^{m+1} \frac{m}{n} q_k^{n(\frac{2}{\pi}|\alpha|+m)} \frac{1}{\|f_{2n}\|^2} \left(\sum_{p=1}^{\infty} (-1)^p \left\langle f_{2n}, e_{2p}\right\rangle_{\mathfrak{D}}\right)^2 \\ &= \frac{4K(k)}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \frac{q_k^{n(\frac{2}{\pi}|\alpha|+1)}}{(1+q_k^n)^2} \frac{1}{\|f_{2n}\|^2} \left(\sum_{p=1}^{\infty} (-1)^p \left\langle f_{2n}, e_{2p}\right\rangle_{\mathfrak{D}}\right)^2, \end{aligned} \tag{22}$$

where we used that $\left\langle f_n, e_{2p}\right\rangle_{\mathfrak{D}} = 0$ for n odd. Since $f_{2m}(z)$ has radius of convergence larger than one, we have

$$\sum_{p=1}^{\infty} (-1)^p \left\langle f_{2n}, e_{2p}\right\rangle_{\mathfrak{D}} = \sum_{p=1}^{\infty} (-1)^p\, 2p\, [z^{2p}] f_{2n}(z) = i\, \frac{f_{2n}'(i) - f_{2n}'(-i)}{2}.$$

From the definition (1) of v_{k_1} one may read off that $v_{k_1}'(i) = 1/(4K(k_1)(1+k_1)) = 1/(4K(k))$. Together with (2) we then find

$$\sum_{p=1}^{\infty} (-1)^p \langle e_{2p}, f_{2n}\rangle = (-1)^n \frac{\pi}{K(k)} n \sinh(4\pi n T_k) = (-1)^n \frac{\pi}{2K(k)} n (q_k^{-n} - q_k^n).$$

Combining with (22) and (8) we arrive at

$$F^{(\alpha)}(t) = \frac{2\pi}{K(k)} \sum_{n=1}^{\infty} \frac{(q_k^{-n} - q_k^n)^2}{(1+q_k^n)^2} \frac{q_k^{n(\frac{2}{\pi}|\alpha|+1)}}{q_k^{-2n} - q_k^{2n}} = \frac{2\pi}{K(k)} \sum_{n=1}^{\infty} \frac{(1-q_k^n)^2}{1-q_k^{4n}} q_k^{n(\frac{2}{\pi}|\alpha|+1)}.$$

Using the identity

$$\sum_{\alpha \in \frac{\pi}{2}\mathbb{Z}} x^{\frac{2}{\pi}|\alpha|} e^{i\pi b\alpha} = \frac{x^{-1} - x}{x + x^{-1} - 2\cos(\pi b/2)} \qquad \text{for } 0 < x < 1$$

we obtain with some algebraic manipulation

$$F(t,b) = \frac{2\pi}{K(k)} \sum_{n=1}^{\infty} \frac{(1-q_k^n)^2}{1-q_k^{4n}} \frac{1-q_k^{2n}}{q_k^n + q_k^{-n} - 2\cos(\pi b/2)} \tag{23}$$

$$= \frac{2\pi}{K(k)} \sum_{n=1}^{\infty} \left(1 - \frac{2}{q_k^n + q_k^{-n}}\right) \frac{1}{q_k^n + q_k^{-n} - 2\cos(\pi b/2)} \tag{24}$$

$$= \frac{2\pi}{K(k)} \sum_{n=1}^{\infty} \left[\left(1 - \frac{1}{\cos(\pi b/2)}\right) \frac{1}{q_k^n + q_k^{-n} - 2\cos(\pi b/2)} + \frac{1}{\cos(\pi b/2)} \frac{1}{q_k^n + q_k^{-n}}\right], \tag{25}$$

where the last equality only holds for $b \in \mathbb{R} \setminus (2\mathbb{Z}+1)$. The first term in the sum can be handled for $b \in \mathbb{R} \setminus \mathbb{Z}$ by recognizing that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{q_k^n + q_k^{-n} - 2\cos(\pi b/2)} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_k^{mn} \frac{\sin(\pi b m/2)}{\sin(\pi b/2)} = \sum_{m=1}^{\infty} \frac{q_k^m}{1-q_k^m} \frac{\sin(\pi b m/2)}{\sin(\pi b/2)} \\ &= \frac{1}{4\sin(\pi b/2)} \left[\frac{\theta_1'(\pi b/4, \sqrt{q_k})}{\theta_1(\pi b/4, \sqrt{q_k})} - \cot(\pi b/4)\right],\end{aligned}$$

where in the last equality we used [1, 16.29.1]. The second term follows from [1, 17.3.22],

$$\sum_{n=1}^{\infty} \frac{1}{q_k^n + q_k^{-n}} = -\frac{1}{4} + \frac{K(k)}{2\pi}.$$

Hence, for $b \in \mathbb{R} \setminus \mathbb{Z}$ we have

$$F(t,b) = \frac{1}{\cos\left(\frac{\pi b}{2}\right)} \left[1 - \frac{\pi \tan\left(\frac{\pi b}{4}\right)}{2K(k)} \frac{\theta_1'\left(\frac{\pi b}{4}, \sqrt{q_k}\right)}{\theta_1\left(\frac{\pi b}{4}, \sqrt{q_k}\right)}\right].$$

It remains to check the value of $F(t,b)$ at $b = 0, 1, 2$, since $F(t, b+4) = F(t,b)$ and $F(t,-b) = F(t,b)$. Starting from (24) or (25) and using [15, (23) and (26)] we obtain

$$\begin{aligned}F(t,0) &= \frac{2\pi}{K(k)} \sum_{n=1}^{\infty} \frac{1}{q_k^n + q_k^{-n}} = 1 - \frac{\pi}{2K(k)}, \\ F(t,1) &= \frac{4\pi}{K(k)} \left[\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{q_k^n + q_k^{-n}} - \sum_{n=1}^{\infty} \frac{1}{(q_k^n + q_k^{-n})^2}\right] = \frac{4\pi}{K(k)} \left[\frac{K(k)}{4\pi} - \frac{E(k)K(k)}{2\pi^2}\right], \\ F(t,2) &= \frac{2\pi}{K(k)} \left[2\sum_{n=1}^{\infty} \frac{1}{(q_k^{n/2} + q_k^{-n/2})^2} - \sum_{n=1}^{\infty} \frac{1}{q_k^n + q_k^{-n}}\right] \\ &= \frac{2\pi}{K(k)} \left[2\sum_{n=1}^{\infty} \frac{1}{(q_k^n + q_k^{-n})^2} + 2\sum_{n=1}^{\infty} \frac{1}{(q_k^{n-1/2} + q_k^{1/2-n})^2} - \sum_{n=1}^{\infty} \frac{1}{q_k^n + q_k^{-n}}\right] \\ &= \frac{2\pi}{K(k)} \left[2\frac{E(k)K(k)}{\pi^2} - (1-k^2)\frac{K^2(k)}{\pi^2} - \frac{K(k)}{2\pi}\right].\end{aligned}$$

□

The complete elliptic integrals of the first and second kind, $K(k)$ and $E(k)$, are well known not to be represented by algebraic power series in k. Hence, the same is true for $F(t,b)$ at integer values of b. The situation is surprisingly different at other rational values of b:

Corollary 1. *$t \mapsto F(t,b)$ is algebraic for all $b \in \mathbb{Q} \setminus \mathbb{Z}$.*

