CHARACTERIZATION AND ENUMERATION OF 3-REGULAR PERMUTATION GRAPHS

AYSEL EREY, ZACHARY GERSHKOFF, AMANDA LOHSS, AND RANJAN ROHATGI

Abstract. A permutation graph is a graph that can be derived from a permutation, where the vertices correspond to letters of the permutation, and the edges represent inversions. We provide a construction to show that there are infinitely many connected r-regular permutation graphs for $r \geq 3$. We prove that all 3-regular permutation graphs arise from a similar construction. Finally, we enumerate all 3-regular permutation graphs on n vertices.

1. Introduction

The graphs considered here are finite and simple. A graph on n vertices is a *permutation graph* if there is a labeling $v_1, v_2, \ldots, v_n$ of the vertices, and a permutation $\pi = [\pi(1), \pi(2), \ldots, \pi(n)]$, such that v_i and v_j are adjacent in G if and only if $i < j$ and $\pi(i) > \pi(j)$. In this case, the ordered pair $(\pi(i), \pi(j))$ is said to be an *inversion* of π. This definition of permutation graphs was given in 1971 by Pneuli et al. [14]. We note that this is different from the "generalized prisms" [17] notion of permutation graphs given by Chartrand and Harary [4].

Permutation graphs have received a considerable amount of attention in the literature since their introduction (see, for example, [9, 15, 16]). Many algorithmic problems have efficient solutions on permutation graphs. For example, it was shown in [3] that the longest path problem (which is NP-complete on general graphs) can be solved in linear time on permutation graphs.

There has been interest in enumerating various types of permutation graphs. For instance, in [11], Koh and Ree gave a recurrence relation for the number of connected permutation graphs. In [2], the number of permutation trees is shown to be $2n-2$ for $n \geq 2$. It is well known that permutation graphs cannot have induced cycles of length five or greater. Therefore, it is easy to see that the only connected 2-regular permutation graphs are C_3 and C_4 [10]. In this direction, we will consider r-regular permutation graphs. For $r > 2$, we show that the family is infinite.

Theorem 1.1. *For every $r \geq 3$, there are infinitely many connected r-regular permutation graphs.*

In particular, we give a complete characterization of 3-regular permutation graphs. This will be given in terms of the construction mentioned above.

An interesting corollary of our construction is that almost all 3-regular permutation graphs are planar. The family of permutation graphs is closed under induced subgraphs (see, for example, [5]), but a description in terms of minors, as planarity

Date: October 3, 2017.
1991 *Mathematics Subject Classification.* 05C75.
Key words and phrases. permutation graph, graph regularity.

results are normally stated, is not tractable since permutation graphs are not closed under subgraphs.

Corollary 1.2. *Every 3-regular permutation graph except $K_{3,3}$ is planar.*

Finally, we use the characterization of 3-regular permutation graphs to enumerate them with a recursive formula.

Theorem 1.3. *Let $a(n)$ be the number of connected 3-regular permutation graphs on n vertices. If n is an odd integer or if $n \in \{2, 8, 12\}$, then $a(n) = 0$. If $n \in \{4, 6, 10, 14, 16, 18, 20\}$, then $a(n) = 1$. For even $n > 20$, we have*

$$a(n) = \begin{cases} a(n-4) + a(n-6) & \text{if } n \equiv 2 \pmod 4 \\ a(n-4) + a(n-6) - t(\frac{n-20}{4}) & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

where $t(x)$ is the number of compositions of x into parts of size 2 or 3.

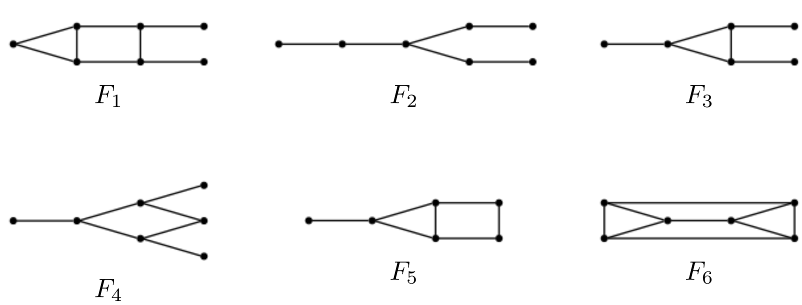
Proofs of Theorem 1.1, Corollary 1.2, and Theorem 1.3 can be found in Sections 3, 4, and 5, respectively.

