On some actions of the 0-Hecke monoids of affine symmetric groups

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Abstract

There are left and right actions of the 0-Hecke monoid of the affine symmetric group $\tilde{S}_n$ on involutions whose cycles are labeled periodically by nonnegative integers. Using these actions we construct two bijections, which are length-preserving in an appropriate sense, from the set of involutions in $\tilde{S}_n$ to the set of $\mathbb{N}$-weighted matchings in the n-element cycle graph. As an application, we show that the bivariate generating function counting the involutions in $\tilde{S}_n$ by length and absolute length is a rescaled Lucas polynomial. The 0-Hecke monoid of $\tilde{S}_n$ also acts on involutions (without any cycle labelling) by Demazure conjugation. The atoms of an involution $z \in \tilde{S}_n$ are the minimal length permutations w which transform the identity to z under this action. We prove that the set of atoms for an involution in $\tilde{S}_n$ is naturally a bounded, graded poset, and give a formula for the set's minimum and maximum elements. Using these properties, we classify the covering relations in the Bruhat order restricted to involutions in $\tilde{S}_n$.

1 Introduction

For each integer $n \geq 1$, let $\tilde{S}_n$ be the *affine symmetric group* of rank n, consisting of the bijections $w : \mathbb{Z} \to \mathbb{Z}$ with $w(i+n) = w(i) + n$ for all $i \in \mathbb{Z}$ and $w(1) + w(2) + \cdots + w(n) = \binom{n+1}{2}$. Note that when $n = 1$ these conditions imply that $\tilde{S}_1 = \{1\}$. Assume $n \geq 2$, and define $s_i \in \tilde{S}_n$ for $i \in \mathbb{Z}$ as the permutation which exchanges $i + mn$ and $i + 1 + mn$ for each $m \in \mathbb{Z}$, while fixing every integer not congruent to i or $i + 1$ modulo n. The elements $s_1, s_2, \ldots, s_n$ generate $\tilde{S}_n$, and with respect to these generators $\tilde{S}_n$ is the Coxeter group of type $\tilde{A}_{n-1}$ [2, §8.3].

If W is any Coxeter group with simple generating set S and length function $\ell : W \to \mathbb{N}$, then there is a unique associative product $\circ : W \times W \to W$ such that $w \circ s = w$ if $\ell(ws) < \ell(w)$ and $w \circ s = ws$ if $\ell(ws) > \ell(w)$ for $w \in W$ and $s \in S$ [16, Theorem 7.1]. The product $\circ$ is often called the *Demazure product*, and the pair $(W, \circ)$ is usually referred to as the *0-Hecke monoid* or *Richardson-Springer monoid* of (W, S). We frame the results of this paper around the discussion of three actions of the 0-Hecke monoid of $\tilde{S}_n$. Each action will be on objects related to the group's involutions, that is, the elements $z \in \tilde{S}_n$ with $z^2 = 1$.

Let I_n be the set of involutions in the finite symmetric group S_n, which we identify with the parabolic subgroup of $\tilde{S}_n$ generated by $s_1, s_2, \ldots, s_{n-1}$. Recall that a *matching* in a graph is a subset of edges with no shared vertices; with slight abuse abuse of notation, a *matching* on a set is a matching in the complete graph on that set. Elements of I_n are permutations whose cycles have length at most two, and so may be viewed as matchings on $\{1, 2, \ldots, n\}$. For example,

$$(1,4)(2,7)(3,6) \in I_8 \qquad \text{corresponds to} \qquad \vcenter{\hbox{}} \; . \tag{1.1}$$

There are several ways to adapt this combinatorial model to the elements of $\tilde{I}_n = \{z \in \tilde{S}_n : z^2 = 1\}$. The simplest method is to represent $z \in \tilde{I}_n$ as the matching on $\mathbb{Z}$ in which i and j are connected by an edge whenever $z(i) = j \neq i = z(j)$. This gives a bijection between $\tilde{I}_n$ and matchings on $\mathbb{Z}$ which are "n-periodic" in the sense of having $\{i, j\}$ as an edge if and only if $\{i + n, j + n\}$ is also an edge. One can make this model more compact by converting n-periodic matchings on $\mathbb{Z}$ to $\mathbb{Z}$-weighted matchings on $\{1, 2, \ldots, n\}$: to represent $z \in \tilde{I}_n$, include the edge $\{i, j\}$ labeled by $m \in \mathbb{Z}$ whenever $i < j$ and $z(i) = j + mn$ and $z(j) = i - mn$. For example,

$$\begin{array}{c} \overset{-1}{\overset{\displaystyle\frown}{}} \\[-2pt] \overset{1\quad 0}{} \\[-4pt] \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \end{array} \tag{1.2}$$

would correspond to $z = \prod_{m \in \mathbb{Z}}(1 + mn, 12 + mn)(7 + mn, 10 + mn)(3 + mn, 6 + mn) \in \tilde{I}_8$. Diagrams of this type are most useful when $\tilde{S}_n$ is viewed as a semidirect product $S_n \ltimes \mathbb{Z}^{n-1}$. When the structure of $\tilde{S}_n$ as a Coxeter group is significant, a better approach is to view n-periodic matchings as *winding diagrams*. To construct the *winding diagram* of $z \in \tilde{I}_n$, arrange $1, 2, \ldots, n$ clockwise on a circle, and whenever $i < z(i) \equiv j \pmod{n}$, connect i to j by an arc winding $\frac{z(i) - i}{n}$ times in the clockwise direction around the circle's exterior. For the involution in (1.2), this produces the picture

$$\tag{1.3}$$
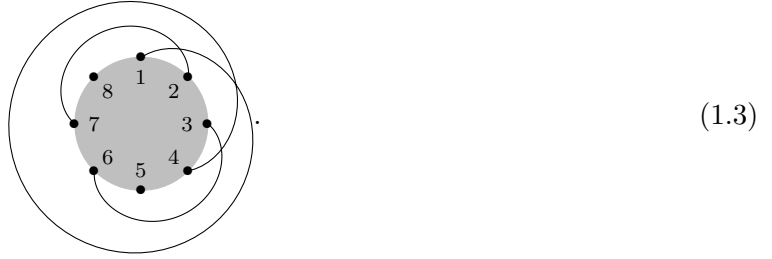
Formally, a winding diagram is a collection of continuous paths between disjoint pairs of marked points on the boundary of the plane minus an open disc, up to homotopy. Each winding diagram corresponds to a unique involution in some affine symmetric group. For our purposes, this construction is the correct generalisation of (1.1) to the affine case.

Write $\ell(w)$ for the usual Coxeter length of $w \in \tilde{S}_n$, and define the *absolute length* $\ell'(z)$ of $z \in \tilde{I}_n$ to be the number of arcs in its winding diagram. Our first main result, Theorem 5.4, identifies two bijections ω_R and ω_L from $\tilde{I}_n$ to the set $\mathcal{M}_n$ of $\mathbb{N}$-weighted matchings in $\mathcal{C}_n$, the cycle graph on n vertices. These bijections preserve length and absolute length, where the absolute length of an $\mathbb{N}$-weighted matching is its number of edges and the length is its number of edges plus twice the sum of their weights. The images of the element $z \in \tilde{I}_8$ in our running example (1.3) are
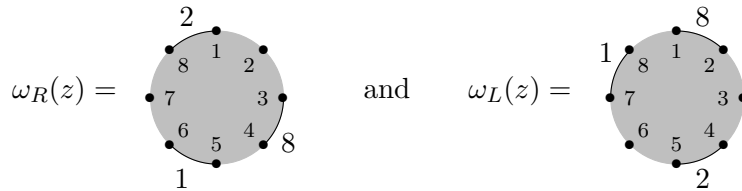
and indeed it holds that $\ell'(z) = 3$ and $\ell(z) = 25$. The proof of Theorem 5.4 relies on the construction of a left and right action of the 0-Hecke monoid of $\tilde{S}_n$ on the set of *weighted involutions*, which

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may be defined informally as n-periodic, $\mathbb{N}$-weighted matchings on $\mathbb{Z}$; see Section 3. Our results provide a fourth model for $\tilde{I}_n$, which makes it easy to count the elements of $\tilde{I}_n$ by length. As an application, we show that $\tilde{I}_n(q, x) = \sum_{z \in \tilde{I}_n} q^{\ell(z)} x^{\ell'(z)} \in \mathbb{N}[[q, x]]$ has the formula

$$\tilde{I}_n(q, x) = \frac{1}{1+q^n} \mathrm{Luc}_n(1 + q, qs)$$

where $\mathrm{Luc}_n(x, s)$ is the nth bivariate *Lucas polynomial*, defined by the recurrence $\mathrm{Luc}_n(x, s) = x\mathrm{Luc}_{n-1}(x, s) + s\mathrm{Luc}_{n-2}(x, s)$ with $\mathrm{Luc}_0(x, s) = 2$ and $\mathrm{Luc}_1(x, s) = x$; see Corollary 5.5. This is an analogue of a more complicated identity proved in [24].

The 0-Hecke monoid of $\tilde{S}_n$ also acts directly on $\tilde{I}_n$ by *Demazure conjugation*: the right action mapping $(z, w) \mapsto w^{-1} \circ z \circ w$ for $z \in \tilde{I}_n$ and $w \in \tilde{S}_n$. This monoid action is a degeneration of the Iwahori-Hecke algebra representation studied by Lusztig and Vogan in [21, 22]. The orbit of the identity under Demazure conjugation is all of $\tilde{I}_n$, and we define $\mathcal{A}(z)$ for $z \in \tilde{I}_n$ as the set of elements $w \in \tilde{S}_n$ of minimal length such that $z = w^{-1} \circ w$. Following [7, 8], we call these permutations the *atoms* of z. There are a few reasons why these elements merit further study, beyond their interesting combinatorial properties. The sets $\mathcal{A}(z)$ may be defined for involutions in any Coxeter group and, in the case of finite Weyl groups, are closely related to the sets $W(Y)$ which Brion [3] attaches to B-orbit closures Y in a spherical homogeneous space G/H (where G is a connected complex reductive group, B a Borel subgroup, and H a spherical subgroup). Results of Hultman [14, 15], extending work of Richardson and Springer [25, 26], show the atoms to be intimately connected to the Bruhat order of a Coxeter group restricted to its involutions. Finally, the atoms of involutions in finite symmetric groups play a central role in recent work of Can, Joyce, Wyser, and Yong on the geometry of the orbits of the orthogonal group on the type A flag variety; see [4, 5, 28, 29].

Our object in the second half of this paper is to generalise a number of results about the atoms of involutions in finite symmetric groups to the affine case. In Section 6, extending results in [8, 13], we show that there is a natural partial order which makes $\mathcal{A}(z)$ for $z \in \tilde{I}_n$ into a bounded, graded poset. We conjecture that this poset is a lattice. Generalising results of Can, Joyce, and Wyser [4, 5], we show in Section 7 that there is an explicit set of inequalities governing the "one-line" representation of a permutation in $\tilde{S}_n$ which determines whether it belongs to $\mathcal{A}(z)$. This result translates to a "local" criterion for membership in $\mathcal{A}(z)$ involving a notion of *(affine) standardisation*; see Corollary 7.11. In Section 8, extending [9, 17], we describe all covering relations in the Bruhat order of $\tilde{S}_n$ restricted to $\tilde{I}_n$. Using this information, we prove that involutions in $\tilde{S}_n$ have what we call the *Bruhat covering property*:

Theorem 1.1 (Bruhat covering property). *If $y \in \tilde{I}_n$ and $t \in \tilde{S}_n$ is a reflection, then there exists at most one $z \in \tilde{I}_n$ such that $\{wt : w \in \mathcal{A}(y) \text{ and } \ell(wt) = \ell(w) + 1\} \cap \mathcal{A}(z) \neq \varnothing$.*

The analogue of this result for involutions in S_n was shown in [9], and served as a key lemma in proofs of "transition formulas" for certain *involution Schubert polynomials*. We conjecture that the same property holds for arbitrary Coxeter systems, in the following sense.

Let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{N}$ and Demazure product $\circ : W \times W \to W$. Suppose $w \mapsto w^*$ is an automorphism of W with $S^* = S$. The corresponding set of *twisted involutions* is $I_* = \{w \in W : w^{-1} = w^*\}$. For $y \in I_*$ let $\mathcal{A}_*(y)$ be the set of elements of minimal length with $(w^*)^{-1} \circ w = y$, and write $T = \{wsw^{-1} : w \in W, \ s \in S\}$.

Conjecture 1.2. *If $y \in I_*$ is a twisted involution in an arbitrary Coxeter group and $t \in T$, then there exists at most one $z \in I_*$ such that $\{wt : w \in \mathcal{A}_*(y) \text{ and } \ell(wt) = \ell(w) + 1\} \cap \mathcal{A}_*(z) \neq \varnothing$.*

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This statement is analogous to [20, Lemma 21], but seems harder to prove. A useful consequence of the new methods in this paper is that we are able to replace certain computer dependent proofs for type A_n in [9] by simpler and more general arguments for type $\tilde{A}_n$. Concerning future work, we anticipate that our results will be useful in developing a theory of affine involution Stanley symmetric functions, simultaneously generalising [19, 20] and [7, 9, 10, 11].

Acknowledgements

Notation

Throughout, $\mathbb{Z}$ denotes the integers, $\mathbb{N}$ the nonnegative integers, and $[n]$ the set $\{1, 2, \ldots, n\}$. We write $\tilde{S}_n$ for the rank n affine symmetric group and $\tilde{I}_n$ for its subset of involutions. Having fixed $n \geq 2$, we define $s_i \in \tilde{I}_n$ for $i \in \mathbb{Z}$ as the permutation interchanging $i + mn$ and $i + 1 + mn$ for $m \in \mathbb{Z}$ while fixing all numbers outside of $\{i, i + 1\} + n\mathbb{Z}$. Let ℓ denote the length function of $\tilde{S}_n$ relative to the generating set $\{s_i : i \in [n]\}$, and write $<$ for the (strong) Bruhat order on $\tilde{S}_n$.

2 Preliminaries

This section recalls some basic facts about affine symmetric groups. We omit most proofs, since the properties we mention are either well-known (see, e.g., [2, §8.3] or [16, §4]) or follow as simple exercises. For any map $w : \mathbb{Z} \to \mathbb{Z}$, write $\mathrm{Inv}(w)$ for the set of pairs $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ with $i < j$ and $w(i) > w(j)$. If $w \in \tilde{S}_n$ and $(i, j) \in \mathrm{Inv}(w)$ then $(i + mn, j + mn) \in \mathrm{Inv}(w)$ for all $m \in \mathbb{Z}$.

Proposition 2.1. Let $w \in \tilde{S}_n$. Then $\ell(w)$ is the number of equivalence classes in $\mathrm{Inv}(w)$ under the relation on $\mathbb{Z} \times \mathbb{Z}$ generated by $(i, j) \sim (i + n, j + n)$.

Let $\mathrm{Des}_R(w)$ and $\mathrm{Des}_L(w)$ denote the right and left descents sets of a permutation $w \in \tilde{S}_n$, consisting of the elements $s \in \{s_1, s_2, \ldots, s_n\}$ with $\ell(ws) < \ell(w)$ and $\ell(sw) < \ell(w)$, respectively.

Corollary 2.2. Let $w \in \tilde{S}_n$. Then $\mathrm{Des}_R(w) = \{s_i : i \in \mathbb{Z} \text{ such that } w(i) > w(i + 1)\}$.

For $i < j \not\equiv i \pmod{n}$, let $t_{ij} = t_{ji} \in \tilde{S}_n$ be the permutation which interchanges $i + mn$ and $j + mn$ for each $m \in \mathbb{Z}$ and which fixes all integers not in $\{i, j\} + n\mathbb{Z}$. Note that $t_{i,i+1} = s_i$. The elements t_{ij} are precisely the *reflections* in $\tilde{S}_n$. The following is [2, Proposition 8.3.6].

Lemma 2.3. Let $w \in \tilde{S}_n$ and $i, j \in \mathbb{Z}$ with $i < j \not\equiv i \pmod{n}$. Suppose $w(i) < w(j)$. Then $\ell(wt_{ij}) \geq \ell(w) + 1$, with equality if and only if no $e \in \mathbb{Z}$ satisfies $i < e < j$ and $w(i) < w(e) < w(j)$.

Note for $z \in \tilde{I}_n = \{w \in \tilde{S}_n : w^2 = 1\}$ and $i \in \mathbb{Z}$ that $z(i) \equiv i \pmod{n}$ if and only if $z(i) = i$.

Lemma 2.4. If $i \in \mathbb{Z}$ and $z \in \tilde{I}_n$ and $i \neq z(i) < z(i + 1) \neq i + 1$ then $\ell(s_i z s_i) = \ell(z) + 2$.

For $z \in \tilde{I}_n$, let $\mathcal{C}(z) = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : i < j = z(i)\}$, so that $\mathcal{C}(z)$ is the set of ordered 2-cycles of z. Note that if $(i, j) \in \mathcal{C}(z)$ then $(i + mn, j + mn) \in \mathcal{C}(z)$ for all $m \in \mathbb{Z}$.

Definition 2.5. Define $\ell'(z)$ for $z \in \tilde{I}_n$ as the number of equivalence classes in $\mathcal{C}(z)$ under the relation on $\mathbb{Z} \times \mathbb{Z}$ generated by $(i, j) \sim (i + n, j + n)$.

Hultman shows in [14] that ℓ' is the *absolute length function* on $\tilde{S}_n$ restricted to $\tilde{I}_n$: the function which returns the minimum number of reflections whose product is a given permutation.

Lemma 2.6. If i and $i+1$ are fixed points of $z \in \tilde{I}_n$ then $zs_i \in \tilde{I}_n$ and $\ell'(zs_i) = \ell'(z) + 1$.

Lemma 2.7. Let $z \in \tilde{I}_n$. Then $\ell(z) = \ell'(z)$ if and only if z is a product of commuting simple generators, i.e., $z = s_{i_1} s_{i_2} \cdots s_{i_l}$ for distinct indices satisfying $i_j \not\equiv i_k \pm 1 \pmod{n}$ for all $j, k \in [l]$.

Say that $j \in \mathbb{Z}$ is a *left endpoint* of $z \in \tilde{I}_n$ if $j < z(j)$, a *right endpoint* if $z(j) < j$, and a *fixed point* if $z(j) = j$. Let $w \in \tilde{S}_n$ act on $\mathbb{Z} \times \mathbb{Z}$ by $w(a, b) = (w(a), w(b))$.

Lemma 2.8. Let $z \in \tilde{I}_n$. If $(i, i+1) \notin \mathcal{C}(z)$ then $\mathcal{C}(s_i z s_i) = s_i \mathcal{C}(z)$, and otherwise $\mathcal{C}(z) = \mathcal{C}(s_i z s_i)$.

Thus $\ell'(wzw^{-1}) = \ell'(z)$ for all $w \in \tilde{S}_n$ and $z \in \tilde{I}_n$. Our last two lemmas are more technical:

Lemma 2.9. Let $z \in \tilde{I}_n$ and $i, j \in \mathbb{Z}$ with $i < j \not\equiv i \pmod{n}$. Assume $j \neq z(i) < z(j) \neq i$. If i is a right endpoint of z or if j is left endpoint then $\ell(t_{ij} z t_{ij}) > \ell(t_{ij} z) = \ell(z t_{ij}) > \ell(z)$.

