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Zero pattern matrix rings, reachable pairs in digraphs, and Sharp's topological invariant τ

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Abstract

Let R be an associative commutative ring with unity and $M_n(R)$ the ring of $n \times n$ matrices over R. A zero pattern matrix ring is a subring of $M_n(R)$ defined by the location of zero and nonzero entries of matrices in the subring. It is known that the set of zero pattern matrix rings of $M_n(R)$ is in bijective correspondence with the set of transitive directed graphs on n vertices, and also with the set of topologies on a set with n elements.

We define the weight of a zero pattern matrix ring $\mathcal{P}$ to be the maximum number of nonzero entries of a matrix in $\mathcal{P}$, which is the rank of $\mathcal{P}$ as an R-module. In this paper, we study the set $W(n)$ of possible weights of $n \times n$ zero pattern matrix rings. We prove that $W(n)$ can be determined recursively and describe the integers in the set. In particular, we prove that most values in $W(n)$ lie in a interval of $\mathbb{N}$ that can be determined recursively and independently of $W(n)$ itself. By establishing asymptotic bounds on this interval, we are able to provide effective estimates of $|W(n)|$. Finally, we describe techniques to determine those elements of $W(n)$ that lie outside of the aforementioned interval. Since the weight corresponds to the number of edges in a transitive directed graph, as well as to the number of containments among the minimal open sets of a topology, our theorems are applicable to digraphs and topologies as well as to matrix rings.

1 Introduction

Notation 1.1. Throughout, we shall use the following notations.

- $\mathbb{N}$ denotes the positive integers.
- n always denotes an element of $\mathbb{N}$.
- For any $n, m \in \mathbb{N}$, $[n, m] = \{x \in \mathbb{N} \mid n \leqslant x \leqslant m\}$. Similarly, $[0, n] = \{x \in \mathbb{Z} \mid 0 \leqslant x \leqslant n\}$.
- For sets $A, B \subseteq \mathbb{N}$ and any $c \in \mathbb{N}$, we let $A + B = \{a + b \mid a \in A, b \in B\}$ and $c + A = \{c + a \mid a \in A\}$.
- R is an associative commutative ring with unity.

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- $M_{n\times m}(R)$ is the set of $n \times m$ matrices with entries from R, and $M_n(R)$ is the ring of $n \times n$ matrices with entries from R.

The purpose of this paper is to examine a class of subrings of $M_n(R)$ that generalize the familiar rings of upper and lower triangular matrices of $M_n(R)$.

Definition 1.2. Let $\mathcal{U} \subseteq [1,n]\times[1,n]$ be a set of ordered pairs containing (i,i) for all $i \in [1,n]$. For each $i,j \in [1,n]$, let E_{ij} be the matrix whose (i,j)-entry is 1 and all of whose other entries are 0. Let $\mathcal{P} = \sum_{(i,j)\in\mathcal{U}} E_{ij}R \subseteq M_n(R)$. If $\mathcal{P}$ is closed under matrix multiplication, we say that $\mathcal{P}$ is a *zero pattern matrix ring* (zpmr, for short).

Other names for these rings are common in the literature. The above definition comes from [5, Sec. 3], where a zpmr is called a "partial matrix ring." When $R = \mathbb{C}$, a zero pattern matrix ring is often called a "zero pattern matrix algebra". Other terms that have been used are "structural matrix algebra" [3] or "zero-one matrix" [6, 10].

When $\mathcal{U} \subseteq [1,n]\times[1,n]$ is known, the corresponding zpmr $\mathcal{P}$ can be denoted by an $n \times n$ matrix where each entry (i,j) is either $*$ (if $(i,j)\in\mathcal{U}$) or 0 (if $(i,j)\notin\mathcal{U}$). For instance, with $n = 3$ and $\mathcal{U} = \{(1,1),(1,3),(2,2),(2,3),(3,3)\}$ we have

$$\mathcal{P} = \sum_{(i,j)\in\mathcal{U}} E_{ij}R = \begin{bmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}.$$

The condition that $(i,i)\in\mathcal{U}$ for all i guarantees that a zpmr always contains all the diagonal matrices in $M_n(R)$. When $n = 1$, the only zpmr is the 1×1 matrix ring $[*]$. When $n = 2$, there are four possible choices for $\mathcal{U}$, all of which yield zpmrs:

$$\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}, \text{ and } \begin{bmatrix} * & * \\ * & * \end{bmatrix}. \tag{1.3}$$

However, once $n \geqslant 3$, not every choice for $\mathcal{U}$ gives a ring. For instance,

$$\mathcal{P} = \begin{bmatrix} * & * & 0 \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

is not a ring, because

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin \mathcal{P}.$$

Recall that a *preorder* (or *quasiorder*) on a set is a binary relation that is reflexive and transitive. It is well known that $\mathcal{P}$ is a zpmr exactly when $\mathcal{U}$ is a preorder on $[1,n]$. This condition establishes a connection between zpmrs and other structures related to preorders such as posets, transitive directed graphs, and finite topological spaces. The relationship among these structures is formalized in Section 2.

Our main concern in this paper is with the combinatorial aspects of zero pattern matrix rings. We are interested in questions such as:

- How many zpmrs are contained in $M_n(R)$ for each n?
- How many isomorphism classes of zpmrs are contained in $M_n(R)$ for each n?
- If $\mathcal{P}$ is a zpmr, then what are the possible values of $|\mathcal{U}|$?

n	$W(n)$	$[n, n^2] \setminus W(n)$
1	1	$\varnothing$
2	$[2, 4]$	$\varnothing$
3	$[3, 7]$, 9	8
4	$[4, 13]$, 16	14, 15
5	$[5, 19]$, 21, 25	20, $[22, 24]$
6	$[6, 28]$, 31, 36	29, 30, $[32, 35]$
7	$[7, 35]$, $[37, 39]$, 43, 49	36, $[40, 42]$, $[44, 48]$
8	$[8, 52]$, 57, 64	$[53, 56]$, $[58, 63]$
9	$[9, 61]$, 63, $[65, 67]$, 73, 81	62, 64, $[68, 72]$, $[74, 80]$
10	$[10, 77]$, 79, $[82, 84]$, 91, 100	78, 80, 81, $[85, 90]$, $[92, 99]$
11	$[11, 95]$, 97, $[101, 103]$, 111, 121	96, $[98, 100]$, $[104, 110]$, $[112, 120]$
12	$[12, 109]$, $[111, 115]$, 117, $[122, 124]$, 133, 144	110, 116, $[118, 121]$, $[125, 132]$, $[134, 143]$

Table 1: Possible weights of zpmrs in $M_n(R)$, for $n \in [1, 12]$.

The first two questions above are equivalent to questions about, respectively, the number of topologies on an n-element set, and the number of homeomorphism classes of topologies on an n-element set. In the context of topologies, both of these questions are open despite thorough investigation, and are known to be computationally difficult; see, for instance, [2]. We will give more details on the connections among matrices, digraphs, and topologies in Section 2. However, the third question has received far less attention, and it is on this problem that we will concentrate.

Notation 1.4. Given $A \in M_{n\times m}(R)$, for each $(i, j) \in [1, n]\times[1, m]$ we let A_{ij} be the (i, j)-entry of A. When $\mathcal{P} \subseteq M_{n\times m}(R)$, we define $\mathcal{P}_{ij} = \{A_{ij} \mid A \in \mathcal{P}\}$ to be the set of all the (i, j)-entries of the matrices in $\mathcal{P}$. Note that if $\mathcal{P}$ corresponds to $\mathcal{U} \subseteq [1, n] \times [1, n]$ as in Definition 1.2, then each $\mathcal{P}_{ij}$ is equal to either R (if $(i, j) \in \mathcal{U}$) or 0 (if $(i, j) \notin \mathcal{U}$).

Definition 1.5. Let $\mathcal{P} \subseteq M_{n\times m}(R)$ be such that $\mathcal{P}_{ij}$ is 0 or R for each $i \in [1, n]$, $j \in [1, m]$. The *weight* of $\mathcal{P}$, denoted by $w(\mathcal{P})$, is equal to the number of components $\mathcal{P}_{ij}$ that are equal to R. We say that $\mathcal{P}$ is *full* or has *full weight* if $\mathcal{P} = M_{n\times m}(R)$, so that $w(\mathcal{P}) = nm$. We define $W(n) = \{w(\mathcal{P}) \mid \mathcal{P} \subseteq M_n(R) \text{ is a zpmr}\}$ to be the set of possible weights of $n \times n$ zero pattern matrix rings.

There are several equivalent ways to interpret the weight. When $\mathcal{P}$ is a zpmr corresponding to $\mathcal{U} \subseteq [1, n]\times[1, n]$, we have $w(\mathcal{P}) = |\mathcal{U}|$. Algebraically, $w(\mathcal{P})$ is the rank of $\mathcal{P}$ as an R-module, and when $\mathcal{P}$ is written in matrix form, the weight is the number of entries equal to $*$. If $\mathcal{P}$ is a zpmr with associated loopless directed graph Γ and topology $\mathcal{T}$, then $w(\mathcal{P}) - n$ is equal to the number of edges of Γ, and $w(\mathcal{P})$ is equal to the number of containments $U_i \supseteq U_j$ among the minimal open sets $U_i, i \in [1, n]$ (see Section 2).

All of our major theorems deal with the weight sets $W(n)$. For small values of n these sets can be computed by hand. Clearly, $W(1) = \{1\}$, and the list of 2×2 zpmrs in (1.3) shows that $W(2) = [2, 4]$. Since the diagonal components of a zpmr $\mathcal{P}$ are always equal to R and $\mathcal{P} \subseteq M_n(R)$, we will always have $W(n) \subseteq [n, n^2]$. However, once $n > 2$ there exist values $k \in [n, n^2]$ such that $k \notin W(n)$. For instance, there is no 3×3 zpmr of weight 8, although $W(3)$ contains every number in $[3, 9]$ except 8. Table 1 lists $W(n)$ for $n \in [1, 12]$, as well as the complement of $W(n)$ in $[n, n^2]$ (for readability, we have omitted union symbols and set braces).

From Table 1, one can see that $W(n)$ becomes more fragmented as n increases. Nevertheless, there are intriguing patterns in this data. For instance, $W(n)$ never includes integers in the range $[n^2-n+2, n^2-1]$, and for $n \geqslant 5$, $W(n)$ does not meet $[n^2-2n+5, n^2-n]$. Moreover, $W(n)$ always begins with a single interval that contains the majority of the elements of the set.

Definition 1.6. For each $n \in \mathbb{N}$, we define $b(n)$ to be the least integer such that $b(n) \geqslant n$ and there does not exist an $n \times n$ zero pattern matrix ring $\mathcal{P}$ with $w(\mathcal{P}) = b(n)+1$. Equivalently, $b(n)$ is the largest positive integer such that $[n, b(n)] \subseteq W(n)$.

The first occurrence of a concept related to $W(n)$ in the literature seems to be in the context of finite topologies in [10]. As we will see in Section 2, the set of zero pattern matrix rings in $M_n(R)$ are in one-to-one correspondence with finite topologies on n points. Sharp [10] noted this connection to matrices, and, given a finite topology $\mathcal{T}$ with corresponding zpmr $\mathcal{P}$, introduced the topological invariant $\tau(\mathcal{T})$, which is the same as $w(\mathcal{P})$. Sharp also proved that, if $\mathcal{T}$ is a nontrivial topology on n points, then $n \leqslant \tau(\mathcal{T}) \leqslant n^2-n+1$.

The weight set $W(n)$ and the integer $b(n)$ are independently studied in the context of digraphs in [7]. There is a natural bijection between the weight set $W(n)$ and the set $S(n)$ of the possible numbers of *reachable pairs* for a digraph on n vertices (see Section 2). Furthermore, the function $f(n)$ studied in [7] is related to $b(n)$ by $f(n) = b(n) - n$. Indeed, [7, Theorem 6], when translated into our notation, gives a lower bound of

$$b(n) \geqslant n^2 - n \cdot \lfloor n^{0.57} \rfloor + \lfloor n^{0.57} \rfloor.$$

On the other hand, this bound is not asymptotically tight, as it is noted that the lower bound holds for large enough n if the exponent 0.57 is replaced by 0.53. The set $S(n)$ is calculated for $n \leq 208$ in [7], although an efficient method for calculating this set in general or even estimating its size is left as an open problem.