2. PRELIMINARIES

If G is a permutation graph with corresponding permutation π, we say that π is a *realizer* of G. When discussing a realizer and its graph, we will sometimes refer to a vertex in the graph and an entry in the permutation with the same label. It is well known (for example, in [14]) that G is a permutation graph if and only if its complement $\overline{G}$ is also a permutation graph.

There are many known characterizations of permutation graphs. Recent characterizations include one by Gervacio et al. [8] in terms of cohesive vertex-set orders, and one by Limouzy [12] in terms of Seidel minors. Here we rely on the 1967 characterization by Gallai [7] in terms of forbidden induced subgraphs (see also [6,13]). All cycle graphs on five for more vertices are forbidden induced subgraphs. We will refer to these as *large holes*. Table 1 illustrates all other forbidden induced subgraphs with maximum degree 3.

TABLE 1. Forbidden induced subgraphs for permutation graphs with $\Delta \leq 3$



Throughout this paper, we use K_i and I_i to denote the complete graph on i vertices and the empty graph on i vertices, respectively. We will also use $\oplus$ to denote graph disjoint union, and $\otimes$ to denote a Cartesian product of graphs.

3. Infinitely many r-regular permutation graphs for $r \geq 3$

Let G be a graph of order n with vertices $v_1, v_2, \ldots, v_n$. Given n graphs $H_1, H_2, \ldots, H_n$, we define the *composition of* $H_1, H_2, \ldots, H_n$ *into* G, denoted $G[H_1, H_2, \ldots, H_n]$, as the graph which is obtained from G by replacing the vertex v_i with the graph H_i. More precisely, the vertex set of $G[H_1, H_2, \ldots, H_n]$ is the disjoint union of the vertex sets of every H_i, and uv is an edge of $G[H_1, H_2, \ldots, H_n]$ if and only if either $uv \in E(H_i)$ for some i, or there are distinct indices i and j such that $u \in V(H_i)$, $v \in V(H_j)$ and $v_i v_j \in E(G)$. If each graph H_i is a complete graph or empty graph then $G[H_1, H_2, \ldots, H_n]$ is called a *blow-up* of G, and we say that vertex v_i is *blown up* into H_i, or *replaced* with H_i.

Lemma 3.1. *Let G be a permutation graph of order n and $(H_1, H_2, \ldots, H_n)$ be permutation graphs. Then $G^* = G[H_1, H_2, \ldots, H_n]$ is also a permutation graph.*

Proof. Let $\sigma \in \mathcal{S}_n$ be a realizer of G. Let the permutation $\tau_i = (\tau_i(1), \tau_i(2), \ldots, \tau_i(|V(H_i)|))$ be a realizer of H_i for $i = 1, \ldots, n$. We construct a permutation σ^* from σ by replacing $\sigma(i)$ in σ with the list

$$\tau_i(1) + t_i,\ \tau_i(2) + t_i,\ \ldots,\ \tau_i(|V(H_i)|) + t_i$$

where

$$t_1 = 0 \text{ and } t_i = \sum_{j\colon j<i} |V(H_j)|.$$

To see that σ^* is a realizer for G^*, let $\tau_i(a)$ and $\tau_i(b)$ be vertices of H_i. Then $(\tau_i(a) + t_i, \tau_i(b) + t_i)$ is an inversion of σ^* (and thus the vertices are adjacent) if and only if $(\tau_i(a), \tau_i(b))$ is an inversion of τ. Moreover, if u is a vertex of G^* that comes from H_i, and v is a vertex of G^* that comes from a distinct H_j, then u and v are adjacent if and only if $(\sigma(i), \sigma(j))$ is an inversion of σ, which implies that (u, v) is an inversion of σ^* by our construction. $\square$

Using the above lemma, we prove that there are infinitely many connected r-regular permutation graphs for every $r \geq 3$.

Proof of Theorem 1.1. Let $r \geq 3$. For every $n \geq 0$, we construct an r-regular permutation graph G_n of order $2nr + r + 1$ by taking a blow-up of a path. Let $m = 4n+2$ and take a path graph P_m with vertices $v_1, v_2, \ldots, v_m$ in standard order. Note that P_m is a permutation graph because its maximum degree is 2 and it does not have an induced subgraph from Table 1. Replace the first vertex v_i with K_2 and the last vertex v_m with K_{r-1}. For vertices v_i with $i \equiv 2 \pmod 4$, replace them with I_{r-1}; with $i \equiv 3 \pmod 4$, replace v_i with I_{r-2}; with $i \equiv 0 \pmod 4$, replace v_i with I_1; and for $i \equiv 1 \pmod 4$, replace v_i with I_2. The resulting graph G_n is r-regular, and since complete graphs and empty graphs are permutation graphs, by Lemma 3.1, G_n is a permutation graph. Hence, we obtain an infinite list of r-regular permutation graphs

