STATISTICS ON SMALL GRAPHS

RICHARD J. MATHAR

ABSTRACT. We create the unlabeled or vertex-labeled graphs with up to 10 edges and up to 10 vertices and classify them by a set of standard properties: directed or not, vertex-labeled or not, connectivity, presence of isolated vertices, presence of multiedges and presence of loops. We present tables of how many graphs exist in these categories.

1. CLASSIFICATIONS

A finite graph on V vertices with E edges may be classified by some properties, which it either does have or does not:

- Each edge in a directed graph has one of two orientations. Edges in unoriented graphs do not have orientations. We reserve the tag `d` for the directed and `-d` for the undirected graphs. The adjacency matrices of undirected graphs are symmetric.
- Graphs may have at least one loop (loops are defined as edges that start and end at the same vertex), or may be loopless. The adjacency matrices of loopless graphs have zero trace. We reserve the tag `l` for the graphs with at least one loop and the tag `-l` for the loopless graphs.
- A multiedge is a collection of two ore more edges having identical endpoints [4, D7]. This implies that in a directed graph two edges of opposite sense do not yet establish a multiedge. We reserve the tag `m` for the graphs with at least one multiedge and `-m` for the others.
- A undirected graph is connected if one can walk from any vertex to any other vertex of the graph along edges. A directed graph is (weakly) connected if replacing each arc with an undirected edge (defining the underlying graph [4, D24]) reduces to a connected undirected graph. This implies that for the sake of weak connectivity it is not required that all arcs are traversed along their orientation to walk from one vertex to the other. We reserve the tag `c` for the graphs which are (weakly) connected and `-c` for the others.

  A directed graph is strongly connected if one can walk from any vertex to any other vertex of the graph along edges in the directions demanded by their orientation. We reserve the tag `C` for the digraphs which are strongly connected and `-C` for the others.

Date: September 27, 2017.
2010 *Mathematics Subject Classification.* Primary 05C30; Secondary 05C20, 05C75.
Key words and phrases. Graph Enumeration, Combinatorics.

- An isolated vertex is a vertex with no edge to any other vertex (so all its edges are loops). There is a loose relation with connectivity, because an isolated vertex in a graph with two or more vertices means the graph is disconnected. (There are disconnected graphs without isolated vertices... where each component contains at least two vertices.) We reserve the tag `i` for the graphs which have at least one isolated vertex and `-i` for the others. Therefore all graphs with $V = 1$ are getting the `i` tag.

There are no graphs with the following combinations of tags

- `d-Cci` A directed, weakly connected graph with at least one isolated vertex has only this vertex (because with two or more vertices the graph could not be connected), and therefore the graph must also be strongly connected. So the `-C` contradicts the other tags.
- `dC-c` If the directed graph is strongly connected, it is also weakly connected, so the `-c` tag contradicts the `C` tag.

There are some non-interesting cases, which are not tabulated explicitly:

- There is the case with the tags `dCci`: A directed strongly-connected graph with isolated vertices has only one vertex, so a table with these graphs counts at most 1 graph for any number of edges (which all are loops).
- Similarly there is the case with the tags `-dci`: An undirected connected graph with isolated vertices has only one vertex, so a table with these graphs counts at most 1 graph for any number of edges (where all edges are loops).

There are many other characterizations of graphs concerning cycles, paths, diameters, transitivity and so on which are not dealt with here.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	1	0	0	0	0	0	0
3	0	0	0	1	3	1	0	0	0	0
4	0	0	0	1	7	15	3	1	0	0
5	0	0	0	0	8	43	58	15	3	
6	0	0	0	0	5	82	244	257	68	
7	0	0	0	0	2	103	674			
8	0	0	0	0	1	102				
9	0	0	0	0	0					
10	0	0	0	0						

TABLE 1. `d-C-c-i-m-l` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	12	0	0	0	0	0	0
3	0	0	0	12	240	120	0	0	0	0
4	0	0	0	3	520	5460	5040	1680	0	0
5	0	0	0	0	500	19770	151200	191520	120960	
6	0	0	0	0	270	37135	795368	5021912	7761600	
7	0	0	0	0	80	46560	2359224			
8	0	0	0	0	10	42450				
9	0	0	0	0	0					
10	0	0	0	0						

TABLE 2. `d-C-c-i-m-l` vertex-labeled

2. STATISTICS

Tables 1–32 collect the statistics of directed graphs; Tables 33–56 collect the statistics of unndirected graphs. Rows and columns are sorted along the number E of edges and along the number V of vertices. There are always successive tables referring to the unlabeled graphs and referring to the vertex-labeled graphs. (The latter count is obtained by weighting each unlabeled graph by the number of distinct adjancency matrices that are created by row-column permutations of the adjacency matrix. This weight is a divisor of the order of the permutation group on V elements [5, Thm 15.2].)

2.1. **Directed Graphs.**

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0
2	0	0	3	0	0	0	0	0	0	0
3	0	0	3	8	0	0	0	0	0	0
4	0	0	2	21	27	0	0	0	0	0
5	0	0	0	33	107	91	0	0	0	
6	0	0	0	31	319	581	350	0	0	
7	0	0	0	16	609	2422	3023			
8	0	0	0	5	887	7529				
9	0	0	0	2	912					
10	0	0	0	0						

TABLE 3. `d-Cc-i-m-l` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	2	0	0	0	0	0	0	0	0
2	0	0	12	0	0	0	0	0	0	0
3	0	0	18	128	0	0	0	0	0	0
4	0	0	6	426	2000	0	0	0	0	0
5	0	0	0	684	11080	41472	0	0	0	
6	0	0	0	604	33160	337800	1075648	0	0	
7	0	0	0	300	67040	1529520	11967984			
8	0	0	0	78	96610	4954230				
9	0	0	0	8	101580					
10	0	0	0	0						

TABLE 4. `d-Cc-i-m-l` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	1	1	1	1	1	1	1	1	1
1	0	0	1	1	1	1	1	1	1	1
2	0	0	1	4	5	5	5	5	5	5
3	0	0	0	4	13	16	17	17	17	17
4	0	0	0	4	27	61	76	79	80	80
5	0	0	0	1	38	154	288	346	361	
6	0	0	0	1	48	379	1043	1637	1894	
7	0	0	0	0	38	707	3242			
8	0	0	0	0	27	1155				
9	0	0	0	0	13					
10	0	0	0	0						

TABLE 5. `d-C-ci-m-l` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	1	1	1	1	1	1	1	1	1
1	0	0	6	12	20	30	42	56	72	90
2	0	0	3	54	190	435	861	1540	2556	4005
3	0	0	0	80	900	3940	11480	27720	59640	117480
4	0	0	0	60	2325	21945	106890	365610	1028790	2555190
5	0	0	0	24	3900	81264	699468	3628296	13870584	
6	0	0	0	4	4610	218720	3374770	27446524	148477308	
7	0	0	0	0	3960	453240	12650400			
8	0	0	0	0	2475	748395				
9	0	0	0	0	1100					
10	0	0	0	0						

TABLE 6. `d-C-ci-m-l` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	1	0	0	0	0	0	0
4	0	0	0	4	7	1	0	0	0	0
5	0	0	0	7	35	42	7	1	0	
6	0	0	0	12	101	271	234	48	7	
7	0	0	0	16	230	1057	1848			
8	0	0	0	24	462	3285				
9	0	0	0	30	855					
10	0	0	0	41						

TABLE 7. `d-C-c-im-l` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	24	0	0	0	0	0	0
4	0	0	0	72	720	360	0	0	0	0
5	0	0	0	132	3520	22560	20160	6720	0	
6	0	0	0	210	10100	154650	806400	974400	604800	
7	0	0	0	312	23120	630360	7141008			
8	0	0	0	441	46970	1991325				
9	0	0	0	600	88280					
10	0	0	0	792						

TABLE 8. `d-C-c-im-l` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0
3	0	1	4	0	0	0	0	0	0	0
4	0	1	16	18	0	0	0	0	0	0
5	0	1	30	109	80	0	0	0	0	
6	0	1	53	391	694	367	0	0	0	
7	0	1	77	1042	3574	4207	1708			
8	0	1	116	2402	14093	29082				
9	0	1	156	5001	46144					
10	0	1	215	9737						

TABLE 9. `d-Cc-im-l` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	2	0	0	0	0	0	0	0	0
3	0	2	24	0	0	0	0	0	0	0
4	0	2	90	384	0	0	0	0	0	0
5	0	2	180	2472	8000	0	0	0	0	
6	0	2	300	8960	75400	207360	0	0	0	
7	0	2	462	24324	405160	2648880	6453888			
8	0	2	672	56322	1623440	19251960				
9	0	2	936	118168	5394560					
10	0	2	1260	230760						

TABLE 10. `d-Cc-im-l` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	1	1	1	1	1	1	1	1
3	0	0	2	6	7	7	7	7	7	7
4	0	0	3	20	42	49	50	50	50	50
5	0	0	3	41	158	273	315	322	323	
6	0	0	4	82	506	1302	1940	2174	2222	
7	0	0	4	132	1330	5174	10439			
8	0	0	5	222	3213	18293				
9	0	0	5	335	7097					
10	0	0	6	511						

TABLE 11. `d-C-cim-l` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0
2	0	0	6	12	20	30	42	56	72
3	0	0	12	120	400	900	1764	3136	5184
4	0	0	15	414	3290	13155	37065	87836	186660
5	0	0	18	948	15480	113190	499926	1634976	4483296
6	0	0	21	1802	52720	667375	4685387	22082536	80250072
7	0	0	24	3120	147320	3031920	33055848		
8	0	0	27	5094	362655	11463930			
9	0	0	30	7948	818780				
10	0	0	33	11946					

Table 12. `d-C-cim-l` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	2	0	0	0	0	0	0
4	0	0	0	7	13	2	0	0	0	0
5	0	0	0	7	52	70	13	2	0	
6	0	0	0	6	106	373	362	82	13	
7	0	0	0	2	137	1092	2392			
8	0	0	0	1	125	2262				
9	0	0	0	0	83					
10	0	0	0	0						

Table 13. `d-C-c-i-ml` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	48	0	0	0	0	0	0
4	0	0	0	120	1200	720	0	0	0	0
5	0	0	0	132	5000	34560	35280	13440	0	
6	0	0	0	78	10100	202920	1164240	1579200	1088640	
7	0	0	0	24	12750	630360	8919176			
8	0	0	0	3	10940	1314405				
9	0	0	0	0	6570					
10	0	0	0	0						