Proof. Using the Landen transformation $\theta_1'(u,q)/\theta_1(u,q)+\theta_4'(u,q)/\theta_4(u,q) = \theta_1'(u,\sqrt{q})/\theta_1(u,\sqrt{q})$, which follows e.g. from the series representations [1, 16.29.1 & 16.29.4], we may rewrite $F(t,b)$ as

$$F(t,b) = \frac{1}{\cos\left(\frac{\pi b}{2}\right)}\left[1 - \frac{\pi\tan\left(\frac{\pi b}{4}\right)}{2K(k)}\left(\frac{\theta_1'\left(\frac{\pi b}{4},q_k\right)}{\theta_1\left(\frac{\pi b}{4},q_k\right)} + \frac{\theta_4'\left(\frac{\pi b}{4},q_k\right)}{\theta_4\left(\frac{\pi b}{4},q_k\right)}\right)\right],$$

which in turn can be expressed using Jacobi's zeta function $Z(u,k)$ (see [1, 16.34.1 & 16.34.4]) as[1]

$$F(t,b) = \frac{1}{\cos\left(\frac{\pi b}{2}\right)}\left[1-\tan\left(\frac{\pi b}{4}\right)\left(2Z(u,k)+\frac{\operatorname{cn}(u,k)\operatorname{dn}(u,k)}{\operatorname{sn}(u,k)}\right)\right], \qquad u = K(k)\,b/2. \tag{26}$$

The trigonometric functions as well as the elliptic functions $\operatorname{cn}(\cdot,k)$, $\operatorname{dn}(\cdot,k)$ and $\operatorname{sn}(\cdot,k)$ are well-known to be algebraic at rational multiples of their period, due to the existence of algebraic multiple-angle formulas. The same is true for $Z(u,k)$ since it satisfies an addition formula [1, 17.4.35],

$$Z(u+v,k) = Z(u,k)+Z(v,k) - k^2\operatorname{sn}(u,k)\operatorname{sn}(v,k)\operatorname{sn}(u+v,k), \tag{27}$$

and $Z(K(k)n,k) = 0$ for all $n\in\mathbb{Z}$. Hence, for any rational value of b one can express $Z(K(k)b/2,k)$ as a polynomial in $\operatorname{sn}(\cdot,k)$ evaluated at rational multiples of its period, which are algebraic in k. □

Corollary 2. *Consider a simple random walk on $\mathbb{Z}^2$ started at the origin. The probability that its winding angle around the origin upon its first return equals $\pi m/2$ for $m\in\mathbb{Z}$ is*

$$\frac{1}{\pi}\left(-\psi\left(\frac{|m|+1}{4}\right)+2\psi\left(\frac{|m|+2}{4}\right)-\psi\left(\frac{|m|+3}{4}\right)\right),$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function.

Proof. Let $m\in\mathbb{Z}$ be fixed. First we rewrite the sum in (19) as

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(1-q_k^n)^2}{1-q_k^{4n}}q_k^{n(|m|+1)} &= \sum_{n=1}^{\infty}\sum_{p=0}^{\infty} q_k^{n(|m|+4p+1)}(1-q_k^n)^2\\ &= \sum_{p=0}^{\infty}\left[\frac{1}{1-q_k^{|m|+4p+1}} - \frac{2}{1-q_k^{|m|+4p+2}} + \frac{1}{1-q_k^{|m|+4p+3}}\right]\end{aligned}$$

As $q_k\to 1$ the latter is asymptotically equal to

$$\begin{aligned}&\sum_{p=0}^{\infty}\left[\frac{1}{|m|+4p+1}-\frac{2}{|m|+4p+2}+\frac{1}{|m|+4p+3}\right]\frac{1}{1-q_k} + O(1)\\ &= \frac{1}{4}\left(-\psi\left(\frac{|m|+1}{4}\right)+2\psi\left(\frac{|m|+2}{4}\right)-\psi\left(\frac{|m|+3}{4}\right)\right)\frac{1}{1-q_k}+O(1),\end{aligned}$$

where we used the series representation $\psi(x+1)-\psi(1) = \sum_{p=1}^{\infty}\left(\frac{1}{p}-\frac{1}{x+p}\right)$ of the digamma function. Since $K(k) = -\pi K(k')/(\log q_k) = \frac{1}{2}\pi^2/(1-q_k)+O(1)$, we find from (19) that as $t\to 1/4$,

$$F^{(m\pi/2)}(t) \to \frac{1}{\pi}\left(-\psi\left(\frac{|m|+1}{4}\right)+2\psi\left(\frac{|m|+2}{4}\right)-\psi\left(\frac{|m|+3}{4}\right)\right).$$

From the definition (17) it is easily seen that this is precisely the desired probability. □

[1] Thanks to Kilian Raschel for pointing out this relation!

Before me move on let us have a look at the asymptotics of $F(t,b)$

Lemma 7. *For fixed $b \in [0,2]$ the coefficients of $t \mapsto F(t,b)$ satisfy the estimate*

$$[t^{2l}]F(t,b) \sim \begin{cases} \sin^2\left(\frac{\pi b}{4}\right)\frac{\Gamma(1+b)}{\pi}\frac{4^{2(l+1-b)}}{l^{b+1}} & \text{for } b \in (0,2) \\ \frac{\pi}{l\log^2 l}4^{2l} & \text{for } b = 0 \\ \frac{1}{4\pi l^3}4^{2l} & \text{for } b = 2. \end{cases}$$

The analogous results for other $b \in \mathbb{R}$ follow from $F(t,b) = F(t,b+4) = F(t,4-b)$.

Proof. General properties of the various functions involved indicate that for fixed $b \in \mathbb{R}$, $k \mapsto F(k/4,b)$ is analytic in a domain that includes $\{z \in \mathbb{C} : |z| < 1+\epsilon\} \setminus \{z \in \mathbb{R} : |z| \geq 1\}$ for some $\epsilon > 0$. Hence, we should focus on the behaviour of $F(k/4,b)$ as $k^2 \to 1$. For b fixed we may use Jacobi's identity (see e.g. Akhiezer, §22 (6))

$$\theta_1\left(\frac{\pi b}{4}, \sqrt{q_k}\right) = \frac{i}{\sqrt{2T_k}}e^{-\frac{\pi b^2}{32T_k}}\theta_1\left(-i\frac{\pi b}{8T_k}, q_{k'}^2\right).$$

Taking logarithmic derivatives in b on both sides, we find

$$\begin{aligned}\frac{\pi}{4K(k)}\frac{\theta_1'\left(\frac{\pi b}{4},\sqrt{q_k}\right)}{\theta_1\left(\frac{\pi b}{4},\sqrt{q_k}\right)} &= -\frac{1}{K(k)}\left(\frac{i\pi}{8T_k}\frac{\theta_1'\left(-i\frac{\pi b}{8T_k}, q_{k'}^2\right)}{\theta_1\left(-i\frac{\pi b}{8T_k}, q_{k'}^2\right)} + \frac{\pi b}{16T_k}\right) \\ &= -\frac{\pi}{2K(k')}\left(i\frac{\theta_1'\left(-i\frac{\pi b}{8T_k}, q_{k'}^2\right)}{\theta_1\left(-i\frac{\pi b}{8T_k}, q_{k'}^2\right)} + \frac{b}{2}\right).\end{aligned}$$