$$G_0 = P_2[K_2, K_{r-1}]$$
$$G_1 = P_6[K_2, I_{r-1}, I_{r-2}, I_1, I_2, K_{r-1}]$$
$$G_2 = P_{10}[K_2, I_{r-1}, I_{r-2}, I_1, I_2, I_{r-1}, I_{r-2}, I_1, I_2, K_{r-1}]$$
$$G_3 = P_{14}[K_2, I_{r-1}, I_{r-2}, I_1, I_2, I_{r-1}, I_{r-2}, I_1, I_2, I_{r-1}, I_{r-2}, I_1, I_2, K_{r-1}]$$
$$\vdots$$

and the result follows. □

4. Characterization of 3-regular permutation graphs

Table 2 shows subgraphs we use in our construction of 3-regular permutation graphs. Let S_1 be G_1 from Table 2, and let S_2 be one of $\{G_2, G_3, G_4\}$. Take the rightmost vertex of S_1 and identify it with the leftmost vertex of S_2. Set $S_1 = S_2$ and repeat the above process, stopping when S_1 is G_4. We will call graphs with such a structure *boxcar graphs*.

Table 2. Some induced subgraphs of boxcar graphs

Lemma 4.1. *A 3-regular graph that is a blow-up of a path is isomorphic to K_4, $K_{3,3}$, or a boxcar graph.*

Proof. Let G be a path graph P_n with vertices $(v_1, v_2, \ldots, v_n)$ in standard order, and consider a blow-up $G^* = G[H_1, H_2, \ldots, H_n]$. There are four possibilities for the graph H_1.

Suppose the first vertex v_1 is blown up into K_k or I_k, with $k \geq 4$. If v_2 exists, then the vertices resulting from blowing up v_2 will have degree at least 4. Thus to obtain a 3-regular graph, v_1 must be the only vertex of G, and it must be blown up into K_4.

Now suppose $H_1 \cong K_3$. Then v_2 must be blown up into a graph of order 1 because the vertices from K_3 require one more neighbor to have degree 3. Since all the vertices have degree 3, we see that G must be P_2, and it blows up into $P_2[K_3, K_1] \cong K_4$.

Suppose $H_1 \cong I_k$, where $k \leq 3$. Since the vertices of H_1 require 3 neighbors, v_2 must be blown up into a graph of order 3. If $H_2 \cong K_3$, then $k = 1$ and we have K_4 as in the case above. If $H_2 \cong I_3$, then the vertices of H_2 have k neighbors on left and they requre $3 - k$ neighbors on the right. In order to not exceed degree 3, we must have $H_3 \cong I_{k-3}$. Thus we obtain $P_2[I_3, I_3]$, $P_3[I_1, I_3, I_2]$, or $P_3[I_2, I_3, I_1]$, all of which are isomorphic to $K_{3,3}$.

The only remaining cases are when H_1 is isomorphic to K_2. If $H_1 \cong K_2$, then H_2 must have order 2. If $H_2 \cong K_2$, then we have $G^* = P_2[K_2, K_2] \cong K_4$. If

$H_2 \cong I_2$, then H_3 must have order 1, so $H_3 \cong K_1$ and we see that G^* must begin with G_1 from Table 2. Then $H_4 \cong K_1$, and H_5 must have order 2. If $H_5 \cong K_2$, then H_6 must have order 1, and we have G_2 as an induced subgraph on $\cup_{i=3}^{6} V(H_i)$. If instead $H_5 \cong I_2$, then we must have either $H_6 \cong I_2$ and $H_7 \cong K_1$, giving us G_3, or $H_6 \cong K_2$, giving us G_4. In the former case, we can continue building our graph and we will get another one of $\{G_2, G_3, G_4\}$. In the latter case, our graph is 3-regular. □

The following lemmas will be useful in our characterization of 3-regular permutation graphs. We say that vertices v_1 and v_2 are *twins* if $N(v_1) - \{v_2\} = N(v_2) - \{v_1\}$, where $N(v_i)$ is the set of vertices that neighbor v_i. We do not distinguish between twins that are adjacent and those that are not.