Table 14. `d-C-c-i-ml` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	2	0	0	0	0	0	0	0	0
3	0	1	7	0	0	0	0	0	0	0
4	0	0	16	26	0	0	0	0	0	0
5	0	0	16	111	107	0	0	0	0	
6	0	0	7	262	702	458	0	0	0	
7	0	0	2	372	2663	4251	2058			
8	0	0	0	361	6936	22925				
9	0	0	0	240	13442					
10	0	0	0	115						

TABLE 15. `d-Cc-i-ml` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	4	0	0	0	0	0	0	0	0
3	0	2	36	0	0	0	0	0	0	0
4	0	0	90	512	0	0	0	0	0	0
5	0	0	84	2472	10000	0	0	0	0	
6	0	0	36	5804	75400	248832	0	0	0	
7	0	0	6	8352	296600	2648880	7529536			
8	0	0	0	7986	787600	15073560				
9	0	0	0	5212	1542450					
10	0	0	0	2304						

TABLE 16. `d-Cc-i-ml` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1	1
2	0	1	4	4	4	4	4	4	4	4
3	0	0	6	17	20	20	20	20	20	20
4	0	0	3	35	83	100	103	103	103	103
5	0	0	1	46	236	457	548	565	568	
6	0	0	0	40	504	1659	2756	3210	3313	
7	0	0	0	25	833	4986	12171			
8	0	0	0	10	1064	12330				
9	0	0	0	3	1084					
10	0	0	0	1						

Table 17. `d-C-ci-ml` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0
1	0	2	3	4	5	6	7	8	9
2	0	1	21	54	110	195	315	476	684
3	0	0	28	292	1160	3080	6944	13944	25680
4	0	0	15	693	6605	30780	99946	268086	634950
5	0	0	3	948	22626	199926	1020936	3791256	11630052
6	0	0	0	830	52720	902265	7573790	40926732	167212668
7	0	0	0	480	89990	3031920	42473544		
8	0	0	0	180	117425	7978770			
9	0	0	0	40	119900				
10	0	0	0	4					

Table 18. `d-C-ci-ml` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	6	0	0	0	0	0	0
5	0	0	0	34	46	6	0	0	0	
6	0	0	0	107	347	314	52	6	0	
7	0	0	0	250	1473	2869	1995			
8	0	0	0	527	4731	15676				
9	0	0	0	994	12883					
10	0	0	0	1797						

Table 19. `d-C-c-iml` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	144	0	0	0	0	0	0
5	0	0	0	768	4800	2880	0	0	0	
6	0	0	0	2340	37000	180000	176400	67200	0	
7	0	0	0	5568	159600	1750320	7514640			
8	0	0	0	11634	518350	9908640				
9	0	0	0	22368	1427320					
10	0	0	0	40392						

TABLE 20. `d-C-c-iml` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	4	0	0	0	0	0	0	0	0
4	0	9	19	0	0	0	0	0	0	0
5	0	14	98	94	0	0	0	0	0	
6	0	20	286	761	479	0	0	0	0	
7	0	27	645	3522	5398	2480	0			
8	0	35	1290	12111	34960	36619				
9	0	44	2372	34847	167682					
10	0	54	4110	89361						

TABLE 21. `d-Cc-iml` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	8	0	0	0	0	0	0	0	0
4	0	18	108	0	0	0	0	0	0	0
5	0	28	576	2048	0	0	0	0	0	
6	0	40	1680	17480	50000	0	0	0	0	
7	0	54	3816	82144	602400	1492992	0			
8	0	70	7644	284788	4009600	23767632				
9	0	88	14112	824480	19528000					
10	0	108	24480	2121756						

TABLE 22. `d-Cc-iml` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	1	1	1	1	1	1	1	1	1
3	0	2	8	8	8	8	8	8	8	8
4	0	3	27	55	63	63	63	63	63	63
5	0	3	55	224	402	460	468	468	468	
6	0	4	97	671	1956	3051	3444	3508	3516	
7	0	4	154	1661	7607	17024	23868			
8	0	5	235	3670	25207	81289				
9	0	5	342	7505	74029					
10	0	6	483	14483						

TABLE 23. `d-C-ciml` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0
2	0	2	3	4	5	6	7	8	9
3	0	4	45	112	225	396	637	960	1377
4	0	5	150	1042	3815	9831	21784	43268	79101
5	0	6	315	4744	33825	142386	442715	1157920	2696625
6	0	7	553	14864	191335	1339211	6175162	21778968	64759989
7	0	8	894	37768	805875	9075456	62861757		
8	0	9	1368	84739	2785270	48311766			
9	0	10	2005	175140	8398505				
10	0	11	2838	340402					

TABLE 24. `d-C-ciml` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0
4	0	0	2	1	0	0	0	0	0	0
5	0	0	1	4	1	0	0	0	0	
6	0	0	1	16	7	1	0	0	0	
7	0	0	0	22	58	10	1			
8	0	0	0	22	240	165				
9	0	0	0	11	565					
10	0	0	0	5						

TABLE 25. `dCc-i-m-l` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0
3	0	0	2	0	0	0	0	0	0	0
4	0	0	9	6	0	0	0	0	0	0
5	0	0	6	84	24	0	0	0	0	
6	0	0	1	316	720	120	0	0	0	
7	0	0	0	492	6440	6480	720			
8	0	0	0	417	26875	107850				
9	0	0	0	212	65280					
10	0	0	0	66						

TABLE 26. `dCc-i-m-l` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	1	0	0	0	0	0	0	0	0
4	0	2	1	0	0	0	0	0	0	0
5	0	2	8	1	0	0	0	0	0	
6	0	3	25	21	1	0	0	0	0	
7	0	3	51	140	40	1	0			
8	0	4	101	565	525	69				
9	0	4	174	1731	3719					
10	0	5	290	4602						

TABLE 27. `dCc-im-l` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	2	0	0	0	0	0	0	0	0
4	0	3	6	0	0	0	0	0	0	0
5	0	4	48	24	0	0	0	0	0	
6	0	5	140	480	120	0	0	0	0	
7	0	6	306	3276	4680	720	0			
8	0	7	588	13230	61040	47880				
9	0	8	1036	41024	437320					
10	0	9	1710	109152						

TABLE 28. `dCc-im-l` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	1	0	0	0	0	0	0	0	0
4	0	1	1	0	0	0	0	0	0	0
5	0	0	6	1	0	0	0	0	0	
6	0	0	9	17	1	0	0	0	0	
7	0	0	6	78	34	1	0			
8	0	0	2	185	346	60				
9	0	0	1	259	1775					
10	0	0	0	252						

TABLE 29. `dCc-i-ml` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	2	0	0	0	0	0	0	0	0
4	0	1	6	0	0	0	0	0	0	0
5	0	0	33	24	0	0	0	0	0	
6	0	0	47	372	120	0	0	0	0	
7	0	0	30	1792	3840	720	0			
8	0	0	9	4206	39640	40680				
9	0	0	1	5968	206095					
10	0	0	0	5634						

TABLE 30. `dCc-i-ml` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	3	0	0	0	0	0	0	0	0
5	0	8	4	0	0	0	0	0	0	
6	0	16	38	5	0	0	0	0	0	
7	0	25	151	110	6	0	0			
8	0	40	431	898	250	7				
9	0	56	1040	4475	3665					
10	0	80	2252	17039						

TABLE 31. `dCc-iml` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	6	0	0	0	0	0	0	0	0
5	0	16	24	0	0	0	0	0	0	
6	0	30	225	120	0	0	0	0	0	
7	0	50	897	2592	720	0	0			
8	0	77	2562	21196	29400	5040				
9	0	112	6190	106336	431360					
10	0	156	13437	405552						

TABLE 32. `dCc-iml` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	1	0	0	0	0	0	0
3	0	0	0	0	1	1	0	0	0	0
4	0	0	0	0	1	3	1	1	0	0
5	0	0	0	0	0	3	6	3	1	
6	0	0	0	0	0	2	9	15	7	
7	0	0	0	0	0	1	8			
8	0	0	0	0	0	0				
9	0	0	0	0	0					
10	0	0	0	0						

TABLE 33. `-d-c-i-m-l` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	3	0	0	0	0	0	0
3	0	0	0	0	30	15	0	0	0	0
4	0	0	0	0	10	330	315	105	0	0
5	0	0	0	0	0	285	4410	5880	3780	
6	0	0	0	0	0	100	6797	71078	116550	
7	0	0	0	0	0	15	5460			
8	0	0	0	0	0	0				
9	0	0	0	0	0					
10	0	0	0	0						

TABLE 34. `-d-c-i-m-l` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0
2	0	0	1	0	0	0	0	0	0	0
3	0	0	1	2	0	0	0	0	0	0
4	0	0	0	2	3	0	0	0	0	0
5	0	0	0	1	5	6	0	0	0	
6	0	0	0	1	5	13	11	0	0	
7	0	0	0	0	4	19	33			
8	0	0	0	0	2	22				
9	0	0	0	0	1					
10	0	0	0	0						

TABLE 35. `-dc-i-m-l` unlabeled

2.2. **Undirected Graphs.**

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0
2	0	0	3	0	0	0	0	0	0	0
3	0	0	1	16	0	0	0	0	0	0
4	0	0	0	15	125	0	0	0	0	0
5	0	0	0	6	222	1296	0	0	0	
6	0	0	0	1	205	3660	16807	0	0	
7	0	0	0	0	120	5700	68295			
8	0	0	0	0	45	6165				
9	0	0	0	0	10					
10	0	0	0	0						

TABLE 36. `-dc-i-m-l` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	1	1	1	1	1	1	1	1	1
1	0	0	1	1	1	1	1	1	1	1
2	0	0	0	1	2	2	2	2	2	2
3	0	0	0	1	3	4	5	5	5	5
4	0	0	0	0	2	6	9	10	11	11
5	0	0	0	0	1	6	15	21	24	
6	0	0	0	0	1	6	21	41	56	
7	0	0	0	0	0	4	24			
8	0	0	0	0	0	2				
9	0	0	0	0	0					
10	0	0	0	0						

