

QUASITRIVIAL SEMIGROUPS: CHARACTERIZATIONS AND ENUMERATIONS

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ABSTRACT. We investigate the class of quasitrivial semigroups and provide various characterizations of the subclass of quasitrivial and commutative semigroups as well as the subclass of quasitrivial and order-preserving semigroups. We also determine explicitly the sizes of these classes when the semigroups are defined on finite sets. As a byproduct of these enumerations, we obtain several new integer sequences.

1. INTRODUCTION

Let X be an arbitrary nonempty set. We use the symbol X_n if X is finite and contains $n \geq 1$ elements, in which case we assume w.l.o.g. that $X_n = \{1, \dots, n\}$.

In this paper we investigate the class of binary operations $F: X^2 \rightarrow X$ that are associative and quasitrivial, where quasitriviality means that F always outputs one of its input values. In the algebraic language, the pair (X, F) is then called a *quasitrivial semigroup* (for general background, see, e.g., [9, 11, 14] and for a recent reference, see [1]). We also investigate certain subclasses of quasitrivial semigroups by adding properties such as commutativity, order-preservation, and the existence of neutral elements. The case where the semigroups are defined on finite sets (i.e., $X = X_n$ for any integer $n \geq 1$) is of particular interest as it enables us to address and solve various enumeration issues. We remark that most of our results rely on a simple known theorem (Theorem 2.1) that provides a descriptive characterization of the class of quasitrivial semigroups.

After presenting some definitions and preliminary results (including Theorem 2.1) in Section 2, we provide in Section 3 different characterizations of the class of quasitrivial and commutative (i.e., Abelian) semigroups on both arbitrary sets and finite sets (Theorem 3.3). In the latter case we illustrate some of our results by showing the contour plots of the operations. When X is endowed with a total ordering we also characterize the subclass of quasitrivial, commutative, and order-preserving semigroups (Theorem 3.7) by means of the single-peakedness property, which is a generalization to arbitrary totally ordered sets of a notion introduced 70 years ago in social choice theory. In Section 4 we introduce the “weak single-peakedness” property (Definition 4.3) as a further generalization of single-peakedness to arbitrary weakly ordered sets to characterize the class of quasitrivial and order-preserving semigroups (Theorem 4.5). In the special case where the semigroups are defined on finite sets, the class of quasitrivial semigroups is also finite. This raises the problem of computing the size of this class as well as the sizes of all subclasses discussed in this paper. We tackle this problem in Section 4 where we arrive at some known integer sequences as well as new ones. The number of quasitrivial

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semigroups on X_n for any integer $n \geq 1$ (Theorem 4.1) gives rise to a sequence that was previously unknown in the Sloane's On-Line Encyclopedia of Integer Sequences (OEIS, see [18]). All the (old and new) sequences that we consider are given in explicit forms (i.e., closed-form expressions) and/or through their generating functions or exponential generating functions (Theorem 4.1 and Propositions 4.2, 4.8, 4.9, 4.11, and 4.12). In Section 5 we further investigate the single-peakedness and weak single-peakedness properties and provide a graphical characterization of weakly single-peaked weak orderings.

2. PRELIMINARIES

Recall that a binary relation R on X is said to be

- *total* if $\forall x, y: xRy$ or yRx ;
- *transitive* if $\forall x, y, z: xRy$ and yRz implies xRz ;
- *antisymmetric* if $\forall x, y: xRy$ and yRx implies $x = y$.

A *total ordering* on X is a binary relation \leq on X that is total, transitive, and antisymmetric. A *weak ordering* on X is a binary relation \lesssim on X that is total and transitive. We denote the symmetric and asymmetric parts of \lesssim by \sim and $<$, respectively. Recall that \sim is an equivalence relation on X and that $<$ induces a total ordering on the quotient set X/\sim . Thus, defining a weak ordering on X amounts to defining an ordered partition of X . For any $a \in X$, we use the notation $[a]_{\sim}$ to denote the equivalence class of a , i.e., $[a]_{\sim} = \{x \in X : x \sim a\}$.

For any total ordering \leq on X , the pair (X, \leq) is called a *totally ordered set* or a *chain*. Similarly, for any weak ordering \lesssim on X , the pair (X, \lesssim) is called a *weakly ordered set*. For any integer $n \geq 1$, we assume w.l.o.g. that the pair (X_n, \leq_n) represents the set $X_n = \{1, \dots, n\}$ endowed with the total ordering relation \leq_n defined by $1 <_n \dots <_n n$.

If (X, \lesssim) is a weakly ordered set, an element $a \in X$ is said to be *maximal* (resp. *minimal*) for \lesssim if $x \lesssim a$ (resp. $a \lesssim x$) for all $x \in X$. We denote the set of maximal (resp. minimal) elements of X for \lesssim by $\max_{\lesssim} X$ (resp. $\min_{\lesssim} X$). Note that this set need not be nonempty (consider, e.g., the set of nonnegative integers endowed with the usual total ordering \leq).

An operation $F: X^2 \rightarrow X$ is said to be

- *associative* if $F(F(x, y), z) = F(x, F(y, z))$ for all $x, y, z \in X$;
- *idempotent* if $F(x, x) = x$ for all $x \in X$;
- *quasitrivial* (or *conservative*) if $F(x, y) \in \{x, y\}$ for all $x, y \in X$;
- *commutative* if $F(x, y) = F(y, x)$ for all $x, y \in X$;
- *\leq -preserving* for some total ordering \leq on X if for any $x, y, x', y' \in X$ such that $x \leq x'$ and $y \leq y'$, we have $F(x, y) \leq F(x', y')$.

Given a weak ordering \lesssim on X , the *maximum* (resp. *minimum*) *operation* on X w.r.t. \lesssim is the commutative operation \max_{\lesssim} (resp. \min_{\lesssim}) defined on $X^2 \setminus \{(x, y) \in X^2 : x \sim y, x \neq y\}$ by $\max_{\lesssim}(x, y) = y$ (resp. $\max_{\lesssim}(x, y) = x$) whenever $x \lesssim y$. We observe that if \lesssim reduces to a total ordering, then the operation \max_{\lesssim} (resp. \min_{\lesssim}) is defined everywhere on X^2 .

Also, the *projection operations* $\pi_1: X^2 \rightarrow X$ and $\pi_2: X^2 \rightarrow X$ (also called *left-* and *right-semigroups*) are respectively defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for all $x, y \in X$.

An element $e \in X$ is said to be a *neutral element* of $F: X^2 \rightarrow X$ if $F(x, e) = F(e, x) = x$ for all $x \in X$. An element $a \in X$ is said to be an *annihilator element* of $F: X^2 \rightarrow X$ if $F(x, a) = F(a, x) = a$ for all $x \in X$.

For any integer $n \geq 1$, any $F: X_n^2 \rightarrow X_n$, and any $z \in X_n$, the *F-degree* of z , denoted $\deg_F(z)$, is the number of points $(x, y) \in X_n^2 \setminus \{(z, z)\}$ such that $F(x, y) = F(z, z)$.

Also, the *degree sequence of F* , denoted \deg_F , is the nondecreasing n -element sequence of the numbers $\deg_F(x)$, $x \in X_n$.

We now state the key theorem on which most of our results rely. It provides a descriptive characterization of the class of associative and quasitrivial operations on X . As observed by Ackerman [1, Section 1.2], this result is a simple consequence of two papers on idempotent semigroups (see Kimura [12] and McLean [17]). It was also independently presented by various authors (see, e.g., Kepka [11, Corollary 1.6] and Länger [14, Theorem 1]). For the sake of completeness we provide a direct elementary proof.

Theorem 2.1. *$F: X^2 \rightarrow X$ is associative and quasitrivial iff there exists a weak ordering \lesssim on X such that*

$$(1) \quad F|_{A \times B} = \begin{cases} \max_{\lesssim}|_{A \times B}, & \text{if } A \neq B, \\ \pi_1|_{A \times B} \text{ or } \pi_2|_{A \times B}, & \text{if } A = B, \end{cases} \quad \forall A, B \in X / \sim.$$

Proof. (Sufficiency) Trivial.

(Necessity) We observe that the binary relation \lesssim defined on X as

$$x \lesssim y \iff F(x, y) = y \text{ or } F(y, x) = y, \quad x, y \in X,$$

is a weak ordering on X . Indeed, this relation is clearly total. Let us show that it is transitive. Let $x, y, z \in X$ be pairwise distinct and such that $x \lesssim y$ and $y \lesssim z$. Let us assume for instance that $F(x, y) = y$ and $F(z, y) = z$ (the other three cases can be dealt with similarly). Then we have $F(x, z) = z$ and hence $x \lesssim z$. Indeed, otherwise we would have $x = F(x, z) = F(x, F(z, y)) = F(F(x, z), y) = F(x, y) = y$, a contradiction.