From the definition (21) it is clear that $\theta_1'(z,q_{k'}^2)/\theta_1(z,q_{k'}^2) = \cot(z) + O(\sin(2z)q_{k'}^4)$ and $K(k') = \pi/2 + O(1-k^2)$, therefore for $0 < b < 2$ we find

$$\begin{aligned}\frac{\pi}{4K(k)}\frac{\theta_1'\left(\frac{\pi b}{4},\sqrt{q_k}\right)}{\theta_1\left(\frac{\pi b}{4},\sqrt{q_k}\right)} &= -\frac{b}{2} + \coth\left(\frac{\pi b}{8T_k}\right) + O\left(\sinh\left(\frac{\pi b}{4T_k}\right)q_{k'}^4\right) \\ &= -\frac{b}{2} + \frac{q_{k'}^{-b/2} + q_{k'}^{b/2}}{q_{k'}^{-b/2} - q_{k'}^{b/2}} + O\left((q_{k'}^{-b} - q_{k'}^{b})q_{k'}^4\right) = 1 - \frac{b}{2} + 2q_{k'}^b + O\left(q_{k'}^{2b} + q_{k'}^{4-b}\right).\end{aligned}$$

Hence, for $b \in (0,1) \cup (1,2)$ as $k \to \pm 1$ we find[2]

$$F(t,b) = \frac{1 + (b-2)\tan\left(\frac{\pi b}{4}\right)}{\cos\left(\frac{\pi b}{2}\right)} - 4\frac{\tan\left(\frac{\pi b}{4}\right)}{\cos\left(\frac{\pi b}{2}\right)}q_{k'}^b + O\left(q_{k'}^{2b} + q_{k'}^{4-b}\right).$$

Since $q_{k'} = (1-k^2)/16 + \mathcal{O}((1-k^2)^2)$, the dominant singularity is proportional to $(1-k^2)^b$ and then transfer theorems tells us that

$$[t^{2l}]F(t,b) \sim -4^{1-2b}\frac{\tan\left(\frac{\pi b}{4}\right)}{\cos\left(\frac{\pi b}{2}\right)\Gamma(-b)}\frac{4^{2l}}{l^{b+1}} = \sin^2\left(\frac{\pi b}{4}\right)\frac{\Gamma(1+b)}{\pi}\frac{4^{2(l+1-b)}}{l^{b+1}},$$

where we used the reflection formula $\Gamma(1-z)\Gamma(z) = \pi/\sin(\pi z)$.

[2]The constant term is exactly the characteristic function of the probability distribution in Corollary 2.

One may check that this formula is valid for $b = 1$ as well, since $F(t, b) = 1 - 2E(k)/\pi$ has dominant singularity $(1-k^2)\log(1-k^2)/(2\pi)$, and therefore $[t^{2l}]F(t,1) \sim 4^{2l}/(2\pi l^2)$. For $b = 2$, however $F(t,2)$ has singularity $-(1-k^2)^2\log(1-k^2)/(8\pi)$ and $[t^{2l}]F(t,2) \sim 4^{2l}/(4\pi l^3)$, which differs by a factor of two from the general formula. Finally, for $b = 0$ we may use that $[k^{2l}]F(k,0)$ is the probability that a simple random walk first returns to the origin at time $2l$, which is well known to be asymptotically proportional to $\pi/(l\log^2 l)$. □

3.1 Excursions restricted to an angular interval

We turn to the problem of enumerating excursions w with winding angle sequence $(\theta_i^w)_{i=0}^{|w|}$ restricted to lie fully within an interval $I \subset \mathbb{R}$. For convenience we let $\mathcal{E}' = \{w \in \mathcal{E} : w_1 = (1,1)\}$ be the set of excursions that leave the origin in a fixed direction. Then we let $F^{(\alpha,I)}(t)$ be the generating function

$$F^{(\alpha,I)}(t) = \sum_{w\in\mathcal{E}'} t^{|w|}\mathbf{1}_{\{\theta^w=\alpha \text{ and } \theta_i^w\in I \text{ for } 0\le i\le |w|\}}.$$

Notice that with this definition $F^{(\alpha,\mathbb{R})}(t) = F^{(\alpha)}(t)/4$.

Theorem 2. *For $I = (\beta_-, \beta_+)$, $\beta_\pm \in \frac{\pi}{4}\mathbb{Z}$ such that $0 \in I$, and $\alpha \in I \cap \frac{\pi}{2}\mathbb{Z}$ the generating function $F^{(\alpha,I)}$ is given by the finite sum*

$$F^{(\alpha,I)}(t) = \frac{\pi}{8\delta}\sum_{\sigma\in(0,\delta)\cap\frac{\pi}{2}\mathbb{Z}}\left(\cos\left(\frac{4\sigma\alpha}{\delta}\right) - \cos\left(\frac{4\sigma(2\beta_+-\alpha)}{\delta}\right)\right)F\left(t, \frac{4\sigma}{\delta}\right), \qquad \delta := 2(\beta_+-\beta_-) \tag{28}$$

It is algebraic if $\beta_+ - \beta_- \in \frac{\pi}{2}\mathbb{Z} + \frac{\pi}{4}$, or if $\beta_\pm \in \frac{\pi}{2}\mathbb{Z}$ and either $\beta_+ - \beta_- \in \pi\mathbb{Z} + \frac{\pi}{2}$ or $\alpha \in \pi\mathbb{Z} + \frac{\pi}{2}$ or $\beta_+ - \alpha \in \pi\mathbb{Z}$. Moreover, its coefficients satisfy the asymptotic estimate

$$[t^{2l}]F_{n,m,p}(t) \sim \left(\cos\left(\frac{2\pi\alpha}{\delta}\right) - \cos\left(\frac{2\pi(2\beta_+-\alpha)}{\delta}\right)\right)\sin^2\left(\frac{\pi^2}{2\delta}\right)\frac{\Gamma\left(1+\frac{2\pi}{\delta}\right)}{4\delta}\frac{4^{2(l+1-2\pi/\delta)}}{l^{1+2\pi/\delta}}$$

Proof. By a reflection principle that is completely analogous to that used in Lemma 5 we observe that $F^{(\alpha,I)}(t)$ is given by the sum

$$F^{(\alpha,I)}(t) = \frac{1}{4}\sum_{n=-\infty}^{\infty}\left(F^{(\alpha+n\delta)}(t) - F^{(2\beta_+-\alpha+n\delta)}(t)\right), \qquad \delta := 2(\beta_+-\beta_-),$$

where the factor $1/4$ is due to the fourfold difference between $\mathcal{E}$ and $\mathcal{E}'$.

Then the discrete Fourier transform

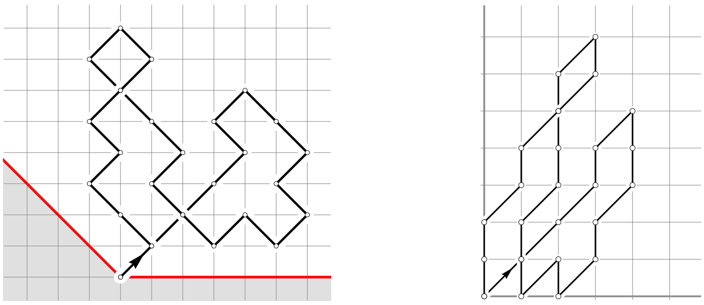
$$\frac{\pi}{2\delta}\sum_{\sigma\in(0,\delta)\cap\frac{\pi}{2}\mathbb{Z}}\left(e^{-4i\sigma\alpha/\delta} - e^{-4i\sigma(2\beta_+-\alpha)/\delta}\right)e^{4i\sigma\alpha'/\delta} = \mathbf{1}_{\{\alpha-\alpha'\in\delta\mathbb{Z}\}} - \mathbf{1}_{\{2\beta_+-\alpha-\alpha'\in\delta\mathbb{Z}\}}$$

leads to

$$F^{(\alpha,I)}(t) = \frac{\pi}{8\delta}\sum_{\sigma\in(0,\delta)\cap\frac{\pi}{2}\mathbb{Z}}\left(e^{-4i\sigma\alpha/\delta} - e^{-4i\sigma(2\beta_+-\alpha)/\delta}\right)\sum_{\alpha'\in\frac{\pi}{2}\mathbb{Z}}F^{(\alpha')}(t)e^{4i\sigma\alpha'/\delta}.$$

Using that $F(t, 4\sigma/\delta) = F(t, 4(\delta-\sigma)/\delta)$, the latter is seen to agree exactly with (28).