TABLE 37. `-d-ci-m-l` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	1	1	1	1	1	1	1	1	1
1	0	0	3	6	10	15	21	28	36	45
2	0	0	0	12	45	105	210	378	630	990
3	0	0	0	4	90	440	1330	3276	7140	14190
4	0	0	0	0	75	1035	5670	20370	58905	148995
5	0	0	0	0	30	1422	15939	92400	373212	
6	0	0	0	0	5	1245	30660	305662	1831242	
7	0	0	0	0	0	720	42525			
8	0	0	0	0	0	270				
9	0	0	0	0	0					
10	0	0	0	0						

TABLE 38. `-d-ci-m-l` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	1	0	0	0	0	0	0
4	0	0	0	2	2	1	0	0	0	0
5	0	0	0	2	6	8	2	1	0	
6	0	0	0	3	10	25	21	9	2	
7	0	0	0	3	16	53	80			
8	0	0	0	4	23	102				
9	0	0	0	4	32					
10	0	0	0	5						

TABLE 39. `-d-c-im-l` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	6	0	0	0	0	0	0
4	0	0	0	9	90	45	0	0	0	0
5	0	0	0	12	220	1410	1260	420	0	
6	0	0	0	15	400	4875	25200	30450	18900	
7	0	0	0	18	650	11700	113232			
8	0	0	0	21	980	24045				
9	0	0	0	24	1400					
10	0	0	0	27						

TABLE 40. `-d-c-im-l` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0
3	0	1	1	0	0	0	0	0	0	0
4	0	1	3	3	0	0	0	0	0	0
5	0	1	4	10	6	0	0	0	0	
6	0	1	6	21	29	16	0	0	0	
7	0	1	7	37	81	91	37			
8	0	1	9	61	191	326				
9	0	1	11	95	395					
10	0	1	13	141						

TABLE 41. `-dc-im-l` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0
3	0	1	6	0	0	0	0	0	0	0
4	0	1	12	48	0	0	0	0	0	0
5	0	1	18	156	500	0	0	0	0	
6	0	1	25	340	2360	6480	0	0	0	
7	0	1	33	636	7060	41400	100842			
8	0	1	42	1092	17290	162120				
9	0	1	52	1764	37740					
10	0	1	63	2718						

TABLE 42. `-dc-im-l` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	1	1	1	1	1	1	1	1
3	0	0	1	2	3	3	3	3	3	3
4	0	0	1	4	9	11	12	12	12	12
5	0	0	1	5	17	29	37	39	40	
6	0	0	1	7	31	70	111	132	141	
7	0	0	1	8	48	145	289			
8	0	0	1	10	75	289				
9	0	0	1	12	111					
10	0	0	1	14						

TABLE 43. `-d-cim-l` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	3	6	10	15	21	28	36	45
3	0	0	3	30	100	225	441	784	1296	2025
4	0	0	3	54	415	1650	4641	10990	23346	45585
5	0	0	3	78	1030	7215	31521	102676	281016	
6	0	0	3	106	2035	22400	150766	700378	2529696	
7	0	0	3	138	3610	56745	557676			
8	0	0	3	174	5995	127170				
9	0	0	3	214	9470					
10	0	0	3	258						

TABLE 44. `-d-cim-l` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	1	0	0	0	0	0	0
4	0	0	0	2	3	1	0	0	0	0
5	0	0	0	1	7	10	3	1	0	
6	0	0	0	1	8	28	28	11	3	
7	0	0	0	0	6	42	91			
8	0	0	0	0	3	48				
9	0	0	0	0	1					
10	0	0	0	0						

TABLE 45. `-d-c-i-ml` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	12	0	0	0	0	0	0
4	0	0	0	18	150	90	0	0	0	0
5	0	0	0	12	350	2205	2205	840	0	
6	0	0	0	3	400	6960	37485	49980	34020	
7	0	0	0	0	250	11700	151214			
8	0	0	0	0	80	12330				
9	0	0	0	0	10					
10	0	0	0	0						

TABLE 46. `-d-c-i-ml` vertex-labeled

E\V	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0
3	0	1	2	0	0	0	0	0	0	0
4	0	0	3	4	0	0	0	0	0	0
5	0	0	2	10	9	0	0	0	0	
6	0	0	1	12	30	20	0	0	0	
7	0	0	0	10	57	93	48			
8	0	0	0	5	73	240				
9	0	0	0	2	67					
10	0	0	0	1						

TABLE 47. `-dc-i-ml` unlabeled

E\V	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	2	0	0	0	0	0	0	0	0
3	0	1	9	0	0	0	0	0	0	0
4	0	0	12	64	0	0	0	0	0	0
5	0	0	6	156	625	0	0	0	0	
6	0	0	1	178	2360	7776	0	0	0	
7	0	0	0	116	4495	41400	117649			
8	0	0	0	45	5495	115020				
9	0	0	0	10	4710					
10	0	0	0	1						

TABLE 48. `-dc-i-ml` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1	1	1
2	0	1	3	3	3	3	3	3	3	3
3	0	0	3	7	9	9	9	9	9	9
4	0	0	1	9	20	25	27	27	27	27
5	0	0	0	6	30	58	74	79	81	
6	0	0	0	3	32	104	183	226	243	
7	0	0	0	1	27	149	381			
8	0	0	0	0	16	175				
9	0	0	0	0	7					
10	0	0	0	0						

TABLE 49. `-d-ci-ml` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	2	3	4	5	6	7	8	9	10
2	0	1	12	30	60	105	168	252	360	495
3	0	0	10	88	335	875	1946	3864	7050	12045
4	0	0	3	113	1005	4530	14490	38430	90090	192060
5	0	0	0	78	1776	15141	75726	277872	844767	
6	0	0	0	28	2035	34523	284991	1521072	6163248	
7	0	0	0	4	1570	56745	798897			
8	0	0	0	0	815	69705				
9	0	0	0	0	275					
10	0	0	0	0						

TABLE 50. `-d-ci-ml` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	3	0	0	0	0	0	0
5	0	0	0	11	10	3	0	0	0	
6	0	0	0	27	51	44	11	3	0	
7	0	0	0	51	157	236	153			
8	0	0	0	93	386	850				
9	0	0	0	150	838					
10	0	0	0	241						

TABLE 51. `-d-c-iml` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	36	0	0	0	0	0	0
5	0	0	0	144	600	360	0	0	0	
6	0	0	0	360	3250	11925	11025	4200	0	
7	0	0	0	738	10650	76635	257985			
8	0	0	0	1365	27650	308385				
9	0	0	0	2352	62940					
10	0	0	0	3834						

Table 52. `-d-c-iml` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	2	0	0	0	0	0	0	0	0
4	0	5	5	0	0	0	0	0	0	0
5	0	8	19	13	0	0	0	0	0	
6	0	11	45	70	35	0	0	0	0	
7	0	15	87	227	245	95	0			
8	0	19	153	579	1029	840				
9	0	24	252	1302	3346					
10	0	29	394	2681						

Table 53. `-dc-iml` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	4	0	0	0	0	0	0	0	0
4	0	9	27	0	0	0	0	0	0	0
5	0	14	102	256	0	0	0	0	0	
6	0	20	240	1420	3125	0	0	0	0	
7	0	27	471	4688	23535	46656	0			
8	0	35	840	12250	102900	453096				
9	0	44	1400	28080	345730					
10	0	54	2214	58914						

Table 54. `-dc-iml` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	1	1	1	1	1	1	1	1	1
3	0	2	6	6	6	6	6	6	6	6
4	0	3	15	24	29	29	29	29	29	29
5	0	3	26	66	107	122	127	127	127	
6	0	4	40	142	318	454	520	536	541	
7	0	4	57	269	800	1464	1967			
8	0	5	79	474	1813	4224				
9	0	5	106	793	3810					
10	0	6	138	1273						

TABLE 55. `-d-ciml` unlabeled

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	2	3	4	5	6	7	8	9	10
3	0	4	27	64	125	216	343	512	729	1000
4	0	5	69	358	1190	2946	6349	12356	22239	37630
5	0	6	123	1104	6275	23796	70315	177920	404109	
6	0	7	193	2554	22585	130286	543837	1813568	5197044	
7	0	8	285	5102	64340	538614	3165841			
8	0	9	402	9363	158520	1829799				
9	0	10	547	16176	354905					
10	0	11	723	26626						

TABLE 56. `-d-ciml` vertex-labeled

$E\backslash V$	1	2	3	4	5	6	7	8	9		
0	1										
1	0	1									
2	0	0	1								
3	0	0	1	2							
4	0	0	0	2	3						
5	0	0	0	1	5	6					
6	0	0	0	1	5	13	11				
7	0	0	0	0	4	19	33	23			
8	0	0	0	0	2	22	67	89	47		
9	0	0	0	0	1	20	107	236	240	106	
10	0	0	0	0	1	14	132	486	797	657	235

TABLE 57. (`-dc.*-m-l`) unlabeled. The number of connected undirected graphs without multiedges or loops [10, A054923,A054924,A046742][11, Vol. 1, Sec. 7, Table 1]. With the exception of the value of 1 at $E = 0$, $V = 1$ this is the same as Table 35.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0
2	0	0	3	0	0	0	0	0	0	0
3	0	0	1	16	0	0	0	0	0	0
4	0	0	0	15	125	0	0	0	0	0
5	0	0	0	6	222	1296	0	0	0	
6	0	0	0	1	205	3660	16807	0		
7	0	0	0	0	120	5700				
8	0	0	0	0	45					
9	0	0	0	0						

TABLE 58. (`-dc.*-m-l`) vertex-labeled [10, A062734].

3. ACCUMULATED MARGINAL STATISTICS

Adding the contents of one or more of the previous arrays defines the union of their graph sets, and regards some of the properties as irrelevant in these tables. If we look on the flags as defining a hypertable along five or six axes, these sums are the marginal sums; they create the Tables in Section 3.

The properties that are not taken into account while counting the graphs are either replaced by the filler `.*` or not tagged at all, using regular expressions of the usual programming languages as the tags.