Proof. Note that $\ell(t_{ij}z) = \ell(z t_{ij}) > \ell(z)$. Assume $z(i) < i$. Since $z(i) < i$, we cannot have $z(i) \equiv i \pmod{n}$. If $z(i) \equiv j \pmod{n}$ then $z(j) \equiv i \pmod{n}$ so $t_{ij} z(i) < z(i) < z(j) < t_{ij} z(j)$. If $z(i) \not\equiv j \pmod{n}$ then $z(j) \not\equiv i \pmod{n}$, so either $z(j) = j$ and $t_{ij} z(i) = z(i) < i = t_{ij} z(j)$, or $z(j) \not\equiv j \pmod{n}$ and $t_{ij} z(i) = z(i) < z(j) = t_{ij} z(j)$. We deduce that $\ell(t_{ij} z t_{ij}) > \ell(t_{ij}z) > \ell(z)$. When $j < z(j)$, the same conclusion follows by a similar argument. $\square$

Lemma 2.10. Let $z \in \tilde{I}_n$ and $i, j \in \mathbb{Z}$ with $i < j \not\equiv i \pmod{n}$. Suppose that either $i, i+1, \ldots, j-1$ are all right endpoints of z or $i+1, \ldots, j-1, j$ are all left endpoints of z. Then:

(a) $\ell(t_{ij} z t_{ij}) > \ell(t_{ij}z) > \ell(z)$ if $z(i) < z(j)$ and $\ell(t_{ij} z t_{ij}) < \ell(t_{ij}z) < \ell(z)$ if $z(i) > z(j)$.

(b) $\ell(t_{ij} z t_{ij}) = \ell(z) + 2$ if and only if $\ell(z t_{ij}) = \ell(z) + 1$.

Proof. The first half of part (a) follows from the previous lemma. The second half follows from the first (with z replaced by $t_{ij} z t_{ij}$) on noting that $i, i+1, \ldots, j-1, j$ must be all left or all right endpoints of z if $z(i) > z(j)$. Part (a) implies that if $\ell(z t_{ij}) > \ell(z) + 1$ then $\ell(t_{ij} z t_{ij}) > \ell(z) + 2$. Assume $\ell(z t_{ij}) = \ell(z) + 1$ so that $z(i) < z(j)$. By symmetry, it suffices to consider the case when $i, i+1, \ldots, j-1$ are all right endpoints of z. Fix $e \in \mathbb{Z}$ with $i < e < j$, so that either $z(e) < z(i)$ or $z(j) < z(e)$. If $z(j) < z(e)$, then since i and e are right endpoints, j must also be a right endpoint of z, so neither $z(e)$ nor $z(j)$ belongs to $\{i, j\} + n\mathbb{Z}$ and $t_{ij} z(j) = z(j) < z(e) = t_{ij} z(e)$. Suppose instead that $z(e) < z(i)$. We claim that $t_{ij} z(e) < t_{ij} z(i)$. We cannot have $z(e) \equiv i \pmod{n}$ since i and e are right endpoints, so if $t_{ij} z(i) < t_{ij} z(e)$ then we must have $i - mn < z(e) < j - mn = z(i)$ for some $m > 0$. But if this occurred then $f = z(e) + mn$ would have $i < f < j$ and $f < z(f) = e + mn$, contradicting our assumption that $i, i+1, \ldots, j-1$ are right endpoints. This proves our claim, and we conclude that no $i < e < j$ has $t_{ij} z(i) < t_{ij} z(e) < t_{ij} z(j)$, so $\ell(t_{ij} z t_{ij}) = \ell(t_{ij}z) + 1 = \ell(z) + 2$. $\square$
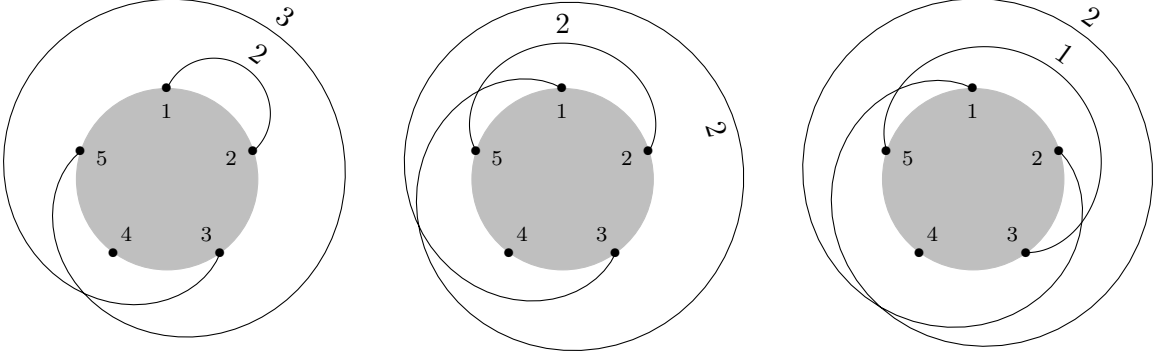
3   Weighted involutions

The goal of the next three sections is to construct a "length-preserving" bijection between $\tilde{I}_n$ and the set of $\mathbb{N}$-weighted matchings of the cycle graph on n vertices. Our description of this correspondence will rely on an action of the Richardson-Springer monoid of $\tilde{S}_n$ on pairs of the following type:

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Definition 3.1. A *weighted involution* in $\tilde{S}_n$ is a pair (w, ϕ) where $w \in \tilde{I}_n$ and ϕ is a map $\mathcal{C}(w) \to \mathbb{N}$ with $\phi(i,j) = \phi(i+n, j+n)$ for all $(i,j) \in \mathcal{C}(w)$. We refer to ϕ as the *weight map* of (w, ϕ). Define the *weight* of (w, ϕ) as the number $\mathrm{wt}(w, \phi) = \sum_\gamma \phi(\gamma)$ where the sum is over a set of cycles γ representing the distinct equivalence classes in $\mathcal{C}(w)$ under the relation $(i,j) \sim (i+n, j+n)$.

Let $\mathcal{W}_n$ be the set of all weighted involutions in $\tilde{S}_n$.

Example 3.2. We can represent a weighted involution $(w, \phi) \in \mathcal{W}_n$ graphically by drawing the winding diagram of w with its arcs labeled by the values of ϕ. For example, if $\theta_1, \theta_2, \theta_3 \in \mathcal{W}_5$ are

and we write $\theta_i = (w_i, \phi_i)$, then $w_1 = t_{1,2}t_{3,10}$, $w_2 = t_{0,2}t_{3,11}$, and $w_3 = t_{0,3}t_{2,11}$, while $\phi_1(1,2) = 2$ and $\phi_1(3,10) = 3$, $\phi_2(3,11) = \phi_2(5,7) = 2$, and $\phi_3(2,11) = 2$ and $\phi_3(5,8) = 1$.

We identify $\tilde{I}_n$ with the subset of weighted involutions of the form $(w, 0) \in \mathcal{W}_n$ with 0 denoting the unique weight map $\mathcal{C}(w) \to \{0\}$. We extend $\ell : \tilde{S}_n \to \mathbb{N}$ and $\ell' : \tilde{I}_n \to \mathbb{N}$ to $\mathcal{W}_n$ by setting

$$\ell(\theta) = \ell(w) + 2\,\mathrm{wt}(\theta) \qquad \text{and} \qquad \ell'(\theta) = \ell'(w) \qquad \text{for } \theta = (w, \phi) \in \mathcal{W}_n.$$

Given $(w, \phi) \in \mathcal{W}_n$, define the *right form* of ϕ to be the map $\phi_R : \mathbb{Z} \to \mathbb{N}$ with $\phi_R(i) = \phi(w(i), i)$ if $w(i) < i$ and with $\phi_R(i) = 0$ otherwise. Likewise, define the *left form* of ϕ to be the map $\phi_L : \mathbb{Z} \to \mathbb{N}$ with $\phi_L(i) = \phi(i, w(i))$ if $i < w(i)$ and with $\phi_L(i) = 0$ otherwise. Clearly ϕ_L and ϕ_R each determine ϕ, given w. We now define operators $\pi_1, \pi_2, \ldots, \pi_n$ which act on $\mathcal{W}_n$ on the right and left.

Definition 3.3. Let $\theta = (w, \phi) \in \mathcal{W}_n$ and $i \in \mathbb{Z}$.

(a) If $\phi_R(i) > \phi_R(i+1)$ then let $\theta\pi_i = (s_i w s_i, \psi) \in \mathcal{W}_n$ where ψ is the unique weight map with

$$\psi_R(j) = \begin{cases} \phi_R(i) - 1 & \text{if } j \equiv i+1 \pmod{n} \\ \phi_R(i+1) & \text{if } j \equiv i \pmod{n} \\ \phi_R(j) & \text{otherwise} \end{cases} \qquad \text{for } j \in \mathbb{Z}.$$

If $\phi_R(i) \le \phi_R(i+1)$ then let $\theta\pi_i = \theta$.

(b) If $\phi_L(i+1) > \phi_L(i)$ then let $\pi_i\theta = (s_i w s_i, \chi) \in \mathcal{W}_n$ where χ is the unique weight map with

$$\chi_L(j) = \begin{cases} \phi_L(i+1) - 1 & \text{if } j \equiv i \pmod{n} \\ \phi_L(i) & \text{if } j \equiv i+1 \pmod{n} \\ \phi_L(j) & \text{otherwise} \end{cases} \qquad \text{for } j \in \mathbb{Z}.$$

If $\phi_L(i+1) \le \phi_L(i)$ then let $\pi_i\theta = \theta$.

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Example 3.4. Define $\theta_1, \theta_2, \theta_3 \in \mathcal{W}_5$ as in Example 3.2. Then $\theta_1 \pi_5 = \theta_2 \pi_1 = \theta_2$ and $\theta_2 \pi_2 = \theta_3$. Form $\theta_2' \in \mathcal{W}_5$ from θ_2 by replacing the label of the short arc in the picture in Example 3.2 by 1 and the label of the long arc by 3. Then $\pi_5 \theta_1 = \theta_2'$ and $\pi_2 \theta_2' = \theta_3$.

It may hold that $(\pi_i \theta)\pi_j \neq \pi_i(\theta \pi_j)$; for example, if $\theta = (w, \phi)$ where $w = s_1 \in \tilde{S}_2$ and $\phi(1,2) = 1$, then $\pi_0 \theta = (\pi_0 \theta)\pi_2 \neq \pi_0(\theta \pi_2) = \theta \pi_2$. Note that $\pi_i = \pi_{i+n}$, as a right and left operator.

Proposition 3.5. The left (respectively, right) operators π_i satisfy (a) $\pi_i^2 = \pi_i$, (b) $\pi_i \pi_j = \pi_j \pi_i$ if $i \not\equiv j \pm 1 \pmod{n}$, and (c) $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$ for all $i, j \in \mathbb{Z}$.

Proof. It is clear that $\pi_i^2 = \pi_i$ and $\pi_i \pi_j = \pi_j \pi_i$ if $i \not\equiv j \pm 1 \pmod{n}$. Fix $\theta = (w, \phi) \in \mathcal{W}_n$. Define $a = \phi_R(i)$, $b = \phi_R(i+1)$, and $c = \phi_R(i+2)$. We check part (c) carefully as follows:

- If $a > b > c$ then $\theta \pi_i \pi_{i+1} \pi_i = \theta \pi_{i+1} \pi_i \pi_{i+1}$ since $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

- If $a > c \geq b$ then $\theta \pi_i \pi_{i+1} \pi_i = \theta \pi_{i+1} \pi_i \pi_{i+1}$ is $\theta \pi_i \pi_{i+1}$ if $a > c+1$ and $\theta \pi_i$ if $a = c+1$.

- If $b \geq a > c$ then $\theta \pi_i \pi_{i+1} \pi_i = \theta \pi_{i+1} \pi_i \pi_{i+1} = \theta \pi_{i+1} \pi_i$.

- If $b > c \geq a$ then $\theta \pi_i \pi_{i+1} \pi_i = \theta \pi_{i+1} \pi_i \pi_{i+1} = \theta \pi_{i+1}$.

- If $c \geq a > b$ then $\theta \pi_i \pi_{i+1} \pi_i = \theta \pi_{i+1} \pi_i \pi_{i+1} = \theta \pi_i$.

- If $c \geq b \geq a$ then $\theta \pi_i \pi_{i+1} \pi_i = \theta \pi_{i+1} \pi_i \pi_{i+1} = \theta$.

One of these cases must occur, so we conclude that $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$ as a right operator. The argument that $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$ as a left operator is symmetric. $\square$

By Matsumoto's theorem, it follows that for each $g \in \tilde{S}_n$, we may define a right (respectively, left) operator π_g on $\mathcal{W}_n$ by setting $\pi_g = \pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$ where $g = s_{i_1} s_{i_2} \cdots s_{i_k}$ is any reduced expression. Recall the definition of the Demazure product $\circ : \tilde{S}_n \times \tilde{S}_n \to \tilde{S}_n$ from the introduction.

Corollary 3.6. The map $g \mapsto \pi_g$ defines a right (also left) monoid action of $(\tilde{S}_n, \circ)$ on $\mathcal{W}_n$.

Define $\tau : \mathbb{Z} \to \mathbb{Z}$ by $\tau(i) = n + 1 - i$ and let $w^* = \tau w \tau$ for $w \in \tilde{S}_n$. Then $w \mapsto w^*$ is an automorphism of $\tilde{S}_n$ with $s_i^* = s_{n-i}$ for $i \in [n]$, so $s_n^* = s_0 = s_n$. Note that if $w \in \tilde{I}_n$ then

$$\mathcal{C}(w^*) = \{(\tau(j), \tau(i)) : (i,j) \in \mathcal{C}(w)\}.$$

For $\phi : \mathcal{C}(w) \to \mathbb{N}$ let ϕ^* be the map $\mathcal{C}(w^*) \to \mathbb{N}$ given by $(\tau(j), \tau(i)) \mapsto \phi(i,j)$. Extend $*$ to $\mathcal{W}_n$ by setting $\theta^* = (w^*, \phi^*)$ for $\theta = (w, \phi) \in \mathcal{W}_n$. Clearly $(\theta^*)^* = \theta$. The following is easy to check:

Lemma 3.7. Let $i \in \mathbb{Z}$ and $\theta \in \mathcal{W}_n$. Then $\mathrm{wt}(\theta) = \mathrm{wt}(\theta^*)$ and $(\pi_i \theta)^* = \theta^* \pi_{n-i}$.

By definition $\pi_i \theta$ (also $\theta \pi_i$) is either θ or has weight one less than θ. For $\theta \in \mathcal{W}_n$, let $\mathrm{Des}_L(\theta)$ and $\mathrm{Des}_R(\theta)$ be the sets of generators s_i for $i \in [n]$ such that $\pi_i \theta \neq \theta$ and $\theta \pi_i \neq \theta$, respectively. The map which fixes s_n and maps $s_i \mapsto s_{n-i}$ is a bijection $\mathrm{Des}_L(\theta) \leftrightarrow \mathrm{Des}_R(\theta^*)$. This means that we only need to prove the right-handed version of the following lemma:

Lemma 3.8. Suppose $\theta \in \mathcal{W}_n$ and $\mathrm{wt}(\theta) > 0$. Then $\mathrm{Des}_L(\theta)$ and $\mathrm{Des}_R(\theta)$ are both nonempty.

Proof. Write $\theta = (w, \phi)$. We have $\mathrm{Des}_R(\theta) = \varnothing$ only if no right endpoints i of w have $\phi(w(i), i) > 0$ or if whenever i is right endpoint it holds that $i + 1$ is also a right endpoint and $\phi(w(i), i) \leq \phi(w(i+1), i+1)$. The second case is impossible, and if $\mathrm{wt}(\theta) > 0$ then the first case is excluded. $\qquad\square$

Recall that $\leq$ denotes the Bruhat order on $\tilde{S}_n$.

Theorem-Definition 3.9. Let $\theta = (w, \phi) \in \mathcal{W}_n$. The following statements then hold:

(a) There are unique elements $g, h \in \tilde{S}_n$ with $\ell(g) = \ell(h) = \mathrm{wt}(\theta)$ and $\mathrm{wt}(\pi_g \theta) = \mathrm{wt}(\theta \pi_h) = 0$.

(b) For the elements in part (a), we have $\pi_g \theta = g w g^{-1}$ and $\theta \pi_h = h^{-1} w h$.

(c) Any other elements $g', h' \in \tilde{S}_n$ with $\mathrm{wt}(\pi_{g'} \theta) = \mathrm{wt}(\theta \pi_{h'}) = 0$ satisfy $g \leq g'$ and $h \leq h'$.

Define $g_L(\theta) = g^{-1}$ and $g_R(\theta) = h$, and set $\omega_L(\theta) = \pi_g \theta = g w g^{-1}$ and $\omega_R(\theta) = \theta \pi_h = h^{-1} w h$.

Note that $g_L(\theta), g_R(\theta) \in \tilde{S}_n$ while $\omega_L(\theta), \omega_R(\theta) \in \tilde{I}_n \subset \mathcal{W}_n$.

Proof. Induction and Lemma 3.8 imply that $\mathrm{wt}(\theta \pi_{h'}) = 0$ for some $h' \in \tilde{S}_n$ with $\mathrm{wt}(\theta) \leq \ell(h')$. Let $h' = s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression. Let $h_0 = 1$, for $j \in [k]$ define h_j to be either $h_{j-1} s_{i_j}$ if $s_{i_j} \in \mathrm{Des}_R(\theta \pi_{h_{j-1}})$ or h_{j-1} otherwise, and set $h = h_k$. By construction $h \leq h'$, $\theta \pi_h = \theta \pi_{h'}$, $\ell(h) = \mathrm{wt}(\theta)$, and $\theta \pi_h = h^{-1} w h$. It remains to show that $h \in \tilde{S}_n$ is the unique element satisfying both $\mathrm{wt}(\theta \pi_h) = 0$ and $\ell(h) = \mathrm{wt}(\theta)$. This obviously holds if $\mathrm{wt}(\theta) = 0$. Assume $g, h \in \tilde{S}_n$ are such that $\mathrm{wt}(\theta \pi_g) = \mathrm{wt}(\theta \pi_h) = 0$ and $\ell(g) = \ell(h) = \mathrm{wt}(\theta) > 0$. If s_i is a left descent of both g and h, then $s_i \in \mathrm{Des}_R(\theta)$ and it follows by induction that $s_i g = s_i h$, so $g = h$. Assume $\mathrm{Des}_L(g)$ and $\mathrm{Des}_L(h)$ are disjoint, and choose $s_i \in \mathrm{Des}_L(g)$ and $s_j \in \mathrm{Des}_L(h)$. Both s_i and s_j must belong to $\mathrm{Des}_R(\theta)$. It is not hard to check that (i) if $i \not\equiv j \pm 1 \pmod{n}$ then $\theta \pi_i \pi_j = \theta \pi_j \pi_i$ has weight $\mathrm{wt}(\theta) - 2$, while (ii) if $i \equiv j \pm 1 \pmod{n}$ then $\theta \pi_i \pi_j \pi_i = \theta \pi_j \pi_i \pi_j$ has weight $\mathrm{wt}(\theta) - 3$.