The goal of this article is to prove general statements about $W(n)$, $b(n)$, and related phenomena of the weight sets. First, we are able to calculate the set $W(n)$ recursively.

Theorem 1.7. *For each $n \geqslant 1$, let $\beta_n = \bigcup_{k=1}^{n-1}(n(n-k)+W(k))$. Then,*

$$W(7) = [7, 35] \cup \beta_7 \cup \{49\},$$

and if $n \neq 7$ then

$$W(n) = [n, \lceil (3/4)n^2 \rceil] \cup \beta_n \cup \{n^2\}.$$

We are also able to establish the following recursive formula for $b(n)$, which can be computed independently of $W(n)$.

Theorem 1.8. *Define $L(z) := b(z)-z+3$. Let $z \geqslant 1$ and let n be such that $L(z) \leqslant n < L(z+1)$. If $n \neq 8$, then $b(n) = n^2 - zn + b(z)$.*

Theorem 1.8 then allows us to provide the following alternative recursive procedure for calculating $W(n)$.

Theorem 1.9. *Define $L(z) := b(z) - z + 3$. If $n \geqslant 1$, $z \in \mathbb{N}$ is such that $L(z) \leqslant n < L(z+1)$, and $\omega_n = \bigcup_{k=1}^{z}(n(n-k)+W(k))$, then*

$$W(n) = [n, b(n)] \cup \omega_n \cup \{n^2\}.$$

Finally, we are able to provide bounds on $|W(n)|$.

Theorem 1.10. *Define $L(z) := b(z) - z + 3$. Let $n \geqslant 1$, $n \neq 8$, and let $z \in \mathbb{N}$ be such that $L(z) \leqslant n < L(z+1)$. For each $r \in [1, z]$, let c_r be the cardinality of the set $\{k \in W(r) \mid b(r)+1 \leqslant k \leqslant n+r-2\}$. Then, the following hold:*

(1) $|W(n)| = n^2 - (z+1)n - (z-1) + \sum_{r=1}^{z} (L(r) + c_r)$.

(2) $|W(n)| \geqslant n^2 - (z+1)n - (z-1) + \sum_{r=1}^{z} L(r)$.

(3) $|W(n)| \leqslant n^2 - (z+1)(n-1) + \sum_{r=1}^{z} |W(r)|$.

Throughout, our theorems are stated and proved in terms of matrices. However, each of Theorems 1.7, 1.8, 1.9, and 1.10 can be translated into a theorem on the number of reachable pairs possible in a digraph or a theorem on Sharp's topological invariant τ. These alternate interpretations are stated as consequences of the main theorems; see Corollaries 4.7, 5.13, 5.15 and Remark 6.9.

This paper is organized as follows. Section 2 makes explicit the connections among zero pattern matrix rings, reachable pairs in digraphs, and Sharp's topological invariant τ. In Section 3, we prove that every zero pattern matrix ring is similar to a zero pattern matrix ring in block diagonal form, a structural result that is crucial in later proofs. Section 4 is devoted to the proof of Theorem 1.7, a recursive calculation of the set $W(n)$. Section 5 contains the proofs of Theorems 1.8 and 1.9. Finally, in Section 6, we analyze the gaps in the set $W(n)$ (i.e., those integers that are in $[n, n^2] \setminus W(n)$) and prove Theorem 1.10, which provides bounds on the cardinality of $W(n)$.

2 Connection to Graphs and Topologies

As mentioned in the introduction, $\mathcal{P} \subseteq M_n(R)$ is a zero pattern matrix ring if and only if the associated subset $\mathcal{U} \subseteq [1,n] \times [1,n]$ forms a preorder on $[1,n]$. Such preorders can also be used to define directed graphs and topologies on the n-point set $[1,n]$. The results of this section—especially the material on topologies—are well known. We direct the reader toward references such as [1, 4, 6, 8, 10, 11] for proofs and details.

A *directed graph* or *digraph* $\Gamma = (V, E)$ consists of a set V of vertices along with a set E of ordered pairs of distinct vertices (referred to as directed edges). For a given digraph Γ, a natural question is the *reachability* of the vertex j from the vertex i; that is, whether or not there exists a sequence (or *path*) of directed edges which starts with i and ends with j. If there does exist a path of directed edges from i to j, then we say that (i,j) is a *reachable pair*. The concept of reachability has natural applications to the study of communication within a network and has received significant attention; see [9] for a recent example. If one is concerned only with determining or enumerating the reachable pairs in a digraph Γ, then it suffices to consider the *transitive closure* of $\Gamma = (V, E)$, i.e., the graph whose vertex set is V but whose directed edge set is $\mathscr{R}$, where $\mathscr{R}$ is the smallest binary relation on V that contains E and is transitive. A digraph $\Gamma = (V, E)$ is called a *transitive digraph* if E is a transitive relation on the set V, and so questions about reachability may be reduced to questions about transitive digraphs.

Given a preorder $\mathcal{U}$ on $[1,n]$, we construct a directed graph Γ with vertex set $[1,n]$ and with an edge $i \to j$ if and only if $(i,j) \in \mathcal{U}$ and $i \neq j$. We want our graphs to be loopless, so even though $\mathcal{U}$ contains (i,i) for all $i \in [1,n]$, the graph Γ will not include edges corresponding to such ordered pairs. The transitivity of $\mathcal{U}$ implies that if Γ has edges $i \to j$ and $j \to k$, then there is also an edge $i \to k$ (as long as $i \neq k$), and so the graph constructed in this manner is a transitive digraph. With this setup, it is clear that the corresponding zpmr $\mathcal{P}$ functions like the adjacency matrix of Γ, in that $\mathcal{P}_{ij} \neq 0$ if and only if $i \neq j$ and Γ contains an edge $i \to j$.

The construction of a topology from a preorder is slightly more complicated. Given a topology $\mathcal{T}$ on $[1,n]$, for each i we let $U_i \in \mathcal{T}$ be the minimal open set containing i. That is,

$$U_i = \bigcap_{U \in \mathcal{T}, i \in U} U.$$

The set U_i is open since $\mathcal{T}$ is finite, and the collection $\{U_i\}_i$ forms an open basis of $\mathcal{T}$. Moreover, given distinct $i,j \in [1,n]$, the minimality of U_i and U_j means that either $U_i \cap U_j = \varnothing$, or one of the sets contains the other. This allows us to define a preorder $\mathcal{U}$ on $[1,n]$ by defining $(i,j) \in \mathcal{U}$ if and only if $U_i \supseteq U_j$. In other words, $(i,j) \in \mathcal{U}$ if and only if j is in every open set of $\mathcal{T}$ that contains i.

It is also possible to construct the topology $\mathcal{T}$ directly from the zpmr $\mathcal{P}$. To do this, for each $i \in [1,n]$ let $V_i = \{j \in [1,n] \mid \mathcal{P}_{ij} = R\}$. So, V_i indicates the columns of row i that have nonzero entries. We can use $\{V_i\}_{1 \leqslant i \leqslant n}$ as an open basis for $\mathcal{T}$, and the resulting topology is the same one arising from the underlying preorder $\mathcal{U}$. For instance, let

$$\mathcal{P} = \begin{bmatrix} * & 0 & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}.$$

Then, $V_1 = \{1,3,4\}$, $V_2 = \{2,4\}$, $V_3 = \{3\}$, and $V_4 = \{4\}$. The corresponding topology is

$$\mathcal{T} = \{\varnothing, \{1,3,4\}, \{2,4\}, \{3\}, \{4\}, \{3,4\}, \{2,3,4\}, \{1,2,3,4\}\}.$$

As this discussion illustrates, the use of preorders allows us to formalize relationships among zero pattern matrix rings, digraphs, and topologies.

Lemma 2.1. *Let $n \geqslant 1$. Then, the following sets are in one-to-one correspondence:*

- $\{$*zero pattern matrix rings* $\mathcal{P}$ *of* $M_n(R)$ $\}$
- $\{$*Preorders* $\mathcal{U}$ *on* $[1,n]$ $\}$
- $\{$*Transitive digraphs* Γ *on* $[1,n]$ $\}$
- $\{$*Topologies* $\mathcal{T}$ *on* $[1,n]$ $\}$

Moreover, let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two $n \times n$ zero pattern matrix rings with corresponding graphs Γ_1 and Γ_2, and corresponding topologies $\mathcal{T}_1$ and $\mathcal{T}_1$. Then, $\mathcal{P}_1$ and $\mathcal{P}_2$ are isomorphic as rings if and only if Γ_1 and Γ_2 are isomorphic as graphs, if and only if $\mathcal{T}_1$ and $\mathcal{T}_2$ are homeomorphic as topologies.

Lemma 2.1 means that counting the number of zpmrs (respectively, isomorphism classes) in $M_n(R)$ is the same as counting the number of topologies (respectively, homeomorphism classes) on $[1,n]$, and the latter are well-studied problems. There are no known general formulas or closed forms for either the number of topologies on $[1,n]$, nor for the number of homeomorphism classes, but counts have been made for small values of n. These can be found in the On-line Encyclopedia of Integer Sequences (OEIS). The number of topologies on $[1,n]$ is sequence A000798 in OEIS (`https://oeis.org/A000798`), and the number of homeomorphism classes is sequence A001930 (`https://oeis.org/A001930`).

We close this section by showing how the weight of a zpmr is reflected in the corresponding digraph and topology.

Definition 2.2. Let $\mathcal{T}$ be a topology on $[1,n]$ with minimal open sets $U_1, \ldots, U_n$. Define $\tau(T) \in [n, n^2]$ to be the number of ordered pairs $(i,j) \in [1,n] \times [1,n]$ such that $U_i \supseteq U_j$. In other words, $\tau(T)$ equals the number of containments among the minimal open sets $U_1, \ldots, U_n$.

Lemma 2.3.

(1) *Let $\mathcal{P} \subseteq M_n(R)$ be a zero pattern matrix ring with corresponding transitive digraph Γ and topology $\mathcal{T}$. Then, the following quantities are equal:*

(a) *$w(\mathcal{P})$.*

(b) *The sum of n and the number of edges of Γ.*

(c) *$\tau(\mathcal{T})$.*

(2) *Let $S(n)$ be the set of integers m for which there exists a digraph (not necessarily transitive) on n vertices with m reachable pairs. Then, $S(n) = W(n) - n$ and $|S(n)| = |W(n)|$.*

(3) *Let $T(n)$ be the set of integers m for which there exists a topology $\mathcal{T}$ on n points with $\tau(\mathcal{T}) = m$. Then, $T(n) = W(n)$.*

Proof. (1) The fact that $w(\mathcal{P}) - n$ is the number of edges of Γ is obvious from the construction of Γ. For the other equality, note that $\mathcal{P}_{ij} \neq 0$ if and only if j is in every open set of $\mathcal{T}$ containing i, if and only if $U_i \supseteq U_j$.

(2) A vertex pair (i, j) is reachable in a digraph if and only if there is an edge $i \to j$ in the transitive closure of the digraph. Thus, a digraph on n vertices with m reachable pairs corresponds to a zpmr with weight $n+m$. It follows that $S(n) = W(n)-n$ and $|S(n)| = |W(n)|$.

(3) This is clear, since $\tau(\mathcal{T}) = w(\mathcal{P})$. $\square$

The topological invariant τ was introduced by Sharp in [10], but it seems not to have been studied since. We will not pursue it here, but it would be interesting to know if it could be useful in the enumeration of finite topologies or homeomorphism classes of finite topologies.

3 Triangular Forms

This section is devoted to proving (Theorem 3.3) that any zero pattern matrix ring is similar to a zpmr in block triangular form. This form is not unique, but it provides a standard way to construct zpmrs, and we use it often in later sections. We begin by giving some rules for working with the components $\mathcal{P}_{ij}$ of a zpmr that were defined in Notation 1.4; recall also Definition 1.5. We will take these for granted throughout the remainder of the paper.