Since $F^{(\alpha,I)}(t)$ is expressed in terms of trigonometric functions at rational angles and $F(t,b)$ at rational values of b, it follows from Corollary 1 that $F^{(\alpha,I)}(t)$ is algebraic if the sum does not involve $F(t,b)$ at integer values of b. This is certainly the case when $\delta \in \pi\mathbb{Z} + \frac{\pi}{2}$, since $4\sigma/\delta \notin \mathbb{Z}$ for $\sigma \in (0,\delta) \cap \frac{\pi}{2}\mathbb{Z}$. Suppose now $\beta_\pm \in \frac{\pi}{2}\mathbb{Z}$. Then $\cos(4\sigma\alpha/\delta) - \cos(4\sigma(2\beta_+-\alpha)/\delta) = 0$ for $\sigma = \delta/2$, meaning that $F(t,2)$ does not contribute to $F^{(\alpha,I)}(t)$. Hence, the only remaining obstruction for algebraicity is the case that



(a) (b)

Figure 13: Excursions (a) of length $2n+2$ staying in the angular interval $(-\pi/4, \pi/2)$ are related to walks (b) in the quadrant starting and ending at the origin with $2n$ steps in $\{(0,1),(0,-1),(1,1),(-1,-1)\}$.

$\delta = 4\sigma$ and $\cos(\alpha) - \cos(2\beta_+ - \alpha) \neq 0$, which is evaded by having either $\delta \in 2\pi\mathbb{Z} + \pi$ or $\alpha \in \pi\mathbb{Z} + \frac{\pi}{2}$ or $\beta_+ - \alpha \in \pi\mathbb{Z}$.

According to Lemma 7 the asymptotics of $F^{(\alpha,I)}(t)$ is determined by the terms $\sigma = \pi/2$ and $\sigma = \delta - \pi/2$. If $\delta > \pi$ then these are distinct and equal, while in the case $\delta = \pi$ we noticed that the asymptotics for $F(t,2)$ includes an additional factor of two compared to the formula for $b \in (0,2)$. Hence, in general as $l \to \infty$,

$$[t^{2l}]F^{(\alpha,I)}(t) \sim \frac{\pi}{4\delta}\left(\cos\left(\frac{2\pi\alpha}{\delta}\right) - \cos\left(\frac{2\pi(2\beta_+ - \alpha)}{\delta}\right)\right)\sin^2\left(\frac{\pi^2}{2\delta}\right)\frac{\Gamma\left(1+\frac{2\pi}{\delta}\right)}{\pi}\frac{4^{2(l+1-2\pi/\delta)}}{l^{1+2\pi/\delta}},$$

in accordance with the claimed result. □

As a special case we look at excursions that stay in the angular interval $(-\pi/4, \pi/2)$, see Figure 13.

Corollary 3 (Gessel's lattice path conjecture)**.** *The generating function of excursions that stay in the angular interval $(-\pi/4, \pi/2)$ with winding angle $\alpha = 0$ is*

$$F^{(0,(-\pi/4,\pi/2))}(t) = \frac{1}{4}F\left(t, \frac{4}{3}\right) = \frac{1}{2}\left[\frac{\sqrt{3}\pi}{2K(4t)}\frac{\vartheta_1'\left(\frac{\pi}{3}, \sqrt{q_k}\right)}{\vartheta_1\left(\frac{\pi}{3}, \sqrt{q_k}\right)} - 1\right] = \sum_{n=0}^{\infty} t^{2n+2}\, 16^n \frac{(5/6)_n(1/2)_n}{(2)_n(5/3)_n}. \tag{29}$$

Proof. The first two equalities follow from Theorem 2 and Proposition 4 respectively. It remains to show that our generating function reproduces the known formula

$$\sum_{n=0}^{\infty} t^{2n+2}\, 16^n \frac{(5/6)_n(1/2)_n}{(2)_n(5/3)_n} = \frac{1}{2}\left[{}_2F_1\left(-\frac{1}{2}, -\frac{1}{6}; \frac{2}{3}; (4t)^2\right) - 1\right],$$

which we will do by showing they both solve the same algebraic equation (and checking that the first few terms in the expansion agree).

Denoting $y = \frac{1}{2}F(t, 4/3) + 1$ and using (26) we get (with $k = 4t$)

$$y = \sqrt{3}\left(2Z(u,k) + \frac{\mathrm{cn}\,(u,k)\,\mathrm{dn}\,(u,k)}{\mathrm{sn}\,(u,k)}\right), \qquad u = \frac{2K(k)}{3}.$$

Applying the addition formula (27) to $Z(u+u+u,k)$ twice one finds

$$\begin{aligned} 0 = Z(3u,k) &= 3Z(u,k) - k^2\left(\mathrm{sn}(2u,k)\,\mathrm{sn}^2(u,k) + \mathrm{sn}(2u,k)\,\mathrm{sn}(3u,k)\,\mathrm{sn}(u,k)\right) \\ &= 3Z(u,k) - k^2\,\mathrm{sn}(2u,k)\,\mathrm{sn}^2(u,k), \end{aligned}$$

where we used that $Z(2K(k), k) = \mathrm{sn}(2K(k), k) = 0$. Using the double argument formula [1, 16.18.1]

$$\mathrm{sn}(2u, k) = \frac{2\,\mathrm{sn}(u, k)\,\mathrm{cn}(u, k)\,\mathrm{dn}(u, k)}{1 - k^2\,\mathrm{sn}^4(u, k)}$$

as well as $k^2 - \mathrm{dn}^2(u, k) = -k^2\,\mathrm{cn}^2(u, k) = k^2\,\mathrm{sn}^2(u, k) - k^2$ (see [1, 16.9.1]), one may express y in terms of $x := \mathrm{dn}(u, k)$ as

$$y = -x\,\frac{(1 - x^2)^2 + 3k^2}{(1 - x^2)^2 - k^2}\sqrt{\frac{x^2 + k^2 - 1}{3(1 - x^2)}}. \tag{30}$$

Using the various addition theorems applied to $\mathrm{sn}(u+u+u, k) = 0$ and rewriting in terms of $x = \mathrm{dn}(u, k)$ one finds after a slightly tedious calculation that x solves

$$k^2 = \frac{(1 - x)(1 + x)^3}{1 + 2x}.$$

Eliminating k from (30), y is then seen to be given by

$$y = \frac{3 - (1 - x)^2}{\sqrt{3(1 + 2x)}}.$$

Finally one may check that these solve

$$\begin{aligned}27y^8 &- \left(4608t^4 + 4032t^2 + 18\right) y^4 + \left(-32768t^6 + 67584t^4 + 4224t^2 - 8\right) y^2 \\ &- 65536t^8 - 114688t^6 - 50688t^4 - 448t^2 - 1 = 0,\end{aligned}$$

which is equivalent to the polynomial in [8, Corollary 2] (after substituting $\mathtt{t} \to t^2, \mathtt{T} \to (y-1)/(2t^2)$). □