Example 1. `-d.*-m-l` *flags graphs that are undirected, have any type of isolated vertices or connectivity, but have no multiedges or loops.*

Example 2. `-dc` *flags graphs that are undirected and connected, but have any type of isolated vertices, multiedges or loops.*

3.1. Undirected Graphs.

Tables 57–70 summarize statistics of undirected graphs.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	1	2	2	2	2	2	2	2	2	2
2	1	4	6	7	7	7	7	7	7	7
3	1	6	14	20	22	23	23	23	23	23
4	1	9	28	53	69	76	78	79	79	79
5	1	12	52	125	198	245	264	271	273	
6	1	16	93	287	550	782	915	973	993	
7	1	20	152	606	1441	2392	3111			
8	1	25	242	1226	3611	7118				
9	1	30	370	2358	8608					
10	1	36	546	4356						

TABLE 59. `( -d)` unlabeled. The number of undirected graphs allowing loops and multiedges [10, A290428]. Sum of Table 63 and Table 67.

$E\backslash V$	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1
1	1	3	6	10	15	21	28	36	45
2	1	6	21	55	120	231	406	666	1035
3	1	10	56	220	680	1771	4060	8436	16215
4	1	15	126	715	3060	10626	31465	82251	194580
5	1	21	252	2002	11628	53130	201376	658008	1906884
6	1	28	462	5005	38760	230230	1107568	4496388	15890700
7	1	36	792	11440	116280	888030	5379616		
8	1	45	1287	24310	319770	3108105			
9	1	55	2002	48620	817190				
10	1	66	3003	92378					

TABLE 60. `( -d)` vertex-labeled [10, A098568]. Sum of Table 64 and Table 68.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10	
0	1										
1	0	1									
2	0	1	1								
3	0	1	2	2							
4	0	1	3	5	3						
5	0	1	4	11	11	6					
6	0	1	6	22	34	29	11				
7	0	1	7	37	85	110	70	23			
8	0	1	9	61	193	348	339	185	47		
9	0	1	11	95	396	969	1318	1067	479	106	
10	0	1	13	141	771	2445	4457	4940	3294	1729	235

TABLE 61. `-dc.*-l` unlabeled. Undirected loopless connected multigraphs with E edges and V vertices [10, A191646]. With the exception of the 1 at $E = 0$, $V = 1$, the sum of Table 35 and Table 41.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0
2	0	1	3	0	0	0	0	0	0	0
3	0	1	7	16	0	0	0	0	0	0
4	0	1	12	63	125	0	0	0	0	0
5	0	1	18	162	722	1296	0	0	0	
6	0	1	25	341	2565	10140	16807	0	0	
7	0	1	33	636	7180	47100	169137			
8	0	1	42	1092	17335	168285				
9	0	1	52	1764	37750					
10	0	1	63	2718						

TABLE 62. `( -dc.*-l)` vertex-labeled [10, A290776].

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0
2	1	2	1	0	0	0	0	0	0	0
3	1	4	4	2	0	0	0	0	0	0
4	1	6	11	9	3	0	0	0	0	0
5	1	9	25	34	20	6	0	0	0	
6	1	12	52	104	99	49	11	0	0	
7	1	16	94	274	387	298	118			
8	1	20	162	645	1295	1428				
9	1	25	263	1399	3809					
10	1	30	407	2823						

TABLE 63. `( -dc)` unlabeled. Row sums in [10, A007719]

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0
2	1	3	3	0	0	0	0	0	0	0
3	1	6	16	16	0	0	0	0	0	0
4	1	10	51	127	125	0	0	0	0	0
5	1	15	126	574	1347	1296	0	0	0	
6	1	21	266	1939	8050	17916	16807	0	0	
7	1	28	504	5440	35210	135156	286786			
8	1	36	882	13387	125730	736401				
9	1	45	1452	29854	388190					
10	1	55	2277	61633						

TABLE 64. `( -dc)` vertex-labeled.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	1	1	1	1	1	1	1	1	
1	0	1	1	1	1	1	1	1	1	1	1	
2	0	0	1	2	2	2	2	2	2	2	2	
3	0	0	1	3	4	5	5	5	5	5	5	
4	0	0	0	2	6	9	10	11	11	11	11	
5	0	0	0	1	6	15	21	24	25	26	26	
6	0	0	0	1	6	21	41	56	63	66	67	
7	0	0	0	0	4	24	65	115	148	165	172	
8	0	0	0	0	2	24	97	221	345	428	467	
9	0	0	0	0	1	21	131	402	771	1103	1305	1405
10	0	0	0	0	1	15	148	663	1637	2769	3664	4191

TABLE 65. (-d.*-m-l) unlabeled. Simple graphs with E edges and V vertices [10, A008406][11, vol. 4, Tables 2.2–2.2g]. Sum of tables 57, 81–84 and contributions by more than 5 components.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	0	1	3	6	10	15	21	28	36	45
2	0	0	3	15	45	105	210	378	630	990
3	0	0	1	20	120	455	1330	3276	7140	14190
4	0	0	0	15	210	1365	5985	20475	58905	
5	0	0	0	6	252	3003	20349	98280	376992	
6	0	0	0	1	210	5005	54264	376740		
7	0	0	0	0	120	6435				
8	0	0	0	0	45					
9	0	0	0							

TABLE 66. (-d.*-m-l) vertex-labeled [10, A084546].

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	1	1	1	1	1	1	1	1	1
1	0	1	2	2	2	2	2	2	2	2
2	0	2	5	7	7	7	7	7	7	7
3	0	2	10	18	22	23	23	23	23	23
4	0	3	17	44	66	76	78	79	79	79
5	0	3	27	91	178	239	264	271	273	
6	0	4	41	183	451	733	904	973	993	
7	0	4	58	332	1054	2094	2993			
8	0	5	80	581	2316	5690				
9	0	5	107	959	4799					
10	0	6	139	1533						

TABLE 67. (-d-c) unlabeled. See [10, A007717] for the limit $V \to \infty$.

$E\backslash V$	1	2	3	4	5	6	7	8	9
0	0	1	1	1	1	1	1	1	1
1	0	2	6	10	15	21	28	36	45
2	0	3	18	55	120	231	406	666	1035
3	0	4	40	204	680	1771	4060	8436	16215
4	0	5	75	588	2935	10626	31465	82251	194580
5	0	6	126	1428	10281	51834	201376	658008	1906884
6	0	7	196	3066	30710	212314	1090761	4496388	15890700
7	0	8	288	6000	81070	752874	5092830		
8	0	9	405	10923	194040	2371704			
9	0	10	550	18766	429000				
10	0	11	726	30745					

TABLE 68. `( -d-c)` vertex-labeled.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	0	1	1	1	1	1	1	1	1	1
2	0	1	2	3	3	3	3	3	3	3
3	0	1	3	6	7	8	8	8	8	8
4	0	1	4	11	17	21	22	23	23	23
5	0	1	5	18	35	52	60	64	65	
6	0	1	7	32	76	132	173	197	206	
7	0	1	8	48	149	313	471			
8	0	1	10	75	291	741				
9	0	1	12	111	539					
10	0	1	14	160						

TABLE 69. `( -d.*-l)` unlabeled. Undirected loopless multigraphs with E edges and V vertices [10, A192517]. Sum of tables 61, 85–88 and so forth.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	0	1	3	6	10	15	21	28	36	45
2	0	1	6	21	55	120	231	406	666	1035
3	0	1	10	56	220	680	1771	4060	8436	16215
4	0	1	15	126	715	3060	10626	31465	82251	194580
5	0	1	21	252	2002	11628	53130	201376	658008	
6	0	1	28	462	5005	38760	230230	1107568	4496388	
7	0	1	36	792	11440	116280	888030			
8	0	1	45	1287	24310	319770				
9	0	1	55	2002	48620					
10	0	1	66	3003						

TABLE 70. `( -d.*-l)` vertex-labeled.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0
2	0	4	3	0	0	0	0	0	0	0
3	0	8	15	8	0	0	0	0	0	0
4	0	16	57	66	27	0	0	0	0	0
5	0	25	163	353	295	91	0	0	0	
6	0	40	419	1504	2203	1407	350	0	0	
7	0	56	932	5302	12382	13372	6790			
8	0	80	1940	16549	58237	96456				
9	0	105	3743	46566	237904					
10	0	140	6867	121111						

TABLE 71. `( d.*Cc-i)` unlabeled. The number of connected directed multigraphs with loops and no isolated vertex, with E arcs and V vertices [10, A139621].

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0
1	0	2	0	0	0	0	0	0	0	0
2	0	7	12	0	0	0	0	0	0	0
3	0	16	80	128	0	0	0	0	0	0
4	0	30	315	1328	2000	0	0	0	0	0
5	0	50	951	7808	29104	41472	0	0	0	
6	0	77	2429	34136	234920	794112	1075648	0	0	
7	0	112	5517	123272	1386880	8328192	25952128			
8	0	156	11475	388223	6674205	63248832				
9	0	210	22275	1101408	27706645					
10	0	275	40887	2875224						

TABLE 72. `( d.*Cc-i)` vertex-labeled

3.2. **Directed Graphs.** Tables 71–80 summarize statistics of oriented/directed graphs.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0
3	1	2	1	0	0	0	0	0	0	0
4	1	6	4	1	0	0	0	0	0	0
5	1	10	19	6	1	0	0	0	0	
6	1	19	73	59	9	1	0	0	0	
7	1	28	208	350	138	12	1			
8	1	44	534	1670	1361	301				
9	1	60	1215	6476	9724					
10	1	85	2542	21898						

TABLE 73. `( dC)` unlabeled. The number of strongly connected directed multigraphs with loops and no vertex of degree zero, with n arcs and k vertices [10, A139622].