Lemma 3.1. *Let $\mathcal{P} \subseteq M_n(R)$ be a zero pattern matrix ring. Then, the following hold for all $i, j \in [1, n]$.*

(1) *$\mathcal{P}_{ij} = 0$ if and only if for all $k \in [1, n]$, at least one of $\mathcal{P}_{ik}$ or $\mathcal{P}_{kj}$ is 0.*

(2) *$\mathcal{P}_{ij} = R$ if and only if there exists $k \in [1, n]$ such that $\mathcal{P}_{ik} = \mathcal{P}_{kj} = R$.*

Proof. By definition, each $\mathcal{P}_{ij}$ is either 0 or R, and so the $\mathcal{P}_{ij}$ can be added or multiplied using the following rules for the operations:

$+$	0	R
0	0	R
R	R	R

$\cdot$	0	R
0	0	0
R	0	R

Since $\mathcal{P}$ is a zpmr, the rules for ordinary matrix multiplication imply that $\mathcal{P}_{ij} = \sum_{k=1}^{n} \mathcal{P}_{ik}\mathcal{P}_{kj}$, and both (1) and (2) follow from this relation. $\square$

Lemma 3.2. *Let $\mathcal{P} \subseteq M_n(R)$ be a zero pattern matrix ring of the form*

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}' & C \\ 0 & \mathcal{P}'' \end{bmatrix}$$

where both $\mathcal{P}'$ and $\mathcal{P}''$ are full matrix rings. Then, either $C = 0$ or C is full.

Proof. There is nothing to prove if $C = 0$, so assume that $C \neq 0$. Let $m \in [1, n-1]$ be such that $\mathcal{P}' = M_m(R)$. Since $C \neq 0$, there exist $k \in [1, m]$ and $\ell \in [m+1, n]$ such that $\mathcal{P}_{k\ell} = R$. Let $i \in [1, m]$ and $j \in [m+1, n]$. Then, $\mathcal{P}_{ik}$, $\mathcal{P}_{k\ell}$, and $\mathcal{P}_{\ell j}$ are all nonzero, so $\mathcal{P}_{ij} = \mathcal{P}_{ik}\mathcal{P}_{k\ell}\mathcal{P}_{\ell j} = R$. It follows that C is full. □

It is clear that if $\mathcal{P} \subseteq M_n(R)$ is a zpmr, then so is $\sigma\mathcal{P}\sigma^{-1}$ for any permutation matrix $\sigma \in M_n(R)$. The matrix $\sigma\mathcal{P}\sigma^{-1}$ is often called a *similarity permutation* of $\mathcal{P}$. Computing $\sigma\mathcal{P}\sigma^{-1}$ is equivalent to starting with the matrix form of $\mathcal{P}$ and performing the row switches determined by left multiplication by σ, and then the column switches determined by right multiplication by σ^{-1}.

Theorem 3.3. *Let $\mathcal{P} \subseteq M_n(R)$ be a zero pattern matrix ring. Then, there exists a permutation matrix $\sigma \in M_n(R)$ such that $\sigma\mathcal{P}\sigma^{-1}$ has the form*

$$\begin{bmatrix} \mathcal{P}_1 & C_{12} & C_{13} & \cdots & C_{1t} \\ 0 & \mathcal{P}_2 & C_{23} & \cdots & C_{2t} \\ 0 & 0 & \mathcal{P}_3 & \cdots & C_{3t} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \mathcal{P}_t \end{bmatrix}$$

where for each $k \in [1, t]$, $\mathcal{P}_k = M_{n_k}(R)$ for some $n_k \in \mathbb{N}$. Moreover, for each $k \in [1, t-1]$ and each $\ell \in [k+1, t]$, the matrix $C_{k\ell}$ is either 0 or full.

Proof. We begin by proving that $\mathcal{P}$ is similar to a matrix of the form

$$\begin{bmatrix} \mathcal{P}' & C \\ 0 & \mathcal{P}'' \end{bmatrix}$$

where $\mathcal{P}' = M_m(R)$ for some m and $\mathcal{P}''$ is a zpmr. Then, the same steps can be repeated for $\mathcal{P}''$ and its submatrices until we have achieved the form stated in the theorem.

First, switch the rows of $\mathcal{P}$ so that they are in decreasing order of their weights. Explicitly, for each $i \in [1, n]$, let w_i be the weight of row i (that is, w_i is the cardinality of $\{j \in [1, n] \mid \mathcal{P}_{ij} = R\}$). Perform similarity permutations—which will reindex the w_i—so that $w_1 \geqslant w_2 \geqslant \cdots \geqslant w_n$. Call the matrix produced by this process $\mathcal{Q}$.

We claim that if $\mathcal{Q}_{i1} = R$, then row i of $\mathcal{Q}$ is the same as row 1 of $\mathcal{Q}$. Indeed, if $\mathcal{Q}_{i1} = R$ and $\mathcal{Q}_{1k} = R$, then $\mathcal{Q}_{ik} = \mathcal{Q}_{i1}\mathcal{Q}_{1k} = R$. It follows that $w_i \geqslant w_1$; but, the rows of $\mathcal{Q}$ have been ordered so that $w_i \leqslant w_1$, so we must have $w_i = w_1$. Thus, if $\mathcal{Q}_{i1} = R$ then $\mathcal{Q}_{ij} = R$ if and only if $\mathcal{Q}_{1j} = R$, and so row i is identical to row 1.

Next, perform additional row and columns switches so that the first column of the matrix has the form $[* \cdots *\ 0 \cdots 0]^T$. At this stage, we have produced a matrix $\mathcal{R} = \tau\mathcal{P}\tau^{-1}$, where τ is a permutation matrix corresponding to our sequence of row switches. Let m be the weight of the first column of $\mathcal{R}$. Express $\mathcal{R}$ in block matrix form as

$$\mathcal{R} = \begin{bmatrix} \mathcal{P}' & C \\ D & \mathcal{P}'' \end{bmatrix}$$

where $\mathcal{P}'$ is $m \times m$.

We claim that $\mathcal{P}' = M_m(R)$ and $D = 0$. By construction, the rows of $\mathcal{P}'$ are identical. So, for all $j \in [1, m]$, $\mathcal{R}_{1j} = \mathcal{R}_{jj} = R$. Hence, the first row of $\mathcal{P}'$ has full weight, and therefore $\mathcal{P}' = M_m(R)$.

To show that $D = 0$, we use contradiction. Suppose that $\mathcal{R}_{ij} = R$ for some $i \in [m+1, n]$ and some $j \in [1, m]$. Then, $\mathcal{R}_{i1} = \mathcal{R}_{ij}\mathcal{R}_{j1} = R$ because $\mathcal{R}_{ij} = \mathcal{R}_{j1} = R$. However, this is a contradiction because the first column of $\mathcal{R}$ has the form $[* \cdots * \;\; 0 \cdots 0]^T$ and so $\mathcal{R}_{i1} = 0$. Thus, $\mathcal{R}_{ij} = 0$ for all $i \in [m+1, n]$ and all $j \in [1, m]$. Hence, $D = 0$.

Our work so far proves that $\mathcal{R}$ has the form

$$\mathcal{R} = \begin{bmatrix} \mathcal{P}' & C \\ 0 & \mathcal{P}'' \end{bmatrix}.$$

Since $\mathcal{R}$ is subring of $M_n(R)$, $\mathcal{P}''$ must be a subring of $M_{n-m}(R)$, and so $\mathcal{P}''$ is a zpmr. We may now apply a sequence of row and column switches to $\mathcal{P}''$ to place it in the same kind of form as $\mathcal{R}$. Note that these switches will not affect $\mathcal{P}'$. Continuing in this manner will produce a matrix similar to $\mathcal{P}$ and having the form

$$\begin{bmatrix} \mathcal{P}_1 & C_{12} & C_{13} & \cdots & C_{1t} \\ 0 & \mathcal{P}_2 & C_{23} & \cdots & C_{2t} \\ 0 & 0 & \mathcal{P}_3 & \cdots & C_{3t} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \mathcal{P}_t \end{bmatrix}$$

with each $\mathcal{P}_k$ equal to $M_{n_k}(R)$ for some $n_k \in \mathbb{N}$.

It remains to prove that each C_{ij} is either 0 or has full weight. For this, fix $k \in [1, t-1]$ and $\ell \in [k+1, t]$. We may apply Lemma 3.2 to the zpmr

$$\begin{bmatrix} \mathcal{P}_k & C_{k\ell} \\ 0 & \mathcal{P}_\ell \end{bmatrix}$$

and conclude that $C_{k\ell}$ is either 0 or is full. This completes the proof. □

We will not always need the full strength of Theorem 3.3, so we record as a corollary the following weaker form of the theorem.

Corollary 3.4. *Let $\mathcal{P} \subseteq M_n(R)$ be a zero pattern matrix ring. Then, there exists a zero pattern matrix ring $\mathcal{Q} \subseteq M_n(R)$ such that $w(\mathcal{Q}) = w(\mathcal{P})$ and $\mathcal{Q}$ has the form*

$$\mathcal{Q} = \begin{bmatrix} \mathcal{P}' & C \\ 0 & \mathcal{P}'' \end{bmatrix}$$

where $\mathcal{P}' = M_m(R)$ for some $m \in [1, n]$, $\mathcal{P}'' \subseteq M_{n-m}(R)$ is a zero pattern matrix ring, and the rows of C are identical. Moreover, we may assume that m is the maximum dimension of a full diagonal block.

Proof. Construct the block triangular form $\sigma\mathcal{P}\sigma^{-1}$ for $\mathcal{P}$ as in Theorem 3.3. Then, permute the rows and columns of blocks so that the dimension of $\mathcal{P}_1$ is maximal. Take $\mathcal{P}' = \mathcal{P}_1$, $C = [C_{12} \cdots C_{1t}]$, and $\mathcal{P}''$ to be the submatrix below C. □

We close this section by giving several examples demonstrating the limitations of Theorem 3.3 and Corollary 3.4.

Example 3.5.

1. Two zpmrs can have the same weight but not be isomorphic. Take

$$\mathcal{P}_1 = \begin{bmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \quad \text{and} \quad \mathcal{P}_2 = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix}.$$

Then, $w(\mathcal{P}_1) = w(\mathcal{P}_2) = 5$, but the rings are not isomorphic. Indeed, the center of $\mathcal{P}_1$ consists of only the scalar matrices, but the center of $\mathcal{P}_2$ contains the matrix E_{33}. So, $\mathcal{P}_1 \ncong \mathcal{P}_2$. This could also be seen by constructing the corresponding transitive digraphs Γ_1 and Γ_2. Then, $\Gamma_1 \ncong \Gamma_2$ because Γ_1 is connected while Γ_2 is not.

2. The form given by Theorem 3.3 is not unique. This is because different sequences of row and column switches can produce different matrices that still satisfy the conclusion of the theorem. Consider the 4×4 zpmrs

$$\mathcal{P}_1 = \begin{bmatrix} * & 0 & * & 0 \\ 0 & * & 0 & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \quad \text{and} \quad \mathcal{P}_2 = \begin{bmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}.$$

Let σ be the permutation matrix switching row 1 and row 2. Then, both zpmrs satisfy the conclusion of Theorem 3.3, but $\mathcal{P}_1 = \sigma \mathcal{P}_2 \sigma^{-1}$.

3. Theorem 3.3 works by using similarity permutations to put the row weights of $\mathcal{P}$ in decreasing order. The sequence of row weights obtained in this manner is an isomorphism invariant, but two zpmrs can have identical sequences of row weights (and column weights) and yet not be isomorphic. The following example of this is based on two matrices given in [10, p. 1346].

Let

$$\mathcal{P}_1 = \begin{bmatrix} * & 0 & 0 & * & * & * \\ 0 & * & 0 & 0 & 0 & * \\ 0 & 0 & * & 0 & * & 0 \\ 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix} \quad \text{and} \quad \mathcal{P}_2 = \begin{bmatrix} * & 0 & 0 & * & * & * \\ 0 & * & 0 & 0 & 0 & * \\ 0 & 0 & * & 0 & 0 & * \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}.$$

Then, the sequence of row weights is $4, 2, 2, 2, 1, 1$ and the sequence of column weights is $1, 1, 1, 2, 3, 4$, and these are the same for both zpmrs. It is easiest to see that $\mathcal{P}_1 \ncong \mathcal{P}_2$ by looking at the associated transitive digraphs Γ_1 and Γ_2. Both graphs have two vertices of outdegree 1 (vertices 2 and 3 in both Γ_1 and Γ_2). In Γ_1, these vertices are adjacent to different vertices (2 is adjacent to 5, and 3 is adjacent to 6), whereas in Γ_2 the vertices are both adjacent to vertex 6.

4. The matrix $\mathcal{Q}$ obtained in Corollary 3.4 need not be similar to $\mathcal{P}$. This is because permuting the rows and columns of blocks will preserve the weight of the zpmr, but cannot always be accomplished via similarity permutations. For instance, the zpmrs

$$\mathcal{P}_1 = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \quad \text{and} \quad \mathcal{P}_2 = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix}$$

have the same weight, but they are not similarity permutations of one another.