Assume case (i) occurs. By induction, we may assume that unique elements $g', g'' \in \tilde{S}_n$ exist with $\mathrm{wt}(\theta' g') = \mathrm{wt}(\theta'' g'') = 0$, $\ell(g') = \mathrm{wt}(\theta') = \mathrm{wt}(\theta) - 1$, and $\ell(g'') = \mathrm{wt}(\theta'') = \mathrm{wt}(\theta) - 2$ for $\theta' = \theta \pi_i$ and $\theta'' = \theta \pi_i \pi_j = \theta \pi_j \pi_i$. Uniqueness implies that $s_i g = g'$ and $g' = s_j g''$, so $g = s_i s_j g'' = s_j s_i g''$ where $\ell(g) = \ell(g'') + 2$. But this means that $s_j \in \mathrm{Des}_L(g)$, contradicting our assumption otherwise. One reaches a similar contradiction in case (ii). This proves the right-handed version of theorem. The left-handed version follows by symmetric arguments. $\qquad\square$

Example 3.10. If $\theta_1, \theta_2, \theta_3 \in \mathcal{W}_5$ are as in Example 3.2, then we have

$$\omega_R(\theta_1) = \omega_R(\theta_2) = \omega_R(\theta_3) = t_{1,13} t_{5,9} = \quad$$
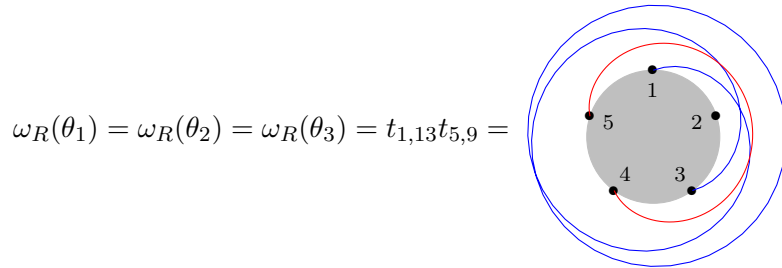
The arcs in the winding diagram are coloured red and blue to make them easier to distinguish.

The following is clear by induction from Lemma 3.7:

Lemma 3.11. If $\theta \in \mathcal{W}_n$ then $g_L(\theta^*) = g_R(\theta)^*$ and $\omega_L(\theta^*) = \omega_R(\theta)^*$.

4 Admissibility

Recall that $\ell(\theta) = \ell(w) + 2\operatorname{wt}(\theta)$ for $\theta = (w, \phi) \in \mathcal{W}_n$. The following terminology identifies a class of weighted involutions whose lengths are unaffected by the operators π_i.

Definition 4.1. Given a weighted involution $\theta = (w, \phi) \in \mathcal{W}_n$, we say that:

(a) θ is *right-admissible* if no sequence of cycles $(x, y), (a_0, b_0), (a_1, b_1), \ldots, (a_k, b_k) \in \mathcal{C}(w)$ has $x < a_k < b_k < \cdots < b_1 < b_0 < y$ and $\phi(a_i, b_i) + b_i + i \geq \phi(x, y) + y$ for all i.

(b) θ is *left-admissible* if no sequence of cycles $(x, y), (a_0, b_0), (a_1, b_1), \ldots, (a_k, b_k) \in \mathcal{C}(w)$ has $x < a_0 < a_1 < \cdots < a_k < b_k < y$ and $\phi(a_i, b_i) - a_i + i \geq \phi(x, y) - x$ for all i.

If w is *non-nesting* in the sense of having no cycles $(x, y), (a, b) \in \mathcal{C}(w)$ with $x < a < b < y$ then $\theta = (w, \theta) \in \mathcal{W}_n$ is both right- and left-admissible. The following is also easy to see:

Lemma 4.2. If θ is right-admissible (respectively, left-admissible) then θ^* is left-admissible (respectively, right-admissible).

Say that $\theta' \in \mathcal{W}_n$ is a *descendant* of $\theta \in \mathcal{W}_n$ if $\theta' = \theta \pi_g$ for some $g \in \tilde{S}_n$. The idea behind right-admissibility is to give a condition ensuring that no descendant $\theta' = (w', \phi')$ of θ allows w' to have nesting cycles $(i, k+1), (j, k) \in \mathcal{C}(w')$ with $i < j < k$ and $\phi'(j, k) > \phi'(i, k+1)$, since such a weighted involution would have $\ell(\theta' \pi_k) < \ell(\theta')$. Our formulation of left-admissibility is motivated by symmetric considerations. This is enough to make the operators π_i length-preserving:

Theorem 4.3. Suppose $\theta \in \mathcal{W}_n$ and $i \in \mathbb{Z}$.

(a) If θ is right-admissible then $\theta \pi_i$ is right-admissible and $\ell(\theta \pi_i) = \ell(\theta)$.

(b) If θ is left-admissible then $\pi_i \theta$ is left-admissible and $\ell(\pi_i \theta) = \ell(\theta)$.

Proof. Write $\theta = (w, \phi) \in \mathcal{W}_n$. We only prove part (a), since part (b) is equivalent by Lemmas 3.7 and 4.2. Assume θ is right-admissible and $s_i \in \operatorname{Des}_R(\theta)$. Then $w(i) < i$ and $\phi_R(i) > \phi_R(i+1)$. It follows $w(i) < w(i+1)$ since otherwise we would have $x < a_0 < b_0 < y$ for the cycles $(x, y) = (w(i+1), i+1)$ and $(a_0, b_0) = (w(i), i)$, and it would hold that $\phi_R(b_0) + b_0 = \phi_R(i) + i \geq \phi_R(i+1) + i + 1 = \phi_R(y) + y$. We conclude by Lemma 2.4 that $\ell(s_i w s_i) = \ell(w) + 2$, so $\ell(\theta \pi_i) = \ell(\theta)$. It remains to check that $\theta \pi_i$ is right-admissible.

Relative to an arbitrary weighted involution $\theta = (w, \phi)$, call a sequence of cycles satisfying the conditions in Definition 4.1(a) *inadmissible*. If such a sequence exists then θ is not right-admissible. If $(x, y), (a_0, b_0), (a_1, b_1), \ldots, (a_k, b_k) \in \mathcal{C}(w)$ is an inadmissible sequence with $b_k < y - n$, and j is the first index with $b_j < y - n$, then $j \leq n$ and the sequence $(x - n, y - n), (a_j, b_j), (a_{j+1}, b_{j+1}), \ldots, (a_k, b_k) \in \mathcal{C}(w)$ is also inadmissible.

Now suppose $s_i \in \operatorname{Des}_R(\theta)$ but $\theta \pi_i$ is not right-admissible. Write $\theta' = (w', \phi') = \theta \pi_i$ so that $w' = s_i w s_i$ and recall by Lemma 2.8 that $\mathcal{C}(w') = s_i \mathcal{C}(w)$. Note that i must be a right endpoint of w so $i+1$ must be a right endpoint of w'. Suppose $(x', y'), (a_0', b_0'), (a_1', b_1'), \ldots, (a_k', b_k') \in \mathcal{C}(w')$ is an inadmissible sequence for θ'. By the observations in the previous paragraph, we may assume that $y' - n < b_k' < \cdots < b_1' < b_0' < y'$. Let $(x, y) = (s_i(x'), s_i(y'))$ and $(a_j, b_j) = (s_i(a_j'), s_i(b_j')) \in \mathcal{C}(w)$ for each j. If at most one number among $b_k' < \cdots < b_1' < b_0' < y'$ is congruent to i or $i+1$ modulo n and $y' \not\equiv i \pmod{n}$, then the obvious sequence of cycles $(x, y), (a_0, b_0), (a_1, b_1), \ldots, (a_k, b_k)$ is

inadmissible for θ. If $y' \equiv i \pmod{n}$ and $b'_k \equiv i+1 \pmod{n}$ then $(x-n, y-n), (a_k, b_k)$ is an inadmissible sequence for θ. If $y' \equiv i \pmod{n}$ and $b'_k \not\equiv i+1 \pmod{n}$ then

$$(x, y), (w(y'), y'), (a_0, b_0), (a_1, b_1), \dots, (a_k, b_k)$$

is inadmissible for θ. If $b'_j \equiv i \pmod{n}$ and $b'_{j-1} \equiv i+1 \pmod{n}$ for $j \in [k-1]$, then the sequence

$$(x, y), (a_0, b_0), (a_1, b_1), \dots, (a_j, b_j), (a_{j-1}, b_{j-1}), \dots, (a_k, b_k)$$

is inadmissible for θ. Similarly, if $k > 0$ and $b'_k \equiv i \pmod{n}$ and $b'_{k-1} \equiv i+1 \pmod{n}$ then

$$(x, y), (a_0, b_0), (a_1, b_1), \dots, (a_{k-2}, b_{k-2}), (a_k, b_k)$$

is inadmissible for θ. Finally, it cannot happen that $b'_0 \equiv i \pmod{n}$ and $y' \equiv i+1 \pmod{n}$, since then the condition $\phi'_R(b'_0) + b'_0 \geq \phi'_R(y') + y'$ would imply that $\phi'_R(i) > \phi'_R(i+1)$ and $\phi_R(i) \leq \phi_R(i+1)$, contradicting the fact that $s_i \in \mathrm{Des}_R(\theta)$. In this way, we deduce the contrapositive of part (a), i.e., that if $\theta\pi_i$ is not right-admissible then θ is also not right-admissible. $\square$

It is obvious from Theorem-Definition 3.9 that $\ell'(\omega_R(\theta)) = \ell'(\omega_L(\theta)) = \ell'(\theta)$ for all $\theta \in \mathcal{W}_n$. The following statement is immediate from the previous theorem by induction.

Corollary 4.4. Let $\theta \in \mathcal{W}_n$. If θ is right-admissible then $\ell(\omega_R(\theta)) = \ell(\theta)$. If θ is left-admissible then $\ell(\omega_L(\theta)) = \ell(\theta)$.

One natural set of "extremal" elements on $\mathcal{W}_n$ is given by the subset $\tilde{I}_n$. Another is this:

Definition 4.5. Define $\mathcal{M}_n$ as the set of weighted involutions $\theta = (w, \phi) \in \mathcal{W}_n$ with $\ell'(w) = \ell(w)$, i.e., such that w is a product of commuting simple reflections.

Every element of $\mathcal{M}_n$ is both left- and right-admissible. The elements of $\mathcal{M}_n$ are in bijection with $\mathbb{N}$-weighted matchings in $\mathcal{C}_n$, the cycle graph on n vertices, which explains our notation.

Example 4.6. The set $\mathcal{M}_4$ consists of the weighted involutions of the form
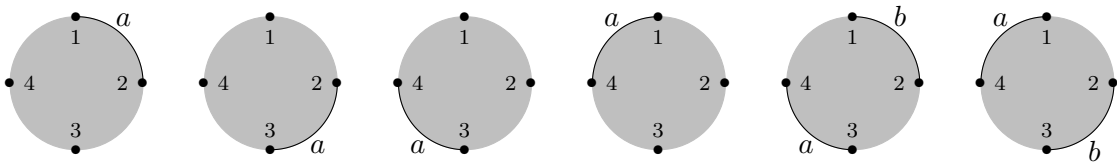
where $a, b \in \mathbb{N}$ are arbitrary natural numbers.

Proposition 4.7. There are $\frac{n}{n-k}\binom{n-k}{k}$ distinct k-element matchings in $\mathcal{C}_n$.

Proof. This is well-known (see [27, A034807]) and also follows as an instructive exercise. $\square$

In the next section we show that the length-preserving maps $\omega_R : \mathcal{M}_n \to \tilde{I}_n$ and $\omega_L : \mathcal{M}_n \to \tilde{I}_n$ are bijections. Here we construct the maps which will turn out to be their inverses.

Proposition-Definition 4.8. Fix $\theta = (w, \phi) \in \mathcal{W}_n$ and $m \in \mathbb{Z}$. Let $k = \ell'(w)$ and suppose

$$a_1 < a_2 < \cdots < a_k \qquad \text{and} \qquad a'_1 < a'_2 < \cdots < a'_k$$

are the respective sequences of left and right endpoints of w in $m + [n]$. For $j \in [k]$, define p_j, p'_j, q_j, and q'_j as the numbers of cycles $(x, y) \in \mathcal{C}(w)$ respectively satisfying

$$x < a_j < y, \qquad x < a'_j < y, \qquad x < a_j < y < w(a_j), \qquad \text{and} \qquad w(a'_j) < x < a'_j < y.$$

Let $i_j = a_j + p_j$ and $i'_j = a'_j - p'_j - 1$, and let $u = s_{i_1} s_{i_2} \cdots s_{i_k}$ and $v = s_{i'_1} s_{i'_2} \cdots s_{i'_k}$. Finally:

(a) Let $\lambda_R(\theta) = (u, \psi) \in \mathcal{W}_n$ where $\psi : \mathcal{C}(u) \to \mathbb{N}$ is the weight map with

$$\psi(i_j, i_j + 1) = \phi(a_j, w(a_j)) + w(a_j) - a_j - q_j - 1 \qquad \text{for } j \in [k].$$
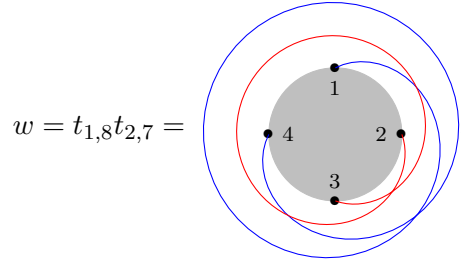
(b) Let $\lambda_L(\theta) = (v, \chi) \in \mathcal{W}_n$ where $\chi : \mathcal{C}(v) \to \mathbb{N}$ is the weight map with

$$\chi(i'_j, i'_j + 1) = \phi(w(a'_j), a'_j) + a'_j - w(a'_j) - q'_j - 1 \qquad \text{for } j \in [k].$$

The weighted involutions $\lambda_R(\theta)$ and $\lambda_L(\theta)$ then both belong $\mathcal{M}_n$, and do not depend on m.

Note that if $w \in \tilde{I}_n \subset \mathcal{W}_n$ then $\lambda_R(w) = \lambda_R(w, 0)$ and $\lambda_L(w) = \lambda_L(w, 0)$. We delay the proof of the proposition to give an example and state a lemma.

Example 4.9. Suppose $n = 4$, $k = 2$, $m = 0$, and

$$w = t_{1,8} t_{2,7} =$$

Then $(a_1, a_2) = (1, 2)$ and $(a'_1, a'_2) = (3, 4)$ and the numbers $p_j, p'_j, q_j, q'_j, i_j, i'_j$ are computed as follows. First, we have $(p_1, p_2) = (2, 3)$ and $(q_1, q_2) = (2, 2)$ so $(i_1, i_2) = (3, 5)$ and $\lambda_R(w) = (s_1 s_3, \psi)$ where $\psi(3, 4) = 4$ and $\psi(1, 2) = 2$. We have $(p'_1, p'_2) = (3, 2)$ and $(q'_1, q'_2) = (2, 2)$ so $(i'_1, i'_2) = (-1, 1)$ and $\lambda_L(w) = (s_1 s_3, \chi)$ where $\chi(3, 4) = 2$ and $\chi(1, 2) = 4$. In terms of winding diagrams,

$$\lambda_R(w) = \qquad \text{and} \qquad \lambda_L(w) =$$

Lemma 4.10. If $\theta \in \mathcal{W}_n$ then $\lambda_L(\theta^*) = \lambda_R(\theta)^*$.

Proof. This is evident from the symmetric definitions of $\lambda_R(\theta)$ and $\lambda_L(\theta)$. $\qquad \square$

Proof of Proposition-Definition 4.8. The fact that $\lambda_R(\theta)$ and $\lambda_L(\theta)$ do not depend on m holds since $w(i+n) = w(i) + n$ for all $i \in \mathbb{Z}$. We must show that $\lambda_R(\theta)$ and $\lambda_L(\theta)$ actually belong to $\mathcal{M}_n$. It is enough to prove that $\lambda_R(\theta) \in \mathcal{M}_n$. For this, we need to check that $i_j + 1 < i_{j+1}$ for $j \in [k-1]$ and $i_k + 1 < i_1 + n$. Let a be any left endpoint of w and let $\tilde{a}$ be the smallest left endpoint greater than a. Define p and $\tilde{p}$ as the respective numbers of cycles $(x, y) \in \mathcal{C}(w)$ with $x < a < y$ and $x < \tilde{a} < y$. It is enough to verify that $a + p + 1 < \tilde{a} + \tilde{p}$. This holds if and only if $a + r + 1 < \tilde{a} + \tilde{r}$ where r and $\tilde{r}$ are the respective numbers of cycles $(x, y) \in \mathcal{C}(w)$ with $x < a < y < \tilde{a}$ and $a \le x < \tilde{a} < y$. Clearly $a + r + 1 \le \tilde{a}$, with equality only if $a < \tilde{a} < w(a)$ in which case $\tilde{r} > 0$. $\qquad\square$

Given $\theta = (w, \phi) \in \mathcal{W}_n$, recall from Section 3 the definition of $\phi_R, \phi_L : \mathbb{Z} \to \mathbb{N}$.

Lemma 4.11. Let $\theta = (w, \phi) \in \mathcal{W}_n$. If $s_i \notin \mathrm{Des}_R(w)$ then $\lambda_R(\theta \pi_i) = \lambda_R(\theta)$ and $\lambda_L(\pi_i \theta) = \lambda_R(\theta)$.

Proof. Choose $i \in \mathbb{Z}$ with $s_i \notin \mathrm{Des}_R(w)$. By Lemmas 3.7 and 4.10 it suffices to prove that $\lambda_R(\theta \pi_i) = \lambda_R(\theta)$. Assume $\theta \pi_i \ne \theta$. Let $\tilde{w} = s_i w s_i$ and define $\tilde{\phi} : \mathcal{C}(\tilde{w}) \to \mathbb{N}$ such that $\theta \pi_i = (\tilde{w}, \tilde{\phi})$. Choose $m \in \mathbb{Z}$ such that $\{i, i+1\} \subset m + [n]$, and define Define a_j, b_j, p_j, q_j, i_j, and ψ relative to w and ϕ as in Proposition-Definition 4.8. Define $\tilde{a}_j, \tilde{b}_j, \tilde{p}_j, \tilde{q}_j, \tilde{i}_j$, and $\tilde{\psi}$ analogously relative to $\tilde{w}$ and $\tilde{\phi}$. We deduce that $\lambda_R(\theta \pi_i) = \lambda_R(\theta)$ by comparing these quantities as follows:

- If $w(i) < w(i+1) < i < i+1$ then $a_j = \tilde{a}_j$ and $p_j = \tilde{p}_j$ for all j, and there are two indices j for which $\phi_R(b_j)$, b_j, q_j and $\tilde{\phi}_R(\tilde{b}_j)$, $\tilde{b}_j$, $\tilde{q}_j$ are different: for one index $\tilde{\phi}_R(\tilde{b}_j) = \phi_R(b_j) - 1$ and $\tilde{b}_j = b_j + 1$ and $\tilde{q}_j = q_j$, and for the other $\tilde{\phi}_R(\tilde{b}_j) = \phi_R(b_j)$ and $\tilde{b}_j = b_j - 1$ and $\tilde{q}_j = q_j - 1$.

- If $w(i) < i < i+1 = w(i+1)$ then $a_j = \tilde{a}_j$ and $p_j = \tilde{p}_j$ for all j, and the numbers $\phi_R(b_j)$, b_j, q_j are the same as $\tilde{\phi}_R(\tilde{b}_j)$, $\tilde{b}_j$, $\tilde{q}_j$ except when j is such that $b_j = i$, in which case it holds that $\tilde{\phi}_R(\tilde{b}_j) = \phi_R(b_j) - 1$ and $\tilde{b}_j = b_j + 1$ and $\tilde{q}_j = q_j$.