Let us now show that Eq. (1) holds. It is easy to see that for any $x, y \in X$ such that $x < y$, we have $F|_{\{x, y\}^2} = \max_{\lesssim}|_{\{x, y\}^2}$. Similarly, for any distinct $x, y \in X$ such that $x \sim y$ we have $F|_{\{x, y\}^2} = \pi_1|_{\{x, y\}^2}$ or $F|_{\{x, y\}^2} = \pi_2|_{\{x, y\}^2}$. Finally, let us show that for any pairwise distinct $x, y, z \in X$ such that $x \sim y \sim z$, we cannot have both $F|_{\{x, y\}^2} = \pi_1|_{\{x, y\}^2}$ and $F|_{\{x, z\}^2} = \pi_2|_{\{x, z\}^2}$. Indeed, otherwise

- if $F(y, z) = y$, then $z = F(x, z) = F(F(x, y), z) = F(x, F(y, z)) = F(x, y) = x$,
- if $F(y, z) = z$, then $y = F(y, x) = F(y, F(z, x)) = F(F(y, z), x) = F(z, x) = x$.

We reach a contradiction in each of these cases. \square

It is not difficult to see that for any associative and quasitrivial operation $F: X^2 \rightarrow X$, there is exactly one weak ordering \lesssim on X for which F is of the form (1). Thus, \lesssim is completely determined by F . If $X = X_n$ for some integer $n \geq 1$, then \lesssim can be as well defined as follows: $x \lesssim y$ iff $\deg_F(x) \leq \deg_F(y)$.¹ This latter equivalence can be easily derived from the following proposition.

Proposition 2.2. *If $F: X_n^2 \rightarrow X_n$ is of the form (1) for some weak ordering \lesssim on X_n , then for any $x \in X_n$, we have*

$$\deg_F(x) = 2 \times |\{z \in X_n : z < x\}| + |\{z \in X_n : z \sim x, z \neq x\}|.$$

Proof. Let $x \in X_n$. By quasitriviality, only points of the form (x, z) or (z, x) for some $z \in X_n$ may have the same value as (x, x) .

- If $z < x$, then $F(x, z) = F(z, x) = x = F(x, x)$.
- If $x < z$, then $F(x, z) = F(z, x) = z \neq F(x, x)$.
- If $z \sim x$ and $z \neq x$, then either $F(x, z) = \pi_1(x, z)$ or $F(x, z) = \pi_2(x, z)$. In the first case, we have $F(x, z) = x = F(x, x) \neq z = F(z, x)$. The other case is similar.

¹Thus, when $X = X_n$ the weak ordering \lesssim is completely determined by a set of n integers (actually $n - 1$ integers since $\sum_{x \in X_n} \deg_F(x) = n(n - 1)$).

This completes the proof of Proposition 2.2. \square

We also have the following fact, which immediately follows from the properties of the maximum operation in (1).

Fact 2.3. *If $F: X^2 \rightarrow X$ is of the form (1) for some weak ordering \lesssim on X , then F has a neutral element $e \in X$ (resp. an annihilator $a \in X$) iff the weakly ordered set (X, \lesssim) has a unique minimal element denoted by x^\perp (resp. a unique maximal element denoted by x^\top). In this case we have $e = x^\perp$ (resp. $a = x^\top$).*

Remark 1. If $F: X^2 \rightarrow X$ is of the form (1) for some weak ordering \lesssim on X , then, by replacing \lesssim with its inverse relation \lesssim^{-1} (defined by $a \lesssim^{-1} b \Leftrightarrow b \lesssim a$), we see that F is again of the form (1), except that the maximum operation is changed to the minimum operation. Thus, choosing the maximum or the minimum operation is just a matter of convention.

The following lemma can be obtained by following the first few steps of the proof of [6, Theorem 3], which was stated in the special case where X is an arbitrary closed real interval. For the sake of self-containedness we provide a short proof.

Lemma 2.4 (see [6, Theorem 3]). *If $F: X^2 \rightarrow X$ is associative, idempotent, \leq -preserving for some total ordering \leq on X , and has a neutral element, then F is quasitrivial.*

Proof. Let $\min: X^2 \rightarrow X$ and $\max: X^2 \rightarrow X$ denote respectively the minimum and maximum operations w.r.t. \leq and let e denote the neutral element of F . By idempotency and \leq -preservation we clearly have

$$\min(x, y) \leq F(x, y) \leq \max(x, y), \quad x, y \in X.$$

If $x, y \leq e$, then by \leq -preservation we obtain $F(x, y) \leq \min(F(x, e), F(e, y)) = \min(x, y)$. Thus $F(x, y) = \min(x, y)$ whenever $x, y \leq e$. We show dually that $F(x, y) = \max(x, y)$ whenever $x, y \geq e$. Assume now that $x < e < y$ (the case $y < e < x$ can be dealt with dually). If $F(x, y) \leq e$, then $F(x, y) = F(F(x, x), y) = F(x, F(x, y)) = \min(x, F(x, y)) = x$. We prove similarly that $F(x, y) = y$ whenever $F(x, y) \geq e$. It follows that F is quasitrivial. \square

When X_n is endowed with \leq_n , the operations $F: X_n^2 \rightarrow X_n$ can be visualized through their contour plots, where we connect points in X_n^2 having the same F -values by edges or paths. For instance, the operation $F: X_6^2 \rightarrow X_6$ whose contour plot is shown in Figure 1 is associative, quasitrivial, commutative, and \leq_6 -preserving.

Actually, connecting the points having the same F -values also enables us to use the following graphical tests (see [5]), where Δ_{X_n} denotes the set $\{(x, x) : x \in X_n\}$.

- An $F: X_n^2 \rightarrow X_n$ is quasitrivial iff it is idempotent and every point $(x, y) \in X_n^2 \setminus \Delta_{X_n}$ is connected to either (x, x) or (y, y) .
- If $F: X_n^2 \rightarrow X_n$ is quasitrivial, then $e \in X$ is a neutral element of F iff the point (e, e) is not connected to another point, i.e., $\deg_F(e) = 0$.
- If $F: X_n^2 \rightarrow X_n$ is quasitrivial, then it is associative iff for every rectangle in X_n^2 that has only one vertex on Δ_{X_n} , at least two of the remaining three vertices are connected.

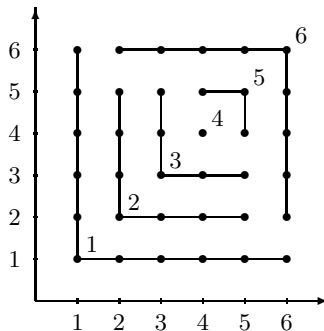


FIGURE 1. An associative and quasitrivial operation on X_6 (contour plot)

3. QUASITRIVIAL AND COMMUTATIVE SEMIGROUPS

In this section we provide characterizations of the class of associative, quasitrivial, and commutative operations $F: X^2 \rightarrow X$, or equivalently, the class of quasitrivial and commutative semigroups on X . We also characterize the subclass of those operations that are order-preserving with respect to some total ordering on X .

The first characterization is given in the following theorem, which immediately follows from Theorem 2.1. Note that Ackerman (see [1, Corollary 4.10]) generalized this result to n -ary semigroups for any integer $n \geq 2$.

Theorem 3.1. $F: X^2 \rightarrow X$ is associative, quasitrivial, and commutative iff there exists a total ordering \leq on X such that $F = \max_{\leq}$.

Theorem 3.3 below provides alternative characterizations of the class of associative, quasitrivial, and commutative operations. We first consider the following auxiliary lemma.

Lemma 3.2. If $F: X^2 \rightarrow X$ is quasitrivial, commutative, \leq -preserving for some total ordering \leq on X , then F is associative.

Proof. This result was established in the special case where X is the real unit interval $[0, 1]$ in [16, Proposition 2]. The proof therein is purely algebraic and hence it applies to any nonempty totally ordered set. \square

Theorem 3.3. Let $F: X^2 \rightarrow X$ be an operation. The following assertions are equivalent.

- (i) F is associative, quasitrivial, and commutative.
- (ii) $F = \max_{\leq}$ for some total ordering \leq on X .
- (iii) F is quasitrivial, commutative, and \leq -preserving for some total ordering \leq on X .

If $X = X_n$ for some integer $n \geq 1$, then any of the assertions (i)–(iii) above is equivalent to any of the following ones.

- (iv) F is quasitrivial and $\deg_F = (0, 2, 4, \dots, 2n - 2)$.
- (v) F is associative, idempotent, commutative, \leq -preserving for some total ordering \leq on X , and has a neutral element.

Moreover, there are exactly $n!$ operations $F: X_n^2 \rightarrow X_n$ satisfying any of the assertions (i)–(v). Furthermore, the total ordering \leq considered in assertion (ii) is uniquely defined as follows: $x \leq y$ iff $\deg_F(x) \leq \deg_F(y)$. In particular, each of these operations has a unique neutral element $e = \min_{\leq} X_n$ and a unique annihilator $a = \max_{\leq} X_n$.

Proof. The equivalence (i) \Leftrightarrow (ii) \Leftrightarrow (iii) follows from Theorem 3.1 and Lemma 3.2. We have (ii) \Rightarrow (v) by Fact 2.3 and (v) \Rightarrow (iii) by Lemma 2.4. Also, it is clear that (ii) \Rightarrow (iv).