4 The Weight Set of a Zero Pattern Matrix Ring

Our focus in this section is the determination of the weight sets $W(n)$. In Theorem 1.7, we prove that $W(n)$ can be computed recursively, which is to say that we can construct $W(n)$ by using $W(1), \ldots, W(n-1)$. A number of results also concern the interval $[n, b(n)]$ that is found at the beginning of each weight set.

Lemma 4.1. *For each $n \geqslant 1$, we have $[n, n(n+1)/2] \subseteq W(n)$.*

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$b(n)$ lower bound	1	4	7	13	19	28	35	52	61	77	95	109	130	153	178	205	223	253	285	319	355	393	433	475
$\lceil (3/4)n^2 \rceil$	1	3	7	12	19	27	37	48	61	75	91	108	127	147	169	192	217	243	271	300	331	363	397	432

Table 2: Lower bounds for $b(n)$, using Lem. 4.2

Proof. We will use induction on n and prove that every $k \in [n, n(n+1)/2]$ occurs as the weight of some upper triangular zpmr. The base case $n = 1$ is trivial, so assume that $n \geqslant 2$ and the result holds for $n - 1$.

Let $k \in [n-1, (n-1)n/2]$. By induction, there exists a zpmr $\mathcal{P}'' \subseteq M_{n-1}(R)$ such that $w(\mathcal{P}'') = k$. We can form the $n \times n$ zpmr

$$\mathcal{P} = \begin{bmatrix} * & 0 \\ 0 & \mathcal{P}'' \end{bmatrix}$$

which has $w(\mathcal{P}) = k + 1$. This shows that $[n, (n-1)n/2 + 1] \subseteq W(n)$. To complete the proof, let $m \in [1, n-1]$ and let $\mathscr{T}_{n-1}(R)$ be the ring of $(n-1) \times (n-1)$ upper triangular matrices. Consider the matrix

$$\mathcal{Q} = \begin{bmatrix} * & C \\ 0 & \mathscr{T}_{n-1}(R) \end{bmatrix}$$

where

$$C = [0 \cdots 0 \ \underbrace{* \cdots *}_{m\text{-times}}].$$

Then, one may easily verify that $\mathcal{Q}$ is a zpmr and $w(\mathcal{Q}) = (n-1)n/2 + 1 + m$. This construction works for any $m \in [1, n-1]$, so we have $[(n-1)n/2 + 1, n(n+1)/2] \subseteq W(n)$, as desired. □

By Lemma 4.1, $b(n) \geqslant n(n+1)/2$ for all $n \geqslant 1$, but Table 1 shows that this bound is insufficient for $n \geqslant 2$. In Proposition 4.3, we will prove that $b(n) \geqslant (3/4)n^2$ as long as $n \neq 7$. Most of the work in that proof is accomplished by using a theorem of Rao [7] on transitive digraphs, but small cases need to be checked by hand, and to do this we need to be able to form the sets $W(n)$. Lemma 4.2 shows how to form a subset of $W(n)$ that is defined recursively. This allows us to find lower bounds for $b(n)$ (see Table 2), which will be used in later proofs.

Lemma 4.2. *For each $n \geqslant 1$, let $\alpha_n = \bigcup_{k=1}^{n-1}(W(n-k) + W(k))$ and $\beta_n = \bigcup_{k=1}^{n-1}(n(n-k) + W(k))$. Then, $W(n) \supseteq \alpha_n \cup \beta_n \cup \{n^2\}$.*

Proof. Certainly, $n^2 \in W(n)$. The values in α_n come from $n \times n$ zpmrs of the form

$$\begin{bmatrix} \mathcal{P}' & 0 \\ 0 & \mathcal{P}'' \end{bmatrix}$$

where $\mathcal{P}''$ is $k \times k$, and the values in β_n come from $n \times n$ zpmrs of the form

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{\text{full}} \\ \hline 0 & \mathcal{P}'' \end{array}\right]$$

where $\mathcal{P}''$ is $k \times k$. □

In [7, p. 1597], Rao defines $f(n)$ to be the largest integer such that for all $k \in [0, f(n)]$, there exists a transitive digraph on n vertices with k edges. It is then shown in [7, Thm. 6] that $f(n) \geqslant (n - \lfloor n^{0.57} \rfloor)(n-1)$ for all $n \geqslant 1$. In the notation of this paper, we have $b(n) = f(n) + n$, and consequently Rao's theorem provides a lower bound for $b(n)$. However, the presence of the floor function and the exponent of 0.57 make this bound somewhat awkward to work with. The next proposition gives a coarser—but more convenient—lower bound for $b(n)$.

Proposition 4.3. *Let $n \geqslant 1$ such that $n \neq 7$. Then, $b(n) \geqslant (3/4)n^2$.*

Proof. The theorem can be verified by inspection for $n \leqslant 22$ by using Lemma 4.2 to compute a subset of $W(n)$, and then using this subset to obtain a lower bound for $b(n)$. The results of this process are summarized in Table 2. Note that the proposition fails for $n = 7$, because $b(7) = 35$ while $(3/4) \cdot 7^2 = 36.75$. So, assume that $n \geqslant 23$.

Now, by [7, Thm. 6], $b(n) \geqslant (n - \lfloor n^{0.57} \rfloor)(n-1) + n$ for all $n \geqslant 1$. So,

$$\begin{aligned} b(n) &\geqslant (n - \lfloor n^{0.57} \rfloor)(n-1) + n \\ &= n^2 - \lfloor n^{0.57} \rfloor (n-1) \\ &\geqslant n^2 - n^{0.57}(n-1) \\ &= n^2 - n^{1.57} + n^{0.57}. \end{aligned}$$

We will show that $n^2 - n^{1.57} + n^{0.57} \geqslant (3/4)n^2$ for all $n \geqslant 23$. Let $g(x) = (1/4)x^2 - x^{1.57} + x^{0.57}$ for a real variable x. Then, the derivative of g is

$$\begin{aligned} g'(x) &= \tfrac{1}{2}x - 1.57x^{0.57} + 0.57x^{-0.43} \\ &> \tfrac{1}{2}x - 1.57x^{0.57} \\ &= x^{0.57}(\tfrac{1}{2}x^{0.43} - 1.57) \end{aligned}$$

and this is positive for all $x > (3.14)^{1/0.43} \approx 14.31$. Moreover, one may check that $g(22) < 0$ and $g(23) > 0$. It follows that $g(n) > 0$ for all $n \geqslant 23$, which completes the proof. $\square$

Proposition 4.3 shows that $W(n) \supseteq [n, \lceil (3/4)n^2 \rceil]$ for all $n \neq 7$. The next two propositions will be used to prove that a weight in $W(n)$ that is greater than $(3/4)n^2$ can be realized by a zpmr of a certain form.

Proposition 4.4. *Let $\mathcal{P} \subseteq M_n(R)$ be a zero pattern matrix ring of the form*

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_1 & C_{12} & C_{13} & \cdots & C_{1t} \\ 0 & \mathcal{P}_2 & C_{23} & \cdots & C_{2t} \\ 0 & 0 & \mathcal{P}_3 & \cdots & C_{3t} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \mathcal{P}_t \end{bmatrix}$$

where for each $k \in [1, t]$, $\mathcal{P}_k = M_{n_k}(R)$ for some $n_k \in \mathbb{N}$. Assume that the blocks $\mathcal{P}_k$ have been ordered so that $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_t$. If $n_1 \leqslant n/2$, then $w(\mathcal{P}) \leqslant (3/4)n^2$.

Proof. Assume that $n_1 \leqslant n/2$. Then, $t \geqslant 2$. Most cases can be handled by induction on t. The base case is $t = 2$, for which we must have $n_1 = n/2$. In this case, the column of zero blocks below $\mathcal{P}_1$ guarantees that $w(\mathcal{P}) \leqslant (3/4)n^2$. So, assume that $t \geqslant 3$ and that the result holds for $t - 1$.

Let $\mathcal{P}''$ be the $(n - n_1) \times (n - n_1)$ zpmr

$$\mathcal{P}'' = \begin{bmatrix} \mathcal{P}_2 & C_{23} & \cdots & C_{2t} \\ 0 & \mathcal{P}_3 & \cdots & C_{3t} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mathcal{P}_t \end{bmatrix}.$$

If $n_2 \leqslant (n - n_1)/2$, then by induction we have $w(\mathcal{P}'') \leqslant (3/4)(n - n_1)^2$. Also, $n_1 < n/2$, because $t \geqslant 3$. So,

$$\begin{aligned} w(\mathcal{P}) &\leqslant n_1 n + w(\mathcal{P}'') \\ &\leqslant n_1 n + \tfrac{3}{4}(n - n_1)^2 \\ &= \tfrac{3}{4}n^2 - \tfrac{1}{2}n_1 n + \tfrac{3}{4}n_1^2 \\ &= \tfrac{3}{4}n^2 - \tfrac{3}{4}n_1(\tfrac{2}{3}n - n_1) \\ &< \tfrac{3}{4}n^2. \end{aligned}$$

We are left with the case where $n_2 > (n - n_1)/2$. Since $n_2 \leqslant n_1$, this means that $n_1 > n/3$. Thus, we can be sure that $n/3 < n_1 < n/2$.

Let $\mathcal{P}''$ be as above. The minimum number of zeros in $\mathcal{P}''$ occurs when $t = 3$, so that the only zeros we can be certain of are those in the column of blocks below $\mathcal{P}_2$. The number of such zeros is $n_2(n - n_1 - n_2)$. We claim that this quantity is minimized when $n_2 = n_1$. Indeed, let $g(x) = x(n - n_1 - x)$ for a real variable x. Then, the maximum value of g occurs at $x = (n - n_1)/2$, and g is decreasing for $x > (n - n_1)/2$. Since $(n - n_1)/2 < n_2 \leqslant n_1$, the minimum value of $n_2(n - n_1 - n_2)$ occurs when $n_2 = n_1$.

With these assumptions, the number of zero entries below $\mathcal{P}_1$ and $\mathcal{P}_2$ is equal to

$$n_1(n - n_1) + n_1(n - 2n_1) = 2n_1 n - 3n_1^2.$$

Let $h(x) = 2nx - 3x^2$ for a real variable x. Then, h is decreasing for $x > n/3$. Since we are assuming that $n/3 < n_1 < n/2$, we conclude that

$$2n_1 n - 3n_1^2 > 2\Big(\frac{n}{2}\Big)n - 3\Big(\frac{n}{2}\Big)^2 = \frac{n^2}{4}.$$

Thus, $w(\mathcal{P}) < (3/4)n^2$, as desired. □

Proposition 4.5. *Let $\mathcal{P} \subsetneqq M_n(R)$ be a zero pattern matrix ring such that $w(\mathcal{P}) \geqslant (3/4)n^2$. Then, there exists a zero pattern matrix ring $\mathcal{Q} \subseteq M_n(R)$ such that $w(\mathcal{Q}) = w(\mathcal{P})$ and $\mathcal{Q}$ has the form*

$$\mathcal{Q} = \left[\begin{array}{c|c} \multicolumn{2}{c}{\text{full}} \\ \hline 0 & \mathcal{Q}'' \end{array}\right]$$

where $\mathcal{Q}''$ is a zero pattern matrix ring of dimension less than n.

Proof. If $w(\mathcal{P}) = (3/4)n^2$, then we may take $\mathcal{Q}'' = M_{n/2}(R)$ and we are done. So, assume that $w(\mathcal{P}) > (3/4)n^2$. By Corollary 3.4, we may assume that $\mathcal{P}$ has the form

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}' & C \\ 0 & \mathcal{P}'' \end{bmatrix}$$

where $\mathcal{P}'$ and $\mathcal{P}''$ are zpmrs, $\mathcal{P}'$ has full weight, the rows of C are identical, and the dimension of $\mathcal{P}'$ is at least as large as that of any other full diagonal block. Since $w(\mathcal{P}) > (3/4)n^2$, by Proposition 4.4 the dimension of $\mathcal{P}'$ is greater than $n/2$.