- If $w(i) < i < i+1 < w(i+1)$ then $a_j = \tilde{a}_j$ and $p_j = \tilde{p}_j$ except when $a_j = i+1$, in which case $\tilde{a}_j = a_j - 1$ and $\tilde{p}_j = p_j + 1$. Likewise, the numbers $\phi_R(b_j)$, b_j, q_j are the same as $\tilde{\phi}_R(\tilde{b}_j)$, $\tilde{b}_j$, $\tilde{q}_j$, except when $b_j = i$, in which case $\tilde{\phi}_R(\tilde{b}_j) = \phi_R(b_j) - 1$ and $\tilde{b}_j = b_j + 1$ and $\tilde{q}_j = q_j$.

One of these cases must occur as since it cannot hold that $w(i+1) < w(i) < i < i+1$. We deduce that $i_j = \tilde{i}_j$ for all j and $\psi = \tilde{\psi}$, so $\lambda_R(\theta \pi_i) = \lambda_R(\theta)$. $\qquad\square$

Corollary 4.12. If $\theta \in \mathcal{W}_n$ is right-admissible (respectively, left-admissible) then $\lambda_R(\theta \pi_g) = \lambda_R(\theta)$ (respectively, $\lambda_L(\pi_g \theta) = \lambda_L(\theta)$) for all $g \in \tilde{S}_n$.

Proof. This follows from the previous lemma and Theorem 4.3. $\qquad\square$

Corollary 4.13. If $\theta \in \mathcal{M}_n$ then $\lambda_R(\omega_R(\theta)) = \lambda_R(\theta) = \theta$ and $\lambda_L(\omega_L(\theta)) = \lambda_L(\theta) = \theta$.

Proof. Apply the previous corollary after checking that any $\theta \in \mathcal{M}_n$ has $\lambda_R(\theta) = \lambda_L(\theta) = \theta$. $\qquad\square$

5 Order isomorphisms

Recall the definition of $t_{ij} \in \tilde{S}_n$ for $i < j \not\equiv i \pmod{n}$. Write $u \lessdot v$ if v covers u in the Bruhat order on $\tilde{S}_n$, i.e., if $v = u t_{ij}$ for some i, j and $\ell(v) = \ell(u) + 1$. Let $\prec$ be the partial order on $\mathcal{M}_n$ with $(w, \phi) \preceq (w', \phi')$ if and only if $w = w'$ and $\phi(a, b) \le \phi'(a, b)$ for all $(a, b) \in \mathcal{C}(w)$.

Lemma 5.1. Let $\theta \in \mathcal{M}_n$ and $z, z' \in \tilde{I}_n$ with $\omega_R(\theta) = z$. The following are then equivalent:

(a) There exist $i, j \in \mathbb{Z}$ with $z(i) < i < j = \min\{e \in \mathbb{Z} : i < e \text{ and } z(i) < z(e)\}$ and $z' = t_{ij} z t_{ij}$.

(b) There exists $\theta' \in \mathcal{M}_n$ which covers θ in the order $\prec$ such that $z' = \omega_R(\theta')$.

Moreover, if these conditions hold then $g_R(\theta) \lessdot g_R(\theta) t_{ij} = g_R(\theta')$.

This lemma has a left-handed version, given by applying the $*$-operator and Lemma 3.11.

Proof. Let $w \in \tilde{I}_n$ and $\phi : \mathcal{C}(w) \to \mathbb{N}$ be such that $\theta = (w, \phi)$. The theorem will derive from simple arguments involving the following specialised notation. Consider an integer sequence $I = (i_1, i_2, \ldots, i_l) \in \mathbb{Z}^l$. For each $0 \leq t \leq l$, define $w^{I,t} \in \tilde{I}_n$ and $\phi^{I,t} : \mathcal{C}(w^{I,t}) \to \mathbb{N}$ such that $\theta \pi_{i_1} \pi_{i_2} \cdots \pi_{i_t} = (w^{I,t}, \phi^{I,t})$, so that $(w, \phi) = (w^{I,0}, \phi^{I,0})$. As usual, write $\phi_R^{I,t}$ for the right form of $\phi^{I,t}$ as given after Definition 3.1. Let $\alpha^{I,0} = 1 \in \tilde{S}_n$ and for $i \in [l]$, define $\alpha^{I,t} \in \tilde{S}_n$ to be $\alpha^{I,t} s_{i_t}$ if $w^{I,t} \neq w^{I,t-1}$, and otherwise set $\alpha^{I,t} = \alpha^{I,t-1}$. Note that if $\phi^{I,t} = 0$ for any t then we have $z = w^{I,t}$ and $g_R(\theta) = \alpha^{I,t}$. As an abbreviation, write $w^I = w^{I,l}$ and $\phi^I = \phi^{I,l}$ and $\alpha^I = \alpha^{I,l}$. We refer to $\alpha^{I,t}(x)$ as the *trajectory* of $x \in \mathbb{Z}$ at time t, relative to the index sequence I. By construction, a given integer x is a left endpoint, right endpoint, or fixed point of w if and only if its trajectory at any time t is respectively a left endpoint, right endpoint, of fixed point of $w^{I,t}$. Moreover, if X is the set of numbers $i \in \mathbb{Z}$ with $\phi_R^{I,t}(x) = 0$ for $x = \alpha^{I,t}(i)$, then the trajectories of any two $i, j \in X$ have the same relative order at all times greater than or equal to t.

Choose a right endpoint $a \in \mathbb{Z}$ of w. Corresponding to this choice, we define a particular integer sequence I. This sequence will be the concatenation of three subsequences J, K, and L, given as follows. If $\phi_R(a) = 0$ then define J to be the empty sequence. Otherwise, let i_0 be an arbitrary integer and define $J = (i_1, i_2, \ldots, i_l)$ as the sequence characterised by the following properties:

- For each $t \geq 0$, i_{t+1} is the smallest integer $x > i_t$ such that $\phi_R^{J,t}(x-1) = 0 < \phi_R^{J,t}(x)$.

- The first time t at which $\phi^{J,t}(x) = 0$ for $x = \alpha^{J,t}(a)$ is $t = l$.

Let $k = \alpha^J(a)$. Consider the sequence given by the infinite repetition of $k+1, k+2, \ldots, k+n-2$. Define K as the shortest initial subsequence of this sequence such that $\phi^{JK}(k+1) = 0$, where JK denotes the concatenation of J and K. We then have $k = \alpha^{JK}(a)$. Define $b \in \mathbb{Z}$ as the integer with $k+1 = \alpha^{JK}(b)$. Note that if $w^{JK}(k+1) < w^{JK}(k)$, then we must also have $w^{J,l-1}(\alpha^{J,l-1}(b)) < w^{J,l-1}(\alpha^{J,l-1}(a))$ and $\phi^{J,l-1}(\alpha^{J,l-1}(b)) = 0 < \phi^{J,l-1}(\alpha^{J,l-1}(a)) = 1$. But this is impossible, since the way we have constructed J means that the weighted involution $(w^{J,t}, \phi^{J,t})$ never has nesting cycles in which the outer cycle has weight zero while the inner cycle has a positive weight. We conclude, therefore, that $w^{JK}(k) < w^{JK}(k+1)$. Finally, let L be any sequence such that $\phi^I = 0$ and $z = \omega_R(\theta) = w^I$, where $I = JKL$. Define $i = \alpha^I(a)$ and $j = \alpha^I(b)$ and note that $z(i) < i < j$ and $z(i) < z(j)$.

Since θ is right-admissible and since the right operators $\pi_1, \pi_2, \ldots$ preserve right-admissibility, each $t \in \{i+1, i+2, \ldots, j-1\}$ must be a left endpoint of z with $z(t) < z(i)$. Thus $j = \min\{e \in \mathbb{Z} : i < j \text{ and } z(j) < z(i)\}$. Moreover, it is easy to see that any pair $(i, j) \in \mathbb{Z}$ with $j = \min\{e \in \mathbb{Z} : i < j \text{ and } z(j) < z(i)\}$ arises in this way for some choice of a: given i, the desired value of a is $w(z(i) + p)$ where p is the number of pairs $(x, y) \in \mathcal{C}(z)$ with $x < z(i) < y$.

Now consider the unique weighted involution $\theta' = (w, \phi') \in \mathcal{W}_n$ which covers θ is the order $\prec$ and has $\phi'_R(a) = \phi_R(a) + 1$. Define $z' \in \tilde{I}_n$ such that $\omega_R(\theta') = z'$. If we define K' by adding the index k to the end of K and set $I' = JK'L$, then it holds by construction that $z = \omega_R(\theta') = w^{I'}$ and $g_L(\theta') = \alpha^{I'} = gsg^{-1}h$ for $g = \alpha^J$, $s = s_k$, and $h = \alpha^I = g_R(\theta)$. Since $g^{-1}h(k) = i$ and

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$g^{-1}h(k+1) = j$, it follows that $g_L(\theta') = h(g^{-1}h)^{-1}s(g^{-1}h) = g_R(\theta)t_{ij}$, and therefore $z' = t_{ij}zt_{ij}$. This suffices to show the equivalence of (a) and (b). For the last assertion, note that it follows from Theorem-Definition 3.9 and Corollary 4.4 that $\ell(g_R(\theta')) = \ell(g_R(\theta)) + 1$, so $g_R(\theta) \lessdot g_R(\theta)t_{ij}$. $\qquad\square$

Define $\prec_R$ as the transitive closure of the relation on $\tilde{I}_n$ with $z \prec_R t_{ij}zt_{ij}$ whenever $z(i) < i$ and $j = \min\{e \in \mathbb{Z} : i < e \text{ and } z(i) < z(e)\}$. Define $\prec_L$ similarly as the transitive closure of the relation on $\tilde{I}_n$ with $z \prec_L t_{ij}zt_{ij}$ whenever $j < z(j)$ and $i = \max\{e \in \mathbb{Z} : e < j \text{ and } z(e) < z(j)\}$. Recall that a partially ordered set is *graded* if all maximal chains between two elements have the same length. A *rank function* for a graded poset $\mathcal{P}$ is a map $\mathcal{P} \to \mathbb{Z}$, the difference of whose values gives the common length of all maximal chains between two comparable elements.

Proposition 5.2. The posets $(\tilde{I}_n, \prec_R)$ and $(\tilde{I}_n, \prec_L)$ are isomorphic via the map $z \mapsto z^*$. Both are graded subposets of $(\tilde{I}_n, <)$ with rank function $z \mapsto \frac{1}{2}\ell(z)$.

Proof. The first assertion is clear from the definitions. For the second claim, note by Lemma 2.10 that if z covers y in $\prec_R$ or $\prec_L$ then $y < z$ and $\ell(z) = \ell(y) + 2$. $\qquad\square$

Lemma 5.3. An element $z \in \tilde{I}_n$ is minimal in $\prec_R$ (respectively, $\prec_L$) if and only if $\ell(z) = \ell'(z)$.

Proof. We may just consider $(\tilde{I}_n, \prec_R)$ since the other poset is isomorphic. Assume $z \in \tilde{I}_n$ is minimal with respect to $\prec_R$. We cannot have $z(i+1) < i+1$ and $z(i+1) < z(i)$ for any $i \in \mathbb{Z}$ since then $s_i z s_i \prec_R z$. This implies that if $i+1$ is a right endpoint of z then either i is also a right endpoint with $z(i) < z(i+1)$, or $z(i) = i+1$. Applying this property in succession, we deduce that in fact $z(i) = i+1$ whenever $i+1$ is a right endpoint of z, so $\ell(z) = \ell'(z)$. Conversely, any involution $z \in \tilde{I}_n$ with $\ell(z) = \ell(z')$ is evidently minimal with respect to $\prec_R$. $\qquad\square$

Putting everything together gives the following.

Theorem 5.4. The maps $\omega_R : (\mathcal{M}_n, \prec) \to (\tilde{I}_n, \prec_R)$ and $\omega_L : (\mathcal{M}_n, \prec) \to (\tilde{I}_n, \prec_L)$ are isomorphisms of partially ordered sets which preserve ℓ and ℓ', and have respective inverses λ_R and λ_L.

Proof. By Corollaries 4.4 and 4.13, the maps ω_R and ω_L preserve ℓ and ℓ', are injective with left inverses λ_R and λ_L, and restrict to bijections $\{\theta \in \mathcal{M}_n : \mathrm{wt}(\theta) = 0\} \to \{w \in \tilde{I}_n : \ell(w) = \ell'(w)\}$. Given these facts, the maps' surjectivity follows from Lemmas 5.1 and 5.3 by induction. $\qquad\square$

Slightly abusing notation, we write $\tilde{I}_n(q,x) = \sum_{w \in \tilde{I}_n} q^{\ell(w)}x^{\ell'(w)} \in \mathbb{N}[[q,x]]$.

Corollary 5.5. If $n \geq 1$ then $\tilde{I}_n(q,x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k}\left(\frac{qx}{1-q^2}\right)^k$.

Proof. Corollary 4.4 and Theorem 5.4 imply that $\tilde{I}_n(q,x) = \sum_{\theta \in \mathcal{M}_n} q^{2\,\mathrm{wt}(\theta)}(qx)^{\ell'(\theta)}$. By Proposition 4.7, the coefficient of x^k in the latter power series is $\frac{n}{n-k}\binom{n-k}{k}q^k(1 + q^2 + q^4 + q^6 + \dots)^k$. $\qquad\square$

Corollary 5.6. If $n \geq 3$ then $\tilde{I}_n(q,x) = \tilde{I}_{n-1}(q,x) + \frac{qx}{1-q^2}\tilde{I}_{n-2}(q,x)$.

Define $\hat{\ell}(w) = \frac{1}{2}(\ell(w) + \ell'(w))$ for $w \in \tilde{I}_n$. Corollary 5.5 shows that $\hat{\ell}(w) \in \mathbb{N}$.

Corollary 5.7. For each $n \geq 2$ and $m \geq 1$, let $N_n(m)$ and $\hat{N}_n(m)$ be the number of involutions $w \in \tilde{I}_n$ with $\ell(w) = m$ and $\hat{\ell}(w) = m$, respectively. Then

$$N_n(m) = \sum_{\substack{1 \leq j \leq \lfloor n/2 \rfloor \\ j \equiv m \pmod 2}} \frac{n}{n-j} \binom{n-j}{j} \binom{\frac{j+m}{2}-1}{j-1} \quad \text{and} \quad \hat{N}_n(m) = \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{n}{n-j} \binom{n-j}{j} \binom{m-1}{j-1}.$$

Proof. Extract the coefficients of q^m in $\tilde{I}_n(q,1)$ and q^{2m} in $\tilde{I}_n(q,q) = \sum_{w \in \tilde{I}_n} q^{2\hat{\ell}(w)}$. $\qquad\square$

Remark. The sequence $\{\hat{N}_n(n)\}_{n=1,2,3,\dots} = (0, 2, 3, 10, 25, 71, 196, 554, 1569, \dots)$ coincides with [27, A246437], which gives the "type B analog for Motzkin sums," while $\{\hat{N}_n(2n)\}_{n=1,2,3,\dots} = (0, 2, 3, 18, 50, 215, 735, 2898, \dots)$ gives [27, A211867], which counts certain Motzkin paths.

Define $\zeta_R, \zeta_L : \tilde{I}_n \to \tilde{I}_n$ as the maps $\zeta_R = {*} \circ \omega_L \circ \lambda_R = \omega_R \circ {*} \circ \lambda_R = \omega_R \circ \lambda_L \circ {*}$ and $\zeta_L = {*} \circ \omega_R \circ \lambda_L = \omega_L \circ {*} \circ \lambda_L = \omega_L \circ \lambda_R \circ {*}$. Corollary 4.4 and Theorem 5.4 imply this property:

Corollary 5.8. The maps ζ_R and ζ_L are involutions of $(\tilde{I}_n, \prec_R)$ and $(\tilde{I}_n, \prec_L)$, respectively, which preserve length ℓ and absolute length ℓ', and it holds that $\zeta_L \circ {*} = {*} \circ \zeta_R$.

Consider the following variation of the permutations $g_L(\theta)$ and $g_R(\theta)$ for $\theta \in \mathcal{W}_n$.

Definition 5.9. For each $z \in \tilde{I}_n$ let $\alpha_R(z) = g_R(\lambda_R(z)) \in \tilde{S}_n$ and $\alpha_L(z) = g_L(\lambda_L(z)) \in \tilde{S}_n$.

As our final result in this section, we derive a more explicit formula for these elements.

Proposition-Definition 5.10. If $a_1, a_2, \dots, a_n \in \mathbb{Z}$ represent the distinct congruence classes modulo n then there is a unique $m \in \mathbb{Z}$ and a unique $w \in \tilde{S}_n$ such that $w(m+i) = a_i$ for $i \in [n]$. Moreover, it holds that $m = \frac{1}{n} \sum_{i=1}^{n} (a_i - i)$. Define $[a_1, a_2, \dots, a_n] = w \in \tilde{S}_n$.

Proof. If an affine permutation exists with the desired property, it must be given by the map $w : \mathbb{Z} \to \mathbb{Z}$ with $w(m+i+jn) = a_i + jn$ for $i \in [n]$ and $j \in \mathbb{Z}$. As this map is in $\tilde{S}_n$ if and only if $\sum_{i \in [n]} i = \sum_{i \in [n]} w(i) = \sum_{i \in [n]} w(m+i) - nm = \sum_{i \in [n]} a_i - nm$, the result follows. $\qquad\square$

If $N > n$ and $a_1, a_2, \dots, a_N \in \mathbb{Z}$ represent all congruence classes modulo n, then we define $[a_1, a_2, \dots, a_N] \in \tilde{S}_n$ inductively by the condition that $[a_1, a_2, \dots, a_N] = [a_1, \dots, \hat{a}_j, \dots, a_N]$ if there are indices $1 \leq i < j \leq N$ with $a_i \equiv a_j \pmod n$. As usual, here $\hat{}$ denotes omission.

Theorem 5.11. Let $z \in \tilde{I}_n$ and $m \in \mathbb{Z}$. Suppose $a_1 < a_2 < \cdots < a_l$ and $d_1 < d_2 < \cdots < d_l$ are the elements of $m + [n]$ with $a_i \leq z(a_i)$ and $z(d_i) \leq d_i$. Define $b_i = z(a_i)$ and $c_i = z(d_i)$. Then

$$\alpha_R(z) = [a_1, b_1, a_2, b_2, \dots, a_l, b_l]^{-1} \quad \text{and} \quad \alpha_L(z) = [c_1, d_1, c_2, d_2, \dots, c_l, d_l]^{-1}.$$

In particular, these formulas do not depend on the choice of m.

Note that l is necessarily $n - \ell'(z)$. Before the proof, we give an example.

Example 5.12. One has $\alpha_R(1) = \alpha_L(1) = [1, 1, 2, 2, \dots, n, n]^{-1} = 1$. If $z = t_{1,8} t_{2,7} \in \tilde{I}_4$ then $\alpha_R(z) = [1, 8, 2, 7]^{-1} = [3, 5, 2, 0]$ and $\alpha_L(z) = [-2, 3, -3, 4]^{-1} = [5, 3, 0, 2]$.