Let us now show by induction on n that (iv) \Rightarrow (ii). The result clearly holds for $n = 1$. Suppose that it holds for some $n \geq 1$ and let us show that it still holds for $n + 1$. Assume that $F: X_{n+1}^2 \rightarrow X_{n+1}$ is quasitrivial and that $\deg_F = (0, 2, \dots, 2n)$. Let \leq be the unique total ordering on X_{n+1} defined by $x \leq y$ iff $\deg_F(x) \leq \deg_F(y)$ and let $z = \max_{\leq} X_{n+1}$. Clearly, the operation $F' = F|_{(X_{n+1} \setminus \{z\})^2}$ is quasitrivial and such that $\deg_{F'} = (0, 2, \dots, 2n-2)$. By induction hypothesis we have $F' = \max_{\leq'}$, where \leq' is the restriction of \leq to $(X_{n+1} \setminus \{z\})^2$. Since $\deg_F(z) = 2n$ we necessarily have $F = \max_{\leq}$.

To complete the proof of the theorem, we observe that there are exactly $n!$ total orderings on X_n and hence exactly $n!$ operations $F: X_n^2 \rightarrow X_n$ satisfying assertion (ii). The rest of the statement is immediate. \square

Remark 2. The existence of a neutral element in assertion (v) of Theorem 3.3 cannot be replaced with the existence of an annihilator element. Indeed, the operation $F: X_3^2 \rightarrow X_3$ whose contour plot is depicted in Figure 2 is associative, idempotent, commutative, \leq_3 -preserving, and has the annihilator element $a = 2$. However it is not quasitrivial.

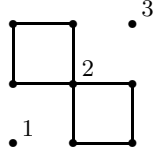


FIGURE 2. An associative operation on X_3 that is not quasitrivial

We now consider the subclass of associative, quasitrivial, and commutative operations $F: X^2 \rightarrow X$ that are \leq -preserving for some total ordering \leq on X . To this extent we recall the single-peakedness property for arbitrary totally ordered sets. This notion was first introduced for finite totally ordered sets (i.e., finite chains) in social choice theory by Black [3, 4].

Definition 3.4 (see [7, Definition 3.8]). Let \leq and \leq be total orderings on X . We say that \leq is *single-peaked w.r.t.* \leq if for any $a, b, c \in X$ such that $a < b < c$, we have $b < a$ or $b < c$.

Example 3.5. There are four total orderings \leq on X_3 that are single-peaked w.r.t. \leq_3 , namely $1 < 2 < 3$, $2 < 1 < 3$, $2 < 3 < 1$, and $3 < 2 < 1$.

For arbitrary total orderings \leq and \leq on X , the operation $F = \max_{\leq}$ need not be \leq -preserving. The following proposition characterizes those total orderings \leq on X for which $F = \max_{\leq}$ is \leq -preserving.

Proposition 3.6 (see [7, Proposition 3.9]). *Let \leq be a total ordering on X and let $F: X^2 \rightarrow X$ be given by $F = \max_{\leq}$ for some total ordering \leq on X . Then F is \leq -preserving iff \leq is single-peaked w.r.t. \leq .*

The following theorem is an immediate consequence of Lemma 3.2, Theorem 3.3, Proposition 3.6 and the known fact (see also Section 5) that there are exactly 2^{n-1} total orderings on X_n that are single-peaked w.r.t. \leq_n .

Theorem 3.7. *Let \leq be a total ordering on X . The following assertions are equivalent.*

- (i) $F: X^2 \rightarrow X$ is quasitrivial, commutative, and \leq -preserving.
- (ii) $F = \max_{\leq}$ for some total ordering \leq on X that is single-peaked w.r.t. \leq .

If $(X, \leq) = (X_n, \leq_n)$ for some integer $n \geq 1$, then any of the assertions (i)–(ii) above is equivalent to any of the following ones.

- (iii) F is quasitrivial, \leq -preserving, and $\deg_F = (0, 2, 4, \dots, 2n - 2)$.
- (iv) F is associative, idempotent, commutative, \leq -preserving, and has a neutral element.

Moreover, there are exactly 2^{n-1} operations $F: X_n^2 \rightarrow X_n$ satisfying any of the assertions (i)–(iv). Furthermore, the total ordering \leq considered in assertion (ii) is uniquely defined as follows: $x \leq y$ iff $\deg_F(x) \leq \deg_F(y)$. In particular, each of these operations has a unique neutral element $e = \min_{\leq} X_n$ and a unique annihilator $a = \max_{\leq} X_n$.

Example 3.8. In Figure 3 we present the $3! = 6$ associative, quasitrivial, and commutative operations on X_3 . Only the first $2^{3-1} = 4$ operations are \leq_3 -preserving. All these operations have neutral and annihilator elements.

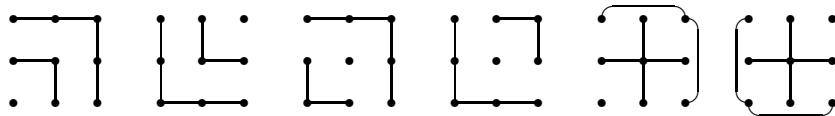


FIGURE 3. The six associative, quasitrivial, and commutative operations on X_3

- Remark 3.* (a) To better illustrate Theorem 3.7 when X is finite, consider the operation $F: X_6^2 \rightarrow X_6$ whose contour plot is shown in Figure 4 (left). This operation is clearly quasitrivial, \leq_6 -preserving, and is such that $\deg_F = (0, 2, \dots, 10)$. By Theorem 3.7 we then have $F = \max_{\leq}$, where \leq is the total ordering on X_6 obtained by sorting the numbers $\deg_F(x)$, $x \in X_6$, in increasing order, that is, $4 < 3 < 5 < 2 < 1 < 6$; see Figure 4 (right). This total ordering is single-peaked w.r.t. \leq_6 (see also Example 5.2).
- (b) The equivalence between assertions (i) and (ii) of Theorem 3.7 was established in [7, Theorem 3.13]. When X is finite, the equivalence among assertions (i), (ii), and (iv) of Theorem 3.7 was established in [5, Theorems 12 and 17].

4. ENUMERATIONS OF ARBITRARY QUASITRIVIAL SEMIGROUPS

This section is devoted to the arbitrary associative and quasitrivial operations that need not be commutative. Recall that a characterization of this class of operations is given in Theorem 2.1. However, to our knowledge a generalization of Theorem 3.3 to noncommutative operations is not known and hence remains an open problem. On this issue we make the following two observations.

- An associative and quasitrivial operation $F: X^2 \rightarrow X$ need not have a neutral element, even if X is finite. For instance, the projection operations π_1 and π_2 have no neutral element.
- An associative and quasitrivial operation $F: X^2 \rightarrow X$ need not be \leq -preserving for some total ordering \leq on X , even if X is finite. To illustrate, consider $F: X_4^2 \rightarrow X_4$ whose contour plot is depicted in Figure 5. This operation is associative and

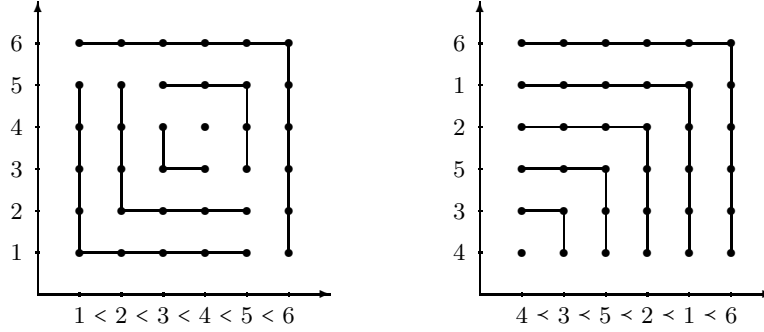


FIGURE 4. An operation $F: X_6^2 \rightarrow X_6$ defined by $F = \max_{\leq}$, where \leq is single-peaked w.r.t. \leq_6

quasitrivial. However, it can be shown that it is not \leq -preserving for any of the 24 total orderings \leq on X_4 .

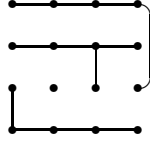


FIGURE 5. An operation that is not \leq -preserving for any total ordering \leq

In the rest of this section we mainly confine ourselves to enumeration issues regarding quasitrivial semigroups on finite sets. For instance, for any integer $n \geq 1$, we provide in Theorem 4.1 the exact number of associative and quasitrivial operations $F: X_n^2 \rightarrow X_n$. We posted the corresponding sequence in Sloane's On-Line Encyclopedia of Integer Sequences (OEIS, see [18]) as sequence Axxxxxx.

In this section we often consider either the (ordinary) generating function (GF) or the exponential generating function (EGF) of a given sequence $(s_n)_{n \geq 0}$. Recall that, when these functions exist, they are respectively defined by the power series

$$S(z) = \sum_{n \geq 0} s_n z^n \quad \text{and} \quad \hat{S}(z) = \sum_{n \geq 0} s_n \frac{z^n}{n!}.$$

Recall also that for any integers $0 \leq k \leq n$, the *Stirling number of the second kind* $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ is defined by

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = \frac{1}{k!} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i^n.$$

For any integer $n \geq 0$, let $p(n)$ denote the number of weak orderings on X_n , or equivalently, the number of ordered partitions of X_n . Setting $p(0) = 1$ by convention, the number $p(n)$ is explicitly given by

$$p(n) = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} k!, \quad n \geq 0.$$

Actually, the corresponding sequence $(p(n))_{n \geq 0}$ consists of the *ordered Bell numbers* (Sloane's A000670) and satisfies the following recurrence equation

$$p(n+1) = \sum_{k=0}^n \binom{n+1}{k} p(k), \quad n \geq 0,$$

with $p(0) = 1$. Moreover, its EGF is given by $\hat{P}(z) = 1/(2 - e^z)$.