4 Jacobi elliptic functions are characteristic functions

Theorem 3. *For a simple random walk on $\mathbb{Z}^2$ started at the origin let $\theta_{j+1/2}$ be its winding angle around $(-\frac{1}{2}, -\frac{1}{2})$ up to half-way between its jth and $(j+1)$th site (and let $\theta_{-1/2} = 0$ by convention). Let also ζ_k be a geometric random variable with parameter k, i.e. having distribution $j \mapsto k^j(1 - k)$ on $\mathbb{Z}_{\geq 0}$. Then we have the "hyperbolic secant laws"*

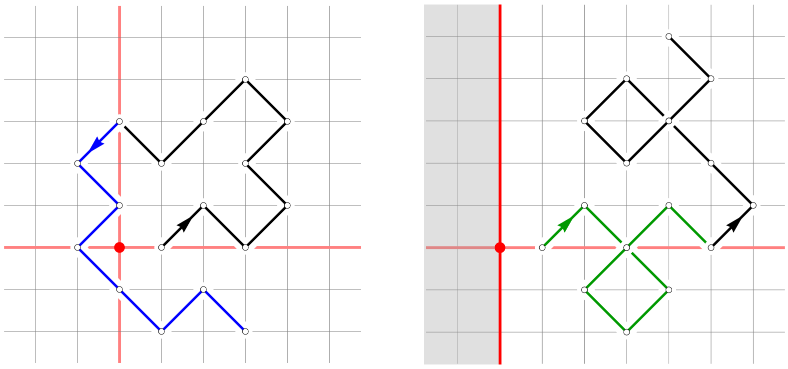
$$\mathbb{P}\left[(n - \tfrac{1}{2})\pi \leq \theta_{\zeta_k - 1/2} < (n + \tfrac{1}{2})\pi\right] = \frac{\pi}{2K(k)}\,\mathrm{sech}\left(n\pi\frac{K(k')}{K(k)}\right),$$
$$\mathbb{P}\left[n\pi \leq \theta_{\zeta_k + 1/2} < (n + 1)\pi\right] = \frac{\pi}{2kK(k)}\,\mathrm{sech}\left((n + \tfrac{1}{2})\pi\frac{K(k')}{K(k)}\right).$$

If we denote by $\{\cdot\}_A : \mathbb{R} \to A$ rounding to the closest element of $A \subset \mathbb{R}$ (conflicts do not arise here) then the corresponding characteristic functions are given by

$$\mathbb{E}\exp\left[ib\{\theta_{\zeta_k - 1/2}\}_{\pi\mathbb{Z}}\right] = \mathrm{dn}(K(k)b, k), \tag{31}$$
$$\mathbb{E}\exp\left[ib\{\theta_{\zeta_k + 1/2}\}_{\pi\mathbb{Z} + \frac{\pi}{2}}\right] = \mathrm{cn}(K(k)b, k), \tag{32}$$

where $\mathrm{cn}(\cdot, k)$ and $\mathrm{dn}(\cdot, k)$ are Jacobi elliptic functions with modulus k.

In order to approach this problem we require a new building block, in the sense of Section 2, that involves walks that start on an axis but end at a general point. In particular we will consider the set $\mathcal{C}_n$ of (possibly empty) walks w starting at $(n, 0)$ and staying strictly inside the positive quadrant, i.e. $w_i \in \mathbb{Z}^2_{>0}$ for $1 \leq i \leq |w|$, with generating function $C_n(t)$.



(a) (b)

Figure 14: (a) An unconstrained walk starting at $(1,0)$ that visits the vertical axis decomposes into a walk in $\mathcal{J}_{3,1}$ (in black) and another unconstrained walk (in blue). (b) A walk starting at $(1,0)$ and avoiding the vertical axis decomposes into a walk in $\mathcal{B}_{5,1}$ (in green) and a walk in $\mathcal{C}_5$ (in black).

Lemma 8. *For m positive and odd we have*

$$\sum_{n=1}^{\infty} C_n(t) \langle e_n, f_m \rangle_{\mathcal{D}} = -\frac{\pi}{8(1-k)K(k)} \frac{m(q_k^{-m} - q_k^m)}{(q_k^{m/4} + q_k^{-m/4})^2}$$

Proof. Let us consider the set of all (possibly empty) diagonal walks starting at $(p,0)$ for p positive and odd, which has generating function $1/(1-k)$. Some of these walks visit the vertical axis and this necessarily happens away from the origin because p is odd. By decomposing such walks at their first visit (see Figure 14a) we find that they have generating function

$$\frac{1}{1-k} \sum_{l=1}^{\infty} 2J_{l,p}(t) = \frac{1}{1-k} \sum_{l=1}^{\infty} \frac{1}{l} \left\langle e_l, 2\mathbf{J}_k e_p \right\rangle_{\mathcal{D}},$$

where the factor 2 takes into account that the first visit may occur at the positive or negative side of the vertical axis. Hence, the remaining walks, those starting at $(p,0)$ and avoiding the vertical axis, have generating function

$$\frac{1}{1-k} - \frac{1}{1-k} \sum_{l=1}^{\infty} \frac{1}{l} \left\langle e_l, 2\mathbf{J}_k e_p \right\rangle_{\mathcal{D}} = \frac{1}{1-k} \sum_{l=1}^{\infty} \frac{1}{l} \left\langle e_l, (I - 2\mathbf{J}_k) e_p \right\rangle_{\mathcal{D}}.$$

By decomposing the latter walks at their last intersection with the horizontal axis (see Figure 14b), we may also express their generating function as

$$\frac{1}{1-k} \sum_{l=1}^{\infty} \frac{1}{l} \left\langle e_l, (I - 2\mathbf{J}_k) e_p \right\rangle_{\mathcal{D}} = 2 \sum_{n=1}^{\infty} C_n(k) \left\langle e_n, \mathbf{B}_k e_p \right\rangle_{\mathcal{D}}.$$

Using that $\left\langle f_m, e_p \right\rangle_{\mathcal{D}} \neq 0$ only when $m+p$ is even, this implies with the help of Propositions 3 and 2 that

$$\begin{aligned}
\sum_{n=1}^{\infty} C_n(k) \langle e_n, f_m \rangle_{\mathcal{D}} &= \frac{\pi}{2K(k)} \frac{m(1+q_k^m)}{1-q_k^m} \sum_{n=1}^{\infty} C_n(k) \langle e_n, \mathbf{B}_k f_m \rangle_{\mathcal{D}} \\
&= \frac{\pi}{4(1-k)K(k)} \frac{m(1+q_k^m)}{1-q_k^m} \sum_{l=1}^{\infty} \frac{1}{l} \langle e_l, (I-2\mathbf{J}_k) f_m \rangle_{\mathcal{D}} \\
&= \frac{\pi}{4(1-k)K(k)} \frac{m(1+q_k^m)}{1-q_k^m} \left(1 - \frac{2}{q_k^{m/2} + q_k^{-m/2}} \right) \sum_{l=1}^{\infty} \frac{1}{l} \langle e_l, f_m \rangle_{\mathcal{D}}.
\end{aligned}$$