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0
3	1	4	2	0	0	0	0	0	0	0
4	1	10	21	6	0	0	0	0	0	0
5	1	20	111	132	24	0	0	0	0	
6	1	35	413	1288	960	120	0	0	0	
7	1	56	1233	8152	15680	7920	720			
8	1	84	3159	39049	156955	201450				
9	1	120	7227	153540	1140055					
10	1	165	15147	520404						

TABLE 74. `( dC)` vertex-labeled.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0
2	0	1	3	0	0	0	0	0	0	0
3	0	0	4	8	0	0	0	0	0	0
4	0	0	4	22	27	0	0	0	0	0
5	0	0	1	37	108	91	0	0	0	
6	0	0	1	47	326	582	350	0	0	
7	0	0	0	38	667	2432	3024			
8	0	0	0	27	1127	7694				
9	0	0	0	13	1477					
10	0	0	0	5						

TABLE 75. `( d.*Cc.*-m-l)` unlabeled. The number of weakly connected directed graphs without multiedges or loops [10, A054733,A283753]. The undirected variants are in Table 57.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0
1	0	2	0	0	0	0	0	0	0	0
2	0	1	12	0	0	0	0	0	0	0
3	0	0	20	128	0	0	0	0	0	0
4	0	0	15	432	2000	0	0	0	0	0
5	0	0	6	768	11104	41472	0	0	0	
6	0	0	1	920	33880	337920	1075648	0	0	
7	0	0	0	792	73480	1536000	11968704			
8	0	0	0	495	123485	5062080				
9	0	0	0	220	166860					
10	0	0	0	66						

TABLE 76. `( d.*Cc.*-m-l)` vertex-labeled. [10, A062735]

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0
4	0	0	2	1	0	0	0	0	0	0
5	0	0	1	4	1	0	0	0	0	
6	0	0	1	16	7	1	0	0	0	
7	0	0	0	22	58	10	1			
8	0	0	0	22	240	165				
9	0	0	0	11	565					
10	0	0	0	5						

TABLE 77. `( dCc.*-m-l)` unlabeled. The number of strongly connected directed graphs without loops or multiedges. Strongly connected variant of Table 75. With the exception of the 1 at $E = 0$, $V = 1$ this is the same as Table 25.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	1	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0
3	0	0	2	0	0	0	0	0	0	0
4	0	0	9	6	0	0	0	0	0	0
5	0	0	6	84	24	0	0	0	0	
6	0	0	1	316	720	120	0	0	0	
7	0	0	0	492	6440	6480	720			
8	0	0	0	417	26875	107850				
9	0	0	0	212	65280					
10	0	0	0	66						

TABLE 78. `( dCc.*-m-l)` vertex-labeled. With the exception of the 1 at $E = 0$, $V = 1$ this is the same as Table 26.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1	1	1	1	1	1	1
1	1	2	2	2	2	2	2	2	2	2
2	1	6	10	11	11	11	11	11	11	11
3	1	10	31	47	51	52	52	52	52	52
4	1	19	90	198	269	291	295	296	296	296
5	1	28	222	713	1270	1596	1697	1719	1723	
6	1	44	520	2423	5776	8838	10425	10922	11033	
7	1	60	1090	7388	24032	46384	63419			
8	1	85	2180	21003	93067	230848				
9	1	110	4090	55433	333948					
10	1	146	7356	137944						

TABLE 79. `( d)` unlabeled. The number of directed graphs allowing loops and multiedges [10, A138107].

$E\backslash V$	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1
1	1	4	9	16	25	36	49	64
2	1	10	45	136	325	666	1225	2080
3	1	20	165	816	2925	8436	20825	45760
4	1	35	495	3876	20475	82251	270725	766480
5	1	56	1287	15504	118755	658008	2869685	10424128
6	1	84	3003	54264	593775	4496388	25827165	119877472
7	1	120	6435	170544	2629575	26978328	202927725	
8	1	165	12870	490314	10518300	145008513		
9	1	220	24310	1307504	38567100			
10	1	286	43758	3268760				

TABLE 80. `( d)` vertex-labeled. The number of labeled directed graphs allowing loops and multiedges [10, A214398].

4. Connected Multigraphs up to 7 vertices

4.1. **Algorithm.** The columns of the undirected connected multigraphs in Table 61 have rational ordinary generating functions. To compute them, we first classify each multigraph by the number of edges and vertices of the underlying simple graph—in as many ways as counted in Table 57— and then distribute the edges of the multigraph over the edges of the underlying graph using Pólya's counting method to deal with the symmetry of the simple graphs.

The process is illustrated in Section 4.2 for $V = 4$ vertices. Explicit intermediate results are tracked in the files `G.`V`.`E`.txt` in the ancillary directory for $V = 2$–7. Each of these files contains the contributing underlying simple graphs with V vertices and E edges. The file starts with V and E printed in the first line. Then each graph is represented by

(1) a canonical adjacency matrix (binary, symmetric and traceless),
(2) the label as in Section 1 followed by the multiplicity of the graph as if one would create all vertex-labeled graphs by permuting rows and columns (i.e. $V!$ divided by the order of the automorphism group),
(3) the cycle index multinomial. This could also be derived from the table of symmetry groups in [11, Vol. 1, Sec. 7, Table 8].

The minimum number of edges for connected simple graphs is $E \geq V - 1$ (sparsest, trees on V vertices), and the maximum number is $E \leq \binom{V}{2}$ (complete graph on V vertices). Summing over all multinomials over the underlying graphs constitutes the generating function by a finite sum of rational polynomials ([10, A001349] terms).

4.2. **4 vertices.** The ordinary generating function for the number of connected multigraphs on 4 vertices is derived by adding the contributions of the 6 distinct geometries of the underlying connected simple graph.

We consider connected multigraphs with 4 vertices and E edges. The multigraph thus has at least one (unoriented) edge attached to edge vertex, so all degrees are ≥ 1. Loops are not allowed; the vertices are not labeled.

If all multiedges are replaced by a single edge, the underlying simple graph has one of 6 shapes [3]:

(1) The linear chain with 3 edges.
(2) A triangle with an edge to a lone vertex of degree 1 (4 edges).
(3) The star graph with 3 edges.
(4) The quadrangle (cycle of 4 edges).
(5) A quadrangle with a single diagonal, total of 5 edges.
(6) The complete graph on 4 vertices with 6 edges.

Linear Chain. We wish to distribute $E \geq 3$ edges over the 3 edges of the simple graph of the linear chain. This can be done by putting any number of $k \geq 1$ edges in the middle, and distributing the remaining $n-k$ edges over the two edges connected to two endpoints.

Due to the left-right symmetry of the graph, the distribution of the $n - k$ edges can only be done in $\lfloor (n-k)/2 \rfloor$ ways. The total number of graphs of this kind with multiedges is

$$\sum_{k=1}^{E-2} \lfloor \frac{E-k}{2} \rfloor = 0, 0, 0, 1, 2, 4, 6, 9, 12, 16, 20, 25, \ldots (E \geq 0) \tag{1}$$

with generating function [10, A002620]

$$g_1(x) = \sum_{E\ge 0} a_1(E)x^E = \frac{x^3}{(1+x)(1-x)^3}. \tag{2}$$

The generating function is the product of $t_1(x)$ representing the number of ways of placing n vertices at the middle edge, by the factor $x^2/[(1+x)(1-x)^2]$. The latter factor is obtained by considering the symmetry of the cyclic group C_2 that swaps the edges that inhabit the first and last edges of the underlying simple graph without generating a new graph. The cycle index of the group is [2]

$$Z(C_2) = (t_1^2 + t_2^1)/2, \tag{3}$$

where the associated generating functions are the number of ways of placing n edges without imposing symmetry on any of them:

$$t_i(x) = \frac{x}{1-x} \mapsto 0,1,1,1,1,1\ldots, \quad i\ge 1. \tag{4}$$

So the latter factor can be written as [10, A004526]

$$\frac{t_1(x)^2 + t_2(x^2)}{2} = \frac{x^2}{(1+x)(1-x)^2}. \tag{5}$$

Triangle. The contribution from the triangular graph is the number of ways of placing $2\le k\le n-2$ edges on the edge to the lone vertex and the triangle edge opposite to it, and then distributing the residual $n-k$ edges to the remaining two edges under the symmetry constraint of the group C_2 that swaps the other two edges:

$$\sum_{k=2}^{E-2}(k-1)\lfloor\frac{E-k}{2}\rfloor = 0,0,0,0,1,3,7,13,22,34,50,70,95,125,161 (E\ge 0) \tag{6}$$

See [10, A002623]

$$g_2(x) = \frac{x^4}{(1+x)(1-x)^4}. \tag{7}$$

This generating function is the product of $x^2/(1-x)^2$—contribution of two cycles of length 1, fixed points under the symmetry—by $x^2/[(1+x)(1-x)^2]$, where again the latter is (5), the number of ways of distributing E edges symmetrically over two edges of the simple graph.

Star Graph. The contribution from the star graph is the number of ways of partitioning E into 3 positive integers [10, A069905],

$$\mapsto 0,0,0,1,1,2,3,4,5,7,8,10,12,14,16 (E\ge 0), \tag{8}$$

$$g_3(x) = \frac{x^3}{(1+x)(1-x)^2(1-x^3)}. \tag{9}$$

Alternatively this expression is obtained if we consider the symmetry group of order 6 of the underlying simple graph, which can be generated by (i) the group C_3 of the triangle combined with (ii) the mirror symmetry along a diagonal.

```
with(group):
g := permgroup(3, {[[1, 2, 3]], [[2, 3]]}) ;
for i in elements(g) do
    print(i) ;
```

```
end do;
```

The cycle index obtained with this Maple code is [6, p 57]

$$Z(S_3) = \frac{t_1^3 + 3t_1t_2 + 2t_3^1}{6}. \tag{10}$$

Insertion of (4) gives (9).
Square. The symmetry group of the square is the Dihedral Group of order 8 which essentially is generated by rotation by 90 degrees or flips along the horizontal or vertical axes or diagonals.

```
with(group):
g := permgroup(4, {[[1, 2, 3, 4]], [[1, 3]]}) ;
for i in elements(g) do
    print(i) ;
end do;
```

The cycle index is [12, Fig 3][2]

$$Z(D_8) = \frac{t_1^4 + 2t_4^1 + 2t_1^2t_2^1 + 3t_2^2}{8} \tag{11}$$

The enumeration theorem turns this into the generating function [10, A005232]

$$g_4(x) = \frac{x^4(1-x+x^2)}{(1+x^2)(1+x)^2(1-x)^4}. \tag{12}$$

$$\mapsto 0,0,0,0,1,1,3,4,8,10,16,20,29,35,47 (E \geq 0). \tag{13}$$

Square with Diagonal. In the square with a diagonal edge, the diagonal stays inert under the symmetry operations, and contributes a factor $t_1(x)$ to the generating function. The symmetry group of the four other edges allows a flip along any of the two diagonals and generates a symmetry group of order 4:

```
with(group):
g := permgroup(4,{[[2, 4], [1, 3]], [[1, 4], [2, 3]]}) ;
for i in elements(g) do
    print(i) ;
end do;
```

The cycle index is

$$Z(C_2 \times C_2) = \frac{t_1^4 + 3t_2^2}{4}. \tag{14}$$

Insertion of (4) into the enumeration theorem yields $x^4(1-x+x^2)/[(1+x)^2(1-x)^4]$ [10, A053307], and convolved with the inert factor

$$g_5(x) = \frac{x^5(1-x+x^2)}{(1+x)^2(1-x)^5}. \tag{15}$$

This expands to

$$\mapsto 0,0,0,0,0,1,2,6,11,22,36,60,90,135,190,266,357,476 (E \geq 0) \tag{16}$$