For any integer $n \geq 1$, we denote by $q(n)$ the number of associative and quasitrivial operations $F: X_n^2 \rightarrow X_n$ (i.e., the number of quasitrivial semigroups on an n -element set). By convention, we set $q(0) = 1$. Also, for any integer $n \geq 0$, we denote by $q_e(n)$ (resp. $q_a(n)$, $q_{ea}(n)$) the number of associative and quasitrivial operations $F: X_n^2 \rightarrow X_n$ that have neutral elements (resp. annihilator elements, both neutral and annihilator elements). Theorem 4.1 and Proposition 4.2 below provide explicit formulas for these sequences. The first few values of these sequences are shown in Table 1.

Theorem 4.1. *For any integer $n \geq 0$, we have the closed-form expression*

$$(2) \quad q(n) = \sum_{i=0}^n 2^i \sum_{k=0}^{n-i} (-1)^k \binom{n}{k} \left\{ \begin{matrix} n-k \\ i \end{matrix} \right\} (i+k)!, \quad n \geq 0.$$

Moreover, the sequence $(q(n))_{n \geq 0}$ satisfies the recurrence equation

$$q(n+1) = (n+1)q(n) + 2 \sum_{k=0}^{n-1} \binom{n+1}{k} q(k), \quad n \geq 0,$$

with $q(0) = 1$. Furthermore, its EGF is given by $\hat{Q}(z) = 1/(z+3-2e^z)$.

Proof. Using Theorem 2.1 we can easily see that

$$(3) \quad q(n) = \sum_{k=1}^n \sum_{\substack{n_1, \dots, n_k \geq 1 \\ n_1 + \dots + n_k = n}} \binom{n}{n_1, \dots, n_k} \prod_{i=1}^k 2, \quad \forall n \geq 1.$$

Indeed, to compute $q(n)$ we need to consider all the ordered partitions of X_n and count twice each equivalence class containing at least two elements (because two possible projections are to be considered for each such class). In Eq. (3), k represents the number of equivalence classes and n_i represents the cardinality of the i th class.

For any integer $k \geq 1$, define the sequence $(s_n^k)_{n \geq 0}$ as

$$(4) \quad s_n^k = \sum_{\substack{n_1, \dots, n_k \geq 0 \\ n_1 + \dots + n_k = n}} \binom{n}{n_1, \dots, n_k} \prod_{i=1}^k \min\{n_i, 2\}.$$

Thus defined, the sequence $(s_n^k)_{n \geq 0}$ is the k -fold binomial convolution of the sequence $(\min\{n, 2\})_{n \geq 0}$ (for background on convolutions see, e.g., [15, Section 7.2.1]). Since the EGF of the latter sequence is clearly the function $z \mapsto 2e^z - z - 2$, it follows that the EGF of the sequence $(s_n^k)_{n \geq 0}$ is the function $z \mapsto (2e^z - z - 2)^k$, which means that

$$(5) \quad s_n^k = D_z^n (2e^z - z - 2)^k \Big|_{z=0},$$

where D_z denotes the usual differential operator.

Using (3)–(5), for any integer $n \geq 1$, we then obtain

$$q(n) = \sum_{k=1}^n s_n^k = D_z^n \frac{1 - (2e^z - z - 2)^{n+1}}{z + 3 - 2e^z} \Big|_{z=0} = D_z^n \frac{1}{z + 3 - 2e^z} \Big|_{z=0}.$$

Since $q(0) = 1$ by definition, we thus see that the EGF of the sequence $(q(n))_{n \geq 0}$ is given by $\hat{Q}(z) = (z + 3 - 2e^z)^{-1}$.

Now, by taking the $(n + 1)$ st derivative at $z = 0$ of both sides of the identity $(z + 3 - 2e^z)\hat{Q}(z) = 1$ (using the general Leibniz rule) we immediately derive the claimed recurrence equation for the sequence $(q(n))_{n \geq 0}$.

Let us now establish Eq. (2). It is enough to show that the EGF of the sequence $(\tilde{q}(n))_{n \geq 0}$ defined by $\tilde{q}(0) = 1$ and

$$\tilde{q}(n) = \sum_{i=0}^n 2^i \sum_{k=0}^{n-i} (-1)^k \binom{n}{k} \left\{ \begin{matrix} n-k \\ i \end{matrix} \right\} (i+k)!, \quad n \geq 1,$$

is exactly $\hat{Q}(z)$.

For any integer $i \geq 0$, consider the sequences $(f_n^i)_{n \geq 0}$ and $(g_n^i)_{n \geq 0}$ defined by $f_n^i = (-1)^n (n+i)!$ and $g_n^i = \left\{ \begin{matrix} n \\ i \end{matrix} \right\}$. Define also the sequence $(h_n^i)_{n \geq 0}$ by the binomial convolution of $(f_n^i)_{n \geq 0}$ and $(g_n^i)_{n \geq 0}$, that is,

$$h_n^i = \sum_{k=0}^n \binom{n}{k} (-1)^k (i+k)! \left\{ \begin{matrix} n-k \\ i \end{matrix} \right\}.$$

Observing that $\left\{ \begin{matrix} n-k \\ i \end{matrix} \right\} = 0$ if $n - k < i$ we see that

$$(6) \quad \tilde{q}(n) = \sum_{i=0}^n 2^i h_n^i, \quad n \geq 0.$$

Let $\hat{F}_i(z)$, $\hat{G}_i(z)$, and $\hat{H}_i(z)$ be the EGFs of the sequences $(f_n^i)_{n \geq 0}$, $(g_n^i)_{n \geq 0}$, and $(h_n^i)_{n \geq 0}$, respectively. It is known (see, e.g., [8, p. 335, p. 351]) that $\hat{F}_i(z) = i!(z+1)^{-i-1}$ and $\hat{G}_i(z) = (e^z - 1)^i / i!$. We then have

$$\hat{H}_i(z) = \hat{F}_i(z)\hat{G}_i(z) = \frac{(e^z - 1)^i}{(z+1)^{i+1}}.$$

Since $h_n^i = D_z^n \hat{H}_i(z)|_{z=0}$, using (6) we obtain

$$\tilde{q}(n) = D_z^n \frac{1 - \left(2 \frac{e^z - 1}{z+1}\right)^{n+1}}{z + 3 - 2e^z} \Big|_{z=0} = D_z^n \frac{1}{z + 3 - 2e^z} \Big|_{z=0} = (D_z^n \hat{Q})(0).$$

This means that the EGF of $(\tilde{q}(n))_{n \geq 0}$ is given by $\hat{Q}(z)$. This completes the proof. \square

Remark 4. (a) It is clear that the radius r of convergence of the series $\hat{Q}(z)$ is less than or equal to the closest singularity (≈ 0.583) to the origin of the real function $x \mapsto 1/(x + 3 - 2e^x)$. We conjecture that r is given by the classical ratio test and corresponds exactly to that singularity. In mathematical terms, this amounts to proving (or disproving) that

$$\frac{q(n+1)}{(n+1)q(n)} \rightarrow \frac{1}{r} \approx 1.715 \quad \text{as } n \rightarrow \infty,$$

where $r \approx 0.583$ is the unique positive zero of the real function $x \mapsto x + 3 - 2e^x$.

(b) In the proof of Theorem 4.1 we have established Eq. (2) by first searching for the explicit form of $\hat{Q}(z)$ from the definition of the sequence $(q(n))_{n \geq 0}$. In the appendix we provide an alternative proof of (2) that does not make use of $\hat{Q}(z)$.

Proposition 4.2. *For any integer $n \geq 0$, we have $q_e(n) = q_a(n) = nq(n-1)$ and $q_{ea}(n) = n(n-1)q(n-2)$.*

Proof. Let us first show how we can construct an associative and quasitrivial operation $F: X_n^2 \rightarrow X_n$ having a neutral element. There are n ways to choose the neutral element e in X_n . Then we observe that the restriction of F to $(X_n \setminus \{e\})^2$ is still an associative and quasitrivial operation, so we have $q(n-1)$ possible choices to construct this restriction. This shows that $q_e(n) = nq(n-1)$. Using the same reasoning, we also obtain $q_a(n) = nq(n-1)$ and $q_{ea}(n) = n(n-1)q(n-2)$. \square

n	$q(n)$	$q_e(n)$	$q_a(n)$	$q_{ea}(n)$
0	1	0	0	0
1	1	1	1	0
2	4	2	2	2
3	20	12	12	6
4	138	80	80	48
5	1 182	690	690	400
6	12 166	7 092	7 092	4 140
OEIS	Axxxxxxx	Axxxxxxx	Axxxxxxx	Axxxxxxx

TABLE 1. First few values of $q(n)$, $q_e(n)$, $q_a(n)$, and $q_{ea}(n)$

We now consider the subclass of associative and quasitrivial operations $F: X_n^2 \rightarrow X_n$ that are \leq_n -preserving. To this extent, we introduce a generalization of single-peakedness to weak orderings, that we call *weak single-peakedness*. This leads to a generalization of Proposition 3.6 to arbitrary quasitrivial semigroups (see Proposition 4.4). We will further elaborate on this concept in Section 5.