Proof. By Lemma 3.11, we may just prove the assertions for $g_R(z)$. Replacing m by $m+1$ has no effect on the formula for $g_R(z)$ since $[a_2, b_2, \ldots, a_l, b_l, a_1 + n, b_1 + n] = [a_1, b_1, a_2, b_2, \ldots, a_l, b_l]$, and so we deduce that this formula is independent of the choice of m. When $\ell(z) = \ell'(z)$, we have $[a_1, b_1, \ldots, a_l, b_l]^{-1} = [1, 2, \ldots, n]^{-1} = 1 = g_R(z)$. For the general case, suppose $z' \in \tilde{I}_n$ covers z in $\prec_R$, so that $z' = t_{ij} z t_{ij}$ for some $i, j \in \mathbb{Z}$ with $z(i) < i$ and $j = \min\{e \in \mathbb{Z} : i < e : z(i) < z(e)\}$. Let $m = z(i) - 1$, so that $a_1 = z(i)$ and $b_1 = i$, and assume by induction that $g_R(z) = [a_1, b_1, a_2, b_2, \ldots, a_l, b_l]^{-1}$. Note that $i, i+1, \ldots, j-1$ are necessarily all right endpoints of z, and that consequently $i < j < i + n$. Define $\Delta = j - i \in [n-1]$ and note that the only way $\{i, z(i)\} + n\mathbb{Z}$ and $\{j, z(j)\} + n\mathbb{Z}$ can intersect is if $j < z(j) \equiv i \pmod{n}$. We now compute the product $g_R(z)t_{ij}$. There are four cases to consider:

(a) Suppose $z(j) < j$. Let $t \in \{2, 3, \ldots, l\}$ be the unique index with $a_t \equiv z(j) \pmod{n}$ and $b_t \equiv j \pmod{n}$. Then $g_R(z)t_{ij} = [a_1, b_1 + \Delta, a_2, b_2, \ldots, a_{t-1}, b_{t-1}, a_t, b_t - \Delta, a_{t+1}, b_{t+1}, \ldots]^{-1}$.

(b) Suppose $z(j) = j$. Let $t \in \{2, 3, \ldots, l\}$ be the unique index with $a_t = b_t \equiv j \pmod{n}$. Then $g_R(z)t_{ij} = [a_1, b_1 + \Delta, a_2, b_2, \ldots, a_{t-1}, b_{t-1}, a_t - \Delta, a_t - \Delta, a_{t+1}, b_{t+1}, \ldots]^{-1}$.

(c) Suppose $j < z(j) \not\equiv i \pmod{n}$. Let $t \in \{2, 3, \ldots, l\}$ be the unique index with $a_t \equiv j \pmod{n}$. Then $g_R(z)t_{ij} = [a_1, b_1 + \Delta, a_2, b_2, \ldots, a_{t-1}, b_{t-1}, a_t - \Delta, b_t, a_{t+1}, b_{t+1}, \ldots]^{-1}$.

(d) Finally suppose $j < z(j) \equiv i \pmod{n}$. Then $z(j) \equiv a_1 \pmod{n}$ and $j \equiv b_1 \pmod{n}$, and we have $g_R(z)t_{ij} = [a_1 - \Delta, b_1 + \Delta, a_2, b_2, \ldots, a_l, b_l]^{-1}$.

Let $a'_1 < a'_2 < \cdots < a'_l$ be the numbers in $m + [n]$ with $a'_i \leq z'(a'_i)$ and set $b'_i = z'(a'_i)$. In each case (a)-(d), our computations reduce to the identity $g_R(z)t_{ij} = [a'_1, b'_1, \ldots, a'_l, b'_l]^{-1}$. For example, in case (d) we have $a'_i = a_{i+1}$ and $b'_i = b_{i+1}$ for $i \in [l-1]$ while $a'_l = a_1 - \Delta + n$ and $b'_l = a_1 + \Delta + n$, so $g_R(z)t_{ij} = [a'_l - n, b'_l - n, a'_2, b'_2, \ldots, a'_{l-1}, b'_{l-1}]^{-1} = [a'_1, b'_1, \ldots, a'_l, b'_l]^{-1}$. Since $g_R(z') = g_R(z)t_{ij}$ by Lemma 5.1, the desired formula for $g_R(z)$ holds for all $z \in \tilde{I}_n$ by induction. $\square$

6 Demazure conjugation

Recall the definition of the Demazure product $\circ : \tilde{S}_n \times \tilde{S}_n \to \tilde{S}_n$ from the introduction. The operation $(z, w) \mapsto w^{-1} \circ z \circ w$ for $z \in \tilde{I}_n$ and $w \in \tilde{S}_n$ defines another right action of the monoid $(\tilde{S}_n, \circ)$, now on $\tilde{I}_n$. We refer to this action as *Demazure conjugation*. If $z \in \tilde{I}_n$ and $i \in \mathbb{Z}$ then

$$s_i \circ z \circ s_i = \begin{cases} s_i z s_i & \text{if } z(i) < z(i+1) \text{ and } (i, i+1) \notin \mathcal{C}(z) \\ z s_i & \text{if } z(i) < z(i+1) \text{ and } (i, i+1) \in \mathcal{C}(z) \\ z & \text{otherwise.} \end{cases} \tag{6.1}$$

It follows by induction from this formula that the orbit of 1 under Demazure conjugation is all of $\tilde{I}_n$, so every $z \in \tilde{I}_n$ can be expressed as $z = w^{-1} \circ w$ for some $w \in \tilde{S}_n$.

Definition 6.1. For $z \in \tilde{I}_n$ let $\mathcal{A}(z)$ be the set of shortest elements $w \in \tilde{S}_n$ with $z = w^{-1} \circ w$.

Example 6.2. If $z = t_{0,5} = [-4, 2, 3, 9] \in \tilde{I}_4$ then $\mathcal{A}(z) = \{s_1 s_2 s_3 s_4, s_2 s_1 s_3 s_4, s_3 s_2 s_1 s_4\}$.

The set $\mathcal{A}(z)$ is nonempty for all $z \in \tilde{I}_n$, and we refer to its elements as the *atoms* of z. Recall from Corollary 5.7 that $\hat{\ell}(z) = \frac{1}{2}(\ell(z) + \ell'(z))$. By [24, Proposition 2.6] or (6.1), we have:

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Proposition 6.3. If $z \in \tilde{I}_n$ then $\hat{\ell}(z)$ is the common value of $\ell(w)$ for $w \in \mathcal{A}(z)$.

Results in [8], building on work of Can, Joyce, and Wyser [4, 5], show that the set of atoms of an involution in any finite symmetric group naturally has the structure of a bounded, graded poset. Here, we prove that this phenomenon generalises to involutions in the infinite group $\tilde{S}_n$.

Let $\lessdot_{\mathcal{A}}$ be the relation on $\tilde{S}_n$ with $u \lessdot_{\mathcal{A}} v$ if and only if $u < s_{i+1}u = s_i v > v$ for some $i \in \mathbb{Z}$. Note that this relation is empty when $n \in \{1, 2\}$. Given $w \in \tilde{S}_n$, write $w[j:k]$ for the sequence of values $w(j)w(j+1)\cdots w(k)$. We note two additional characterisations of $\lessdot_{\mathcal{A}}$:

Lemma 6.4. Let $u, v \in \tilde{S}_n$. Then $u \lessdot_{\mathcal{A}} v$ if and only if for some $a, b, c, i \in \mathbb{Z}$ with $a < b < c$ we have $u^{-1}[i:i+2] = cab$, $v^{-1}[i:i+2] = bca$, and $u^{-1}(j) = v^{-1}(j)$ for all $j \notin \{i, i+1, i+2\} + n\mathbb{Z}$.

Proof. This follows as a simple exercise from Corollary 2.2. $\square$

Lemma 6.5. Let $u, v \in \tilde{S}_n$. Then $u \lessdot_{\mathcal{A}} v$ if and only if for some $i \in \mathbb{Z}$ and $w \in \tilde{S}_n$ it holds that $u = s_i s_{i+1} w$, $v = s_{i+1} s_i w$, and $\ell(w) + 2 = \ell(u) = \ell(v)$.

Proof. This is easy to derive from the preceding lemma given Lemma 2.3. $\square$

Corollary 6.6. Let $z \in \tilde{I}_n$ and $u, v \in \tilde{S}_n$ with $u \lessdot_{\mathcal{A}} v$. Then $u \in \mathcal{A}(z)$ if and only if $v \in \mathcal{A}(z)$.

Proof. This follows from Lemma 6.5 since $\circ$ is associative and $s_{i+1}s_i \circ s_i s_{i+1} = s_i s_{i+1} \circ s_{i+1} s_i$. $\square$

Recall the maps $\lambda_R, \lambda_L : \tilde{I}_n \to \mathcal{W}_n$ and the elements $\alpha_R(z), \alpha_L(z)$ from Definitions 3.9 and 5.9.

Definition 6.7. Given $z \in \tilde{I}_n$, write $(w_R, \phi_R) = \lambda_R(z)$ and $(w_L, \phi_L) = \lambda_L(z)$ and define

$$\alpha_{\min}(z) = w_R \cdot \alpha_R(z) = \alpha_R(z) \cdot z \qquad \text{and} \qquad \alpha_{\max}(z) = w_L \cdot \alpha_L(z) = \alpha_L(z) \cdot z.$$

Proposition 6.8. For each $z \in \tilde{I}_n$ both $\alpha_{\min}(z)$ and $\alpha_{\max}(z)$ belong to $\mathcal{A}(z)$.

Proof. Let $g = \alpha_R(z)$ and $w = w_R$. By Theorem-Definition 3.9(b) and Corollary 4.4 we have $\ell(z) = \ell\left(g^{-1}wg\right) = \ell(w) + 2\ell(g)$, so $z = g^{-1}wg = g^{-1} \circ w \circ g$. Since $\ell(w) = \ell'(w)$, Lemma 2.7 implies that $w = w^{-1} \circ w$, and therefore $z = g^{-1} \circ w^{-1} \circ w \circ g = (wg)^{-1} \circ wg$. As $\ell(wg) = \ell(w) + \ell(g) = \hat{\ell}(z)$, we conclude that $wg \in \mathcal{A}(z)$. The argument that $w_L \cdot \alpha_L(z) \in \mathcal{A}(z)$ is similar. $\square$

The atoms $\alpha_{\min}(z)$ and $\alpha_{\max}(z)$ have this convenient formula:

Corollary 6.9. Let $z \in \tilde{I}_n$, $m \in \mathbb{Z}$, and $l = n - \ell'(z)$. Define $a_i, b_i, c_i, d_i \in \mathbb{Z}$ for $i \in [l]$ as in Theorem 5.11. Then $\alpha_{\min}(z) = [b_1, a_1, b_2, a_2, \ldots, b_l, a_l]^{-1}$ and $\alpha_{\max}(z) = [d_1, c_1, d_2, c_2, \ldots, d_l, c_l]^{-1}$.

Proof. As $z[a_1, b_1, \ldots, a_l, b_l] = [b_1, a_1, \ldots, b_l, a_l]$ and $z[c_1, d_1, \ldots, c_l, d_l] = [d_1, c_1, \ldots, d_l, c_l]$ by construction, the result follows from Theorem 5.11. $\square$

Example 6.10. If $z = t_{1,8}t_{2,7} \in \tilde{I}_4$ then $\alpha_{\min}(z) = [8, 1, 7, 2]^{-1} = [4, 6, 1, -1]$ and $\alpha_{\max}(z) = [3, -2, 4, -3]^{-1} = [6, 4, -1, 1]$.

Lemma 6.11. Let $z \in \tilde{I}_n$ and $w \in \mathcal{A}(z)$. Then no $i \in \mathbb{Z}$ has $w^{-1}[i:i+2] = cba$ where $a < b < c$.

Proof. If $w^{-1}(i) > w^{-1}(i+1) > w^{-1}(i+2)$ for $i \in \mathbb{Z}$ then we could write $w = s_i s_{i+1} s_i v$ for some $v \in \tilde{S}_n$ with $\ell(w) = \ell(v) + 3$. But then we would have $z = w^{-1} \circ w = (s_i s_{i+1} s_i v)^{-1} \circ (s_i s_{i+1} s_i v) = (s_{i+1} s_i v)^{-1} \circ (s_{i+1} s_i v)$ despite $\ell(s_{i+1} s_i v) = \ell(v) + 2 < \ell(w)$, contradicting the definition of $\mathcal{A}(w)$. $\square$

Lemma 6.12. Let $z \in \tilde{I}_n$ and $v \in \mathcal{A}(z)$.

(a) If no $u \in \mathcal{A}(z)$ exists with $u \lessdot_{\mathcal{A}} v$ then $v = \alpha_{\min}(z)$.

(b) If no $w \in \mathcal{A}(z)$ exists with $v \lessdot_{\mathcal{A}} w$ then $v = \alpha_{\max}(z)$.

Proof. Assume no $u \in \mathcal{A}(z)$ exists with $u \lessdot_{\mathcal{A}} v$. Lemmas 6.4 and 6.11 then imply that if $v^{-1}(i+1) > v^{-1}(i+2)$ for some $i \in \mathbb{Z}$ then $v^{-1}(i) < v^{-1}(i+2)$. Therefore $v = [b_1, a_1, b_2, a_2, \ldots, b_l, a_l]^{-1}$ for some numbers with $a_1 < a_2 < \cdots < a_l$, $a_i \leq b_i$ for all i, and $b_i \neq b_j$ for all $i \neq j$. Thus $v = \alpha_{\min}(y)$ for the involution $y \in \tilde{I}_n$ with $y(a_i) = b_i$ for $i \in [l]$ by Corollary 6.9. Since $\alpha_{\min}(y) \in \mathcal{A}(y)$ we must have $y = z$. This proves part (a). The proof of part (b) is similar, proceeding from the observation if no $w \in \mathcal{A}(z)$ exists with $v \lessdot_{\mathcal{A}} w$ then $v^{-1}(i) > v^{-1}(i+1)$ implies $v^{-1}(i) < v^{-1}(i+2)$. $\square$

Fix $z \in \tilde{I}_n$ and $w \in \tilde{S}_n$. Define two sets

$$\underline{L}_z = \{a \in \mathbb{Z} : a \leq z(a)\} \quad \text{and} \quad R_z = \{b \in \mathbb{Z} : z(b) < b\}.$$

Let $\mathrm{Inv}_{\mathcal{A}}(w; z)$ be the set of pairs $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ with $w(p) > w(q)$ and either $p < q$ and $\{p, q\} \subset \underline{L}_z$, or $p > q$ and $\{p, q\} \subset R_z$. Note that $\mathrm{Inv}_{\mathcal{A}}(w; z)$ is closed under the relation generated by $(i, j) \sim (i + n, j + n)$. Define $\mathrm{rank}_{\mathcal{A}}(w; z)$ as the number of equivalence classes under this relation (which a priori could be infinite) in the set difference $\mathrm{Inv}_{\mathcal{A}}(w; z) \setminus \mathrm{Inv}_{\mathcal{A}}(\alpha_{\min}(z); z)$.

Remark. All of our arguments would go through if we redefined $\underline{L}_z$ to be $\{a \in \mathbb{Z} : a < z(a)\}$ and R_z to be $\{b \in \mathbb{Z} : z(b) \leq b\}$. It makes no difference which set contains the fixed points of z.

Example 6.13. Suppose $z = t_{1,8} t_{2,7} \in \tilde{I}_4$ so that $\alpha_{\min}(z) = [4, 6, 1, -1]$ and $w = [6, 4, -1, 1] \in \mathcal{A}(z)$ by Example 6.10. Then $\underline{L}_z = \{1, 2\} + 4\mathbb{Z}$ and $R_z = \{3, 4\} + 4\mathbb{Z}$, so we have

$$\mathrm{Inv}_{\mathcal{A}}(\alpha_{\min}(z); z) = \{(i + mn, j + ln) : i, j \in \{3, 4\} \text{ and } l < m\}$$

and $\mathrm{Inv}_{\mathcal{A}}(w; z) = \{(1 + mn, 2 + mn) : m \in \mathbb{Z}\} \sqcup \{(4 + mn, 3 + mn) : m \in \mathbb{Z}\} \sqcup \mathrm{Inv}_{\mathcal{A}}(\alpha_{\min}(z); z)$. Therefore $\mathrm{rank}_{\mathcal{A}}(w; z) = 2$.

Finally, let $<_{\mathcal{A}}$ be the transitive closure of $\lessdot_{\mathcal{A}}$. The following generalises [8, Theorem 6.10].

Theorem 6.14. Let $z \in \tilde{I}_n$. Restricted to the set $\mathcal{A}(z)$, the relation $<_{\mathcal{A}}$ is a bounded, graded partial order with rank function $w \mapsto \mathrm{rank}_{\mathcal{A}}(w; z)$. Moreover, it holds that

$$\mathcal{A}(z) = \left\{ w \in \tilde{S}_n : \alpha_{\min}(z) \leq_{\mathcal{A}} w \right\} = \left\{ w \in \tilde{S}_n : w \leq_{\mathcal{A}} \alpha_{\max}(z) \right\}. \tag{6.2}$$

Remark. Let $\sim_{\mathcal{A}}$ be the symmetric closure of $<_{\mathcal{A}}$. The theorem implies that each set $\mathcal{A}(z)$ for $z \in \tilde{I}_n$ is an equivalence class under $\sim_{\mathcal{A}}$. This weaker statement is equivalent to the main results of [12, 23] specialised to type $\tilde{A}$; see [23, Corollary 1.6] or [12, Theorem 1.2].

Proof. Fix $v \in \mathcal{A}(z)$ and let $\mathrm{rank}(v) = \mathrm{rank}_{\mathcal{A}}(v; z)$. Since $\mathcal{A}(z)$ is a subset of the finite set of elements $w \in \tilde{S}_n$ with $\ell(w) = \hat{\ell}(z)$, Lemma 6.12 implies that $\alpha_{\min}(z) \leq_{\mathcal{A}} v \leq_{\mathcal{A}} \alpha_{\max}(z)$. Equation (6.2) therefore follows from Corollary 6.6, though it remains to show that $<_{\mathcal{A}}$ is a partial order.

To prove this, we argue (i) that $\mathrm{rank}(v)$ is finite, (ii) that $\mathrm{Inv}_{\mathcal{A}}(\alpha_{\min}(z)) \subset \mathrm{Inv}_{\mathcal{A}}(v)$, and (iii) that $(a, c) \in \mathcal{C}(z)$ whenever $c = u^{-1}(j) > u^{-1}(j + 1) = a$ for some $j \in \mathbb{Z}$. If $v = \alpha_{\min}(z)$ then these properties hold by Corollary 6.9. Otherwise, there exists an atom $u \in \mathcal{A}(z)$ with $u \lessdot_{\mathcal{A}} v$ by

Lemma 6.12. We may assume by induction that properties (i)-(iii) hold for u. By Lemma 6.4, we know that $u^{-1}[i : i+2] = cab$ and $v^{-1}[i : i+2] = bca$ where $a < b < c$ for some $i \in \mathbb{Z}$. Property (iii) implies that $(a, c) \in \mathcal{C}(z)$, so $\mathrm{Inv}_\mathcal{A}(v; z)$ is either

$$\mathrm{Inv}_\mathcal{A}(u; z) \sqcup \{(a + mn, b + mn) : m \in \mathbb{Z}\} \quad \text{or} \quad \mathrm{Inv}_\mathcal{A}(u; z) \sqcup \{(c + mn, b + mn) : m \in \mathbb{Z}\}$$

according to whether $b \leq z(b)$ or $z(b) < b$. We conclude from (i) and (ii) that $\mathrm{rank}(v) = \mathrm{rank}(u) + 1 < \infty$ and $\mathrm{Inv}_\mathcal{A}(\alpha_{\min}(z)) \subset \mathrm{Inv}_\mathcal{A}(v)$. Finally suppose $v^{-1}(j) > v^{-1}(j + 1)$ for some $j \in \mathbb{Z}$. If $u^{-1}[j : j+1] = v^{-1}[j : j+1]$ then $(v^{-1}(j + 1), v^{-1}(j)) \in \mathcal{C}(z)$ by property (iii) for u. Otherwise, Lemma 6.4 implies that for some integers $a < b < c$ we have one of the following:

- $u^{-1}[j + 1 : j + 3] = cab$ and $v^{-1}[j + 1 : j + 3] = bca$ and $u^{-1}(j) = v^{-1}(j)$.