Lemma 4.2. *Every permutation graph G has a realizer π where, for every pair of twins u and v in G, there is a contiguous, consecutive increasing or decreasing subsequence s of π that contains u and v. Moreover, u and v are adjacent in G if and only if s is decreasing.*

Proof. Let π be a realizer of a graph G, and define G_π to be a graph isomorphic to G with vertex labels corresponding to π. Let u and v be twins in G_π with $u < v$. We will first assume u and v are nonadjacent. If u and v are not part of a contiguous, consecutive increasing subsequence of π, then we can obtain another realizer π' of G by removing v from π, shifting all of the entries greater than u and less than v up by 1, and inserting $u + 1$ to the immediate right of u. Clearly π and π' realize isomorphic graphs, and if a and b are entries of π that belong to a contiguous, consecutive increasing or decreasing subsequence of π, then this transformation does not separate them.

If we assume instead that u and v are adjacent in G_π, then we apply a similar transformation, ultimately placing $u + 1$ to the left of u instead of the right. This results in u and $u+1$ being part of a contiguous, consecutive decreasing subsequence, instead of increasing. □

Lemma 4.3. *If G^* is a graph with maximum degree d, and if G is a graph of minimum order such that G^* is a blow-up of G, then G has no degree d twins.*

Proof. Observe that by our construction of blow-ups given in Lemma 3.1, if G^* is a blow-up of G, than any realizer of G can be used to obtain a realizer for G^* by blow-up. Let u and v be degree d twins of G. By Lemma 4.2, G has a realizer π where u and v are adjacent and consecutive. Let $\{u_1, u_2, \ldots, u_j\}$ and $\{v_1, v_2, \ldots, v_k\}$ be the entries of a realizer π^* for G^* obtained by blowing up u and v, respectively. Then u must be blown up into I_j and v must be blown up into I_k, because if they were blown up into K_j or K_k for $k \geq 2$, then we would have vertices with degree exceeding d. Moreover, unless $j = k = 1$, the vertices u and v must be nonadjacent. In the case that $j = k = 1$, u_1 and v_1 are twins in G^*, and they are adjacent and consecutive in π^*, which means that there is a graph such that $\{u, v\}$ is blown up from a single vertex. In the remaining cases, $\{u_1, u_2, \ldots, u_j, v_1, v_2, \ldots, v_k\}$ is part of a twin class of G^*, and part of a contiguous, consecutive increasing sequence of π^*, so they can also be blown up from a single vertex. This contradicts the assumption that G has minimum order. □

Recall that a *ladder* is a graph $P_2 \otimes P_n$, where n is the number of rungs.

Lemma 4.4. *A 3-regular permutation graph cannot have a ladder with four or more rungs as a subgraph.*

Proof. Suppose G has a ladder as a subgraph, and let u_i and v_i be adjacent vertices on the ith rung of a maximal ladder for i in $\{1, 2, \ldots, k\}$. We will prove the lemma by considering three propositions.

(1) A ladder with three or more rungs cannot have an edge between opposite vertices on the same side of the ladder, such as v_1 and v_k.
(2) A ladder with four or more rungs cannot have an edge between opposite vertices on the different sides of the ladder, such as v_1 and u_k.
(3) There cannot be a ladder with three or more rungs without an edge between the first and last rung of the ladder.

To prove proposition (1), suppose that $i = 3$. Let v_1 and v_k be adjacent, and suppose first that u_1 and u_k are not. Then we have an odd hole using vertices $\{v_1, u_1, u_2, u_3, v_3\}$. However, if u_1 and u_k are also adjacent, then we have F_6. Next suppose $i = 4$. If (v_1, v_k) is an edge and (u_1, u_k) is not, then $\{v_1, u_1, u_3, u_4, v_4\}$ is a large hole. If (u_1, u_k) is also an edge, then the graph is isomorphic to a cube, which has C_6 as an induced subgraph by deleting a pair of opposite vertices. Finally, suppose $i \geq 5$. Then $\{v_1, v_2, \ldots, v_k\}$ is a large hole.

Similarly, for proposition (2), if v_1 and u_k are adjacent, we have an odd hole using $\{v_1, v_2, u_2, u_3, \ldots, u_k\}$.

Finally, for proposition (3), suppose v_1 and u_1 have a common neighbor v. Then v cannot have v_k or u_k as neighbors, or else we have a large hole. So v has another neighbor v', but this gives us F_5 using $\{v', v, v_1, v_2, u_1, u_2\}$. Suppose instead that the third neighbors of v_1 and u_1 are v and u, respectively, with $v \neq u$. Then we have F_4 using $\{v, v_1, v_2, u, u_1, u_2, u_3\}$. □

We now prove that the graphs from Lemma 4.1 are the only 3-regular permutation graphs.

Theorem 4.5. *Every connected 3-regular permutation graph is the blow-up of a path.*

Proof. Suppose G^* is a 3-regular permutation graph that is not a blow-up of a path. Let G be a graph of minimum order such that G^* is a blow-up of G. Then G is either a cycle or G has a degree 3 vertex.