Definition 4.3. Let \leq be a total ordering on X and let \preceq be a weak ordering on X . We say that \preceq is *weakly single-peaked w.r.t. \leq* if for any $a, b, c \in X$ such that $a < b < c$, we have $b < a$ or $b < c$ or $a \sim b \sim c$.

Proposition 4.4. Let \leq be a total ordering on X and let \preceq be a weak ordering on X . Suppose that $F: X^2 \rightarrow X$ is of the form (1). Then F is \leq -preserving iff \preceq is weakly single-peaked w.r.t. \leq .

Proof. (Necessity) We proceed by contradiction. Suppose that there exist $a, b, c \in X$ satisfying $a < b < c$ such that $a \preceq b$ and $c \preceq b$ and $\neg(a \sim b \sim c)$. Suppose that $a < b$ and $c \sim b$. The other two cases can be dealt with similarly.

- If $F|_{[b]^2} = \pi_1|_{[b]^2}$, then by \leq -preservation of F we have $b = F(a, b) \leq F(a, c) \leq F(b, c) = b$.
- If $F|_{[b]^2} = \pi_2|_{[b]^2}$, then by \leq -preservation of F we have $b = F(b, a) \leq F(c, a) \leq F(c, b) = b$.

In the first (resp. second) case we obtain $F(a, c) = b$ (resp. $F(c, a) = b$), which contradicts quasitriviality.

(Sufficiency) We proceed by contradiction. Suppose that \preceq is weakly single-peaked w.r.t. \leq and that F is not \leq -preserving. Then, for instance there exist $x, y, z \in X$ such that

$$(7) \quad y < z \quad \text{and} \quad F(x, y) > F(x, z).$$

Using (7) it is easy to see by contradiction that we necessarily have

$$(x \preceq y \text{ or } x \preceq z) \quad \text{and} \quad (y \preceq x \text{ or } z \preceq x).$$

We then have only the following three mutually exclusive cases to consider.

- If $y < x < z$ or $y \sim x < z$ or $y < x \sim z$, then by (7) we obtain $y < z < x$, which violates weak single-peakedness.
- If $z < x < y$ or $z \sim x < y$ or $z < x \sim y$, then by (7) we obtain $x < y < z$, which violates weak single-peakedness.
- If $x \sim y \sim z$, then we must have $F|_{[x]^2} = \pi_1|_{[x]^2}$ or $F|_{[x]^2} = \pi_2|_{[x]^2}$, which immediately violates (7).

This completes the proof of Proposition 4.4. \square

From Proposition 4.4 we immediately derive the following theorem, which generalizes to the noncommutative case the equivalence between assertions (i) and (ii) of Theorem 3.7.

Theorem 4.5. *Let \leq be a total ordering on X . An $F: X^2 \rightarrow X$ is associative, quasitrivial, and \leq -preserving iff it is of the form (1) for some weak ordering \lesssim on X that is weakly single-peaked w.r.t. \leq .*

We now consider the problem of enumerating associative and quasitrivial operations $F: X_n^2 \rightarrow X_n$ that are \leq_n -preserving. We will make use of the following two auxiliary lemmas.

Lemma 4.6. *Let \leq be a total ordering on X and let \lesssim be a weak ordering on X . If \lesssim is weakly single-peaked w.r.t. \leq , then there are no pairwise distinct $a, b, c, d \in X$ such that $a < b \sim c \sim d$.*

Proof. We proceed by contradiction. Suppose that there exist pairwise distinct $a, b, c, d \in X$ such that $a < b \sim c \sim d$. Assume w.l.o.g. that $b < c < d$. If $b < a < c$, then the set $\{a, c, d\}$ violates weak single-peakedness of \lesssim . In the three other cases the set $\{a, b, c\}$ violates weak single-peakedness of \lesssim . \square

Lemma 4.7. *Let \leq be a total ordering on X and let \lesssim be a weak ordering on X that is weakly single-peaked w.r.t. \leq . Also, let $a = \min_{\leq} X$ and $b = \max_{\leq} X$. If $\max_{\lesssim} X \neq X$, then $\max_{\lesssim} X \subseteq \{a, b\}$.*

Proof. By Lemma 4.6 the set $\max_{\lesssim} X$ contains at most two elements. Now suppose that there exists $x \in (\max_{\lesssim} X) \setminus \{a, b\}$. Then the set $\{a, x, b\}$ violates weak single-peakedness of \lesssim . \square

Assume that X_n is endowed with \leq_n . For any integer $n \geq 0$, we denote by $u(n)$ the number of weak orderings \lesssim on X_n that are weakly single-peaked w.r.t. \leq_n . Also, we denote by $u_e(n)$ (resp. $u_a(n)$, $u_{ea}(n)$) the number of weak orderings \lesssim on X_n that are weakly single-peaked w.r.t. \leq_n and for which X_n has exactly one minimal element (resp. one maximal element, one minimal element and one maximal element) for \lesssim . Propositions 4.8 and 4.9 below provide explicit formulas for these sequences. The first few values of these sequences are shown in Table 2.

Proposition 4.8. *The sequence $(u(n))_{n \geq 0}$ satisfies the second order linear recurrence equation*

$$u(n+2) - 2u(n+1) - u(n) = 1, \quad n \geq 0,$$

with $u(0) = 0$ and $u(1) = 1$, and we have

$$2u(n) + 1 = \frac{1}{2}(1 + \sqrt{2})^{n+1} + \frac{1}{2}(1 - \sqrt{2})^{n+1} = \sum_{k \geq 0} \binom{n+1}{2k} 2^k, \quad n \geq 0.$$

Moreover, its GF is given by $U(z) = z/(z^3 + z^2 - 3z + 1)$.

Proof. We clearly have $u(0) = 0$ and $u(1) = 1$. So let us assume that $n \geq 2$. If \preceq is a weak ordering on X_n that is weakly single-peaked w.r.t. \leq_n , then by Lemma 4.7 either $\max_{\preceq} X_n = X_n$, or $\max_{\preceq} X_n = \{1\}$, or $\max_{\preceq} X_n = \{n\}$, or $\max_{\preceq} X_n = \{1, n\}$. In the three latter cases it is clear that the restriction of \preceq to $X_n \setminus \max_{\preceq} X_n$ is weakly single-peaked w.r.t. the restriction of \leq_n to $X_n \setminus \max_{\preceq} X_n$. It follows that the number $u(n)$ of weakly single-peaked weak orderings on X_n w.r.t. \leq_n satisfies the following second order linear equation

$$u(n) = 1 + u(n-1) + u(n-1) + u(n-2), \quad n \geq 2.$$

The claimed expressions of $u(n)$ and GF of $(u(n))_{n \geq 2}$ follow straightforwardly. \square

Proposition 4.9. *The sequence $(u_e(n))_{n \geq 0}$ satisfies the second order linear recurrence equation*

$$u_e(n+2) - 2u_e(n+1) - u_e(n) = 0, \quad n \geq 0,$$

with $u_e(0) = 0$ and $u_e(1) = 1$, and we have

$$u_e(n) = \frac{\sqrt{2}}{4}(1 + \sqrt{2})^n - \frac{\sqrt{2}}{4}(1 - \sqrt{2})^n = \sum_{k \geq 0} \binom{n}{2k+1} 2^k, \quad n \geq 0.$$

Moreover, its GF is given by $U_e(z) = -z/(z^2 + 2z - 1)$. Furthermore, for any integer $n \geq 1$, we have $u_a(n) = 2u(n-1)$, $u_{ea}(n) = 2u_e(n-1)$, and $u_a(0) = u_{ea}(0) = 0$.

Proof. The formula describing the sequence $(u_e(n))_{n \geq 0}$ is obtained by following the same steps as in the proof of Proposition 4.8, except that in this case we always have $\max_{\preceq} X_n \neq X_n$. As for the sequence $(u_a(n))_{n \geq 0}$ we note that $\max_{\preceq} X_n$ must be either $\{1\}$ or $\{n\}$ and that the restriction of \preceq to $X_n \setminus \max_{\preceq} X_n$ is weakly single-peaked w.r.t. the restriction of \leq_n to $X_n \setminus \max_{\preceq} X_n$. We proceed similarly for the sequence $(u_{ea}(n))_{n \geq 0}$. \square

n	$u(n)$	$u_e(n)$	$u_a(n)$	$u_{ea}(n)$
0	0	0	0	0
1	1	1	0	0
2	3	2	2	2
3	8	5	6	4
4	20	12	16	10
5	49	29	40	24
6	119	70	98	58
OEIS	Axxxxxx	A000129	Axxxxxx	A163271

TABLE 2. First few values of $u(n)$, $u_e(n)$, $u_a(n)$, and $u_{ea}(n)$

Example 4.10. The $u(3) = 8$ weak orderings on X_3 that are weakly single-peaked w.r.t. \leq_3 are: $1 < 2 < 3$, $2 < 1 < 3$, $2 < 3 < 1$, $3 < 2 < 1$, $2 < 1 \sim 3$, $1 \sim 2 < 3$, $2 \sim 3 < 1$, and $1 \sim 2 \sim 3$. $u_e(3) = 5$ of those have exactly one minimal element and $u_a(3) = 6$ of those have exactly one maximal element. $u_{ea}(3) = 4$ of those have one minimal element and one maximal element. These four weak orderings correspond to the $2^{3-1} = 4$ total orderings on X_3 that are weakly single-peaked w.r.t. \leq_3 .