Complete Graph. The cycle index of the complete graph K_4 is [2],

$$Z(S_4) = \frac{t_1^6 + 9t_1^2t_2^2 + 8t_3^2 + 6t_2t_4}{24}. \tag{17}$$

Insertion of (4) into the enumeration theorem yields

$$g_6(x) = \frac{x^6(1-x+x^2+x^4+x^6-x^7+x^8)}{(1-x)^6(1+x)^2(1+x^2)(1+x+x^2)^2}, \tag{18}$$

with [10, A003082]

$$\mapsto 0,0,0,0,0,0,1,1,3,6,11,18,32,48,75,111,160,224,313(E\ge 0) \tag{19}$$

Sum. The generating function contributed by the 6 underlying simple graphs is

$$\sum_{i=1}^{6} g_i(x) = \frac{x^3(-x^{10}+x^9+2x^7-x^6+x^5-3x^4+x^2+x+2)}{(x-1)^6(1+x)^2(1+x^2)(1+x+x^2)^2}, \tag{20}$$

which expands to [10, A290778]

$$\mapsto 0,0,0,2,5,11,22,37,61,95,141,203,288,393,531,704,918,1180,1504(E\ge 0, V=4) \tag{21}$$

4.3. **Up to 7 vertices.** The generating function for the number of connected loopless multigraphs on 2 vertices is

$$\frac{x}{1-x} \mapsto 0,1,1,1,1,1,(E\ge 0, V=2). \tag{22}$$

The generating function for the number of connected loopless multigraphs on 3 vertices is [10, A253186]

$$\frac{\left(x^3-x-1\right)x^2}{\left(-1+x\right)^3\left(x+1\right)\left(x^2+x+1\right)} \mapsto 0,0,1,2,3,4,6,7,9,11,13,15,18,20,23(E\ge 0, V=3). \tag{23}$$

On 5 vertices

$$\frac{x^4p_5(x)}{\left(-1+x\right)^{10}\left(x^2+x+1\right)^3\left(x+1\right)^4\left(x^2-x+1\right)\left(x^4+x^3+x^2+x+1\right)^2\left(x^2+1\right)^2} \mapsto 0,0,0,0,3,11,34,85,193,396,771,1411,2490,4221(E\ge 0, V=5), \tag{24}$$

where

$$\begin{aligned} p_5 \equiv\ & 3+5\,x+12\,x^2+17\,x^3+26\,x^4+27\,x^5+35\,x^6+28\,x^7+38\,x^8+30\,x^9+39\,x^{10} \\ & +37\,x^{11}+34\,x^{12}+24\,x^{13}+15\,x^{14}+3\,x^{15}-7\,x^{16}-9\,x^{17}+4\,x^{20} \\ & +5\,x^{22}+3\,x^{23}-8\,x^{18}-x^{19}+6\,x^{21}-2\,x^{24}-2\,x^{25}-2\,x^{26}-x^{27}+x^{29}. \end{aligned} \tag{25}$$

On 6 vertices

$$-\frac{x^5p_6}{\left(-1+x\right)^{15}\left(x+1\right)^6\left(x^2+1\right)^3\left(x^2+x+1\right)^5\left(x^2-x+1\right)^2\left(x^4+x^3+x^2+x+1\right)^3} \mapsto 0,0,0,0,0,6,29,110,348,969,2445,5746,12736,26843,54256,105669(E\ge 0, V=6), \tag{26}$$

where

(27)

$$p_6 \equiv 6+11\,x+35\,x^2+70\,x^3+134\,x^4+217\,x^5+348\,x^6+533\,x^7+726\,x^8+1038\,x^9+1290\,x^{10} +1629\,x^{11}+1810\,x^{12}+2040\,x^{13}+1976\,x^{14}+1984\,x^{15}+1696\,x^{16}+1542\,x^{17}+1206\,x^{18} +1050\,x^{19}+787\,x^{20}+636\,x^{21}+474\,x^{22}+273\,x^{23}+169\,x^{24}-11\,x^{25}-31\,x^{26}-97\,x^{27}-44\,x^{28} -8\,x^{29}+33\,x^{30}+63\,x^{31}+32\,x^{32}+38\,x^{33}-17\,x^{34}-14\,x^{35}-31\,x^{36}-8\,x^{37}-5\,x^{38}+8\,x^{39} +11\,x^{40}+4\,x^{41}+3\,x^{42}-4\,x^{43}-3\,x^{45}+x^{47}.$$

On 7 vertices

(28)

$$\frac{x^6 p_7(x)}{(-1+x)^{21}(x^4+x^3+x^2+x+1)^4(x^2+x+1)^7(x+1)^9(x^2+1)^4(x^2-x+1)^3} \times \frac{1}{(x^4-x^2+1)(x^4-x^3+x^2-x+1)(x^6+x^5+x^4+x^3+x^2+x+1)^3}$$

$$\mapsto 0,0,0,0,0,0,11,70,339,1318,4457,13572,38201,100622,251078,\ldots(E\geq 0, V=7),$$

where

(29)

$$p_7(x) \equiv -11-48\,x-188\,x^2-570\,x^3-1526\,x^4-3675\,x^5-8284\,x^6-17431\,x^7-35005\,x^8 -66742\,x^9-121908\,x^{10}-213342\,x^{11}-359515\,x^{12}-583522\,x^{13}-916091\,x^{14}-1391716\,x^{15} -2051981\,x^{16}-2938963\,x^{17}-4097420\,x^{18}-5564508\,x^{19}-7373793\,x^{20}-9539279\,x^{21} -12063528\,x^{22}-14919997\,x^{23}-18064473\,x^{24}-21418776\,x^{25}-24890827\,x^{26}-28355984\,x^{27} -31688266\,x^{28}-34742272\,x^{29}-37387611\,x^{30}-39493274\,x^{31}-40963946\,x^{32}-41717383\,x^{33} -41723196\,x^{34}-40973187\,x^{35}-39511812\,x^{36}-37405689\,x^{37}-34764514\,x^{38}-31705308\,x^{39} -28372262\,x^{40}-24898844\,x^{41}-21423490\,x^{42}-18060699\,x^{43}-14913079\,x^{44}-12050303\,x^{45} -9525196\,x^{46}-7357519\,x^{47}-5550815\,x^{48}-4085547\,x^{49}-2932089\,x^{50}-2048825\,x^{51} -1393454\,x^{52}-920594\,x^{53}-590477\,x^{54}-366935\,x^{55}-220705\,x^{56}-128024\,x^{57}-71511\,x^{58} -37993\,x^{59}-18932\,x^{60}-8318\,x^{61}-2668\,x^{62}+247\,x^{63}+1501\,x^{64}+1827\,x^{65}+1523\,x^{66} +980\,x^{67}+357\,x^{68}-99\,x^{69}-369\,x^{70}-387\,x^{71}-247\,x^{72}-23\,x^{73}+152\,x^{74} +230\,x^{75}+205\,x^{76}+118\,x^{77}+15\,x^{78}-61\,x^{79}-88\,x^{80}-74\,x^{81}-33\,x^{82}+3\,x^{83}+26\,x^{84} +28\,x^{85}+19\,x^{86}+5\,x^{87}-4\,x^{88}-7\,x^{89}-5\,x^{90}-x^{91}+x^{92}+x^{93}.$$

As a cross-check on these numbers we note that using a weight $t_i(x) = x$ instead of (4) just counts the underlying simple graphs; it computes the generating functions down columns of Table 57.

5. Multisets of Connected Graphs

If a table of connected graphs as a function of vertex count and edge count is known, the Multiset Transformation generates tables of disconnected graphs with fixed number of components. The calculation involves creating an intermediate multiset of edge-vertex pairs of the components, and looking up a product of multiset coefficients as a function of the number of connected graphs that support the pairs. The technique is demonstrated for simple (undirected, unlabeled, loopless) graphs and for undirected, unlabeled loopless graphs allowing multiedges.

5.1. **The Multiset Coefficient.** The concept of the multiset is based on the concept of the set (a collection of objectes, only one object of a given type), but allows to put more than one object of a type into the collection [9].

Definition 1. *A Multiset is a collection of objects with some individual count (object of type i appearing f_i times in the collection). The objects have no order in the collection.*

The number of ways of assembling a multiset with m objects plugged from a set of n different objects is a variant of Pascal's triangle of binomial coefficients,

$$P(n,m) = \binom{n+m-1}{m}. \tag{30}$$

The equation may be illustrated for small orders m:

- If there is only $n = 1$ type of objects, the multiset has only one choice: it contains m replicates of the unique object. $P(1,m) = 1$.
- If the multiset contains $m = 1$ object, it contains one object of n candidates. $P(n,m) = n$.
- If the multiset contains $m = 2$ objects, it contains either the same type of object twice (n choices), or two different objects ($\binom{n}{2}$ choices), so $P(n,2) = n + \binom{n}{2} = \binom{n+1}{2}$.
- If the multiset contains $m = 3$ objects, it either contains the same type of object thrice (n choices), or one type of object once and another type of object twice ($n(n-1)$ choices), or three different types of objects ($\binom{n}{3}$ choices); so $P(n,3) = n + n(n-1) + \binom{n}{3} = \binom{n+2}{3}$.
- If the multiset contains $m = 4$ objects, we consider all five partitions of m, namely $4^1, 1^1 3^1, 2^2, 1^2 2^1, 1^4$: it either contains the same type of object 4 times (n choices), or one type of object once and another type of object thrice ($n(n-1)$ choices), or two pairs of objects ($\binom{n}{2}$ choices), or two different objects and one pair of objects ($\binom{n}{2}(n-2)$ choices), or four different types of objects ($\binom{n}{4}$ choices); so $P(n,4) = n + n(n-1) + \binom{n}{2} + \binom{n}{2}(n-2) + \binom{n}{4} = \binom{n+3}{4}$.