- $u^{-1}[j - 2 : j] = cab$ and $v^{-1}[j - 2 : j] = bca$ and $u^{-1}(j + 1) = v^{-1}(j + 1)$.

- $u^{-1}[j - 1 : j + 1] = cab$ and $v^{-1}[j - 1 : j + 1] = bca$.

The first two cases cannot occur since they contradict Lemma 6.11, while in the third case we have $(v^{-1}(j + 1), v^{-1}(j)) = (a, c) = (u^{-1}(j), u^{-1}(j - 1)) \in \mathcal{C}(z)$ by induction. We conclude that the desired properties hold for all $v \in \mathcal{A}(z)$.

Since all atoms have properties (i)-(iii), we can repeat the argument in the preceding paragraph to deduce that $\mathrm{rank}(v) = \mathrm{rank}(u) + 1 < \infty$ whenever $u, v \in \mathcal{A}(z)$ have $u \lessdot_\mathcal{A} v$. This shows precisely that $<_\mathcal{A}$ restricted to $\mathcal{A}(z)$ is a graded partial order with rank function $w \mapsto \mathrm{rank}(w)$. $\square$

Corollary 6.15. Let $z \in \tilde{I}_n$ and $w \in \mathcal{A}(z)$.

(a) If $b = w^{-1}(i) > w^{-1}(i + 1) = a$ for some $i \in \mathbb{Z}$ then $(a, b) \in \mathcal{C}(z)$.

(b) It holds that $\mathrm{Inv}_\mathcal{A}(\alpha_{\min}(z); w) \subset \mathrm{Inv}_\mathcal{A}(w; z) \subset \mathrm{Inv}_\mathcal{A}(\alpha_{\max}(z); z)$.

(c) If $(a, b) \in \mathcal{C}(z)$ then $w(b) < w(a)$.

Proof. Parts (a) and (b) were shown in the proof of Theorem 6.14. It follows from part (a) that if part (c) holds for w then the same property holds for any $w' \in \tilde{S}_n$ with $w \lessdot_\mathcal{A} w'$. Since (c) is true for $w = \alpha_{\min}(z)$, we conclude from Theorem 6.14 that this property holds for all atoms of z. $\square$

Figure 1 shows an example of $(\mathcal{A}(z), <_\mathcal{A})$. The lattice structure evident in this picture appears to be typical; we have used a computer to check the following conjecture for $z \in \tilde{I}_n$ in the 333,307 cases when $0 < \hat{\ell}(z)n \leq 100$. As a graded lattice, $(\mathcal{A}(z), <_\mathcal{A})$ is not necessarily Eulerian, distributive, or semi-modular, but its Möbius function seems to always take values in $\{-1, 0, 1\}$.

Conjecture 6.16. The graded poset $(\mathcal{A}(z), <_\mathcal{A})$ is a lattice for all n and $z \in \tilde{I}_n$.

A permutation $w \in \tilde{S}_n$ is *321-avoiding* if no integers $a < b < c$ satisfy $w(a) > w(b) > w(c)$. An element $w \in \tilde{S}_n$ is *fully commutative* if we cannot write $w = us_is_{i+1}s_iv$ for any $u, v \in \tilde{S}_n$ and $i \in \mathbb{Z}$ with $\ell(w) = \ell(u) + \ell(v) + 3$. The following extends [8, Corollary 6.11] to affine type A.

Corollary 6.17. Let $z \in \tilde{I}_n$. The following are equivalent: (a) $|\mathcal{A}(z)| = 1$, (b) $\alpha_{\min}(z) = \alpha_{\max}(z)$, (c) $\alpha_R(z) = \alpha_L(z)$, (d) z is 321-avoiding, and (e) z is fully commutative.

Remark. Biagioli, Jouhet, and Nadeau [1, Proposition 3.3] have derived a length generating function for the involutions in $\tilde{S}_n$ with these equivalent properties.
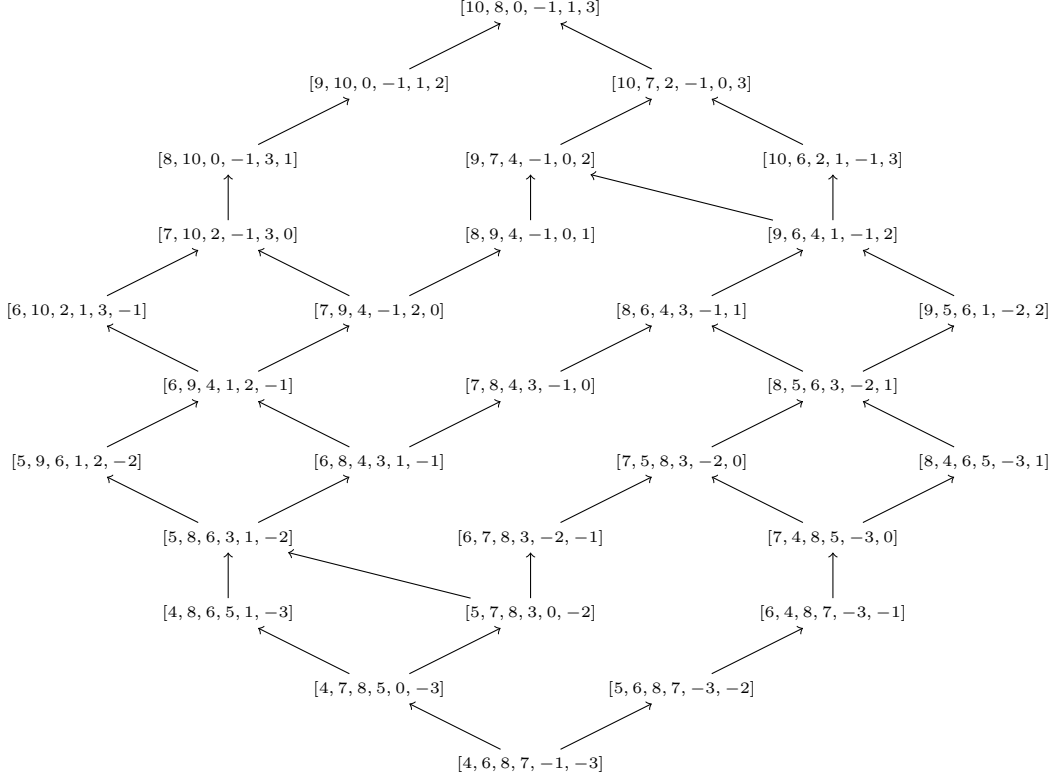
Figure 1: Hasse diagram of $(\mathcal{A}(z), <_{\mathcal{A}})$ for $z = t_{1,12}t_{2,11}t_{3,4} \in \tilde{I}_6$

Proof. The equivalence of (a), (b), and (c) is clear from Theorem 6.14. Properties (d) and (e) are equivalent for all affine permutations $z \in \tilde{S}_n$; see the results of Green [6, Theorem 2.7] or Lam [19, Proposition 35]. If z is fully commutative then it follows from [8, Proposition 7.12] that $|\mathcal{A}(z)| = 1$; this can also be shown by a direct argument. Conversely, if z is not 321-avoiding, then there must exist $a, a' \in \mathbb{Z}$ with $a < a' \leq z(a') < z(a)$, in which case the formulas in Corollary 6.9 show that $\alpha_{\min}(z) \neq \alpha_{\max}(z)$. Thus (e) $\Rightarrow$ (a) and (b) $\Rightarrow$ (d), which completes the proof. $\square$

7 Cycle removal process

A *doubly infinite sequence* $\mathsf{S} = (\ldots a_{-2}a_{-1}a_0a_1a_2 \ldots)$ is an orbit of a map $f : \mathbb{Z} \to X$ under the action of $\mathbb{Z}$ by translation. If for some $m \in \mathbb{Z}$ we have $f(m + i) = a_i$ for all $i \in \mathbb{Z}$, then we call S the *string representation* of f. We sometimes use the term *string* as a shorthand for doubly infinite sequence. By Proposition-Definition 5.10, no two elements of $\tilde{S}_n$ have the same string representation.

For the remainder of this section, we fix the following notation. Let $y \in \tilde{I}_n$ and $w \in \tilde{S}_n$ with $w^{-1} \in \mathcal{A}(y)$, and write S for the string representation of w. We construct a sequence of strings $\mathsf{S}_0, \mathsf{S}_1, \ldots, \mathsf{S}_p$ and pairs $(a_1, b_1), (a_2, b_2), \ldots, (a_p, b_p) \in \mathbb{Z} \times \mathbb{Z}$ by the following algorithm.

Definition 7.1 (Cycle removal process). Start with $\mathsf{S}_0 = \mathsf{S}$. Then, for each $i = 0, 1, 2, \ldots$, define (a_{i+1}, b_{i+1}) to be any pair of integers with $a_{i+1} < b_{i+1}$ such that $b_{i+1}a_{i+1}$ appears as a consecutive

subsequence of $\mathtt{S}_i$. If no such pair exists, so that $\mathtt{S}_i$ is an increasing sequence, then the process terminates with $p = i$. Otherwise, we form $\mathtt{S}_{i+1}$ by removing all numbers congruent to a_{i+1} or b_{i+1} modulo n from $\mathtt{S}_i$ and continue.

We refer to this procedure as the *cycle removal process* for w. The process must terminate in a finite number of steps since each iteration removes two congruence classes of integers modulo n from the current string. By construction the last string $\mathtt{S}_p$ is a strictly increasing sequence.

Example 7.2. Suppose $y = t_{1,8}t_{2,7} \in \tilde{I}_4$ and $w = [3, 4, -3, -2] \in \mathcal{A}(y)^{-1}$, so that

$$\mathtt{S} = (\ldots, -5, -4, -11, -10, -1, 0, -7, -6, 3, 4, -3, -2, 7, 8, 1, 2, 11, 12, 5, 6, \ldots).$$

For our first cycle, choose $(a_1, b_1) = (1, 8)$. Then

$$\mathtt{S}_1 = (\ldots, -5, -10, -1, -6, 3, -2, 7, 2, 11, 6, \ldots).$$

For our second cycle, choose $(a_2, b_2) = (2, 7)$. Removing all numbers congruent to 2 or 7 modulo 4 from $\mathtt{S}_1$ leaves the empty string $\mathtt{S}_2 = \emptyset$, so our process terminates with $p = 2$.

Theorem 7.3. No matter how the cycle removal process for w is carried out, $\ell'(y) = p$ and $\mathcal{C}(y) = \{(a_i + mn, b_i + mn) : i \in [p], \ m \in \mathbb{Z}\}$, and $\mathtt{S}_p$ is the increasing sequence of fixed points of y.

Proof. Fix $j \in [p]$ and suppose $a, b \in \mathbb{Z}$ are such that $a - a_j = b - b_j \in n\mathbb{Z}$, so that $a < b$. Then ba must be a consecutive subsequence of $\mathtt{S}_{j-1}$, so there are integers $c_1, c_2, \ldots, c_m$ such that $bc_1c_2 \cdots c_m a$ is a consecutive subsequence of $\mathtt{S} = \mathtt{S}_0$. The number of such integers must satisfy $m \leq n - 2$, since if $m = n - 1$ then we would have $a = b + n \not< b$, while if $m \geq n$ then $c_n = b + n$ would appear between b and a in $\mathtt{S}_{j-1}$. It follows that $a, b, c_1, c_2, \ldots, c_m$ belong to distinct congruences classes modulo n.

Since none of the numbers between b and a in $\mathtt{S}$ are present in $\mathtt{S}_{j-1}$, there is a subset $I \subset [j-1]$ and integers $g_i, h_i \in \mathbb{Z}$ with $g_i - a_i = h_i - b_i \in n\mathbb{Z}$ for each $i \in I$ such that $\{c_1, c_2, \ldots, c_m\} = \{g_i : i \in I\} \cup \{h_i : i \in I\}$. Note, as such, that $m = 2|I|$. Let $\sim$ be the equivalence relation on finite integer sequences generated by setting $\alpha \sim \beta$ whenever it is possible to obtain β from α by replacing a consecutive subsequence of the form $e_2e_3e_1$ by $e_3e_1e_2$ where $e_1 < e_2 < e_3$. We claim that $bc_1c_2 \cdots c_m a \sim h_{i_1}g_{i_1}h_{i_2}g_{i_2} \cdots h_{i_k}g_{i_k}ba$ where $i_1, i_2, \ldots, i_k$ are the elements of I in some order. Applying this property inductively with (a, b) replaced by the pairs (g_i, h_i), we deduce that at least $bc_1c_2 \cdots c_m a \sim bh_{i_1}g_{i_1}h_{i_2}g_{i_2} \cdots h_{i_k}g_{i_k}a$. Corollary 6.6 and Lemma 6.11 imply that no sequence equivalent to $bc_1c_2 \cdots c_m a$ under $\sim$ contains a consecutive subsequence $e_3e_2e_1$ with $e_1 < e_2 < e_3$. From this property, it is easy to see that $bh_{i_1}g_{i_1}h_{i_2}g_{i_2} \cdots h_{i_k}g_{i_k}a$ is equivalent under $\sim$ to a sequence of the same form with $g_{i_k} = \max\{g_i : i \in I\}$, as well as to a (possibly different) sequence of the same form with $h_{i_1} = \min\{h_i : i \in I\}$. But then it follows that

$$g_i < a < b < h_i \quad \text{for all } i \in I \tag{7.1}$$

so $bh_{i_1}g_{i_1}h_{i_2}g_{i_2} \cdots h_{i_k}g_{i_k}a \sim h_{i_1}g_{i_1}h_{i_2}g_{i_2} \cdots h_{i_k}g_{i_k}ba$ as desired. We conclude by Corollary 6.6 that there exists $v \in \mathcal{A}(y)^{-1}$ whose string representation contains ba as a consecutive subsequence, so $(a, b) \in \mathcal{C}(y)$ by Corollary 6.15(a). Hence $(a_j, b_j) \in \mathcal{C}(y)$ for each $j \in [p]$. Corollary 6.15(c) implies that each element of the increasing sequence $\mathtt{S}_p$ is a fixed point of y, so the theorem follows. $\qed$

Given $a, b, c, \ldots$, we say that a string "has the form $-a-b-c- \ldots -$" as a shorthand for the statement that the string contains $(a, b, c, \ldots)$ as a not necessarily consecutive subsequence. Maintain the notation of Definition 7.1 and Theorem 7.3 in the following corollaries.

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Corollary 7.4. If $\mathtt{S}$ has the form $-b'-b-a-a'-$ where $(a,b),(a',b') \in \mathcal{C}(y)$, then $a < a' < b' < b$.

Proof. This is equivalent to property (7.1) shown in the proof of Theorem 7.3. $\square$

Corollary 7.5. None of $\mathtt{S}_0, \mathtt{S}_1, \ldots, \mathtt{S}_p$ contains a consecutive subsequence cba where $a < b < c$.

Proof. If any string $\mathtt{S}_i$ had a consecutive subsequence of the form cba where $a < b < c$, then Theorem 7.3 would imply that both (a,b) and (b,c) belong to $\mathcal{C}(y)$, which is impossible. $\square$

We can use the preceding results to give a more explicit set of conditions characterising the elements of $\mathcal{A}(y)$. Restricted to involutions in the finite symmetric group $S_n \subset \tilde{S}_n$, the following theorem is equivalent to a result of Can, Joyce, and Wyser [5, Theorem 2.5].

Theorem 7.6. Let $y \in \tilde{I}_n$ and let $\mathtt{S}$ be the string representation of an affine permutation $w \in \tilde{S}_n$. Then $w^{-1} \in \mathcal{A}(y)$ if and only if the following properties hold:

1. If $X, Y \in \mathbb{Z}$ are fixed points of y with $X < Y$ then $\mathtt{S}$ has the form $-X-Y-$.

2. If $(a,b) \in \mathcal{C}(y)$ then $\mathtt{S}$ has the form $-b-a-$.

3. If $X \in \mathbb{Z}$ is a fixed point of y and $(a,b) \in \mathcal{C}(y)$ then:

   (a) $\mathtt{S}$ does not have the form $-b-X-a-$.
   (b) If $X < a < b$ then $\mathtt{S}$ has the form $-X-b-a-$.
   (c) If $a < b < X$ then $\mathtt{S}$ has the form $-b-a-X-$.

4. If $(a,b),(a',b') \in \mathcal{C}(y)$ and $a < a' < b' < b$ then $\mathtt{S}$ has one of the forms

$$-b-a-b'-a'- \qquad \text{or} \qquad -b'-b-a-a'- \qquad \text{or} \qquad -b'-a'-b-a-$$

while if $a < a'$ and $b < b'$ then $\mathtt{S}$ has the form $-b-a-b'-a'-$.

Proof. First suppose $w^{-1} \in \mathcal{A}(y)$. Property 1 holds by Theorem 7.3, and property 2 follows by Corollary 6.15(c). Property 3(a) holds since ba must occur as a consecutive subsequence of some intermediate string $\mathtt{S}_i$ in the cycle removal process for w. Following this observation, one can deduce properties 3(b) and 3(c) from Corollary 6.15(b). By similar reasoning, if $(a,b),(a',b') \in \mathcal{C}(y)$ then $\mathtt{S}$ cannot have the form $-b-b'-a-a'-$, so property 4 follows from Corollary 6.15(b) and Corollary 7.4.