Assume again that X_n is endowed with \leq_n . For any integer $n \geq 0$, we denote by $v(n)$ the number of associative, quasitrivial, and \leq_n -preserving operations $F: X_n^2 \rightarrow X_n$. Also,

we denote by $v_e(n)$ (resp. $v_a(n)$, $v_{ea}(n)$) the number of associative, quasitrivial, and \leq_n -preserving operations $F: X_n^2 \rightarrow X_n$ that have neutral elements (resp. annihilator elements, both neutral and annihilator elements). Propositions 4.11 and 4.12 below provide explicit formulas for these sequences. The first few values of these sequences are shown in Table 3.

Proposition 4.11. *The sequence $(v(n))_{n \geq 0}$ satisfies the second order linear recurrence equation*

$$v(n+2) - 2v(n+1) - 2v(n) = 2, \quad n \geq 0,$$

with $v(0) = 0$ and $v(1) = 1$, and we have

$$3v(n) + 2 = \frac{2+\sqrt{3}}{2}(1+\sqrt{3})^n + \frac{2-\sqrt{3}}{2}(1-\sqrt{3})^n = \sum_{k \geq 0} 3^k (2\binom{n}{2k} + 3\binom{n}{2k+1}), \quad n \geq 0.$$

Moreover, its GF is given by $V(z) = z(z+1)/(2z^3 - 3z + 1)$.

Proof. We clearly have $v(0) = 0$ and $v(1) = 1$. So let us assume that $n \geq 2$. If $F: X_n^2 \rightarrow X_n$ is an associative, quasitrivial, and \leq_n -preserving operation, then by Theorem 4.5 it is of the form (1) for some weak ordering \lesssim on X_n that is weakly single-peaked w.r.t. \leq_n . By Lemma 4.7 either $\max_{\lesssim} X_n = X_n$ or $\max_{\lesssim} X_n = \{1\}$ or $\max_{\lesssim} X_n = \{n\}$ or $\max_{\lesssim} X_n = \{1, n\}$. In the first case we have to consider the two projections $F = \pi_1$ and $F = \pi_2$. In the three latter cases it is clear that the restriction of F to $(X_n \setminus \max_{\lesssim} X_n)^2$ is still associative, quasitrivial, and \leq_n -preserving, where in the latter case we have to consider the two projections $F|_{\{1, n\}^2} = \pi_1|_{\{1, n\}^2}$ and $F|_{\{1, n\}^2} = \pi_2|_{\{1, n\}^2}$. It follows that the number $v(n)$ of associative, quasitrivial, and \leq_n -preserving operations $F: X_n^2 \rightarrow X_n$ satisfies the following second order linear equation

$$v(n) = 2 + v(n-1) + v(n-1) + 2v(n-2), \quad n \geq 2.$$

The claimed expressions of $v(n)$ and GF of $(v(n))_{n \geq 2}$ follow straightforwardly. \square

Proposition 4.12. *The sequence $(v_e(n))_{n \geq 0}$ satisfies the second order linear recurrence equation*

$$v_e(n+2) - 2v_e(n+1) - 2v_e(n) = 0, \quad n \geq 0,$$

with $v_e(0) = 0$ and $v_e(1) = 1$, and we have

$$v_e(n) = \frac{\sqrt{3}}{6}(1+\sqrt{3})^n - \frac{\sqrt{3}}{6}(1-\sqrt{3})^n = \sum_{k \geq 0} \binom{n}{2k+1} 3^k, \quad n \geq 0.$$

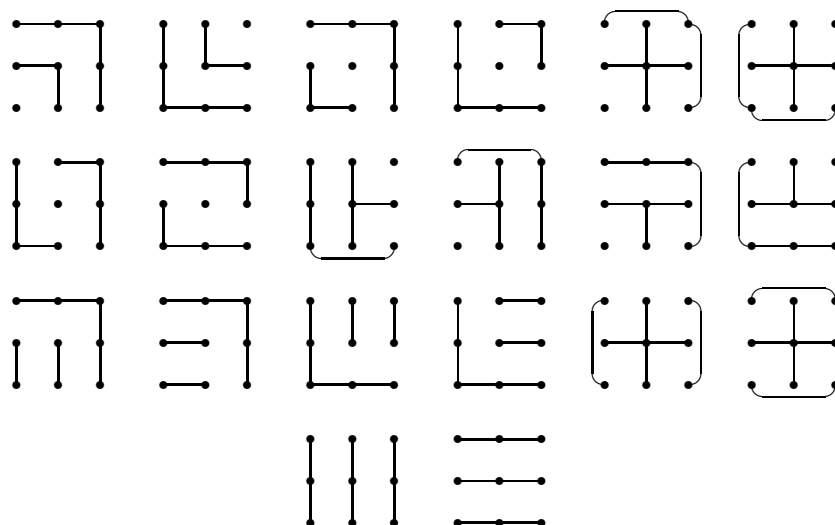
Moreover, its GF is given by $V_e(z) = -z/(2z^2 + 2z - 1)$. Furthermore, for any integer $n \geq 1$, we have $v_a(n) = 2v(n-1)$, $v_{ea}(n) = 2v_e(n-1)$, and $v_a(0) = v_{ea}(0) = 0$.

Proof. The formula describing the sequence $(v_e(n))_{n \geq 0}$ is obtained by following the same steps as in the proof of Proposition 4.11, except that in this case we always have $\max_{\lesssim} X_n \neq X_n$. As for the sequence $(v_a(n))_{n \geq 0}$ we note that $\max_{\lesssim} X_n$ must be either $\{1\}$ or $\{n\}$ and that the restriction of F to $(X_n \setminus \max_{\lesssim} X_n)^2$ is associative, quasitrivial, and \leq_n -preserving. We proceed similarly for the sequence $(v_{ea}(n))_{n \geq 0}$. \square

Example 4.13. We show in Figure 6 the $q(3) = 20$ associative and quasitrivial operations on X_3 . Among these operations, $q_e(3) = 12$ have neutral elements and $v(3) = 12$ are \leq_3 -preserving.

Remark 5. We observe that the explicit expressions of $v(n)$ and $v_e(n)$ as stated in Propositions 4.11 and 4.12 were recently and independently obtained in [13, Section 6] by means of a totally different approach.

n	$v(n)$	$v_e(n)$	$v_a(n)$	$v_{ea}(n)$
0	0	0	0	0
1	1	1	0	0
2	4	2	2	2
3	12	6	8	4
4	34	16	24	12
5	94	44	68	32
6	258	120	188	88
OEIS	Axxxxxx	A002605	Axxxxxx	Axxxxxx

TABLE 3. First few values of $v(n)$, $v_e(n)$, $v_a(n)$, and $v_{ea}(n)$ FIGURE 6. The 20 associative and quasitrivial operations on X_3

5. SINGLE-PEAKEDNESS AND WEAK SINGLE-PEAKEDNESS

In this section we further analyze the single-peakedness and weak single-peakedness properties. In particular, we show how these properties can be easily checked graphically.

Define the *strict convex hull* of $x, y \in X$ w.r.t. \leq by $\text{conv}_{\leq}(x, y) = \{z \in X : x < z < y\}$, if $x < y$, and $\text{conv}_{\leq}(x, y) = \{z \in X : y < z < x\}$, if $y < x$. Using this concept we can rewrite the definitions of single-peakedness and weak single-peakedness in a more symmetric way.

Thus, a total ordering \leq on X is single-peaked w.r.t. a (reference) total ordering \leq on X iff for any $a, b, c \in X$ such that $b \in \text{conv}_{\leq}(a, c)$, we have $b < a$ or $b < c$ (see Definition 3.4). In other words, the condition says that from among three pairwise distinct elements of X , the centrist one w.r.t. \leq is never ranked last by \leq .

A remarkable characterization of single-peakedness is that for any total orderings \leq and \leq on X , the operation $F = \max_{\leq}$ is \leq -preserving iff \leq is single-peaked w.r.t. \leq (cf. Proposition 3.6).

Remark 6. It is natural to define the dual version of single-peakedness by saying that from among three pairwise distinct elements of X , the centrist one w.r.t. \leq is never ranked *first* by \leq . By doing so, it is clear that \leq is single-peaked w.r.t. \leq iff the inverse ordering \leq^{-1} (defined by $a \leq^{-1} b \Leftrightarrow b \leq a$) is dual single-peaked w.r.t. \leq . For instance, we could replace \max_{\leq} with \min_{\leq} and “single-peaked” with “dual single-peaked” in Proposition 3.6. Thus, considering the single-peakedness property or its dual version is simply a matter of convention.