Proof. Formula (30) can be rephrased with [7, (3.8)] (setting $r = m-1$, $k = j$, $n = 1$ there) or with [8, (1.11)]:

$$\binom{n+m-1}{m} = \sum_{j=1}^{m} \binom{n}{j}\binom{m-1}{j-1}. \tag{31}$$

The first factor on the right hand side indicates that in a first step one can create a set of j distinct objects out of n in $\binom{n}{j}$ ways. Consider that set sorted by some lexicographic order. Then the factor $\binom{m-1}{j-1}$ counts in how many ways one can insert separators in the multiset of the same lexicographic ordering to select switch-over from one type of object to the next one. □

Example 3. *To create multisets of $m = 4$ objects given n distinct objects, selecting $j = 1$ type of object gives the multiset $o_1o_1o_1o_1$ (no separator), $\binom{3}{0} = 1$ choices; selecting $j = 2$ types of objects gives $o_1o_1o_1|o_2$ or $o_1o_1|o_2o_2$ or $o_1|o_2o_2o_2$ with $\binom{3}{1} =$*

3 positions of the separator; selecting $j = 3$ types of objects gives $o_1o_1|o_2|o_3$ or $o_1|o_2o_2|o_3$ or $o_1|o_2|o_3o_3$ with $\binom{3}{2} = 3$ positions of the 2 separators; or selecting $j = 4$ types of objects gives $o_1|o_2|o_3|o_4$ with $\binom{3}{3} = 1$ positions of the 3 separators.

5.2. **Multiset Transform of An Integer Sequence.** The Multiset Transform deals with the question: if the objects of type i have some weight (expressed as a positive integer), in how many ways can we assemble a multiset of the objects with some prescribed total weight? The total weight is the usual arithmetic sum of the weights of the objects.

Example 4. *(Money Exchange Problem) In how different ways can you combine coins (weight 5 for type nickels, weight 10 for type dimes and weight 25 for type quarter...) for a wallet worth 200 (2 dollars)?*

Example 5. *In how different ways can you fill a bag of 10 kg (weight 100) with apples of 100 g (weight 1) and oranges of 200 g (weight 2)?*

The Multiset Transform computes the number $T_{n,k}$ of multisets containing k objects, drawn from a set of objects of which there are $T_{n,1}$ of some additive weight n [1]. Given $T_{0,1} = 1$ and an integer sequence $T_{n,1}$ for the number of objects in the weight class n, the T of total weight n are calculated recursively by

$$T_{n,k} = \sum_{f_1n_1+f_2n_2+\cdots+f_kn_k=n} \prod_{i=1}^{k} P(T_{n_i,1}, f_i), \tag{32}$$

where the sum is over the partitions of n into parts n_i which occur with frequencies f_i.

Remark 1. *The row sums $\sum_{k=1}^{n} T_{n,k}$ are obtained by the Euler Transform of the sequence $T_{n,1}$* [1, (25)].

Example 6. *Given two types of nickels (2 types of weight 5, tin and copper), one type of dime (weight 10), and one type of quarter (weight 25), the integer sequence $T_{n,1}$ is* $(1), 0, 0, 0, 0, 2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 25$.

Example 7. *Given three types of apples (brown, yellow and red, each of weight 1), one type of banana (weight 2), and one type melon (weight 4), the integer sequence is* $(1), 3, 1, 0, 1, 0, 0, 0 \ldots$. *The Multiset Transform generates the triangular table*

$n\backslash k$	1	2	3	4	5	6		
1	3							
2	1	6						
3	0	3	10					
4	1	1	6	15				
5	0	3	3	10	21			
6	0	1	7	6	15	28		
7	0	0	3	13	10	21	36	
8	0	1	1	7	21	15	28	45

The row sums in the table are 3, 7, 13, 23, 37, 57, 83, 118... There are $T_{3,2} = 3$ ways of generating a weight of 3 with two objects (a banana and any of the three types of apples). There are $T_{3,3} = 10$ ways of generating a weight of 3 with three objects [three apples (bbb), (yyy), (rrr), (byy), (brr), (bby), (yrr), (bbr), (yyr), (bry)]. There is $T_{6,2} = 1$ way to generate a weight of 6 with two objects (a banana and a melon).

Example 8. *If there is one type of object of each weight, $T_{n,1} = 1$, the Multiset Transform generates the partition numbers* [10, A008284]*, and the row sums are the partition numbers* [10, A000041].

5.3. **Graphs specified by number of components.** If the sequence $T_{n,1}$ enumerates connected graphs of type n (where the weight n is either the vertex count or the edge count), one fundamental way of generating a multiset is putting k of them side by side and considering them a graph with k components. Vertex or edge number are additive, as required.

Example 9. *If $T_{n,1}$ denotes connected graphs with n nodes* [10, A001349]*, the Multiset Transform counts graphs with k components* [10, A201922].

Example 10. *If $T_{n,1}$ denotes connected graphs with n edges* [10, A002905]*, the Multiset Transform counts graphs with n edges and k components* [10, A275421].

Example 11. *If $T_{n,1}$ denotes trees with n nodes* [10, A000055]*, the Multiset Transform counts forests with k trees* [10, A095133].

Example 12. *If $T_{n,1}$ denotes rooted trees with n nodes* [10, A000081]*, the Multiset Transform counts rooted forests with k trees* [10, A033185].

Example 13. *If $T_{n,1}$ denotes connected regular graphs with n nodes* [10, A005177]*, the Multiset Transform counts regular graphs with k components* [10, A275420]*. In the case of cubic graphs the transform pair is* [10, A002851] *and* [10, A275744].

6. Graphs specified by number of edges, vertices and components

6.1. **Union of connected graphs.** Let $G(E,V,k)$ be the number of graphs with E edges, V vertices and k components. $G(E,V,1)$ is the number of connected graphs with E edges and V vertices. The other properties like whether the graphs are labeled, may contain loops or multiedges, are not classified here, but assumed to be fixed while composing graphs with k components from connected graphs. (A multiset of labeled connected graphs is a disconnected labeled graph; a multiset of connected oriented graphs is a disconnected oriented graph; and so on.) The unified graph is the multiset union of connected graphs $\mathcal{G}_1$ which individually have e_i edges and v_i vertices:

$$\mathcal{G}_k(E,V) = \bigcup_{i=1}^{k} \mathcal{G}_1(e_i, v_i), \tag{33}$$

where both the number of edges and the number of vertices are additive:

$$E = \sum_i^k e_i; \quad V = \sum_i^k v_i. \tag{34}$$

Summation over a set of the variables creates marginal sums: $G(.,V,k) = \sum_{E\ge 0} G(E,V,k)$ are the graphs with V vertices and k components. $G(E,,k) = \sum_{V\ge 1} G(E,V,k)$ is the number of graphs with E edges and k components. $G(E,V,.) = \sum_{k\ge 1} G(E,V,k)$ is the number of graphs with E edges and V vertices.

6.2. **Correlated Multiset Transforms.** The particular case we explore here is that constructing a disconnected graph from connected components means building a multiset of connected graphs, where the number of edges and *also* the number of vertices are such an additive weight.

The examples of Section 5.3 illustrated how $G(E,,k)$ is the Multiset Transform of $G(E,,1)$ [10, A076864,A275421,A191970] and $G(,V,k)$ is the Multiset Transform of $G(,V,1)$ [10, A054924,A275420,A281446]. The aim of this paper is to demonstrate a similar technique for $G(E,V,k)$ assuming $G(E,V,1)$ is known.

Each graph which is a component contributing to $G(E,V,k)$ has a specific pair (e_i,v_i) of edge and vertex count; the union of these graphs is a multiset of such pairs—which means in the multiset of graphs contributing to $G(E,V,k)$, each pair may occur more than once, and each pair may represent (in the sense of the weights above) more than one graph because there may be more than one distinct connected graph for one pair of E and V.

The calculation starts by constructing all weak compositions of E into k parts e_i, and all weak compositions of V into k parts v_i of pairs (e_i,v_i) compatible with the requirement (34). This defines a two-dimensional $\binom{E+k-1}{k-1}\times\binom{V+k-1}{k}$ outer product matrix with multisets [13].

Example 14. *If* $E=2$ *and* $V=3$ *and* $k=3$*, the compositions are* 2 =2+0+0 =0+2+0 =0+0+2 =1+1+0 =1+0+1 =0+1+1, 3 =3+0+0 =0+3+0 =0+0+3 =2+1+0 =1+2+0 =2+0+1 =2+1+0 = ..., *and the matrix contains multisets with k pairs:*

$\sum e_i\backslash\sum v_i$	3+0+0	0+3+0	2+1+0	...	1+1+1
2+0+0	(2,3)(0,0)(0,0)	(2,0)(0,3)(0,0)	...		(2,1)(0,1)(0,1)
0+2+0	(0,3)(2,0)(0,0)	(0,0)(2,3)(0,0)	...		(0,1)(2,1)(0,1)
0+0+2	...				
1+1+0	...				
0+1+1	(0,3)(1,0)(1,0)	(0,0)(1,3)(1,0)	...		(0,1)(1,1)(1,1)
	...				

Each element of the matrix is a multiset $(e_1,v_1)(e_2,v_2)\cdots(e_k,v_k)$ of pairs obtained by interleaving the e_i and v_i components of the compositions. At that point we realize that

(1) if any of the v_i is zero, the method of selecting such a null-graph into the disconnected graph would not fulfill the requirement of being k-connected. So actually only the compositions (not the weak compositions) of V need to be considered as table columns.
(2) because the decompositions $(e_1,v_1),(e_2,v_2)\cdots$ are candidates for multiset compositions, the order of the pairs does not matter. In the example, $(2,1)(0,1)(0,1)$ and $(0,1),(2,1)(0,1)$ are the same scheme of selecting connected graphs. So we may sort for example each k-set by the e_i member of the pair without loss of samples, which means, we may build the table by just considering weak partitions (not all weak compositions) of E into parts e_i as table rows.

Example 15. *Continuing Example 14 above, the table reduces to*

$\sum e_i\backslash\sum v_i$	1+1+1
2+0+0	(2,1)(0,1)(0,1)
1+1+0	(1,1)(1,1)(0,1)

$E\backslash V$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	1												
1	0	0	1											
2	0	0	0	2										
3	0	0	0	1	3									
4	0	0	0	0	3	6								
5	0	0	0	0	1	8	11							
6	0	0	0	0	1	7	22	23						
7	0	0	0	0	0	5	27	58	46					
8	0	0	0	0	0	2	28	101	157	99				
9	0	0	0	0	0	1	23	142	358	426	216			
10	0	0	0	0	0	1	15	161	660	1233	1166	488		
11	0	0	0	0	0	0	10	156	1010	2873	4163	3206	1121	
12	0	0	0	0	0	0	5	138	1356	5705	11987	13847	8892	2644
13	0	0	0	0	0	0	2	101	1613	9985	29652	48071	45505	24743

TABLE 81. $G(E, V, 2)$. Simple graphs with 2 components and a total of E edges and V vertices. See [10, A274934] for column sums, [10, A274937] for the diagonal.