If G is a cycle, then G must be C_3 or C_4, because larger cycles are forbidden as induced subgraphs. In C_3, since all the vertices are adjacent to each other and they all have degree 2, only one vertex can be blown up or else we would have a vertex with degree exceeding 3. Moreover, the vertex must be blown up into K_2 in order for every vertex to have degree 3. The resulting graph is K_4. In C_4, since every vertex has degree 2, at most one of the neighbors of every vertex can be blown up. The only possibility that gives a 3-regular graph is blowing up each of two adjacent vertices into I_2. This gives a graph isomorphic to $K_{3,3}$.

Suppose instead that G has a degree 3 vertex v. Note that the maximum degree $\Delta(G) = 3$. We will show that there are four possibilities for the configuration of the induced subgraph H in a neighborhood of v:

(1) H is admissible as the induced subgraph of a blow-up of a path,
(2) H has a forbidden induced subgraph,
(3) H cannot be blown up into a 3-regular graph,

(4) H has degree 3 twins.

Case (3) occurs when there is a vertex u of degree 3 such that neither u nor any of its neighbors can be blown up without having a vertex exceed degree 3. Note that case (4) is forbidden by Lemma 4.3.

Let the neighbors of v $N(v)$ be $\{v_1, v_2, v_3\}$. We will consider the possible subgraphs induced by $N(v)$.

Suppose $N(v)$ induces I_3, that is, none of the vertices in $N(v)$ are adjacent. Observe that if v is adjacent to a leaf, then v must be blown up into I_3 in order to obtain a 3-regular graph. This implies that all the neighbors of v in G must be leaves, and the resulting graph of this blow-up is $K_{3,3}$. Thus we may assume that all vertices adjacent to a degree 3 vertex have degree at least 2.

We will proceed by considering the number of squares that use v. If v is not involved in any squares, then the subgraph induced by $N(v)$ and its neighbors has either F_2 or a large hole as an induced subgraph. Suppose instead that $\{v, v_2, v_3\}$ are used in a square. Let v_4 be the remaining vertex of the square. If each of $\{v_2, v_3, v_4\}$ have degree 2, then this falls under case (1), because v_4 can be blown up into K_2 to realize G_4 from Table 2. Similarly, if v_2 and v_3 have degree 2 and v_4 has degree 3, then v_4 can be blown up into I_2 to realize G_3. If, however, only one of $\{v_2, v_3\}$ has degree 3, or they both have degree 3 and v_4 has degree 2, then we have a situation described in case (3). If all of $\{v_2, v_3, v_4\}$ have degree 3, then depending the configuration of the remaining edges, we either have F_4, F_5, or a large hole as a subgraph.

Suppose v is used in two squares. One possibility is for two neighbors of v to be involved in both squares; say $\{v, v_2, v_3\}$ are involved in two distinct squares. This implies that v_2 and v_3 are degree 3 twins. Another possibility is for the two squares to share a single edge. Observe that if the largest ladder subgraph using v has three rungs, and there is an edge between two opposite vertices in a cycle around the ladder, then v is involved in at least three squares. The remaining possibilities for a ladder on three or more rungs using v contradict cases (1), (2), and (3) from Lemma 4.4.

The final possibility when $N(v)$ induces I_3 is for v to be involved in three or more squares. In this case, either there is an induced 6-cycle around v, or the neigbhorhood around v is admissible a subgraph of G_3. If v is used in more than three squares, then $G \cong K_{3,3}$.

Now suppose $N(v)$ induces $K_2 \oplus I_1$, and suppose that $\{v, v_2, v_3\}$ forms a triangle. If one of $\{v_2, v_3\}$ has degree 2, then the graph cannot be blown up to be 3-regular. If they both have degree 3 and are not in a square with v_1, then either they have a common neighbor other than v, giving us G_2 from Table 2, or they have different neighbors, giving us F_3. Suppose that $\{v, v_2, v_3\}$ is a triangle and $\{v, v_1, v_2, v_4\}$ is a square for a new vertex v_4. If v_3 has degree 2, then G cannot be blown up into a 3-regular graph. If v_3 is adjacent to v_4, then this is isomorphic to G_1. If v_3 is adjacent to a new vertex v_5, then we have F_5 as an induced subgraph.