The following proposition provides an alternative characterization of single-peakedness. Recall first that, for any total ordering \leq on X , a subset C of X is said to be *convex w.r.t.* \leq if for any $a, b, c \in X$ such that $b \in \text{conv}_{\leq}(a, c)$, we have that $a, c \in C$ implies $b \in C$.

Proposition 5.1 (see [7, Proposition 3.10]). *Let \leq and \leq' be total orderings on X . Then \leq is single-peaked w.r.t. \leq' iff for every $t \in X$ the set $\{x \in X : x \leq t\}$ is convex w.r.t. \leq .*

The single-peakedness property of a total ordering \leq on X w.r.t. some total ordering \leq' can often be easily checked (especially if X is finite) by plotting a function, call it f_{\leq} , in a rectangular coordinate system in the following way. Represent the reference ordered set (X, \leq) on the horizontal axis and the reversed version of the ordered set (X, \leq) , that is (X, \leq^{-1}) , on the vertical axis. The function f_{\leq} is defined by its graph $\{(x, x) : x \in X\}$.² We then see that the total ordering \leq is single-peaked w.r.t. \leq' iff the graph of f_{\leq} is “V-free” in the sense that we cannot find three points (i, i) , (j, j) , (k, k) in V-shape. Equivalently, f_{\leq} has only one local maximum.

Example 5.2. Figure 7 gives the functions f_{\leq} and $f_{\leq'}$ corresponding to the total orderings $4 < 3 < 5 < 2 < 1 < 6$ (from Remark 3(a)) and $6 <' 5 <' 2 <' 1 <' 3 <' 4$, respectively, on X_6 . We see that \leq is single-peaked w.r.t. \leq_6 since f_{\leq} has only one local maximum while \leq' is not single-peaked w.r.t. \leq_6 since $f_{\leq'}$ has two local maxima (also, the points $(1, 1)$, $(3, 3)$, $(5, 5)$ for instance are in V-shape).

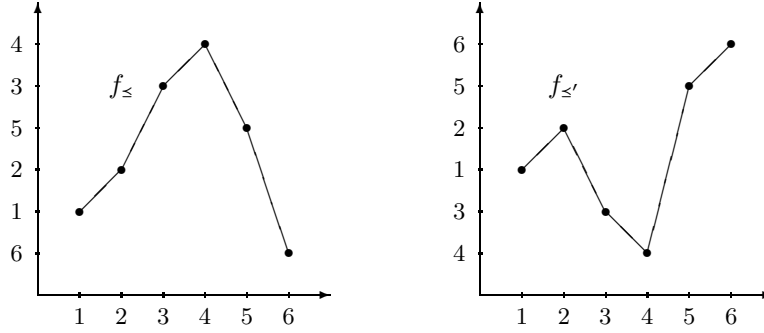


FIGURE 7. \leq is single-peaked (left) while \leq' is not (right)

It is known that (see, e.g., [2]) that there are exactly 2^{n-1} single-peaked total orderings on X_n w.r.t. \leq_n . The proof is a simplified version of that of Proposition 4.8 (just observe that either $\max_{\leq} X_n = \{1\}$ or $\max_{\leq} X_n = \{n\}$).

²When $X = X_n$ for some integer $n \geq 1$, the graphical representation of f_{\leq} is then obtained by joining the points $(1, 1), \dots, (n, n)$ by line segments.

Let us now focus on weak single-peakedness. Recall (cf. Definition 4.3) that a weak ordering \lesssim on X is weakly single-peaked w.r.t. a reference total ordering \leq on X if for any $a, b, c \in X$ such that $b \in \text{conv}_{\leq}(a, c)$, we have $b < a$ or $b < c$ or $a \sim b \sim c$.

In Proposition 4.4 we saw that for any total ordering \leq and weak ordering \lesssim on X , any operation $F: X^2 \rightarrow X$ of the form (1) is \leq -preserving iff \lesssim is weakly single-peaked w.r.t. \leq . This characterization justifies the definition of weak single-peakedness and shows in particular that the condition $a \sim b \sim c$ is necessary in the definition. It is noteworthy that the following equivalence holds

$$b < a \text{ or } b < c \text{ or } a \sim b \sim c \iff \begin{cases} a < b \implies b < c, \\ c < b \implies b < a. \end{cases}$$

We also have the following alternative characterization of weak single-peakedness. We omit the proof for it is straightforward (by contradiction).

Proposition 5.3. *Let \leq be a total ordering on X and let \lesssim be a weak ordering on X . Then \lesssim is weakly single-peaked w.r.t. \leq iff the following conditions hold.*

- (a) *For any $a, b, c \in X$ such that $b \in \text{conv}_{\leq}(a, c)$, we have $b \lesssim a$ or $b \lesssim c$.*
- (b) *For any $a, b, c \in X$ such that $a \neq c$ and $b < a \sim c$, we have $b \in \text{conv}_{\leq}(a, c)$.*

Weak single-peakedness of a weak ordering \lesssim on X w.r.t. some total ordering \leq can be visualized and checked by plotting a function f_{\lesssim} in a rectangular coordinate system. Represent the reference ordered set (X, \leq) on the horizontal axis and the reversed version of the ordered set $(X/\sim, \lesssim)$ on the vertical axis. Here again the function f_{\lesssim} is defined by its graph $\{(x, x) : x \in X\}$. Condition (a) of Proposition 5.3 says that the graph of f_{\lesssim} is V-free, i.e., we cannot find three points $(i, i), (j, j), (k, k)$ in V-shape. Condition (b) is less immediate to interpret graphically. However, Proposition 5.5 below shows how conditions (a) and (b) together can be easily interpreted.

Definition 5.4. Let \leq be a total ordering on X and let \lesssim be a weak ordering on X . We say that a subset P of X of size $|P| \geq 2$ is a *plateau w.r.t. (\leq, \lesssim)* if P is convex w.r.t. \leq and if there exists $x \in X$ such that $P \subseteq [x]_{\sim}$.

Proposition 5.5. *Let \leq be a total ordering on X and let \lesssim be a weak ordering on X . Consider the assertions (a) and (b) of Proposition 5.3 as well as the following one.*

- (b') *If $P \subseteq X, |P| \geq 2$, is a plateau w.r.t. (\leq, \lesssim) , then it is \lesssim -minimal in the sense that for every $a \in X$ satisfying $a \lesssim P$ there exists $z \in P$ such that $z \sim a$.*

Then we have (a) and (b') \implies (b) and (b) \implies (b').

Proof. Let us prove that (a) and (b') implies (b). Let $a, b, c \in X$ such that $a \neq c$ and $b < a \sim c$ and suppose that $b \notin \text{conv}_{\leq}(a, c)$. Assume w.l.o.g. that $b < a$. If $\text{conv}_{\leq}(a, c)$ is a plateau, then it cannot be \lesssim -minimal, which contradicts (b'). Hence $\text{conv}_{\leq}(a, c)$ is not a plateau, which means that there exists $z \in \text{conv}_{\leq}(a, c)$ such that $\neg(z \sim a)$. By (a) we then have $z < a$. But then the set $\{a, b, z\}$ violates condition (a) since the points $(b, b), (a, a), (z, z)$ are in V-shape.

Let us now prove that (b) implies (b'). Let $P \subseteq X, |P| \geq 2$, be a plateau w.r.t. (\leq, \lesssim) and let $a, c \in P, a \neq c$. Suppose that P is not \lesssim -minimal, i.e., there exists $b \in X$ such that $b < a \sim c$. By (b), we have $b \in \text{conv}_{\leq}(a, c)$, which contradicts the fact that P is a plateau. \square

Proposition 5.5 says that conditions (a) and (b) hold iff conditions (a) and (b') hold. As discussed above, condition (a) says that the graph of f_{\lesssim} is V-free. Now, condition (b')

simply says that the graph of f_{\preceq} is both L-free and reversed L-free, which means that the two patterns shown in Figure 8 (L-shape and reversed L-shape), where each horizontal part is a plateau P , are forbidden.



FIGURE 8. The two patterns excluded by condition (b')

Summing up, we have proved the following result.

Theorem 5.6. *Let \leq be a total ordering on X and let \preceq be a weak ordering on X . Then \preceq is weakly single-peaked w.r.t. \leq iff conditions (a) and (b') of Propositions 5.3 and 5.5 hold (i.e., the graph of f_{\preceq} is V-free, L-free, and reversed L-free).*

Example 5.7. Let us consider the operation $F: X_4^2 \rightarrow X_4$ shown in Figure 9 (left). Using the tests given at the end of Section 2 for instance, we can see that this operation is associative and quasitrivial. It is also \leq_4 -preserving and such that $\deg_F = (0, 3, 3, 6)$. Thus, F is of the form (1), where \preceq is the weak ordering on X_4 obtained by ranking the numbers $\deg_F(x)$, $x \in X_4$, in nondecreasing order, that is, $2 < 1 \sim 3 < 4$; see Figure 9 (center). By Proposition 4.4 this weak ordering \preceq is weakly single-peaked w.r.t. \leq_4 . By Theorem 5.6 the graph of f_{\preceq} is V-free, L-free, and reversed L-free; see Figure 9 (right).