The $(2,1)(0,1)(0,1)$ *entry indicates to take one connected graph with 2 edges and one vertex (obviously a double-loop) and two graphs without edges and one vertex (single points). The* $(1,1)(1,1)(0,1)$ *entry indicates to take 2 graphs with an edge and a vertex (2 points, each with a loop) and a graph without edges and a single vertex (a single point).*

The number of k-compositions of graphs then is the sum over all unique remaining multisets in the table, akin to Equation (32):

$$G(E, V, k) = \sum_{(e_i, v_i)^{f_i}} \prod_i P(G(e_i, v_i, 1), f_i) \tag{35}$$

where f_i is the frequency (number of occurrences) of the pair (e_i, v_i) in the multiset.

6.3. **Simple Graphs.** The most important example considers undirected, unlabeled graphs without multiedges or loops. Table 57 shows the base information $G(E, V, 1)$ which is fed into the formula to compute the tables $G(E, V, k)$ 81–84 for $k \geq 1$. (These tables differ from Steinbach's tables [11, Vol. 4, Table 1.2a] because our components may be/have isolated vertices.)

The arithmetic sum over these tables $k \geq 1$ yields Table 65.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	0	1											
1	0	0	0	1										
2	0	0	0	0	2									
3	0	0	0	0	1	4								
4	0	0	0	0	0	3	7							
5	0	0	0	0	0	1	9	14						
6	0	0	0	0	0	1	7	25	29					
7	0	0	0	0	0	0	5	29	68	60				
8	0	0	0	0	0	0	2	29	110	186	128			
9	0	0	0	0	0	0	1	23	149	397	509	284		
10	0	0	0	0	0	0	1	15	164	699	1377	1399	636	
11	0	0	0	0	0	0	0	10	157	1041	3070	4685	3857	1467
12	0	0	0	0	0	0	0	5	139	1375	5919	12899	15646	10706
13	0	0	0	0	0	0	0	2	101	1625	10183	30980	52024	51622

TABLE 82. $G(E, V, 3)$. Simple graphs with 3 components and a total of E edges and V vertices. Column sums are in column 3 of [10, A201922] $\mapsto 1, 1, 3, 9, 32, 154, 1065, 12513, 276114, 12021725\ldots$.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	1											
1	0	0	0	0	1										
2	0	0	0	0	0	2									
3	0	0	0	0	0	1	4								
4	0	0	0	0	0	0	3	8							
5	0	0	0	0	0	0	1	9	15						
6	0	0	0	0	0	0	1	7	26	32					
7	0	0	0	0	0	0	0	5	29	71	66				
8	0	0	0	0	0	0	0	2	29	112	196	143			
9	0	0	0	0	0	0	0	1	23	150	406	539	315		
10	0	0	0	0	0	0	0	1	15	164	706	1417	1486	710	
11	0	0	0	0	0	0	0	0	10	157	1044	3110	4834	4105	1631
12	0	0	0	0	0	0	0	0	5	139	1376	5951	13102	16193	11408
13	0	0	0	0	0	0	0	0	2	101	1626	10202	31198	52966	53519

TABLE 83. $G(E, V, 4)$. Simple graphs with 4 components and a total of E edges and V vertices.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	1										
1	0	0	0	0	0	1									
2	0	0	0	0	0	0	2								
3	0	0	0	0	0	0	1	4							
4	0	0	0	0	0	0	0	3	8						
5	0	0	0	0	0	0	0	1	9	16					
6	0	0	0	0	0	0	0	1	7	26	33				
7	0	0	0	0	0	0	0	0	5	29	72	69			
8	0	0	0	0	0	0	0	0	2	29	112	199	149		
9	0	0	0	0	0	0	0	0	1	23	150	408	549	330	
10	0	0	0	0	0	0	0	0	1	15	164	707	1426	1516	742
11	0	0	0	0	0	0	0	0	0	10	157	1044	3117	4874	4193
12	0	0	0	0	0	0	0	0	0	5	139	1376	5954	13142	16343
13	0	0	0	0	0	0	0	0	0	2	101	1626	10203	31230	53170

TABLE 84. $G(E, V, 5)$. Simple graphs with 5 components and a total of E edges and V vertices.

E\V	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1										
1	0	0	1									
2	0	0	1	2								
3	0	0	1	3	3							
4	0	0	1	5	8	6						
5	0	0	1	6	17	20	11					
6	0	0	1	9	32	58	52	23				
7	0	0	1	10	53	135	185	132	46			
8	0	0	1	13	84	290	548	586	344	99		
9	0	0	1	15	127	565	1441	2108	1829	900	216	
10	0	0	1	18	184	1055	3456	6696	7884	5680	2834	488
11	0	0	1	20	259	1859	7774	19288	29633	28718	17546	6811
12	0	0	1	24	359	3178	16578	51799	100810	126013	102743	54469

TABLE 85. Undirected loopless multigraphs with 2 components and a total of E edges and V vertices.

E\V	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	1									
1	0	0	0	1								
2	0	0	0	1	2							
3	0	0	0	1	3	4						
4	0	0	0	1	5	9	7					
5	0	0	0	1	6	19	23	14				
6	0	0	0	1	9	35	65	62	29			
7	0	0	0	1	10	57	148	214	159	60		
8	0	0	0	1	13	89	313	614	681	421	128	
9	0	0	0	1	15	134	601	1577	2374	2148	1104	284
10	0	0	0	1	18	192	1110	3711	7353	8938	6683	3389
11	0	0	0	1	20	269	1938	8225	20752	32692	32639	20712
12	0	0	0	1	24	371	3289	17332	54847	108802	139316	117082

TABLE 86. Undirected loopless multigraphs with 3 components and a total of E edges and V vertices.

6.4. **Loopless connected Multigraphs.** Another application of the algorithm is to construct [10, A192517] from [10, A191646]. The base information of $G(E, V, k = 1)$ is this time in Table 61, and the Multiset Transformations creates Tables 85–88 and so on ($k \geq 2$). The sum over all $k \geq 1$ converges to Table 69.

REFERENCES

1. Philippe Flajolet and Robert Sedgewick, *Analytic combinatorics*, Cambridge University Press, 2009. MR 2483235
2. F. Freudenstein, *The basic concepts of polya's theory of enumeration, with application to the structual classification of mechanisms*, Jn. Mechanisms **3** (1967), 275–290.
3. E. N. Gilbert, *Enumeration of labelled graphs*, Can. J. Math. **8** (1956), 405–411.
4. Jonathan L. Gross and Jay Yellen, *Handbook of graph theory*, Discrete Mathematics and its applications, CRC Press, 2004.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	1									
1	0	0	0	0	1								
2	0	0	0	0	1	2							
3	0	0	0	0	1	3	4						
4	0	0	0	0	1	5	9	8					
5	0	0	0	0	1	6	19	24	15				
6	0	0	0	0	1	9	35	67	65	32			
7	0	0	0	0	1	10	57	151	221	169	66		
8	0	0	0	0	1	13	89	318	628	711	449	143	
9	0	0	0	0	1	15	134	607	1603	2445	2248	1185	315
10	0	0	0	0	1	18	192	1119	3754	7506	9227	7025	3608
11	0	0	0	0	1	20	269	1949	8294	21049	33426	33790	21799
12	0	0	0	0	1	24	371	3304	17437	55396	110485	142723	121385

TABLE 87. Undirected loopless multigraphs with 4 components and a total of E edges and V vertices.

$E\backslash V$	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	0	0	0	1									
1	0	0	0	0	0	1								
2	0	0	0	0	0	1	2							
3	0	0	0	0	0	1	3	4						
4	0	0	0	0	0	1	5	9	8					
5	0	0	0	0	0	1	6	19	24	16				
6	0	0	0	0	0	1	9	35	67	66	33			
7	0	0	0	0	0	1	10	57	151	223	172	69		
8	0	0	0	0	0	1	13	89	318	631	718	459	149	
9	0	0	0	0	0	1	15	134	607	1608	2459	2278	1213	330
10	0	0	0	0	0	1	18	192	1119	3761	7533	9299	7126	3690
11	0	0	0	0	0	1	20	269	1949	8304	21095	33584	34084	22146
12	0	0	0	0	0	1	24	371	3304	17450	55472	110799	143481	122562

TABLE 88. Undirected loopless multigraphs with 5 components and a total of E edges and V vertices.

5. Frank Harary and Edgar M. Palmer, *Graphical enumeration*, Academic Press, New York, London, 1973. MR 0357214 (50 #9682)
6. G. Pólya, Robert E. Tarjan, and Donald R. Woods, *Notes on introductory combinatorics*, Birkhäuser, 1983. MR 0741680
7. Ranjan Roy, *Binomial identities and hypergeometric series*, Amer. Math. Monthly **94** (1987), no. 1, 36–46. MR 0873603
8. Sharad S. Sane, *Combinatorial techniques*, Text and readings in mathematics, vol. 64, Hindustan Book Agency, 2013. MR 2963397
9. D. Singh, A. M. Ibrahim, T. Yohanna, and J. N. Singh, *An overview of the applications of multisets*, Novi Sad J. Math. **37** (2007), no. 2, 73–92. MR 2401607
10. Neil J. A. Sloane, *The On-Line Encyclopedia Of Integer Sequences*, Notices Am. Math. Soc. **50** (2003), no. 8, 912–915, http://oeis.org/. MR 1992789 (2004f:11151)
11. Peter Steinbach, *Field guide to simple graphs*, 2006.
12. Alan Tucker, *Polya's enumeration formula by example*, Math. Mag. **47** (1974), no. 5, 248–256. MR 0357128

13. Thomas Wieder, *Generation of all possible multiselections from a multiset*, Prog. Appl. Math. **2** (2011), no. 1, 61–66.

URL: `http://www.mpia.de/~mathar`

Max-Planck Institute of Astronomy, Königstuhl 17, 69117 Heidelberg, Germany