Finally, let $\{v, v_2, v_3\}$ be a triangle, and suppose there are squares $\{v, v_1, v_2, v_4\}$ and $\{v, v_1, v_3, v_5\}$. If $v_4 = v_5$, then v_4 and v are twins; a contradiction. Suppose $v_4 \neq v_5$. If v_4 and v_5 are nonadjacent, then $\{v_1, v_2, v_3, v_4, v_5\}$ is a large hole, and if they are adjacent, then our subgraph is isomorphic to F_6.

The remaining possibilities are if $N(v)$ induces P_3 or K_3. In both of these cases, we have twin vertices of degree 3, contradicting Lemma 4.3. □

This theorem and Lemma 4.1 immediately imply the following corollary.

Corollary 4.6. *Every 3-regular permutation graph is isomorphic to K_4, $K_{3,3}$, or a boxcar graph.*

Note that this also implies Corollary 1.2, since every boxcar graph has a planar embedding.

Corollary 4.7. *Every 3-regular permutation graph has a Hamiltonian path.*

Proof. Clearly K_4 and $K_{3,3}$ are Hamiltonian. Observe that every graph in $\{G_1, G_2, G_3, G_4\}$ from Table 2 also has a Hamiltonian path. By merging the degree 1 vertices to obtain a boxcar graph, we find that Hamiltonian path of each of the graphs $\{G_1, G_2, G_3, G_4\}$ connected in sequence give a Hamiltonian path for the boxcar graph. □

5. Enumeration of connected 3-regular permutation graphs

We conclude with a proof of Theorem 1.3, which gives a recursive formula for the number of connected 3-regular permutation graphs on n vertices. Let m be an integer. In the following proof, we will use *sequences for m* to mean equivalence classes of compositions of m into parts of size 2 and 3 where a composition and its reverse are considered to be the same. We will refer to the parts of size 2 or 3 as *symbols.*

Proof of Theorem 1.3. Clearly there cannot be a 3-regular graph on an odd number of vertices.

Using Corollary 4.6, we can easily count the number of connected 3-regular permutation graphs on 20 or fewer vertices, and we know that the ones on more than 20 vertices must be boxcar graphs. Boxcar graphs can be thought of as beginning with G_1 from Table 2, continuing with a sequence of copies of G_2 and G_3 in any order and of any length, and ending with G_4. Note that because we merge vertices, the G_1 and G_4 subgraphs together contribute 10 vertices to the graph, each G_2 contributes 4 vertices, and each G_3 contributes 6 vertices. Let $m = \frac{n-10}{2}$. The problem of enumerating connected 3-regular permutation graphs on n vertices reduces to that of enumerating compositions of $2m$ into parts of size 4 and 6 for every nonnegative integer m, or equivalently, enumerating compositions of m into parts of size 2 and 3. Moreover, because we are only concerned with graphs up to isomorphism, we must count a composition and its reverse as being the same. These are our sequences for m; when working them, we will rely on the following fact.

5.0.1. *Consider all sequences for m. If a sequence has an odd number of symbols consider the sequence obtained by deleting the middle symbol, and if it has an even number of symbols, then delete one of the two symbols closest to the middle. This gives all sequences for $m-2$ and $m-3$. The converse is also true; that is, by considering all sequences for $m-i$ for i in $\{2,3\}$, inserting the symbol i to the middle if there are an even number of symbols, and inserting i to the immediate left or the immediate right of the middle if there are an odd number of symbols, we get all sequences for m.*

We now create an auxiliary bipartite graph B with sides X and Y. Let the vertices on side X represent the sequences for $m-2$ and $m-3$, and let the vertices on side Y represent the sequences for m; we will use the same labels for the vertices and for the sequences that they represent. Place an edge between a vertex x in X and a vertex y in Y if and only if it is possible to get one sequence from the other by using 5.0.1.

Since there are only one or two options for getting one sequence from the other through deleting (or inserting) a symbol according to 5.0.1, each vertex of B has degree 1 or 2.

Consider a vertex y in Y. It will have degree 2 precisely when its sequence has an even number of symbols and the middle two symbols are different. It will have degree 1 otherwise. A vertex x in X, however, will have degree 1 if and only if one of the following conditions hold:

(1) Its sequence has an even number of symbols.
(2) Its sequence has an odd number of symbols, and the middle symbol is the same as the symbol that needs to be inserted to get a sequence for m.
(3) Its sequence has an odd number of symbols, the middle symbol is different from the symbol that needs to be inserted to get a sequence for m, and the sequence and its reverse are the same.