Let r be the dimension of $\mathcal{P}''$ and let $c \in [0, r]$ be the number of nonzero columns of C. If $c = r$, then C is full and we are done. So, assume that $c < r$. With these assignments for r and c, we have

$$\begin{aligned} w(\mathcal{P}) &= w(\mathcal{P}') + w(C) + w(\mathcal{P}'') \\ &= (n - r)^2 + (n - r)c + w(\mathcal{P}'') \\ &= n^2 - 2nr + r^2 + nc - rc + w(\mathcal{P}'') \\ &= n(n - 2r + c) + (r - c)^2 + c(r - c) + w(\mathcal{P}''). \end{aligned}$$

Let $\mathcal{Q}''$ be the matrix formed by deleting the first $n-2r+c$ rows and columns of $\mathcal{P}$. The matrix $\mathcal{Q}''$ is square of dimension $2r-c$, and is equal to

$$\begin{bmatrix} M_{r-c}(R) & C' \\ 0 & \mathcal{P}'' \end{bmatrix}$$

where the columns of C' that are nonzero are the same as in C. Note that $\mathcal{Q}''$ is a zpmr because $\mathcal{P}$ is a zpmr, and $w(\mathcal{Q}'') = (r-c)^2 + c(r-c) + w(\mathcal{P}'')$. Finally, we can take $\mathcal{Q}$ to be

$$\mathcal{Q} = \left[\begin{array}{c|c} \multicolumn{2}{c}{\text{full}} \\ \hline 0 & \mathcal{Q}'' \end{array}\right]$$

where the full block at the top of the matrix is $(n-2r+c) \times n$. The matrix $\mathcal{Q}$ has the desired form, and $w(\mathcal{Q}) = w(\mathcal{P})$, so we are done. $\square$

Because Proposition 4.3 does not apply when $n = 7$, we have to treat $W(7)$ separately from the other weight sets. Fortunately, we have built up enough theory that only one potential weight between 7 and 49 requires a detailed analysis.

Lemma 4.6. *$36 \notin W(7)$.*

Proof. Let $\mathcal{P} \subsetneq M_7(R)$. By Corollary 3.4, we may assume that $\mathcal{P}$ has the form

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}' & C \\ 0 & \mathcal{P}'' \end{bmatrix}$$

where $\mathcal{P}' = M_m(R)$ for some $m \in [1, 6]$, $\mathcal{P}''$ is a zpmr, C has identical rows, and all full diagonal blocks have dimension at most m. We will examine cases depending on m, and will show that $w(\mathcal{P})$ cannot equal 36.

If $m = 1$, then $\mathcal{P}$ must be upper triangular and hence $w(\mathcal{P}) \leqslant 28$. If $m = 2$, then the maximum weight of $\mathcal{P}$ occurs when C is full and

$$\mathcal{P}'' = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix},$$

but such a 7×7 zpmr has weight 31. Similarly, if $m = 3$ then the maximum weight of $\mathcal{P}$ is 34, which occurs when C is full and

$$\mathcal{P}'' = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix}.$$

So, we may assume that $m \in [4, 6]$.

Now, the fact that the rows of C are identical means that $w(C)$ must be a multiple of m. Moreover, since $m \in [4, 6]$, the dimension of $\mathcal{P}''$ is in $[1, 3]$, and it is easy to determine the possible values of $w(\mathcal{P}'')$. The chart below summarizes the possible weights for $\mathcal{P}'$, $\mathcal{P}''$, and C in these cases.

m	$w(\mathcal{P}')$	$w(\mathcal{P}'')$	$w(C)$
4	16	$[3, 7], 9$	0, 4, 8, 12
5	25	$[2, 4]$	0, 5, 10
6	36	1	0, 6

Since $w(\mathcal{P}) = w(\mathcal{P}') + w(\mathcal{P}'') + w(C)$, we see that in each case there is no way for $w(\mathcal{P})$ to equal 36. $\square$

We can now prove Theorem 1.7, which—as mentioned previously—demonstrates how to calculate the sets $W(n)$ recursively.

Proof of Theorem 1.7. We know that $\beta_n \cup \{n^2\} \subseteq W(n)$ for all n from Lemma 4.2, and $[n, \lceil (3/4)n^2 \rceil] \subseteq W(n)$ for $n \neq 7$ by Proposition 4.3. For $n = 7$, one may use Lemma 4.2 to compute that $b(7) \geqslant 35$. So,

$$W(7) \supseteq [7, 35] \cup \beta_7 \cup \{49\},$$

and when $n \neq 7$,

$$W(n) \supseteq [n, \lceil (3/4)n^2 \rceil] \cup \beta_n \cup \{n^2\}.$$

For the reverse containments, first assume that $n \neq 7$ and let $\mathcal{P} \subsetneqq M_n(R)$ be a zpmr. If $w(\mathcal{P}) \leqslant (3/4)n^2$, then we are done, so assume that $w(\mathcal{P}) > (3/4)n^2$. By Proposition 4.5, $w(\mathcal{P}) = w(\mathcal{Q})$ for some zpmr $\mathcal{Q}$ whose weight is in β_n. Thus, $W(n)$ is equal to the union of the stated sets, and we are done.

When $n = 7$, we have $(3/4) \cdot 7^2 = 36.75$, so the arguments of the previous paragraph apply to zpmrs in $M_7(R)$ of weight at least 37. We also know that $[7, 35] \subseteq W(7)$, so the only remaining question is whether $M_7(R)$ contains a zpmr of weight 36. However, this is ruled out by Lemma 4.6. Hence, $W(7) = [7, 35] \cup \beta_7 \cup \{49\}$, and the proof is complete. $\square$

Corollary 4.7. *Let $S(n)$ be the set of integers m for which there exists a digraph on n vertices with m reachable pairs. Let $T(n)$ be the set of possible values of $\tau(\mathcal{T})$, where $\mathcal{T}$ is a finite topology on n points and τ is Sharp's topological invariant τ. For each $n \geqslant 1$, let $\delta_n = \bigcup_{k=1}^{n-1}((n-1)(n-k) + S(k))$, and let $\beta_n = \bigcup_{k=1}^{n-1}(n(n-k) + W(k))$. Then,*

$$S(7) = [0, 28] \cup \delta_7 \cup \{42\},$$

and if $n \neq 7$ then

$$S(n) = [0, \lceil (3/4)n^2 - n \rceil] \cup \delta_n \cup \{n^2 - n\}.$$

Also,

$$T(7) = [7, 35] \cup \beta_7 \cup \{49\},$$

and if $n \neq 7$ then

$$T(n) = [n, \lceil (3/4)n^2 \rceil] \cup \beta_n \cup \{n^2\}.$$

Proof. This follows immediately from Theorem 1.7 and Lemma 2.3. $\square$

Remark 4.8. Using Theorem 1.7 to compute $W(n)$ and $b(n)$ shows that the lower bounds for $b(n)$ given in Table 2 are actually the exact values of $b(n)$. Some of these values will be used later in the proof of Theorem 1.8, where we prove that $b(n)$ can be calculated recursively (without knowing $W(n)$).

5 The Weight Loss Sequence and $b(n)$

Definition 5.1. Recall that $b(n)$ is the largest positive integer such that $[n, b(n)] \subseteq W(n)$. For all $z \in \mathbb{N}$, we define $L(z) := b(z) - z + 3$. The sequence of integers $\{L(z)\}_{z \geqslant 1} = \{3, 5, 7, 12, \ldots\}$ is called the *weight loss sequence.*

We call $\{L(z)\}_{z\geqslant 1}$ the weight loss sequence because it can be used to find gaps in the weight sets $W(n)$. For instance, $L(1) = 3$ and $n = 3$ is the first time that a gap appears in a weight set: $8 \notin W(3)$. This gap is the first instance of a general restriction on $W(n)$, namely that $W(n) \cap [n^2 - n + 2, n^2 - 1] = \varnothing$ for all $n \geqslant 3$. Similarly, $L(2) = 5$, and the fact that $20 \notin W(5)$ is the first occurrence of a new gap. In this case, the general result is $W(n) \cap [n^2 - 2n + 5, n^2 - n] = \varnothing$ for all $n \geqslant 5$. Proofs of these restrictions (and others) on $W(n)$ can be found in Theorem 6.4.

Clearly, we must know $b(z)$ to be able to calculate $L(z)$. However, it turns out that knowing $L(z)$ allows us to compute $b(n)$ for some values of $n \geqslant L(z)$. The relationship (barring some small exceptions) between $b(n)$ and $L(z)$ is that if $L(z) \leqslant n < L(z+1)$, then $b(n) = n^2 - zn + b(z)$. Thus, if we know $b(k)$ for $k \in [1, n-1]$, then we can calculate each $L(k)$, find the appropriate z, and then calculate $b(n)$.

Most of this section is dedicated to proving the statements of the previous paragraphs. The main theorem is Theorem 1.8, and most of the work is done in Propositions 5.4 and 5.9. Computational lemmas are introduced as they are needed to prove the propositions.

Lemma 5.2. *For all $n \geqslant 1$ and all $m \geqslant 1$, $b(n) + m \leqslant b(n+m)$.*

Proof. For any $n \geqslant 1$, we can form $(n+1)$-dimensional zpmrs of the form

$$\begin{bmatrix} * & 0 \\ 0 & \mathcal{P}'' \end{bmatrix}$$

where the weight of $\mathcal{P}'' \subseteq M_n(R)$ varies between n and $b(n)$. Hence, $b(n+1) \geqslant b(n) + 1$. The lemma now follows easily by induction on m. □

Lemma 5.3. *Let $z \geqslant 6$. Then,*

(1) $L(z) > 4z$.

(2) If $n \geqslant L(z)$, then $n^2 - zn + b(z) \geqslant (3/4)n^2$.

Proof. (1) We have $L(6) = 25$ and $L(7) = 31$, so assume that $z \geqslant 8$. By Proposition 4.3, $b(z) \geqslant (3/4)z^2$, so

$$L(z) = b(z) - z + 3 \geqslant (3/4)z^2 - z + 3$$

and it is routine to verify that this is greater than $4z$.

For (2), assume $n \geqslant L(z)$ and let $g(x) = (1/4)x^2 - zx + b(z)$. Solving $g(x) = 0$ for x in terms of z yields $x = 2(z \pm \sqrt{z^2 - b(z)})$. We have

$$2(z + \sqrt{z^2 - b(z)}) < 4z < L(z) \leqslant n$$

so the desired inequality holds. □

Proposition 5.4. *Let $z \geqslant 4$ and let n be such that $L(z) \leqslant n < L(z+1)$. Then, $b(n) \geqslant n^2 - zn + b(z)$.*

Proof. Since $z \geqslant 4$, $n \geqslant L(4) = 12$, so by Proposition 4.3, $b(n) \geqslant (3/4)n^2$. Moreover, by Lemma 5.3(2), $n^2 - zn + b(z) \geqslant (3/4)n^2$. To get the stated result, we need to show that there exists an $n \times n$ zpmr of weight k for every $k \in [\lceil (3/4)n^2 \rceil, n^2 - zn + b(z)]$.

Consider $n \times n$ zpmrs of the form

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{\text{full}} \\ \hline 0 & \mathcal{P}_r \end{array}\right]$$

where $\mathcal{P}_r$ is $r \times r$. We can vary the choice of $\mathcal{P}_r$ so that $w(\mathcal{P}_r)$ is any integer between r and $b(r)$ (inclusive). This allows us to produce $n \times n$ zpmrs with weights in the interval

$$I_r := [n(n-r)+r, \quad n(n-r)+b(r)],$$

and every weight in I_r is achievable. We will prove that the union $\bigcup_{r=z}^{n-1} I_r$ covers the entire interval $[\lceil (3/4)n^2 \rceil, n^2 - zn + b(z)]$. (Note that by our choice of notation, I_{n-1} is the leftmost interval and I_z is the rightmost interval.)