Conversely, suppose $\mathtt{S}$ has the given properties. Then b appears to the left of a in $\mathtt{S}$ whenever $(a,b) \in \mathcal{C}(y)$, and in this case no fixed points of y appear between b and a in $\mathtt{S}$. Moreover, if $(a,b),(a',b') \in \mathcal{C}(y)$ and either a or b appears between b' and a' in $\mathtt{S}$, then $\mathtt{S}$ has the form $-b'-b-a-a'-$ and $a < a' < b' < b$. Recall the definition of the relation $\sim_{\mathcal{A}}$ from the remark following Theorem 6.14. By an inductive argument similar to the one in the proof of Theorem 7.3, we deduce from the preceding observations that $w^{-1} \sim_{\mathcal{A}} (w')^{-1}$ for a permutation $w' \in \tilde{S}_n$ whose string representation $\mathtt{S}'$ also satisfies the given conditions and has that property that ba appears as a consecutive subsequence for each $(a,b) \in \mathcal{C}(y)$. Property 1 implies that all fixed points of y appear in order in $\mathtt{S}'$. If cab is a consecutive subsequence of $\mathtt{S}'$ where $(a,c) \in \mathcal{C}(y)$ and $b = y(b)$, then property 3 implies that $a < b$. If bca is a consecutive subsequence of $\mathtt{S}'$ where $(a,c) \in \mathcal{C}(y)$ and $b = y(b)$, then

property 3 implies that either $b < a$ or $a < b < c$. Finally, if $(a, b), (a', b') \in \mathcal{C}(y)$ where $a < a'$, then S' may contain $bab'a'$ or $b'baa'$ or $b'a'ba$ as a consecutive subsequence, and in the second two cases, property 4 implies that $a < a' < b' < b$. Given these observations, it is straightforward to show that $(w')^{-1} \sim_{\mathcal{A}} (w'')^{-1}$ for a permutation $w'' \in \tilde{S}_n$ whose string representation S'' contains ba as a consequence subsequence for each $(a, b) \in \mathcal{C}(y)$, and has the property that a appears to the left of a' whenever $a < a'$ and $a \leq y(a)$ and $a' \leq y(a')$. By Corollary 6.9, the unique such permutation is $w'' = \alpha_{\min}(y)^{-1}$, so by Theorem 6.14 we have $w^{-1} \in \mathcal{A}(y)$. $\square$

Fix a subset $E \subset [n]$ of size m. Let $\phi_E : [m] \to E$ and $\psi_E : E \to [m]$ be order-preserving bijections. If $w \in S_n$ is a permutation in a finite symmetric group, then its *standardisation* is the permutation $[w]_E = \psi_{w(E)} \circ w \circ \phi_E \in\in S_m$. If $w^2 = 1$ and $w(E) = E$, then $([w]_E)^2 = 1$.

The Demazure product $\circ$ on $\tilde{S}_n$ restricts to an associative product $S_n \times S_n \to S_n$ and each involution $y \in I_n = \tilde{I}_n \cap S_n$ has $\mathcal{A}(y) \subset S_n$. Can, Joyce, and Wyser's description of $\mathcal{A}(y)$ for $y \in I_n$ in [5] implies that $w \in S_n$ belongs to $\mathcal{A}(y)$ if and only if $[w]_E \in \mathcal{A}([y]_E)$ for all subsets $E \subset [n]$ which are invariant under y and contain at most two y-orbits; cf. [9, Corollary 3.19]. This "local" criterion for membership in $\mathcal{A}(y)$ was an important tool in the proofs of the main results in [9].

Example 7.7. The atoms of the reverse permutations in S_2, S_3, and S_4 are given by $\mathcal{A}(21) = \{21\}$, $\mathcal{A}(321) = \{312, 231\}$, and $\mathcal{A}(4321) = \{4213, 3412, 2431\}$. If $y = n \cdots 321 \in I_n$ and $E \subset [n]$ is y-invariant with at most two orbits, then $[y]_E$ is either 21, 321, or 4321, and one can check that requiring $[w]_E \in \mathcal{A}([y]_E)$ imposes precisely the sort of conditions we saw in Theorem 7.6.

This result can be extended to the affine case, provided we give the right definition of the standardisation of an affine permutation. Fix a subset $E \subset \mathbb{Z}$ with $|(E + n\mathbb{Z}) \cap [n]| = m$, and define $\tilde{\phi}_{E,n}$ as the unique order-preserving $\mathbb{Z} \to E + n\mathbb{Z}$ with $\tilde{\phi}_{E,n}([m]) \subset [n]$.

Lemma 7.8. If $w \in \tilde{S}_n$ then there is a unique order-preserving bijection $\tilde{\psi}_{E,w} : w(E) + n\mathbb{Z} \to \mathbb{Z}$ such that $\tilde{\psi}_{E,w} \circ w \circ \tilde{\phi}_{E,n} \in \tilde{S}_m$. If $w \in \tilde{I}_n$ and $w(E) = E$ then $\tilde{\phi}_{E,n}$ and $\tilde{\psi}_{E,w}$ are inverses.

Proof. Let $w \in \tilde{S}_n$. The first assertion follows on checking that the images of $1, 2, \ldots, m$ under $w \circ \tilde{\phi}_{E,n}$ represent the distinct congruences classes in $w(E) + n\mathbb{Z}$ modulo n. For the second assertion, one uses the fact that if w is an involution, then for each $i \in [n]$ with $w(i) \neq i$, there is a unique $j \in [n]$ with $w(i) \equiv j \pmod{n}$ and $w(i) + w(j) = i + j$. The details are left as an exercise. $\square$

Given $w \in \tilde{S}_n$ and $E \subset \mathbb{Z}$ with $|(E + n\mathbb{Z}) \cap [n]| = m$, define $[w]_{E,n} = \tilde{\psi}_{E,w} \circ w \circ \tilde{\phi}_{E,n} \in \tilde{S}_m$. We refer to $[w]_{E,n}$ as the *(affine) standardisation* of w. One has $[w]_{E,n} = [w]_{E+mn,n}$ for all $m \in \mathbb{Z}$. When n is clear from context, we write $[w]_E$ instead of $[w]_{E,n}$. This is consistent with our earlier notation since if $E \subset [n]$ and $w \in S_n \subset \tilde{S}_n$, then $\tilde{\phi}_{E,n}|_{[m]} = \phi_E$ and $\tilde{\psi}_{E,w}|_E = \psi_{w(E)}$.

Corollary 7.9. If $E \subset \mathbb{Z}$, $y \in \tilde{I}_n$, and $y(E) = E$, then $[y]_E \in \tilde{I}_n$.

Proof. By Lemma 7.8, both $\tilde{\psi}_{E,y} \circ \tilde{\phi}_{E,n}$ and $\tilde{\phi}_{E,n} \circ \tilde{\psi}_{E,y}$ are identity maps so $([y]_E)^2 = 1$. $\square$

Example 7.10. Standardisation has a simple interpretation in terms of winding diagrams. If $E = y(E) \subset \mathbb{Z}$ then the winding diagram of $[y]_E$ is formed from that of $y \in \tilde{I}_n$ by erasing the vertices in $[n] \setminus (E + n\mathbb{Z})$ along with their incident edges, and then replacing the numbers that

remain by $1, 2, \ldots, m = |(E + n\mathbb{Z}) \cap [n]|$ in order. For example, if $n = 8$ and $E = \{2, 4, 6, 7, 8\}$, then
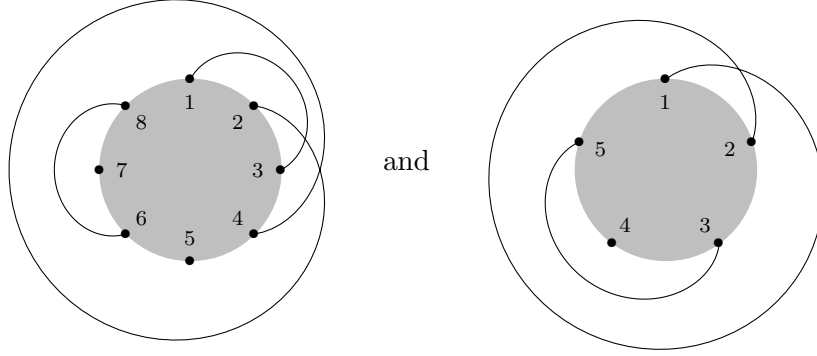
represent $y = t_{1,3} t_{2,12} t_{6,8} \in \tilde{I}_8$ and $[y]_E = t_{1,7} t_{3,5} \in \tilde{I}_5$, respectively.

The following is a corollary of Theorem 7.6 via the preceding lemmas.

Corollary 7.11. Let $y \in \tilde{I}_n$, $w \in \tilde{S}_n$, and $X = \{1, y(1), \ldots, n, y(n)\}$. The following are equivalent:

(a) $w \in \mathcal{A}(y)$.

(b) $[w]_E \in \mathcal{A}([y]_E)$ for each subset $E \subset X$ with $y(E) = E$.

(c) $[w]_E \in \mathcal{A}([y]_E)$ for each subset $E \subset X$ with $y(E) = E$ and containing at most two y-orbits.

Proof. If $E \subset \mathbb{Z}$ has $y(E) = E$, then $\mathcal{C}([y]_E)$ consists of the pairs $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ such that $(a, b) \in \mathcal{C}(y)$ for $a = \tilde{\phi}_{E,n}(i)$ and $b = \tilde{\phi}_{E,n}(j)$. If $w \in \tilde{S}_n$, then applying $\tilde{\phi}_{E,n}$ to each term in the string representation of $([w]_E)^{-1}$ gives the same thing as removing all numbers not in $E + n\mathbb{Z}$ from the string representation of w^{-1}. Since $\tilde{\phi}_{E,n}$ is order-preserving, it follows from Theorem 7.6 that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). The conditions in Theorem 7.6 each apply to the relative ordering, within the string representation of w^{-1}, of numbers from a set of at most two cycles of y, so (c) $\Rightarrow$ (a). $\qquad \square$

8 Covering transformations

The sets $\mathcal{A}(z)$ for $z \in \tilde{I}_n$ are closely related to the restriction of the Bruhat order $<$ on $\tilde{S}_n$ to $\tilde{I}_n$. Recall that if $z \in \tilde{I}_n$ then $\hat{\ell}(z) = \frac{1}{2}(\ell(z) + \ell'(z)) = \ell(w)$ for all $w \in \mathcal{A}(z)$, where ℓ' is given by Definition 2.5. The following statements derive from results of Hultman [14, 15]; see [9, §3.1].

Proposition 8.1. The poset $(\tilde{I}_n, <)$ is graded with rank function $\hat{\ell}$.

Lemma 8.2. If $y, z \in \tilde{I}_n$ and $w \in \mathcal{A}(z)$. Then $y \leq z$ if and only if some $v \in \mathcal{A}(y)$ has $v \leq w$.

In the usual Bruhat order on $\tilde{S}_n$ all covering relations have the form $w \lessdot wt$ where t is a reflection. Describing the covering relations in the subposet $(\tilde{I}_n, <)$ is a more delicate problem, but one we can attack using Lemma 8.2 and the results of the previous two sections.

Incitti [17, 18] solves the analogue of this problem for involutions in the finite Weyl groups of type A, B, and D, by associating to each reflection a *covering transformation* to play the role of right multiplication. In this section, we define our own covering transformations $\tau_{ij}^n : \tilde{I}_n \to \tilde{I}_n$. Restricted to involutions in $S_n \subset \tilde{S}_n$, these operators will coincide with the maps τ_{ij} in [9], which are themselves just a different notation for the maps ct_{ij} in [17].

Definition 8.3. Let $\sim_{\text{graph}}$ be the equivalence relation on vertex-coloured graphs with integer vertices in which $\mathcal{G} \sim_{\text{graph}} \mathcal{H}$ if and only there exists a graph isomorphism $\mathcal{G} \to \mathcal{H}$ which is an order-preserving bijection.

Definition 8.4. Fix $y \in \tilde{I}_n$ and $i, j \in \mathbb{Z}$ with $i < j \not\equiv i \pmod{n}$. Define $\mathcal{G}_{ij}(y)$ as the graph with vertex set $\{i, j, y(i), y(j)\}$ and edge set $\{\{i, y(i)\}, \{j, y(j)\}\} \setminus \{\{i\}, \{j\}\}$, in which the vertices i and j are coloured white and all other vertices are coloured black. Let k be the number of vertices in $\mathcal{G}_{ij}(y)$ and define $\mathcal{D}_{ij}(y)$ as the unique vertex-coloured graph on $[k]$ with $\mathcal{D}_{ij}(y) \sim_{\text{graph}} \mathcal{G}_{ij}(y)$.

We use these definitions to simplify our notation. There are only 20 possibilities for $\mathcal{D}_{ij}(y)$, which we draw by arranging the vertices $1, 2, \ldots, k$ in order from left to right, using $\circ$ for the white vertices and $\bullet$ for the black vertices.

Example 8.5. If $y, z \in \tilde{I}_n$ are such that $y(i) < j = y(j) < i$ and $i < z(j) < j < z(i)$ then

$$\mathcal{D}_{ij}(y) = \bullet\!\frown\!\circ\,\circ \qquad \text{and} \qquad \mathcal{D}_{ij}(z) = \circ\frown\!\bullet\frown\!\circ\,\bullet \ .$$

Definition 8.6. Fix $y \in \tilde{I}_n$ and $i, j \in \mathbb{Z}$ with $i < j \not\equiv i \pmod{n}$. Set $t_{ii} = t_{jj} = 1$ and define

$$(\circ, \circ) = t_{ij}, \quad (\circ, \bullet) = t_{i, y(j)}, \quad (\bullet, \circ) = t_{y(i), j}, \quad \overline{y} = \begin{cases} y \cdot t_{i, y(i)} & \text{if } i \equiv y(j) \pmod{n} \\ y \cdot t_{i, y(i)} \cdot t_{j, y(j)} & \text{if } i \not\equiv y(j) \pmod{n}. \end{cases}$$

Using this notation, let $\tau_{ij}^n(y) \in \tilde{I}_n$ be given as follows:

$$\tau_{ij}^n(y) = \begin{cases} (\circ, \circ) \cdot y \cdot (\circ, \circ) & \text{if } \mathcal{D}_{ij}(y) \text{ is } \circ\ \bullet\!\frown\!\bullet,\ \bullet\!\frown\!\circ\ \circ,\ \bullet\!\frown\!\bullet\ \bullet\!\frown\!\bullet,\ \circ\frown\!\circ\frown\!\bullet\ \text{or}\ \bullet\frown\!\bullet\frown\!\circ\,\circ \\ (\circ, \bullet) \cdot y \cdot (\circ, \bullet) & \text{if } \mathcal{D}_{ij}(y) \text{ is } \circ\ \bullet\!\frown\!\circ \\ (\bullet, \circ) \cdot y \cdot (\bullet, \circ) & \text{if } \mathcal{D}_{ij}(y) \text{ is } \circ\!\frown\!\bullet\ \circ \\ (\circ, \bullet) \cdot y \cdot (\circ, \bullet) & \text{if } \mathcal{D}_{ij}(y) \text{ is } \circ\frown\!\bullet\frown\!\circ\ \text{and } i \not\equiv y(j) \pmod{n} \\ (\circ, \circ) \cdot \overline{y} & \text{if } \mathcal{D}_{ij}(y) \text{ is } \circ\frown\!\bullet\frown\!\circ\ \text{and } i \equiv y(j) \pmod{n} \\ (\circ, \circ) \cdot \overline{y} & \text{if } \mathcal{D}_{ij}(y) \text{ is } \circ\!\frown\!\bullet\ \circ\!\frown\!\bullet \\ (\circ, \bullet) \cdot \overline{y} & \text{if } \mathcal{D}_{ij}(y) \text{ is } \circ\!\frown\!\bullet\ \circ\frown\!\bullet \\ (\bullet, \circ) \cdot \overline{y} & \text{if } \mathcal{D}_{ij}(y) \text{ is } \bullet\!\frown\!\circ\ \bullet\!\frown\!\circ \\ (\circ, \circ) \cdot y & \text{if } \mathcal{D}_{ij}(y) \text{ is } \circ\ \circ \\ y & \text{otherwise.} \end{cases}$$

Remark. There is a lot to unpack here. We include a few remarks about our notation:

(a) Let i' be the number adjacent to i in $\mathcal{G}_{ij}(y)$, if one exists, and define j' similarly. Then $(\circ, \circ)$ transposes i and j (as well as $i + mn$ and $j + mn$ for all $m \in \mathbb{Z}$), while $(\circ, \bullet)$ transposes i and j' and $(\bullet, \circ)$ transposes i' and j. The unique order-preserving graph isomorphism $\mathcal{G}_{ij}(y) \to \mathcal{D}_{ij}(y)$ maps i and j to vertices labeled $\circ$ and i' and j' to vertices labeled $\bullet$.

(b) If $i \equiv y(j) \pmod{n}$ then $t_{i, y(i)} = t_{j, y(j)}$, so $\overline{y}$ is the unique element of $\tilde{S}_n$ which fixes each element of $\{i, j, y(i), y(j)\} + n\mathbb{Z}$ and which agrees with y at all integers outside this set.

(c) If $i \not\equiv y(j) \pmod n$ then both $\mathcal{G}_{ij}(y)$ and $\mathcal{G}_{ij}(\tau_{ij}^n(y))$ have vertex set $V = \{i, j, y(i), y(j)\}$. In this case, given V, the value of $\tau_{ij}^n(y)$ is uniquely determined by the graphs $\mathcal{D}_{ij}(y)$ and $\mathcal{D}_{ij}(\tau_{ij}^n(y))$. If $i \equiv y(j) \pmod n$ then $j \equiv y(i) \pmod n$ also holds, and $\mathcal{D}_{ij}(y)$ must be

$$\text{[diagram]} \quad \text{or} \quad \text{[diagram]} \quad \text{or} \quad \text{[diagram]} \quad \text{or} \quad \text{[diagram]}.$$

When this occurs $\mathcal{G}_{ij}(y)$ and $\mathcal{G}_{ij}(\tau_{ij}^n(y))$ may have different vertex sets; see Table 1.

$\mathcal{D}_{ij}(y)$ for $y \in \tilde{I}_n$ and $i < j \not\equiv i \pmod n$	$\mathcal{D}_{ij}(\tau_{ij}^n(y))$
[diagrams] if $i \equiv y(j) \pmod n$ / if $i \equiv y(j) \pmod n$	[diagrams]
[diagrams]	[diagrams]
[diagrams]	[diagrams]
[diagrams] if $i \not\equiv y(j) \pmod n$	[diagrams]
[diagrams] if $i \not\equiv y(j) \pmod n$	[diagrams]

Table 1: Possible values for $\mathcal{D}_{ij}(y)$ and $\mathcal{D}_{ij}(\tau_{ij}^n(y))$. If $\mathcal{D}_{ij}(y)$ does not match any of the listed cases, then $\tau_{ij}^n(y) = y$. Only in the second two cases do $\mathcal{D}_{ij}(y)$ and $\mathcal{D}_{ij}(\tau_{ij}^n(y))$ have different vertex sets.

Example 8.7. For $z = t_{1,3} t_{5,7} \in \tilde{I}_7$ we have

$$\tau_{1,4}^7 \left(\text{[diagram]} \right) = \text{[diagram]} \quad \text{and} \quad \tau_{1,14}^7 \left(\text{[diagram]} \right) = \text{[diagram]}.$$

Lemma 8.8. Let $y \in \tilde{I}_n$ and $i, j \in \mathbb{Z}$ with $i < j \not\equiv i \pmod n$. Then $y \leq \tau_{ij}^n(y)$.

Proof. Maintain the notation of Definition 8.6. In the cases when $\tau_{ij}^n(y) \in \{y, ty, tyt\}$ for some $t \in \{(\circ, \circ), (\circ, \bullet), (\bullet, \circ)\}$, the relation $y \leq \tau_{ij}^n(y)$ follows from Lemmas 2.3 and 2.9. If $\mathcal{D}_{ij}(y)$ is [diagram] and $i \equiv y(j) \pmod n$, then one can check using Lemma 2.3 that for any integer j' with

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$i < j' < y(j)$ and $j' \equiv j \pmod{n}$, we have $y < y \cdot t_{j'y(j)} < y \cdot t_{j'y(j)} \cdot t_{ij'} = \tau_{ij}^n(y)$. If $\mathcal{D}_{ij}(y)$ is ⌢• ⌢• and $i \equiv y(j) \pmod{n}$, then the same statement holds with $j' = y(i)$. In the remaining cases, the set $\{i, j, y(i), y(j)\}$ has four elements $a < b < c < d$ which represent distinct congruence classes modulo n, and we have $y < y \cdot t_{bc} = t_{ad} \cdot y < t_{ad} \cdot y \cdot t_{ab} < t_{ad} \cdot y \cdot t_{ab} \cdot t_{cd} = \tau_{ij}^n(y)$. $\qquad\square$

The following result is useful for determining when $\hat{\ell}(\tau_{ij}^n(y)) = \hat{\ell}(y) + 1$.