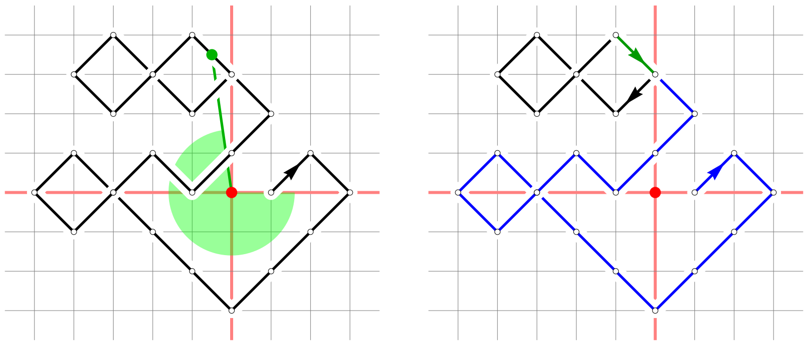


Figure 15: (a) An unconstrained walk w with $\frac{1}{2}(\theta^w_{|w|}+\theta^w_{|w|-1}) \in (-3\pi/2,-\pi)$. (b) Its decomposition into $w' \in \mathcal{W}^{(-3\pi/2)}_{3,1}$ (blue), $w'' \in \mathcal{C}_3$ (black) and $s=(1,-1)$ (green).

Since f_m has a radius of convergence larger than one, we may evaluate

$$\sum_{l=1}^{\infty} \frac{1}{l} \langle e_l, f_m \rangle_{\mathfrak{D}} = \sum_{l=1}^{\infty} [z^l] f_m(z) = f_m(1) = -\cosh(2\pi m T_k) = -\frac{1}{2}\left(q_k^{m/2} + q_k^{-m/2}\right).$$

Combining the last two displays yields the claimed result. ◻

Proof of Theorem 3. Let $\alpha \in \frac{\pi}{2}\mathbb{Z} - \frac{\pi}{4}$. We observe that

$$\mathbb{P}\left[\alpha - \tfrac{\pi}{4} \leq \theta_{\zeta_k+1/2} < \alpha + \tfrac{\pi}{4}\right] = \frac{1-k}{k} G^{(\alpha)}(k/4),$$

where $G^{(\alpha)}(t)$ is the generating function for the set of non-empty simple diagonal walks w that start at $(1,0)$ and have winding angle $\frac{1}{2}(\theta^w_{|w|} + \theta^w_{|w|-1}) \in (\alpha - \frac{\pi}{4}, \alpha + \frac{\pi}{4})$ (see Figure 15a). It is not hard to see that such a walk can be uniquely encoded in a triple (w', w'', s), where $w' \in \mathcal{W}^{(\alpha-\pi/4)}_{n,1} \cup \mathcal{W}^{(\alpha+\pi/4)}_{n,1}$ for some odd $n \geq 1$, $w'' \in \mathcal{C}_n$ and $s \in \{(1,1),(1,-1),(-1,-1),(-1,1)\}$ a single step (see Figure 15b). As a consequence we obtain the relation

$$G^{(\alpha)}(t) = 4t \sum_{n=1}^{\infty} C_n(t) \left(W^{(\alpha-\pi/4)}_{n,1}(t) + W^{(\alpha+\pi/4)}_{n,1}(t)\right).$$

With the help of Theorem 1(i) this evaluates to

$$\begin{aligned} G^{(\alpha)}(t) &= k \sum_{n=1}^{\infty} C_n(t) \left\langle e_n, (\mathrm{Y}_k^{(\alpha-\pi/4)} + \mathrm{Y}_k^{(\alpha+\pi/4)}) e_1 \right\rangle_{\mathfrak{D}} \\ &= k \sum_{m,n=1}^{\infty} C_n(t) \langle e_n, f_m \rangle_{\mathfrak{D}} \frac{\left\langle e_1, (\mathrm{Y}_k^{(\alpha-\pi/4)} + \mathrm{Y}_k^{(\alpha+\pi/4)}) f_m \right\rangle_{\mathfrak{D}}}{\|f_m\|^2_{\mathfrak{D}}} \\ &= k \sum_{m,n=1}^{\infty} C_n(t) \langle e_n, f_m \rangle_{\mathfrak{D}} \frac{8K(k)}{\pi} \frac{q_k^{m|\alpha|/\pi}(q_k^{m/4} + q_k^{-m/4})}{m^2(q_k^{-m} - q_k^{m})} \langle e_1, f_m \rangle_{\mathfrak{D}}. \end{aligned}$$

For m odd we have

$$\langle e_1, f_m \rangle_{\mathfrak{D}} = f'_m(0) = (-1)^{(m+1)/2} 2\pi m\, v'_{k_1}(0) = \frac{(-1)^{(m+1)/2}\pi m}{2\sqrt{k_1} K(k_1)} = \frac{(-1)^{(m+1)/2}\pi m}{kK(k)},$$

while $\langle e_1, f_m\rangle_{\mathcal{D}} = 0$ for m even. Together with Lemma 8 and some manipulation this leads to

$$G^{(\alpha)}(t) = \frac{1}{1-k}\frac{\pi}{K(k)}\sum_{p=0}^{\infty}(-1)^p \frac{q_k^{(2p+1)|\alpha|/\pi}}{q_k^{(2p+1)/4} + q_k^{-(2p+1)/4}}.$$

In particular, the probability that $\theta_{\zeta_k+1/2} \in (n\pi, (n+1)\pi)$ for $n \in \mathbb{Z}$ is then

$$\begin{aligned}\mathbb{P}\left[n\pi \le \theta_{\zeta_k+1/2} < (n+1)\pi\right] &= \frac{1-k}{k}\left(G^{(n\pi+\pi/4)}(k/4) + G^{(n\pi+3\pi/4)}(k/4)\right)\\ &= \frac{\pi}{kK(k)}\sum_{p=0}^{\infty}(-1)^p q_k^{(2p+1)(|n|+1/2)} = \frac{\pi}{kK(k)}\frac{1}{q_k^{n+1/2} + q_k^{-n-1/2}} = \frac{\pi}{2kK(k)}\operatorname{sech}\left(\left(n+\frac{1}{2}\right)\pi\frac{K(k')}{K(k)}\right).\end{aligned}$$

Similarly, for $n \in \mathbb{Z}\setminus\{0\}$ we have

$$\begin{aligned}\mathbb{P}\left[(n-\tfrac{1}{2})\pi \le \theta_{\zeta_k+1/2} < (n+\tfrac{1}{2})\pi\right] &= \frac{1-k}{k}\left(G^{(n\pi-\pi/4)}(k/4) + G^{(n\pi+\pi/4)}(k/4)\right)\\ &= \frac{\pi}{kK(k)}\sum_{p=0}^{\infty}(-1)^p q_k^{(2p+1)|n|} = \frac{\pi}{kK(k)}\frac{1}{q_k^{n} + q_k^{-n}} = \frac{\pi}{2kK(k)}\operatorname{sech}\left(n\pi\frac{K(k')}{K(k)}\right).\end{aligned}$$

According to [15, (19)], $\sum_{n=-\infty}^{\infty}\operatorname{sech}(n\pi K(k')/K(k)) = 2K(k)/\pi$ and therefore

$$\mathbb{P}\left[-\tfrac{1}{2}\pi \le \theta_{\zeta_k+1/2} < \tfrac{1}{2}\pi\right] = 1 - \frac{1}{k} + \frac{\pi}{2kK(k)}.$$