We claim that when $z+1 \leqslant r$, the lower endpoint of I_{r-1} is at most one more than the upper endpoint of I_r. That is, we seek to show that

$$n(n-(r-1)) + (r-1) \leqslant 1 + n(n-r) + b(r) \tag{5.5}$$

which is equivalent to

$$n + r - 1 \leqslant b(r) + 1.$$

Now, by assumption, $n \leqslant L(z+1) - 1$, and by definition $L(z+1) = b(z+1) - (z+1) + 3$. From this, we get that $n + z \leqslant b(z+1) + 1$. Let m be such that $z + m = r - 1$. Using Lemma 5.2, we have

$$\begin{aligned} n + r - 1 &= n + z + m \\ &\leqslant b(z+1) + 1 + m \\ &\leqslant b(z+1+m) + 1 \\ &= b(r) + 1. \end{aligned}$$

Thus, (5.5) holds. This means that the union of the intervals I_r comprises a single interval $[2n-1, n^2 - zn + b(z)]$, which goes from the lower endpoint of I_{n-1} to the upper endpoint of I_z. Clearly, $2n - 1 \leqslant (3/4)n^2$, so we conclude that

$$[n, \ \lceil (3/4)n^2 \rceil] \cup [2n-1, \ \ n(n-z) + b(z)] = [n, \ \ n^2 - zn + b(z)]$$

and the stated result follows. □

Lemma 5.6. *Assume that $z \geqslant 4$, $n \geqslant L(z)$, and $z + 1 \leqslant r < n/2$. Then,*

$$n^2 - rn + r^2 \leqslant n^2 - zn + b(z).$$

Proof. First, we note that we need z to be at least 4 in order to have $L(z)/2 > z + 1$. If $z = 3$, then $L(z) = 7$, and $z + 1 = 4 > 7/2$. However, once $z \geqslant 4$, one can use Proposition 4.3 to easily show that $L(z) > 2z + 2$. Hence, our assumptions on z, n, and r are necessary and can be satisfied.

Now, let $g(r) = n^2 - rn + r^2$. Then, g is decreasing for $r < n/2$, so

$$n^2 - rn + r^2 \leqslant n^2 - (z+1)n + (z+1)^2. \tag{5.7}$$

Next, we claim that $2b(z) \geqslant z^2 + 3z - 2$. This is clear when $z \in [4, 7]$, and for $z \geqslant 8$ one may apply Proposition 4.3 and verify that

$$2b(z) \geqslant (3/2)z^2 \geqslant z^2 + 3z - 2.$$

So,

$$2b(z) \geqslant z^2 + 3z - 2 = (z+1)^2 + z - 3$$

which implies that

$$n + b(z) \geqslant L(z) + b(z) = 2b(z) - z + 3 \geqslant (z+1)^2. \tag{5.8}$$

Combining (5.8) and (5.7) yields

$$n^2 - rn + r^2 \leqslant n^2 - (z+1)n + (z+1)^2 \leqslant n^2 - zn + b(z),$$

as desired. $\square$

The next proposition is the key to most of our remaining results.

Proposition 5.9. *Let $z \geqslant 6$ and let n be such that $n \geqslant L(z)$. Let $\mathcal{P} \subsetneqq M_n(R)$ be a zero pattern matrix ring such that $w(\mathcal{P}) \geqslant n^2 - zn + b(z) + 1$. Then, there exists a zero pattern matrix ring $\mathcal{Q} \subseteq M_n(R)$ such that $w(\mathcal{Q}) = w(\mathcal{P})$ and $\mathcal{Q}$ has the form*

$$\mathcal{Q} = \left[\begin{array}{c|c} \multicolumn{2}{c}{\text{full}} \\ \hline 0 & \mathcal{Q}'' \end{array}\right]$$

where $\mathcal{Q}''$ is $m \times m$ for some $m \in [1, z]$.

Proof. By Corollary 3.4, we may assume that $\mathcal{P}$ has the form

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}' & C \\ 0 & \mathcal{P}'' \end{bmatrix}$$

where $\mathcal{P}'$ and $\mathcal{P}''$ are zpmrs, $\mathcal{P}'$ has full weight, the rows of C are identical, and the dimension of $\mathcal{P}'$ is at least as large as that of any other full diagonal block. Let $r = \dim(\mathcal{P}'')$. Since $z \geqslant 6$, we have $w(\mathcal{P}) > n^2 - zn + b(z) \geqslant (3/4)n^2$ by Lemma 5.3(2). So, $\dim(\mathcal{P}') > n/2$ by Proposition 3.3, and hence $r < n/2$.

Suppose that $z + 1 \leqslant r < n/2$. Then, the maximum weight of $\mathcal{P}$ is

$$w(\mathcal{P}) \leqslant n(n-r) + w(\mathcal{P}'') \leqslant n(n-r) + r^2 \tag{5.10}$$

which by Lemma 5.6 is strictly less than $n^2 - zn + b(z) + 1$. So, we must have $r \leqslant z$.

Let $c \in [0, r]$ be the number of nonzero columns of C and let $m = 2r - c$. Following the steps used in the proof of Proposition 4.5 shows that there exists a zpmr $\mathcal{Q} \subseteq M_n(R)$ such that $w(\mathcal{Q}) = w(\mathcal{P})$ and $\mathcal{Q}$ has the form

$$\mathcal{Q} = \left[\begin{array}{c|c} \multicolumn{2}{c}{\text{full}} \\ \hline 0 & \mathcal{Q}'' \end{array}\right]$$

where $\mathcal{Q}''$ is $m \times m$ and the full block has $n - 2r + c$ rows. By Lemma 5.3(1), $n > 4z$ and we are assuming that $r \leqslant z$, so we get $r < n/4$. This gives $n - 2r + c > n/2$ and $m < n/2$.

Finally, if $z + 1 \leqslant m < n/2$, then using (5.10) and Lemma 5.6 shows that $w(\mathcal{Q}) < n^2 - zn + b(z) + 1$. Thus, $m \leqslant z$, as required. $\square$

We are now ready to prove Theorem 1.8, which asserts that if $z \geqslant 1$ and $L(z) \leqslant n < L(z+1)$ (but $n \neq 8$), then $b(n) = n^2 - zn + b(z)$.

Proof of Theorem 1.8. For $z \in [1, 5]$, the theorem can be proved by inspection using Table 2 (as mentioned in Remark 4.8, the lower bounds for $b(n)$ in Table 2 are actually the exact values of the function). Note that the case $n = 8$ occurs when $z = 3$; see the remark following this theorem for an explanation of what goes wrong in that situation.

So, assume that $z \geqslant 6$. We know that $b(n) \geqslant n^2 - zn + b(z)$ by Proposition 5.4. Suppose by way of contradiction that $b(n) > n^2 - zn + b(z)$. Then, there must exist a zpmr $\mathcal{P} \subseteq M_n(R)$ such that $w(\mathcal{P}) = n^2 - zn + b(z) + 1$. By Proposition 5.9, we may assume that $\mathcal{P}$ has the form

$$\mathcal{P} = \left[\begin{array}{c|c} \multicolumn{2}{c}{\text{full}} \\ \hline 0 & \mathcal{Q}'' \end{array}\right]$$

where $\mathcal{Q}'' \subseteq M_m(R)$ for some $m \in [1, z]$. If $m = z$, then $w(\mathcal{Q}'') = b(z) + 1$, which is impossible by the definition of $b(z)$. So, we must have $m \leqslant z - 1$. In this situation, a lower bound for $w(\mathcal{P})$ is

$$\begin{aligned} w(\mathcal{P}) &= n(n-m) + w(\mathcal{Q}'') \\ &\geqslant n(n-m) + m \\ &= n^2 - m(n-1). \end{aligned} \tag{5.11}$$

Now, $n^2 - m(n-1)$ is decreasing as a function of m, so (5.11) and the fact that $n \geqslant L(z)$ yield

$$\begin{aligned} n^2 - zn + b(z) + 1 &= w(\mathcal{P}) \\ &\geqslant n^2 - (z-1)(n-1) \\ &= n^2 - zn + z - 1 + n \\ &\geqslant n^2 - zn + z - 1 + (b(z) - z + 3) \\ &= n^2 - zn + b(z) + 2, \end{aligned}$$

a contradiction. Thus, $b(n) = n^2 - zn + b(z)$. □

Remark 5.12. When $n = 8$, the corresponding z is $z = 3$, for which $L(z) = 7$. Computing $n^2 - zn + b(z)$ in this case yields 47, but $b(8) = 52$ by Table 2. What goes wrong here is that, when $n = 8$, $(3/4)n^2 = n^2 - zn + b(z) + 1$, and there exists an 8×8 zpmr

$$\mathcal{P} = \begin{bmatrix} M_4(R) & M_4(R) \\ 0 & M_4(R) \end{bmatrix}$$

of weight 48. Lemma 5.3(2) shows that such coincidences cannot happen when z (and n) are sufficiently large.

Corollary 5.13. *Let $z \geqslant 1$ and let n be such that $L(z) \leqslant n < L(z+1)$. If $n \neq 8$, then $n^2 - (z+1)n + b(z) + 1$ is the smallest nonnegative integer m for which there does not exist a digraph on n vertices with exactly m reachable pairs. Furthermore, if $n \neq 8$, then $n^2 - zn + b(z) + 1$ is also the smallest nonnegative integer m for which there does not exist a topology $\mathcal{T}$ on n points with $\tau(\mathcal{T}) = m$.*

Proof. This follows immediately from Lemma 2.3 and Theorem 1.8. □

We obtain the following corollary about the asymptotics of $b(n)$, which is also a consequence of [7, Theorem 6].

Corollary 5.14. *As $n \to \infty$, $b(n)/n^2 \to 1$.*

Proof. Let n and z be positive integers such that $L(z) \leqslant n < L(z+1)$; note that $z \to \infty$ as $n \to \infty$. By Theorem 1.8, $b(n) = n^2 - zn + b(z)$, so it suffices to show that $z/n \to 0$ and $b(z)/n^2 \to 0$ as $n \to \infty$. For $z \geqslant 8$, we have $b(z) \geqslant (3/4)z^2$ by Proposition 4.3, so

$$n \geqslant L(z) = b(z) - z + 3 \geqslant (3/4)z^2 - z + 3$$

and it follows that $z/n \to 0$ as $n \to \infty$. Moreover, for $z \geqslant 6$, Lemma 5.3 gives $z < n/4$, so

$$b(z) = L(z) + z - 3 \leqslant n + z - 3 < (5n/4) - 3$$

and hence $b(z)/n^2 \to 0$ as $n \to \infty$, as required. □

We are also in a position to prove Theorem 1.9 and give a recursive form for $W(n)$ different than the one provided by Theorem 1.7.

Proof of Theorem 1.9. Recall that $\omega_n = \bigcup_{k=1}^{z}(n(n-k) + W(k))$. For $n \in [1, 24]$, $n \neq 7$, one may verify computationally that $[n, b(n)] \cup \omega_n \cup \{n^2\}$ equals the expression for $W(n)$ given in Theorem 1.7. So, assume that $n \geqslant 25$, which means that $z \geqslant 6$.

It is clear that $W(n) \supseteq [n, b(n)] \cup \omega_n \cup \{n^2\}$, so we only need to prove the other containment. Let $\mathcal{P} \subseteq M_n(R)$ be a zpmr. There is nothing to prove if $w(\mathcal{P}) \leqslant b(n)$ or $w(\mathcal{P}) = n^2$, so assume that $b(z) < w(\mathcal{P}) < n^2$. By Proposition 5.9, $w(\mathcal{P}) \subseteq n(n-m) + W(m)$ for some $m \in [1, z]$. Hence, $w(\mathcal{P}) \in \omega_n$, completing the proof. $\square$

Corollary 5.15. *Let $S(n)$ be the set of integers m for which there exists a digraph on n vertices with m reachable pairs. Let $T(n)$ be the set of integers m for which there exists a finite topology $\mathcal{T}$ on n points such that $\tau(\mathcal{T}) = m$. If $n \geqslant 1$, $z \in \mathbb{N}$ is such that $L(z) \leqslant n < L(z+1)$, $\gamma_n = \bigcup_{k=1}^{z}((n-1)(n-k) + S(k))$, and $\omega_n = \bigcup_{k=1}^{z}(n(n-k) + W(k))$, then*

$$S(n) = [0, b(n) - n] \cup \gamma_n \cup \{n^2 - n\}, \text{ and}$$
$$T(n) = [n, b(n)] \cup \omega_n \cup \{n^2\}.$$

Proof. This follows immediately from Theorem 1.9 and Lemma 2.3. $\square$

The formula in Theorem 1.8 and the results obtained subsequently are recursive in nature; if we want to know $b(n)$, we need to know the value of z for which $L(z) \leqslant n < L(z+1)$. The following proposition gives us bounds for z in terms of n, showing that z is approximately $n^{0.5}$ when n is large enough.