Case (1) is clear. To see case (2), observe that if x is a sequence for $m - i$ with i in $\{2, 3\}$, and the middle symbol is i, then the sequences we get from inserting i to the left or the right of the middle are indistinguishable. For case (3), if x is its own reverse, then the two sequences we get from inserting a symbol immediately to the left or to the right of the middle are reverses of each other and are therefore equivalent. If none of these cases hold, that is, x has an odd number of symbols, the middle symbol is different from the symbol to be inserted, and the sequences and its reverse are different, then x has degree 2 because two distinct sequences arise from the insertions.

Note that in cases (1) and (2), the neighbor of x will necessarily have degree 1. We must now show the following.

5.0.2. *Case (3) can only occur when* $m \equiv 1 \pmod 2$*, or equivalently,* $n \equiv 0 \pmod 4$.

Consider a sequence x for $m - 3$ that is the reverse of itself, and suppose x has odd length and that its middle symbol is 2. Because the subsequences on either side of the middle symbol must be reverses of each other, the sum of all the parts of the sequence must be even, so $m - 3$ is even and m is odd. A similar argument holds if x is a sequence for $m - 2$ that is the reverse of itself, x has odd length, and the middle symbol is 3. Recalling that $m = \frac{n-10}{2}$, we see that 5.0.2 holds.

We can now conclude the proof of Theorem 1.3 by counting the number of sequences of m for m even and m odd. Suppose m is even, and let a sequence y in Y have even length. If the middle two symbols of y are the same, then y has degree 1, and its neighbor in X also has degree 1 by case (2) above. Otherwise, y has degree 2, and its neighbors in X also have degree 2 because cases (1)–(3) do not apply. If we asssume instead that y has odd length, then it has degree 1, and its neighbor in X also has degree 1 by case (1). Thus B is the disjoint union of isolated edges and cycles, so by Hall's Marriage Theorem, it has a perfect matching. If $a(m)$ is the number of sequences for m, then when m is even, we have $a(m) = a(m - 2) + a(m - 3)$ for $m > 3$. This gives us $a(n) = a(n - 4) + a(n - 6)$ for $n > 16$.

Now suppose m is odd. If a vertex x in X falls into case (1) or (2), then by the above argument, it can be matched to a vertex in y. Observe that if x is in case (3), then its neighbor y in Y has degree 2, because deleting either of the middle symbols of y will produce different sequences. Both neighbors x_1 and x_2 of y will fall under case (3). Moreover, x_1 and x_2 must be the same except for the symbol in the middle. Therefore one of the sequences, say x_1, must be for $m-3$, and the other sequence x_2 must be for $m-2$.

To count the number of such pairs x_1 and x_2, it suffices to count the number of sequences of $m-3$ that are their own reverse. This is equal to the number of compositions of $\frac{m-5}{2}$ into parts of size 2 and 3 because the subsequences to the left and to the right of the central 2 must be reverses of each other, and these subsequences are precisely the above compositions. If $t(x)$ is the number of compositions of x into size 2 and 3, then when m is odd, we have $a(m) = a(m-2)+a(m-3)-t\left(\frac{m-5}{2}\right)$ for $m > 5$. Since $m = \frac{n-10}{2}$, this is equivalent to $a(n) = a(n-4) + a(n-6) - t\left(\frac{n-20}{4}\right)$ for $n > 20$. □

The number of compositions of m into parts of size 2 or 3 is given in OEIS sequence 000931 [1].

6. Conclusion

We have proven that there are infinitely many r-regular permutation graphs for $r \geq 3$ and given a complete characterization of 3-regular permutation graphs in terms of blow-ups of paths. It is perhaps surprising that all 3-regular permutation graphs are blow-ups of a path. Unfortunately, this is not the case for all r-regular graphs in general. In particular, we found a counterexample when $r = 4$ (see Figure 1).

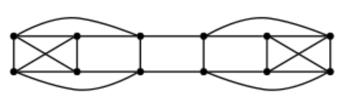


Figure 1. A 4-regular permutation graph that is not a blow-up of a path ($\pi = [5, 4, 7, 2, 1, 10, 3, 12, 11, 6, 9, 8]$).

The graph from Figure 1 can be constructed by blowing up a 4-runged ladder. More specifically, if G is the 4-runged ladder whose vertices are labeled as they appear in a Hamiltonian path starting and ending on a degree 2 vertex, then the graph from Figure 1 is $G[K_2, K_1, K_1, K_2, K_2, K_1, K_1, K_2]$. Note that G is a permutation graph with realizer $[3, 5, 1, 7, 2, 8, 4, 6]$. This observation, along with the lemma below, indicates that the permutation graph from Figure 1 is not the blow-up of a path.