But then for general $n \in \mathbb{Z}$,

$$\begin{aligned}\mathbb{P}\left[(n-\tfrac{1}{2})\pi \le \theta_{\zeta_k-1/2} < (n+\tfrac{1}{2})\pi\right] &= (1-k)\mathbf{1}_{\{n=0\}} + k\,\mathbb{P}\left[(n-\tfrac{1}{2})\pi \le \theta_{\zeta_k+1/2} < (n+\tfrac{1}{2})\pi\right]\\ &= \frac{\pi}{2K(k)}\operatorname{sech}\left(n\pi\frac{K(k')}{K(k)}\right).\end{aligned}$$

With the help of [1, 16.23.2 & 16.23.3] the characteristic functions may finally be expressed as

$$\begin{aligned}\mathbb{E}\exp\left(ib\{\theta_{\zeta_k-1/2}\}_{\pi\mathbb{Z}}\right) &= \frac{\pi}{K(k)}\sum_{n=-\infty}^{\infty}\frac{e^{ibn\pi}}{q_k^{n} + q_k^{-n}} = \operatorname{dn}(K(k)b, k),\\ \mathbb{E}\exp\left(ib\{\theta_{\zeta_k+1/2}\}_{\pi\mathbb{Z}+\frac{\pi}{2}}\right) &= \frac{\pi}{kK(k)}\sum_{n=-\infty}^{\infty}\frac{e^{ib(n+1/2)\pi}}{q_k^{n+1/2} + q_k^{-n-1/2}} = \operatorname{cn}(K(k)b, k).\end{aligned}$$

□

5 Winding angle of loops

Another, rather interesting application of Theorem 1 is the counting of *loops* on $\mathbb{Z}^2$. To be precise, for integer $n \neq 0$, let the set $\mathcal{L}_n$ of *rooted loops of index* n be the set of simple diagonal walks w on $\mathbb{Z}^2\setminus\{(0,0)\}$ that start and end at the same (arbitrary) point and have winding angle $\theta^w = 2\pi n$. The set $\mathcal{L}_n = \mathcal{L}_n^{\text{even}} \cup \mathcal{L}_n^{\text{odd}}$ naturally partitions into the *even loops* $\mathcal{L}_n^{\text{even}}$ supported on $\{(x,y)\in\mathbb{Z}^2 : x+y \text{ even}, (x,y)\neq(0,0)\}$ and the *odd loops* $\mathcal{L}_n^{\text{odd}}$ on $\{(x,y)\in\mathbb{Z}^2 : x+y \text{ odd}\}$.

Theorem 4. *The ("inverse-size biased") generating functions for $\mathcal{L}_n^{even}$ and $\mathcal{L}_n^{odd}$ are given by*

$$\begin{aligned}L_n^{even}(t) &:= \sum_{w\in\mathcal{L}_n^{even}}\frac{t^{|w|}}{|w|} = \frac{1}{|n|}\operatorname{tr}_{\mathcal{D}}\mathbf{P}^{even}\mathbf{J}_k^{(2\pi|n|,-\infty)} = \frac{1}{|n|}\frac{q_k^{4|n|}}{1-q_k^{4|n|}},\\ L_n^{odd}(t) &:= \sum_{w\in\mathcal{L}_n^{odd}}\frac{t^{|w|}}{|w|} = \frac{1}{|n|}\operatorname{tr}_{\mathcal{D}}\mathbf{P}^{odd}\mathbf{J}_k^{(2\pi|n|,-\infty)} = \frac{1}{|n|}\frac{q_k^{2|n|}}{1-q_k^{4|n|}},\end{aligned}$$

where $\mathbf{P}^{even}$ (respectively $\mathbf{P}^{odd}$) is the projection operator onto the even (respectively odd) functions in $\mathcal{D}$.

Proof. Without loss of generality we will take $n > 0$, since the case of negative n then follows from symmetry. Let us consider the subset

$$\hat{\mathcal{L}}_n := \{w \in \mathcal{L}_n : w_0 \in \{(x,0) : x > 0\} \text{ and } \theta^w_i < \theta^w \text{ for } 0 \le i < |w|\}$$

of rooted loops of index $n > 0$ that start on the positive x-axis and that attain the winding angle $2n\pi$ only at the very end. Clearly $\hat{\mathcal{L}}_n = \bigcup_{p\ge1} \mathcal{W}^{(2\pi n,(-\infty,2\pi n))}_{p,p}$ in the notation of Theorem 1. Similarly $\hat{\mathcal{L}}^{\text{even/odd}}_n := \hat{\mathcal{L}}_n \cap \mathcal{L}^{\text{even/odd}} = \bigcup_{p \text{ even/odd}} \mathcal{W}^{(2\pi n,(-\infty,2\pi n))}_{p,p}$. The generating functions of $\hat{\mathcal{L}}^{\text{even}}_n$ and $\hat{\mathcal{L}}^{\text{odd}}_n$ are therefore given by

$$\sum_{w\in\hat{\mathcal{L}}^{\text{even/odd}}_n} t^{|w|} = \sum_{p \text{ even/odd}} \frac{1}{p}\left\langle e_p, \mathbf{J}^{(2\pi n,-\infty)}_k e_p\right\rangle_{\mathcal{D}} = \operatorname{tr}_{\mathcal{D}} \mathbf{P}^{\text{even/odd}}\mathbf{J}^{(2n\pi,-\infty)}_k = \sum_{m \text{ even/odd}} q_k^{2(2m+1)n},$$

where we used that (according to Theorem 1(ii)) $\mathbf{J}^{(2n\pi,-\infty)}_k$ has eigenvalues $(q_k^{2nm})_{m\ge1}$ and that the even and odd subspaces of $\mathcal{D}$ are spanned by the even respectively odd elements of the basis $(f_m)_{m\ge1}$. Hence,

$$\sum_{w\in\hat{\mathcal{L}}^{\text{even}}_n} t^{|w|} = \frac{q_k^{4n}}{1-q_k^{4n}} \qquad \text{and} \qquad \sum_{w\in\hat{\mathcal{L}}^{\text{odd}}_n} t^{|w|} = \frac{q_k^{2n}}{1-q_k^{4n}}. \tag{33}$$

Now suppose we take a general loop $w \in \mathcal{L}_n$. We denote by $w^{(j)} \in \mathcal{L}_n$, $1 \le j \le |w|$, the cyclic permutation of w given by the walk $w^{(j)} := (w_j, w_{j+1}, \ldots, w_{|w|}, w_1, \ldots, w_j)$. We claim that among these $|w|$ cyclic permutations are exactly n elements of $\hat{\mathcal{L}}_n$.

To see this, let $(i_l)_{l=1}^m$ be the sequence of increasing times (in $\{1,2,\ldots,|w|\}$) at which w intersects the positive x-axis. Then $w^{(i_l)}$, $1 \le l \le m$, are potential candidates for walks in $\hat{\mathcal{L}}_n$, since they start on the positive x-axis. For each such walk $w' = w^{(i_l)}$ we may consider the winding angle sequence $(\theta^{w'}_i)_{i=0}^{|w'|}$ as well as the subsequence $(\alpha^{(l)}_j)_{j=0}^m$ of $(\theta^{w'}_i)_{i=0}^{|w'|}$ containing just those angles in $2\pi\mathbb{Z}$. Then $(\alpha^{(l)}_j)_{j=0}^m$ describes a walk on $2\pi\mathbb{Z}$ from 0 to $2\pi n$ with steps in $\{-2\pi, 0, 2\pi\}$, and $w^{(i_l)} \in \hat{\mathcal{L}}_n$ precisely when this walk stays strictly below $2\pi n$ until the very end. Since the walks $(\alpha^{(l)}_j)_{j=0}^m$, $1 \le l \le m$, correspond precisely to an equivalence class under cyclic permutation of the increments, a well-known cycle lemma (or ballot theorem) tells us that the latter condition, hence $w^{(i_l)} \in \hat{\mathcal{L}}_n$, is satisfied for exactly n values of l.