Proposition 5.16. *Let $\delta > 0$. There exists $N = N(\delta) \in \mathbb{N}$ such that if $n \geqslant N$ and z is such that $L(z) \leqslant n < L(z+1)$, then*

$$n^{0.5} < z = n^{0.5+\varepsilon(n)},$$

where

$$\varepsilon(n) < \frac{\ln\left(1 + \frac{1}{n^{0.25-2\delta-\delta^2}} + \frac{1}{n^{0.5-\delta}}\right)}{2\ln(n)}.$$

In particular, for any $n \geq 5$,

$$\varepsilon(n) < \frac{\ln\left(1 + \frac{1}{n^{0.1}} + \frac{1}{n^{0.43}}\right)}{2\ln(n)}.$$

Proof. We will establish the lower bound first. By assumption, $n < L(z+1) = b(z+1) - (z+1) + 3$. Let $t \in \mathbb{N}$ be such that $L(t) \leq z+1 < L(t+1)$. Theorem 1.8 implies that

$$b(z+1) = (z+1)^2 - t(z+1) + b(t),$$

and this in turn gives us

$$n < z^2 - (t-1)z + b(t) - t + 3 = z^2 - (t-1)z + L(t).$$

By assumption, $L(t) \leqslant z + 1 < (t-1)z$ when $t \geqslant 3$, and so

$$n < z^2 - ((t-1)z - L(t)) < z^2,$$

which implies the lower bound on z.

To establish the upper bound, we will assume that $z = n^{0.5+\varepsilon(n)}$, where $\varepsilon(n)$ is a function depending on n. We know already that $\varepsilon(n)$ is bounded below by 0; we will now use Rao's result [7, Theorem 6] to obtain an absolute upper bound on $\varepsilon(n)$. Since $z = n^{0.5+\varepsilon(n)}$, we have

$$\begin{aligned} 0.5 + \varepsilon(n) &= \frac{\ln(z)}{\ln(n)} \\ &\leqslant \frac{\ln(z)}{\ln(L(z))} \\ &= \frac{\ln(z)}{\ln(b(z) - z + 3)} \\ &\leqslant \frac{\ln(z)}{\ln(z^2 - z^{1.57} - z + 3)}, \end{aligned}$$

where the last line comes from Rao's lower bound [7, Theorem 6]. A routine calculation shows that this function is decreasing for $z \geqslant 3$, which implies that $0.5 + \varepsilon(n)$ is bounded above by $\ln(3)/\ln(7)$. Indeed, $0.5 + \varepsilon(n) \leqslant \ln(3)/\ln(7) < 0.57$. Hence $\varepsilon(n) < 0.07$ for all $n \geqslant 7$, and inspection shows that $\varepsilon(n) < 0.07$ when $n = 5, 6$, so such an absolute upper bound on $\varepsilon(n)$ exists.

Let $\delta > 0$, and let $N \in \mathbb{N}$ be such that when $m \geq N$, $\varepsilon(m) < \delta$. Assume that $z \geqslant N$. Now,

$$n \geqslant L(z) = b(z) - z + 3 > z^2 - z^{1.5+\varepsilon(z)} - z,$$

which implies that

$$z^2 < n + z^{1.5+\varepsilon} + z \leqslant n + z^{1.5+\delta} + z.$$

Using the facts that $z = n^{0.5+\varepsilon(n)}$ and $\varepsilon(n) < \delta$, we get

$$n^{1+2\varepsilon(n)} < n + n^{(1.5+\delta)(0.5+\delta)} + n^{0.5+\delta} = n + n^{0.75+2\delta+\delta^2} + n^{0.5+\delta}.$$

Solving for $\varepsilon(n)$, we obtain

$$\varepsilon(n) < \frac{\ln\left(1 + \frac{1}{n^{0.25-2\delta-\delta^2}} + \frac{1}{n^{0.5-\delta}}\right)}{2\ln(n)},$$

as desired.

Finally, when $n \geqslant 5$, we have $\varepsilon(n) < 0.07$ and so

$$\varepsilon(n) < \frac{\ln\left(1 + \frac{1}{n^{0.1}} + \frac{1}{n^{0.43}}\right)}{2\ln(n)}.$$

□

As a corollary, we establish some bounds on $b(n)$ when n is sufficiently large.

Corollary 5.17. *Let $\delta > 0$. There exists $N = N(\delta) \in \mathbb{N}$ such that, if $n \geqslant N$, then*

$$n^2 - n^{1.5+\delta} < b(n) < n^2 - n^{1.5} + n^{1+2\delta}.$$

Proof. This follows from Theorem 1.8 and Proposition 5.16. □

6 Gaps and Cardinalities of Weight Sets

In this section, we analyze the gaps in the weight sets, and obtain bounds on the cardinality of $W(n)$. When we speak of "gaps", we mean integers or runs of integers that are in $[n, n^2] \setminus W(n)$. By the definition of $b(n)$, any $k \in [n, n^2] \setminus W(n)$ must be at least $b(n) + 1$. Hence, Proposition 5.9 and Theorem 1.8 will be relevant in our work. Ultimately, we will be able to use our knowledge of the gaps in $W(n)$ to give bounds on $|W(n)|$.

The gaps in $W(n)$ can always be described by using two polynomials that depend on n and z. These polynomials also shed some light on the mysterious nature of the weight loss function $L(z)$.

Notation 6.1. For positive integers n and z, we define $\phi(n, z) = n^2 - zn + b(z) + 1$ and $\psi(n, z) = n^2 - (z - 1)n + z - 2$.

Lemma 6.2. *Let $z \geqslant 1$.*

(1) If $n \geqslant L(z)$, then $\phi(n, z) \leqslant \psi(n, z)$. In particular, $\phi(L(z), z) = \psi(L(z), z)$.

(2) If $n \geqslant L(z)$ and $r \in [1, z]$, then the cardinality of the interval $[\phi(n, r), \psi(n, r)]$ is equal to $n - L(r) + 1$.

(3) Let $j \in W(z)$. If $\phi(n, z) \leqslant n^2 - zn + j \leqslant \psi(n, z)$, then $b(z) + 1 \leqslant j \leqslant n + z - 2$.

Proof. All parts are straightforward computations. □

Proposition 6.3. *Let $n \geq 3$ and let $z \in \mathbb{N}$ be such that $L(z) \leqslant n < L(z + 1)$. Let $k \in [n, n^2]$. If $k \notin W(n)$, then $k \in [\phi(n, r), \psi(n, r)]$ for some $r \in [1, z]$.*

Proof. For each $r \in [1, z]$, let $I_r = [n^2 - rn + r, n^2 - rn + b(r)]$ and let $J_r = [\phi(n, r), \psi(n, r)]$. By Theorem 1.8, we see that

$$[b(n) + 1, n^2 - 1] = J_z \cup I_{z-1} \cup J_{z-1} \cup I_{z-2} \cup \cdots \cup J_1$$

for $n \neq 8$. If $n = 8$, then $b(8) = 52 > 48 = \phi(8, 3)$, so that $[b(n) + 1, n^2 - 1]$ is a subset of the union of the J_r and I_r.

Now, assume that $k \in [n, n^2] \setminus W(n)$. By the definition of $b(n)$, we must have $k \geqslant b(n) + 1$. As noted above, Theorem 1.8 shows that $b(n) + 1$ is equal to $\phi(n, z)$ when $n \neq 8$, and if $n = 8$, then $b(8) + 1 > \phi(8, 3)$. Thus, in all cases we see that k is greater than the lower endpoint of J_z. By considering $n \times n$ zpmrs of the form

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{\text{full}} \\ \hline 0 & \mathcal{P}'' \end{array}\right]$$

where $\mathcal{P}''$ is $r \times r$, we can obtain all the weights in I_r. Since k is not an obtainable weight, it is not contained in any I_r for $1 \leqslant r \leqslant z$, and so it must lie in one of the intervals J_r, as required. □

For fixed z, the set $[\phi(n, z), \psi(n, z)]$ first becomes nonempty when $n \geqslant L(z)$, and then grows as n increases. By Proposition 6.3, any gap in the weight set must occur in such an interval, but not every number in $[\phi(n, z), \psi(n, z)]$ lies outside of $W(n)$. For instance, when $z = 3$ and $n \geq 7$ the interval is

$$[\phi(n, 3), \psi(n, 3)] = [n^2 - 3n + 8, n^2 - 2n + 1]$$

but we always have $n^2 - 3n + 9 \in W(n)$ because we can form the zpmr

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{\text{full}} \\ \hline 0 & M_3(R) \end{array}\right].$$

However, we can prove that there are always few weights contained in $[\phi(n,z),\psi(n,z)]$, in the sense that the number of possible weights between $\phi(n,z)$ and $\psi(n,z)$ is bounded above by a constant depending only on z.

Theorem 6.4. *Let $z \geqslant 1$ and let n be such that $n \geqslant L(z)$. Additionally, if $z = 3$ then assume that $n \neq 8$. Then,*

$$W(n) \cap [\phi(n,z),\psi(n,z)] = \left\{n^2 - zn + k \mid k \in W(z) \text{ and } b(z)+1 \leqslant k \leqslant n+z-2\right\}.$$

Proof. Assume first that $z \geqslant 6$. By Proposition 5.9, if $\mathcal{P}$ is an $n \times n$ zpmr such that $w(\mathcal{P}) \in [\phi(n,z),\psi(n,z)]$, then we may assume that $\mathcal{P}$ has the form

$$\mathcal{P} = \left[\begin{array}{c|c} \multicolumn{2}{c}{\text{full}} \\ \hline 0 & \mathcal{Q}'' \end{array}\right] \tag{6.5}$$

where $\mathcal{Q}'' \subseteq M_m(R)$ for some $m \in [1,z]$. If $m \leqslant z-1$, then as in the proof of Theorem 1.8 we have

$$\begin{aligned} w(\mathcal{P}) &= n(n-m) + w(\mathcal{Q}'') \\ &\geqslant n^2 - m(n-1) \\ &\geqslant n^2 - (z-1)(n-1) \\ &= n^2 - (z-1)n + z - 1 \end{aligned}$$

which is strictly greater than $\psi(n,z)$. Thus, we must have $m = z$. This means that $w(\mathcal{Q}'')$ is equal to some $k \in W(z)$ such that

$$\phi(n,z) \leqslant n^2 - zn + k \leqslant \psi(n,z).$$

By Lemma 6.2 part (3), k is between $b(z)+1$ and $n-z+2$. This proves that

$$W(n) \cap [\phi(n,z),\psi(n,z)] \subseteq \left\{n^2 - zn + k \mid k \in W(z) \text{ and } b(z)+1 \leqslant k \leqslant n+z-2\right\}.$$

The reverse containment holds because we can always vary the choice of $\mathcal{Q}''$ in (6.5) to produce a weight of $n^2 - zn + k$. Hence, the two sets are equal.

It remains to prove the theorem for smaller values of z. We will handle the cases where $z = 1$ and $z = 2$. The remaining cases where $z \in [3,5]$ can be done using similar logic and are left to the reader.

In what follows, $\mathcal{P} \subsetneqq M_n(R)$ will be a zpmr of the form

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}' & C \\ 0 & \mathcal{P}'' \end{bmatrix}$$

where $\mathcal{P}'$ and $\mathcal{P}''$ are zpmrs, $\mathcal{P}'$ has full weight, and the rows of C are identical. Let $r \in [1, n-1]$ be the dimension of $\mathcal{P}''$. With this setup, we have

$$w(\mathcal{P}) \leqslant (n-r)^2 + (n-r)r + r^2 = n(n-r) + r^2. \tag{6.6}$$

Assume that $z = 1$, so that $n \geqslant L(1) = 3$, $\phi(n,1) = n^2 - n + 2$, and $\psi(n,1) = n^2 - 1$. We will prove that $W(n)$ does not meet $[\phi(n,1),\psi(n,1)]$. As a function of r, the maximum value of $n(n-r) + r^2$ on the interval $[1, n-1]$ is achieved when $r = 1$ or $r = n-1$. Thus, by (6.6)

$$w(\mathcal{P}) \leqslant n(n-1) + 1 < \phi(n,1)$$

and so $W(n) \cap [\phi(n,1),\psi(n,1)] = \varnothing$.