Lemma 6.1. *For each graph G, there is unique graph G' of minimal order such that G is a blow-up of G'.*

Proof. Let $P = (p_1, p_2, \ldots, p_m)$ be the partition of $V(G)$ such that two vertices are in the same part if and only if they are twins. We construct an m-vertex graph G', where distinct vertices v_i, v_j of $V(G')$ are adjacent if and only if the members of p_i and p_j are adjacent in G. Then G is a blow-up of G', obtained by replacing each vertex v_i with the vertices of p_i. We know that G' is minimal because if H is a graph such that G is a blow-up of H, and u_1 and u_2 are vertices of G that arise from the same vertex of H, then u_1 and u_2 must be twins. Moreover, G' is unique because P is unique. □

By taking complements of our graphs from Corollary 4.6 and applying Lemma 6.1, we find other counterexamples for r-regularity for certain even values of r. Counterexamples to show that not every r-regular permutation graph is a blow-up of a path for odd values of $r > 4$ are not known.

7. Acknowledgements

We gratefully acknowledge the financial support from the following grants that made this research possible: NSF-DMS Grants 1604458, 1604773, 1604697 and 1603823, "Rocky Mountain-Great Plains Graduate Research Workshops in Combinatorics" (all authors), and National Security Agency Grant H98230-16-1-0018, "The 2016 Rocky Mountain-Great Plains Graduate Research Workshop in Combinatorics" (Amanda Lohss). Generous support was also given by the Institute for Mathematics and its Applications.

We thank the organizers of the Graduate Research Workshop in Combinatorics (GRWC) 2016 for providing a great environment for this research to occur, and we thank the other participants who give insight into the problem.

References

[1] The On-Line Encyclopedia of Integer Sequences. `http://oeis.org/A000931`, 2017.
[2] H. Acan and P. Hitczenko. On random trees obtained from permutation graphs. *Discrete Math.*, 339(12):2871–2883, 2016.
[3] R.-Y. Chang, C.-H. Hsu, and S.-L. Peng. The longest path problem on permutation graphs. 2012.
[4] G. Chartrand and F. Harary. Planar permutation graphs. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 3:433–438, 1967.
[5] C. Crespelle and C. Paul. Fully dynamic algorithm for recognition and modular decomposition of permutation graphs. *Algorithmica*, 58:405–432, 2010.
[6] H. N. de Ridder et al. Information System on Graph Classes and their Inclusions (ISGCI). `http://www.graphclasses.org`.
[7] T. Gallai. Transitiv orientierbare graphen. *Acta Mathematica Hungarica*, 18:25–66, 1967.
[8] S. V. Gervacio, T. A. Rapanut, and P. C. F. Ramos. Characterization and construction of permutation graphs. *Open Journal of Discrete Mathematics*, 3:33–38, 2013.
[9] P. Keevash, P. Loh, and B. Sudakov. Bounding the number of edges in permutation graphs. *The Electronic Journal of Combinatorics*, 13, 2006.
[10] Y. Koh and S. Ree. Determination of permutation graphs. *Honam Mathematical Journal*, pages 183–194, 2005.
[11] Y. Koh and S. Ree. Connected permutation graphs. *Discrete Math.*, 307(21):2628–2635, 2007.
[12] V. Limouzy. Seidel minor, permutation graphs and combinatorial properties. *Algorithms and Computation*, pages 194–205, 2010.
[13] F. Maffray and M. Preissmann. A translation of Gallai's paper: 'Transitiv orientierbare graphen'. In J. L. R. Alfonsń and B. A. Reed, editors, *Perfect Graphs*, chapter 3, pages 25–66. John Wiley & Sons, Ltd, 2001.
[14] A. Pneuli, A. Lempel, and S. Even. Transitive orientation of graphs and characterization of permutation graphs. *Canadian Journal of Mathematics*, pages 160–175, 1971.
[15] D. Rotem and J. Urritia. Circular permutation graphs. *Networks*, 12(4):429–437, 1982.
[16] J. Spinrad. On comparability and permutation graphs. *SIAM Journal on Computing*, 14(3):658–670, 1985.
[17] E. Yi. The fractional metric dimension of permutation graphs. *Acta Math. Sin. (Engl. Ser.)*, 31(3):367–382, 2015.

Department of Mathematics, University of Denver, Denver, Colorado
E-mail address: aysel.erey@du.edu

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana
E-mail address: zgersh2@math.lsu.edu

Department of Mathematics, Physics, and Statistics, Messiah College, Mechanicsburg, Pennsylvania
E-mail address: alohss@messiah.edu

Department of Mathematics and Computer Science, Saint Mary's College, Notre Dame, Indiana
E-mail address: rrohatgi@saintmarys.edu