Proposition 8.9. Suppose $y \in \tilde{I}_n$ and $z = \tau_{ij}^n(y) \neq y$ for some integers $i < j \not\equiv i \pmod{n}$.

(a) Suppose $y(i) \leq i$ or $j \leq y(j)$. Then $\hat{\ell}(z) = \hat{\ell}(y) + 1$ if and only if $\ell(yt_{ij}) = \ell(y) + 1$.

(b) Suppose $i < y(i) < y(j) < j \equiv y(i) \pmod{n}$. Then $\hat{\ell}(z) = \hat{\ell}(y) + 1$ if and only if $y(j) = i + n$ and no $e \in \mathbb{Z}$ satisfies either pair of conditions

$$j - n < e < i + n \text{ and } i < y(e) < j \quad \text{or} \quad i < e < j - n \text{ and } i - n < y(e) < j.$$

(c) Suppose $i < y(j) < y(i) < j \equiv y(i) \pmod{n}$. Then $\hat{\ell}(z) = \hat{\ell}(y) + 1$ if and only if $y(j) = i + n$ and no $e \in \mathbb{Z}$ satisfies either pair of conditions

$$j - 2n < e < i + n \text{ and } i - n < y(e) < j \quad \text{or} \quad i < e < j - 2n \text{ and } i - 2n < y(e) < j.$$

Remark. If $z = \tau_{kl}^n(y) \neq y$ for any integers k, l, then it is always possible to find some other integers i, j such that $z = \tau_{ij}^n(y) = \tau_{kl}^n(y)$ and the hypotheses of (a), (b), or (c) hold.

Proof. Maintain the notation of Definition 8.6. First assume $y(i) \leq i$ or $j \leq y(j)$. If i and j are both fixed points of y, then $z = yt_{ij}$ and $\hat{\ell}(z) - \hat{\ell}(y) = \frac{1}{2}(1 + \ell(yt_{ij}) - \ell(y))$. If instead

$$\mathcal{D}_{ij}(y) \in \{\, \circ\ \circ\hspace{-0.3em}⌢\hspace{-0.3em}•\ ,\ •\hspace{-0.3em}⌢\hspace{-0.3em}\circ\ \circ\ ,\ •\hspace{-0.3em}⌢\hspace{-0.3em}\circ\ \circ\hspace{-0.3em}⌢\hspace{-0.3em}•\ ,\ \circ\hspace{-0.3em}⌢\hspace{-0.3em}\circ\hspace{-0.3em}•\hspace{-0.3em}•\ ,\ •\hspace{-0.3em}•\hspace{-0.3em}\circ\hspace{-0.3em}⌢\hspace{-0.3em}\circ\,\},$$

then $z = t_{ij}yt_{ij}$ and $\hat{\ell}(z) - \hat{\ell}(y) = \frac{1}{2}(\ell(z) - \ell(y))$, and it is straightforward, using Lemma 2.9, to check that $\ell(z) - \ell(y) = 2$ if and only if $\ell(yt_{ij}) = \ell(y) + 1$. If $i = y(i) < y(j) < j$ then

$$z = t_{ij'}yt_{ij'} = \tau_{ij}^n(y) = \tau_{ij'}^n(y)$$

for $j' = y(j)$, and $\ell(yt_{ij}) = \ell(yt_{ij'})$. Our claim that $\hat{\ell}(z) = \hat{\ell}(y) + 1$ if and only if $\ell(yt_{ij}) = \ell(y) + 1$ therefore follows from the previous case with j replaced by j'. If $i < y(j) < y(j) = j$ then the same conclusion follows likewise. Finally, if $i < i' < j' < j$ for $i' = y(i)$ and $j' = y(j)$, then

$$y < y \cdot t_{i'j} < y \cdot t_{i'j} \cdot t_{ii'} < y \cdot t_{i'j} \cdot t_{ii'} \cdot t_{jj'} = z$$

and $\hat{\ell}(z) - \hat{\ell}(y) = \frac{1}{2}(\ell(z) - \ell(y) - 1)$, and in this case it is again an exercise to check that $\ell(z) = \ell(y) + 3$ if and only if $\ell(yt_{ij}) = \ell(y) + 1$. In the symmetric case when $i' < i < j < j'$, we reach the same conclusion by a similar argument. This sketch suffices to prove part (a).

The proofs of (b) and (c) are similar. Assume we are in either case. Define $i' = y(i)$ and $j' = y(j)$. In case (b) let $i'' = i'$ and in case (c) let i'' be the smallest integer greater than i with $i'' \equiv j \pmod{n}$. As noted in the proof of Lemma 8.8, we then have $y < y \cdot t_{i''j'} < y \cdot t_{i''j'} \cdot t_{ii''} = z$ and $\hat{\ell}(z) - \hat{\ell}(z) = \frac{1}{2}(\ell(z) - \ell(y))$. We can only have $y \lessdot y \cdot t_{i''j'}$ if $j' = i + n$ and no integer e satisfies $i'' < e < i + n$ and $y(i'') < y(e) < j$. This translates to the first set of conditions in (b) and (c). One checks that the second pair of conditions in each case is equivalent to $y \cdot t_{i''j'} \lessdot y \cdot t_{i''j'} \cdot t_{ii''}$. $\qquad\square$

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Let $w \in \tilde{S}_n$ be an affine permutation. Refining the terminology of Theorem 7.6 slightly, we say that "w has the form $-a_1 a_2 \cdots a_k - b_1 b_2 \cdots b_l - c_1 c_2 \cdots c_m - \ldots -$" if the string representation of w contains $a_1 a_2 \cdots a_k$ and $b_1 b_2 \cdots b_l$ and $c_1 c_2 \cdots c_m$ and so forth as consecutive subsequences, with every a_i appearing to left of every b_j, every b_i appearing to the left of every c_j, and so on. The following statement is a constructive version of Theorem 1.1 from the introduction:

Theorem 8.10. Suppose $y, z \in \tilde{I}_n$ and $i, j \in \mathbb{Z}$ are such that $i < j \not\equiv i \pmod{n}$. If $w \in \mathcal{A}(y)$ and $w \lessdot wt_{ij} \in \mathcal{A}(z)$ then $z = \tau_{ij}^n(y)$.

Proof. Let $t = t_{ij}$ and suppose $w \in \tilde{S}_n$ is such that $w^{-1} \in \mathcal{A}(y)$, $\ell(tw) = \ell(w) + 1$, and $(tw)^{-1} \in \mathcal{A}(z)$, so that $\hat{\ell}(z) = \hat{\ell}(y) + 1$. It suffices to show that $z = \tau_{ij}^n(y)$. Our strategy will be to compare the cycle removal processes for w and tw, which differ in predictable ways, and then invoke Theorem 7.3. Note, from Lemma 2.3, that $\ell(tw) = \ell(w) + 1$ if and only if i appears to the left of j in the string representation of w and no integer $e \in \mathbb{Z}$ with $i < e < j$ appears between i and j in this string.

Define $t\gamma = (t(a), t(b))$ for $\gamma = (a, b) \in \mathbb{Z} \times \mathbb{Z}$, so that if $\gamma \in \mathcal{C}(y)$ then $t\gamma \in \mathcal{C}(tyt)$ if and only if $t(a) < t(b)$. Throughout, we let $p = \ell'(y)$ and suppose the cycle removal process for w outputs the sequence $\gamma_1, \gamma_2, \ldots, \gamma_p \in \mathcal{C}(y)$. First assume $i, j \in \{a, b, a', b'\}$ where $(a, b), (a', b') \in \mathcal{C}(y)$. The following useful observation then holds:

Lemma 8.11. If $t\gamma_1, t\gamma_2, \ldots, t\gamma_p$ all belong to $\mathcal{C}(tyt)$ then the cycle removal process for tw can be carried out to have these cycles as output, and it follows from Theorem 7.3 that $z = tyt$.

Continuing the proof of the theorem, suppose w has the form

$$-b-a-b'-a'- \qquad \text{where } a \not\equiv a' \pmod{n} \text{ and } b \not\equiv b' \pmod{n}.$$

One of the following cases must then occur:

- Suppose $i = a < a' = j$, so that tw has the form $-b-a'-b'-a-$ and $a < b'$. If $a' < b$ then it follows from Lemma 8.11 that $z = tyt$, so $a < a' < b < b'$ as otherwise Lemma 2.3 implies that $z = tyt < y$, contradicting Lemma 8.2. In this case we therefore have $\tau_{ij}^n(y) = tyt = z$.

  Assume instead that $b < a'$ so that $a < b < a' < b'$. We claim that one may assume that $\gamma_p = (a, b)$, i.e., that ba is the last subsequence removed in the cycle removal process for w. The only way this can fail is if some intermediate string S_k in the cycle removal process has the form $-YbaX-$ for a cycle $(X, Y) \in \mathcal{C}(y)$, which by Corollary 7.4 must have $a < X < Y < b < a'$. But if this occurs then w must have the form

  $$-Y-b-a-X-b'-a'- \qquad \text{or} \qquad -Y-b-a-b'-a'-X-.$$

  which contradicts either $\ell(tw) = \ell(w) + 1$ or Theorem 7.6. Our claim therefore holds. By similar reasoning, we deduce that no fixed point of y less than a' (respectively, greater than b) can appear to the right of a (respectively, to the left of b) in the string representation of w. It follows that the cycle removal process for tw can be carried out to have output $t\gamma_1, t\gamma_2, \ldots, t\gamma_{p-1}$, which by Theorem 7.3 implies that $z = t_{ab'} \cdot y \cdot t_{ab} \cdot t_{a'b'} = \tau_{ij}^n(y)$.

- If $i = b < b' < j$, then the situation is symmetric to the previous case, and we deduce that $z = \tau_{ij}^n(y)$ by similar arguments.

- If $i = b < a' = j$, so that $a < b < a' < b'$, then Lemma 8.11 implies that $z = tyt = \tau_{ij}^n(y)$.

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- Finally suppose $i = a < b' = j$, so that tw has the form $—b—b'—a—a'—$. By Theorem 7.6, we cannot have both $a' < a$ and $b' < b$, and the cases when $a < a' < b' < b$ or $a' < a < b < b'$ each lead to contradiction. For example, suppose $a < a' < b' < b$. By considering what happens if we try to remove the same consecutive subsequences during the cycle removal process for tw as we remove for w, we deduce from Theorem 7.3 that $(b', b) \in \mathcal{C}(z)$ and that either a is a fixed point of z or $(a, c) \in \mathcal{C}(z)$ for an integer $c < b'$. Both cases contradict Theorem 7.6 since a appears to the right of b' in tw. If $a' < a < b < b'$ then we obtain a similar contradiction by symmetric arguments.

  Instead suppose both $a < a'$ and $b < b'$. We claim that one may assume that some intermediate string S_k in the cycle removal process for w contains $bab'a'$ as a consecutive subsequence. To check this, first note that no fixed points of y can appear in the string representation of w between a and b' by Theorem 7.6 since $\ell(tw) = \ell(w) + 1$. By Corollary 7.4, the only way our claim can fail is if w has the form

  $$—Y—b—a—X—b'—a'— \qquad \text{or} \qquad —b—a—Y'—b'—a'—X'—$$

  for some $(X, Y) \in \mathcal{C}(y)$ with $a < X < Y < b$ or some $(X', Y') \in \mathcal{C}(y)$ with $a' < X' < Y' < b'$. Both cases would contradict the fact that $\ell(tw) = \ell(w) + 1$, so our claim holds.

  If $a' < b$, then it follows from our claim that we may construct the cycle removal processes for w and tw to be identical, except that at steps k and $k + 1$ the process for w removes the subsequences ba and then $b'a'$, while the process for tw removes $b'a$ and then ba'. In this case we deduce by Theorem 7.3 that $z = t_{aa'} \cdot y \cdot t_{aa'} = t_{i,y(j)} \cdot y \cdot t_{i,y(j)} = \tau_{ij}^n(y)$.

  Instead suppose $b < a'$. We then make the further claim that the index k can chosen to be $p - 1$, i.e., so that the subsequences ba and $b'a'$ are the last ones removed in the cycle removal process for w. This can only fail if an intermediate string in the cycle removal process for w has the form $—Ybab'a'X—$ for some $(X, Y) \in \mathcal{C}(y)$, but this never occurs since Corollary 7.4 would imply that $Y < b < a' < X$, contradicting $X < Y$. We can therefore assume $(a_{p-1}, b_{p-1}) = (a, b)$ and $(a_p, b_p) = (a', b')$. Under this hypothesis, it follows using Corollary 7.5 that the cycle removal process for tw can be carried out to have output $\gamma_1, \gamma_2, \ldots, \gamma_{p-2}, (a, b')$, whence by Theorem 7.3 we have $z = t_{ab'} \cdot y \cdot t_{ab} \cdot t_{a'b'} = \tau_{ij}^n(z)$.

This concludes the most complicated portion of our analysis. For the next case, continue to let $(a, b), (a', b') \in \mathcal{C}(y)$ and assume $i, j \in \{a, b, a', b'\}$, but now suppose w has the form

$$—b—a—b'—a'— \qquad \text{where } a \equiv a' \pmod{n} \text{ and } b \equiv b' \pmod{n}.$$

One of the following must then occur:

- Suppose $i = a < b' = j$. Since $\ell(tw) = \ell(w) + 1$, Theorem 7.6 implies that no fixed points of y appear between a and b' in the string representation of w. We may assume that $\gamma_p = (a, b)$, since if some string S_k in the cycle removal process for w had the form $—YbaX—Y'b'a'X'—$ for $(X, Y), (X', Y') \in \mathcal{C}(y)$, then Corollary 7.4 would imply that $a < X < Y < b < b'$, contradicting $\ell(tw) = \ell(w) + 1$. Hence the cycle removal process for tw can be carried out to have output $\gamma_1, \gamma_2, \ldots, \gamma_{p-1}, (a, b')$, so by Theorem 7.3 we have $z = t_{ab'} \cdot y \cdot t_{ab} = \tau_{ij}^n(y)$.

- If $i = b < a' = j$, then it follows from Lemma 8.11 that $z = t_{ba'} \cdot y \cdot t_{ba'} = \tau_{ij}^n(y)$.

Still with $(a,b),(a',b') \in \mathcal{C}(y)$ and $i,j \in \{a,b,a',b'\}$, finally suppose w has the form

$$-b-b'-a'-a-$$

so that $a \not\equiv a' \pmod{n}$ and $b \not\equiv b' \pmod{n}$. Since w has the form $-i-j-$, Theorem 7.6 implies that $a' < a < b < b'$ and either $i = a' < a = j$ or $i = b < b' = j$. It follows by Lemma 8.11 that $z = tyt$ so $(a,b'),(a',b) \in \mathcal{C}(z)$. This contradicts Theorem 7.6, however, since tw has the form $-b-b'-a-a'-$ or $-b'-b-a'-a-$. Hence this last case cannot occur.

By Theorem 7.6, the preceding discussion exhausts the possibilities for i and j when these numbers belong to distinct 2-cycles of y. We next consider the case when $i,j \in \{a,b,e\}$ where $e \in \mathbb{Z}$ is fixed point of y and $(a,b) \in \mathcal{C}(y)$. Suppose this happens and w has the form

$$-e-b-a-.$$

Since i must appear to the left of j, one of the following must occur:

- Suppose $i = e < a = j < b$ so that tw has the form $-a-b-e-$. Since $\ell(tw) = \ell(w)+1$, it follows by Theorem 7.6 that no fixed points of y less than a can appear between to the right of e in the string representation of w, and that likewise no fixed points greater than a can appear to the left of e. Hence, by Theorem 7.3 the cycle removal process for tw can be carried out to have output $t\gamma_1, t\gamma_2, \ldots, t\gamma_p$ so $z = tyt = \tau_{ij}^n(y)$.

- Suppose $a < i = e < b = j$ so that tw has the form $-b-e-a-$. We argue that this leads to contradiction. By considering what happens if we try to remove the same consecutive subsequences during the cycle removal process for tw as we remove for w, we deduce from Theorem 7.3 that $(a,e) \in \mathcal{C}(z)$ and that either b is a fixed point of z or $(e',b) \in \mathcal{C}(z)$ for an integer $e' > e$. Both cases contradict Theorem 7.6 since b appears to the left of e in tw.

- Finally suppose $i = e < a < b = j$ so that tw again has the form $-b-e-a-$. Since $\ell(tw) = \ell(w)+1$, it follows by Theorems 7.3 and 7.6 that no fixed points of y appear between e and a in the string representation of w. Moreover, by Theorem 7.6, all fixed points of y appearing in w to the right of a are bounded below by a, and all fixed points of y appearing to the left of e are bounded above by e. We can assume $\gamma_p = (a,b)$, since otherwise some intermediate string S_k in the cycle removal process of w must have the form $-e-YbaX-$ for a cycle $(X,Y) \in \mathcal{C}(y)$, in which case Corollary 7.4 would imply that $e < a < X < Y < b$, contradicting $\ell(tw) = \ell(w)+1$. Hence the penultimate string S_{p-1} contains eba as a consecutive subsequence, so the cycle removal process for tw can be carried out to have output $\gamma_1, \gamma_2, \ldots, \gamma_{p-1}, (e,b)$. By Theorem 7.3, we therefore have $z = t_{ea} \cdot y \cdot t_{ea} = \tau_{ij}^n(y)$.

If w has the form $-b-a-e-$ then $z = \tau_{ij}^n(y)$ follows by a symmetric argument. The only remaining possibility is that $y(i) = i < j = y(j)$. Since no fixed points can occur between i and j in the string representation of w, the cycle removal process for tw can be carried out to have output $\gamma_1, \gamma_2, \ldots, \gamma_p, (j,i)$, and hence $z = ty = \tau_{ij}^n(y)$. We conclude that $z = \tau_{ij}^n(y)$ in all cases. $\square$

Write $y \lessdot_I z$ for $y, z \in \tilde{I}_n$ if z covers y in the Bruhat order of $\tilde{S}_n$ restricted to $\tilde{I}_n$, i.e., if $\{w \in \tilde{I}_n : y \leq w < z\} = \{y\}$. Note that $y \lessdot z$ implies $y \lessdot_I z$ if $y, z \in \tilde{I}_n$, but not vice versa.

Corollary 8.12. If $y, z \in \tilde{I}_n$ then $y \lessdot_I z$ if and only if $\hat{\ell}(z) = \hat{\ell}(y)+1$ and $z = \tau_{ij}^n(y)$ for some i,j.

Proof. If $\hat{\ell}(z) = \hat{\ell}(y)+1$ and $z = \tau_{ij}^n(y)$ then Lemma 8.8 implies that $y \lessdot_I z$. Suppose conversely that $y \lessdot_I z$. By Proposition 8.1, we then have $y < z$ and $\hat{\ell}(z) = \hat{\ell}(y)+1$, so by Lemma 8.2 there exists $v \in \mathcal{A}(y)$ and $w \in \mathcal{A}(z)$ with $v \lessdot w$, so by Theorem 8.10 we have $z = \tau_{ij}^n(y)$ for some i,j. $\square$

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