Now, assume that $z = 2$. Then, $n \geqslant 5$, $\phi(n,2) = n^2 - 2n + 5$, and $\psi(n,2) = n^2 - n$. We will show that $W(n)$ has empty intersection with $[\phi(n,2), \psi(n,2)]$. Applying (6.6) with $r \in [2, n-2]$ shows that

$$w(\mathcal{P}) \leqslant n(n-2) + 4 < \phi(n,2)$$

so if $w(\mathcal{P})$ is to be in $[\phi(n,2), \psi(n,2)]$, we must have $r = 1$ or $r = n-1$. If $r = 1$, then $w(\mathcal{P}) = (n-1)^2 + w(C) + 1$, and by Lemma 3.2 $w(C)$ is either 0 or $n-1$. If $w(C) = 0$, then $w(\mathcal{P}) = n^2 - 2n + 2$, which is too small; and if $w(C) = n-1$, then $w(\mathcal{P}) = n^2 - n + 1$, which is too large. So, assume that $r = n-1$.

Since $r = n-1$, the two largest possible weights for $\mathcal{P}''$ are $(n-1)^2$ and

$$(n-1)^2 - (n-1) + 1 = n^2 - 3n + 3.$$

If $w(\mathcal{P}'') = n^2 - 3n + 3$, then the maximum weight for $\mathcal{P}$ is $(n^2 - 3n + 3) + n = n^2 - 2n + 3$, which is too small. So, we must have $w(\mathcal{P}'') = (n-1)^2$. But, this returns us to the case of the previous paragraph where $w(C)$ is either 0 or $n-1$. There are no more cases to consider, so $W(n) \cap [\phi(n,2), \psi(n,2)] = \varnothing$. ☐

Corollary 6.7. *Let $z \geqslant 1$ and let n be such that $n \geqslant L(z)$. Additionally, if $z = 3$ then assume that $n \neq 8$. Then, the cardinality of $W(n) \cap [\phi(n,z), \psi(n,z)]$ is bounded above by $|W(z)| - L(z) + 2$.*

Proof. By Theorem 6.4, the cardinality of $W(n) \cap [\phi(n,z), \psi(n,z)]$ is equal to the number of integers k such that $k \in W(z)$ and $b(z) + 1 \leqslant k \leqslant n + z - 2$. The lower bound on k guarantees that $k \in W(z) \setminus [z, b(z)]$, and

$$|W(z) \setminus [z, b(z)]| = |W(z)| - (b(z) - z + 1) = |W(z)| - L(z) + 2,$$

as desired. ☐

We can use Theorem 6.4 to obtain a recursive formula for $|W(n)|$. However, as this formula depends on knowing the cardinalities of all the intersections $W(r) \cap [\phi(n,r), \psi(n,r)]$, it is rather tedious to calculate. In lieu of this, we prove Theorem 1.10 and derive more convenient lower and upper bounds on $|W(n)|$.

Proof of Theorem 1.10. By Proposition 6.3, any gaps in the weight set $W(n)$ must occur within an interval of the form $[\phi(n,r), \psi(n,r)]$ for some $r \in [1,z]$. By Lemma 6.2(2), the cardinality of $[\phi(n,r), \psi(n,r)]$ is equal to $n - L(r) + 1$. So, the number of weights in $[n, n^2]$ that are not contained within any interval $[\phi(n,r), \psi(n,r)]$ is equal to

$$\begin{aligned} n^2 - n + 1 - \sum_{r=1}^{z} (n - L(r) + 1) &= n^2 - n + 1 - nz - z + \sum_{r=1}^{z} L(r) \\ &= n^2 - (z+1)n - (z-1) + \sum_{r=1}^{z} L(r). \end{aligned} \tag{6.8}$$

Recall that c_r denotes the cardinality of the set $\{k \in W(r) \mid b(r) + 1 \leqslant k \leqslant n + r - 2\}$. To obtain the exact value of $|W(n)|$, we can take (6.8) and add the cardinality of each intersection $W(n) \cap [\phi(n,r), \psi(n,r)]$, which by Theorem 6.4 is equal to c_r. The result is the formula stated in part (1) of Theorem 1.10.

Ignoring the intersections $W(n) \cap [\phi(n,r), \psi(n,r)]$ produces the lower bound in part (2) of Theorem 1.10. For part (3), we apply Corollary 6.7 to get

$$L(r) + c_r \leqslant L(r) + |W(r)| - L(r) + 2 = |W(r)| + 2$$

for each r. Combining this inequality with the equation from part (1) of Theorem 1.10 yields

$$\begin{aligned}|W(n)| &\leqslant n^2 - (z+1)n - (z-1) + \sum_{r=1}^{z} \Big(|W(r)| + 2\Big) \\ &= n^2 - (z+1)n + z + 1 + \sum_{r=1}^{z} |W(r)| \\ &= n^2 - (z+1)(n-1) + \sum_{r=1}^{z} |W(r)|,\end{aligned}$$

as claimed. $\square$

Remark 6.9. Let $S(n)$ refer to the set of integers m for which there exists a digraph on n vertices with m reachable pairs, and let $T(n)$ refer to the set of integers m for which there exists a finite topology $\mathcal{T}$ on n points with $\tau(\mathcal{T}) = m$. By Lemma 2.3, $|S(n)| = |T(n)| = |W(n)|$, and so Theorem 1.10 also provides bounds on $|S(n)|$ and $|T(n)|$.

These bounds for $|W(n)|$ are relatively accurate, as we can show that the relative error between the upper and lower bounds goes to 0 as $n \to \infty$.

Proposition 6.10. *Let $W^-(n)$ and $W^+(n)$ denote the lower and upper bounds for $|W(n)|$, respectively, from Theorem 1.10. As $n \to \infty$, $(W^+(n) - W^-(n))/|W(n)| \to 0$.*

Proof. Let $\delta > 0$. By Corollary 5.17, there exists a fixed $N \in \mathbb{N}$ such that, if $m \geqslant N$, then $b(m) > m^2 - m^{1.5+\delta}$. Assuming that $z \geqslant N$ and that $L(z) \leqslant n < L(z+1)$, this implies that

$$\begin{aligned}W^+(n) - W^-(n) &= 2z + \sum_{r=1}^{z} (|W(r)| - L(r)) \\ &\leqslant 2z + \sum_{r=1}^{z} \big(r^2 - b(r) + r - 3\big) \\ &\leqslant 2z + \sum_{r=1}^{N-1} \big(r^2 - b(r) + r - 3\big) + \sum_{r=N}^{z} \Big(r^{1.5+\delta} + r - 3\Big) \\ &< 2z + \sum_{r=1}^{N-1} \big(r^2 - b(r) + r - 3\big) + \sum_{r=N}^{z} \Big(r^{1.5+\delta} + r\Big) \\ &< 2z + \sum_{r=1}^{N-1} \big(r^2 - b(r) + r - 3\big) + \int_{N}^{z+1} \Big(r^{1.5+\delta} + r\Big)\ dr \\ &< 2z + \sum_{r=1}^{N-1} \big(r^2 - b(r) + r - 3\big) + \frac{1}{2.5+\delta}(z+1)^{2.5+\delta} + \frac{1}{2}(z+1)^2,\end{aligned}$$

and so $W^+(n) - W^-(n) = O(z^{2.5+\delta})$.

On the other hand, $|W(n)| \geqslant b(n) - n + 1$, $n \geqslant L(z) = b(z) - z + 3$, and b is an increasing function by Lemma 5.2, and so, again assuming that $z \geqslant N$,

$$\begin{aligned}
|W(n)| &\geqslant b(n) - n + 1 \\
&= n^2 - (z+1)n + b(z) + 1 \\
&\geqslant (b(z) - z + 3)^2 - (z+1)\,(b(z) - z + 3) + b(z) + 1 \\
&= b(z)^2 - 3zb(z) + 6b(z) + 2z^2 - 8z + 7 \\
&> \left(z^2 - z^{1.5+\delta}\right)^2 - 3z(z^2) + 6\left(z^2 - z^{1.5+\delta}\right) + 4z^2 - 8z + 7,
\end{aligned}$$

and so $|W(n)| = z^4 - O(z^{3.5+\delta})$. The results follows.

□

Empirical evidence suggests that the upper and lower bounds from Theorem 1.10 are fairly accurate even for small values of n. Figures 1 and 2 show the relative errors corresponding to $W^-(n)$ and $W^+(n)$ for $n \in [1, 166]$. From these graphs, one can see that $W^+(n)$ is more accurate than $W^-(n)$, but both bounds are consistently within 1% of the true value of $|W(n)|$.

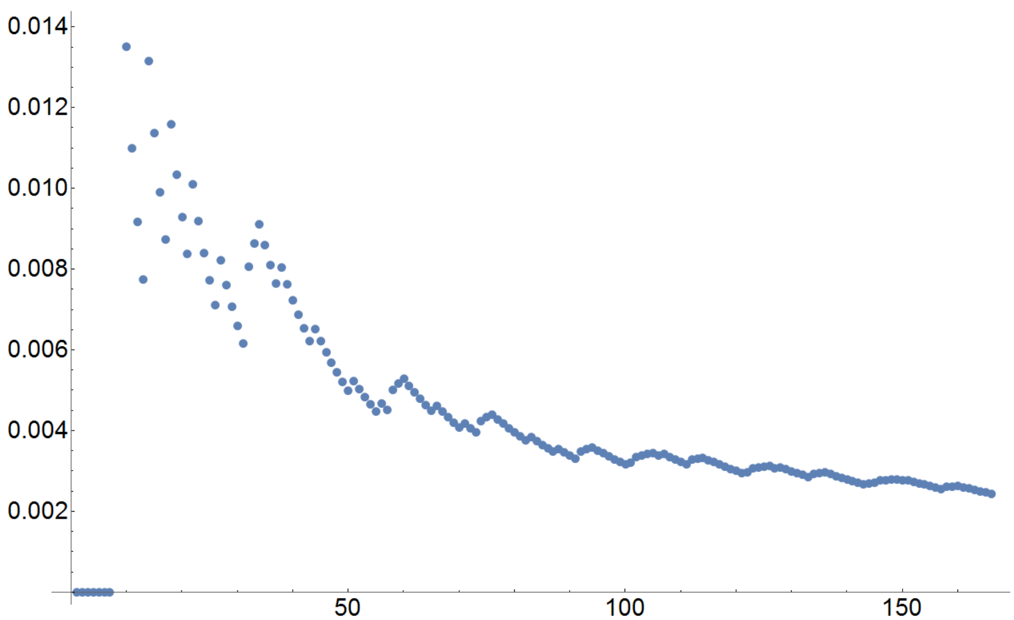

Figure 1: Relative error $\left|W^-(n) - |W(n)|\right| / |W(n)|$

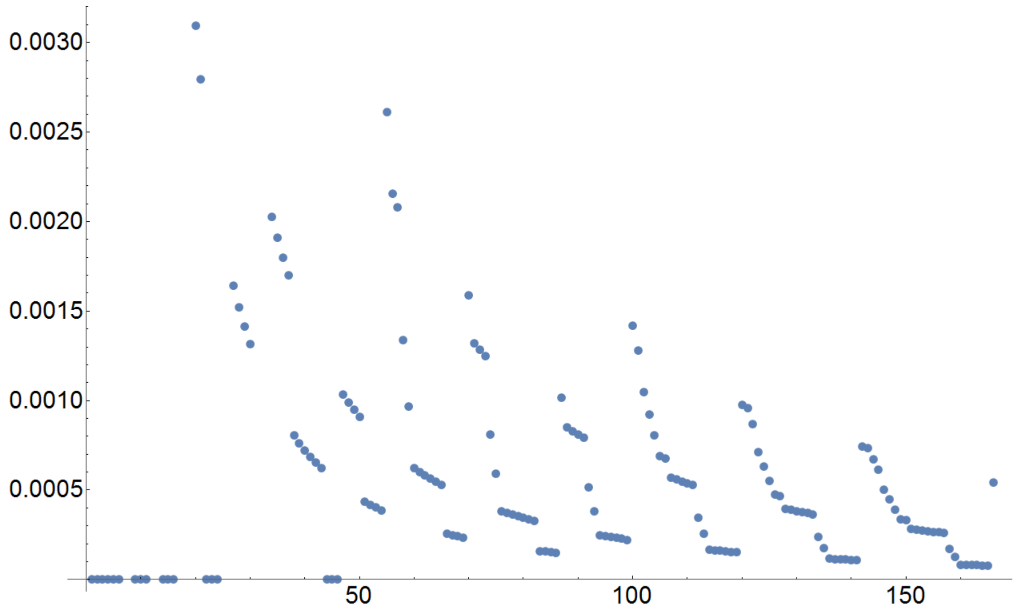

Figure 2: Relative error $\big|W^{+}(n)-|W(n)|\big|/|W(n)|$

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