The claim implies that

$$\sum_{w\in\mathcal{L}_n} \frac{t^{|w|}}{|w|} = \frac{1}{n}\sum_{w\in\mathcal{L}_n}\frac{t^{|w|}}{|w|}\sum_{j=1}^{|w|}\mathbf{1}_{\{w^{(j)}\in\hat{\mathcal{L}}_n\}} = \frac{1}{n}\sum_{w\in\hat{\mathcal{L}}_n}\frac{t^{|w|}}{|w|}\sum_{j=1}^{|w|}\mathbf{1}_{\{w^{(j)}\in\mathcal{L}_n\}} = \frac{1}{n}\sum_{w\in\hat{\mathcal{L}}_n} t^{|w|}.$$

This identity restricted to the even and odd subspaces together with (33) then gives the desired expressions. □

For the rest of this section we switch to simple rectilinear rooted loops on $\mathbb{Z}^2$. For such a loop w we let the *index* $I^w : \mathbb{R}^2 \to \mathbb{Z}$ be defined by setting $I^w(z) = 0$ when z lies on the trajectory of w and otherwise $2\pi I^w(z)$ is the winding angle of w around the point z. By a suitable affine transformation we may now equally think of $\mathcal{L}^{\text{odd}}_n$, $n \ne 0$, as the set of simple rooted loops on $\mathbb{Z}^2$ with index $I^w(z)$ with respect to some fixed off-lattice point, say, $(1/2, 1/2)$. Similarly, $\mathcal{L}^{\text{even}}_n$, $n \ne 0$, is in 1-to-1 correspondence with such loops that have index n with respect to a fixed lattice point, say, the origin. The following probabilistic result takes advantage of this point of view.

Corollary 4. *For $l \ge 1$, let $W = (W_i)_{i=0}^{2l}$ be a simple random walk on $\mathbb{Z}^2$ conditioned to return to the origin* *after $2l$ steps. For $n \neq 0$, let C_n be the set of connected components of $(I^W)^{-1}(n)$. Then*

$$\mathbb{E}\left[\sum_{c\in C_n} |c|\right] = \frac{4^{2l}}{\binom{2l}{l}^2}\frac{2l}{n}[k^{2l}]\frac{q_k^{2n}}{1-q_k^{4n}} \sim \frac{l}{2\pi n^2},$$
$$\mathbb{E}\left[\sum_{c\in C_n} (|\partial c| - 2)\right] = \frac{4^{2l}}{\binom{2l}{l}^2}\frac{4l}{n}[k^{2l}]\frac{q_k^{2n}}{1+q_k^{2n}} \sim \frac{2\pi^3 l}{\log^2 l},$$

where $|c|$ is the area of $c \in C_n$ and $|\partial c|$ the boundary length of c.

Proof. The left and right sides of the identities are all seen to be invariant under $n \to -n$, so we may restrict to the case $n > 0$. A simple counting exercise shows that for any $c \in C_n$ the area and boundary length of c can be expressed in terms of the cardinalities $|c \cap \mathbb{Z}^2|$ and $|c \cap (\mathbb{Z} + 1/2)^2|$ as

$$|c| = |c \cap (\mathbb{Z}+1/2)^2| \qquad \text{and} \qquad |\partial c| = 2|c \cap (\mathbb{Z}+1/2)^2| - 2|c \cap \mathbb{Z}^2| + 2.$$

Hence, we have that

$$\sum_{c\in C_n} |c| = \sum_{z\in(\mathbb{Z}+1/2)^2} \mathbf{1}_{\{I^W(z)=n\}} = \sum_{w} \mathbf{1}_{\{I^w(1/2,1/2)=n\}},$$
$$\sum_{c\in C_n} (|c| + 1 - |\partial c|/2) = \sum_{z\in\mathbb{Z}^2} \mathbf{1}_{\{I^W(z)=n\}} = \sum_{w} \mathbf{1}_{\{I^w(0,0)=n\}},$$

where the last sum on both lines is over all possible translations w of W by a vector in $\mathbb{Z}^2$. The collection of translations that have index $I^w(1/2,1/2) = n$ (respectively $I^w(0,0) = n$) indexed by all possible W precisely determines a partition of the loops of length $2l$ in $\mathcal{L}_n^{\text{odd}}$ (respectively $\mathcal{L}_n^{\text{even}}$). Since the probability of any particular walk W is $\binom{2l}{l}^{-2}$ we find with the help of Theorem 4 that

$$\mathbb{E}\left[\sum_{c\in C_n} |c|\right] = \frac{1}{\binom{2l}{l}^2}[t^{2l}] \sum_{w\in\mathcal{L}_n^{\text{odd}}} t^{|w|} = \frac{1}{\binom{2l}{l}^2} 2l\, [t^{2l}] L_n^{\text{odd}}(t) = \frac{4^{2l}}{\binom{2l}{l}^2}\frac{2l}{n}[k^{2l}]\frac{q_k^{2n}}{1-q_k^{4n}},$$
$$\mathbb{E}\left[\sum_{c\in C_n} (|c| + 1 - |\partial c|/2)\right] = \frac{1}{\binom{2l}{l}^2}[t^{2l}] \sum_{w\in\mathcal{L}_n^{\text{even}}} t^{|w|} = \frac{1}{\binom{2l}{l}^2} 2l\, [t^{2l}] L_n^{\text{even}}(t) = \frac{4^{2l}}{\binom{2l}{l}^2}\frac{2l}{n}[k^{2l}]\frac{q_k^{4n}}{1-q_k^{4n}}.$$

The first line and the difference between the two lines agree with the claimed formulas.

Since $k \mapsto q_k$ is analytic in $\mathbb{C} \setminus \{k \in \mathbb{R} : |k| \geq 1\}$ and $|q_k| < 1$ for $|k| = 1$ and $k^2 \neq 1$, it follows that both $k \mapsto q_k^{2n}/(1 - q_k^{4n})$ and $k \mapsto q_k^{2n}/(1 + q_k^{2n})$ are Δ-analytic with singularities at $k = \pm 1$. Since $q_k = 1 + \frac{\pi^2}{\log(1-k^2)} + O(\log^{-2}(1 - k^2))$ as $k \to \pm 1$, we find

$$\frac{q_k^{2n}}{1-q_k^{4n}} = -\frac{1}{4n}\frac{\log(1-k^2)}{\pi^2} + O(1) \qquad \text{and} \qquad \frac{q_k^{2n}}{1+q_k^{2n}} = \frac{1}{2} + \frac{n}{2}\frac{\pi^2}{\log(1-k^2)} + O(\log^{-2}(1-k^2)).$$

Standard transfer theorems (see [23]) then imply

$$[k^{2l}]\frac{q_k^{2n}}{1-q_k^{4n}} \sim \frac{1}{4n\pi^2 l} \qquad \text{and} \qquad [k^{2l}]\frac{q_k^{2n}}{1+q_k^{2n}} \sim \frac{\pi^2 n}{2l\,\log^2 l} \qquad \text{as } l \to \infty.$$

Together with $4^{2l}/\binom{2l}{l}^2 \sim \pi l$ these give rise to the stated asymptotics. □

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