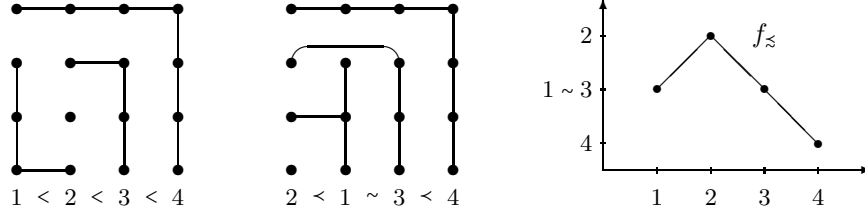


FIGURE 9. Example 5.7

Example 5.8. Let us consider the operation $F: X_4^2 \rightarrow X_4$ shown in Figure 10 (left). Just as in Example 5.7, we can see that this operation is of the form (1), where \preceq is the weak ordering on X_4 defined by $1 < 4 < 2 \sim 3$; see Figure 10 (center). Since F is not \leq_4 -preserving, the weak ordering \preceq is not weakly single-peaked w.r.t. \leq_4 . Here the graph of f_{\preceq} is neither V-free, nor L-free, nor reversed L-free. It has the plateau $P = \{2, 3\}$, which is not \preceq -minimal; see Figure 10 (right).

We now obtain the following extension of Proposition 5.1 to weak orderings.

Proposition 5.9. *Let \leq be a total ordering on X and let \preceq be a weak ordering on X . Then condition (a) of Proposition 5.3 holds iff for every $t \in X$ the set $\{x \in X : x \preceq t\}$ is convex w.r.t. \leq .*

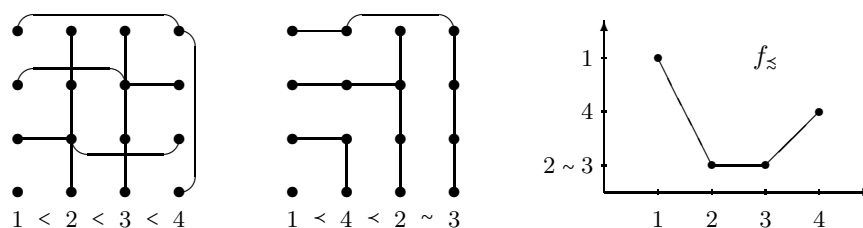


FIGURE 10. Example 5.8

Proof. (Necessity) Let $t \in X$ and let $a, b, c \in X$ such that $a, c \in \{x \in X : x \preceq t\}$ and $b \in \text{conv}_{\preceq}(a, c)$. By condition (a), we have $b \in \{x \in X : x \preceq t\}$.

(Sufficiency) For the sake of a contradiction, suppose that there exist $a, b, c \in X$ such that $b \in \text{conv}_{\preceq}(a, c)$ and $\max_{\preceq}(a, c) < b$. Set $t_0 = c$ if $a < c$, and $t_0 = a$, otherwise. We then have $a, c \in \{x \in X : x \preceq t_0\}$. By convexity w.r.t. \preceq we also have $b \in \{x \in X : x \preceq t_0\}$. Therefore we have $\max_{\preceq}(a, c) < b \preceq t_0$, which contradicts the definition of t_0 . \square

Remark 7. The dual version of weak single-peakedness can be defined exactly as we did for single-peakedness (see Remark 6): just replace the condition $b < a$ or $b < c$ or $a \sim b \sim c$ by $a < b$ or $c < b$ or $a \sim b \sim c$. Here again, considering the weak single-peakedness property or its dual version is simply a matter of convention.

6. CONCLUSION

This paper is rooted on a known characterization of associative and quasitrivial binary operations on an arbitrary set X , which essentially states that they can be thought of as a maximum operation with respect to a weak ordering (Theorem 2.1). We established different characterizations of the subclass of associative, quasitrivial, and commutative operations (Theorem 3.3) and different characterizations of the subclass of associative, quasitrivial, commutative, and \preceq -preserving operations when X is endowed with a total ordering \leq (Theorem 3.7). When commutativity is no longer assumed, finding generalizations of these characterizations remains an interesting open question (see below).

When X is an n -element set we also enumerated

- all associative and quasitrivial operations with or without neutral and/or annihilator elements (Theorem 4.1 and Proposition 4.2),
- all associative, quasitrivial, and \preceq -preserving operations (when X is endowed with a total ordering \leq) with or without neutral and/or annihilator elements (Propositions 4.11 and 4.12).

In order to characterize those that are \preceq -preserving, we made use of single-peakedness, generalized this concept by introducing weak single-peakedness (Definition 4.3 and Theorem 4.5), and provided a graphical characterization of the latter (Theorem 5.6). When X is an n -element set, we also enumerated all weak orderings on X that are weakly single-peaked w.r.t. the reference ordering on X (Propositions 4.8 and 4.9). We posted in the Sloane's OEIS [18] all the new sequences that arose from our results.

In view of these results, some questions emerge naturally and we now list a few below.

- Generalize Theorems 3.3 and 3.7 by removing commutativity in assertion (i).
- Analyze the asymptotic behavior of the sequence $(q(n))_{n \geq 0}$ (see Remark 4(a)).

- The integer sequences A000129, A002605, and A163271 were previously introduced in the OEIS to solve enumeration problems not related to weak single-peakedness and quasitrivial semigroups. It would be interesting to search for possible one-to-one correspondences between those problems and ours.
- Find the number of operations $F: X_n^2 \rightarrow X_n$ that are associative, quasitrivial, and \leq -preserving for some total ordering \leq on X . The values for $1 \leq n \leq 4$ are given by: 1, 4, 20, 130. For instance the operation on X_4 represented in Figure 5 is associative and quasitrivial. However, there is no total ordering \leq on X_4 for which this operation is \leq -preserving.

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APPENDIX: ALTERNATIVE PROOF OF FORMULA (2)

We provide an alternative proof of formula (2) that does not make use of the EGF of the sequence $(q(n))_{n \geq 0}$.

For any integer $n \geq 0$, define $s(n) = \min\{n+1, 2\}$. Also, for any integer $k \geq 1$, let \mathcal{P}_k be the vector space of real polynomial functions of k variables, and let $T_k: \mathcal{P}_k \rightarrow \mathbb{R}$ be the linear transformation defined as

$$T_k(P) = \sum_{I \subseteq \{1, \dots, k\}} (-1)^{|I|+k} 2^{|I|} \int_{[0,1]^k} P(t_1, \dots, t_k) \Big|_{t_i=0 \ \forall i \notin I} dt_1 \cdots dt_k.$$

For instance, if $P(x_1, \dots, x_k) = \prod_{i=1}^k x_i^{m_i}$ for some integers $m_1, \dots, m_k \geq 0$, then we have

$$\begin{aligned} T_k(P) &= \sum_{I \subseteq \{1, \dots, k\} \text{ s.t. } m_i=0 \ \forall i \notin I} (-1)^{|I|+k} 2^{|I|} \prod_{i \in I} \frac{1}{m_i + 1} \\ &= \sum_{I \subseteq \{1, \dots, k\}} \left(\prod_{i \in I} \frac{2}{m_i + 1} \right) \left((-1)^{k-|I|} \prod_{i \in \{1, \dots, k\} \setminus I} (2 - s(m_i)) \right) \\ (8) \quad &= \prod_{i=1}^k \left(\frac{2}{m_i + 1} - 2 + s(m_i) \right) = \prod_{i=1}^k \frac{s(m_i)}{m_i + 1}, \end{aligned}$$

where we have used the multi-binomial theorem

$$\prod_{i=1}^k (x_i + y_i) = \sum_{I \subseteq \{1, \dots, k\}} \prod_{i \in I} x_i \prod_{i \in \{1, \dots, k\} \setminus I} y_i.$$

Similarly, if $P(x_1, \dots, x_k) = (\sum_{i=1}^k x_i)^{n-k}$ for some integer $n \geq k$, then we have

$$\begin{aligned} T_k(P) &= \sum_{I \subseteq \{1, \dots, k\}} (-1)^{|I|+k} 2^{|I|} \int_{[0,1]^k} \left(\sum_{i \in I} t_i \right)^{n-k} dt_1 \cdots dt_j \\ (9) \quad &= \sum_{i=0}^k (-1)^{i+k} \binom{k}{i} 2^i \binom{n-k+i}{i} \binom{n-k+i}{i}^{-1}, \end{aligned}$$

where we have used the formula (see, e.g., [10, p. 202])

$$\int_{[0,1]^n} \left(\sum_{i=1}^n t_i \right)^k dt_1 \cdots dt_n = \binom{k+n}{n} \binom{k+n}{n}^{-1} \quad (k \geq 0, n \geq 1, \text{ integers}).$$

We now use the results above to establish the claimed expression of $q(n)$.

Let us assume that $n \geq 1$. Using Theorem 2.1 we can easily see that (see justification in the proof of Theorem 4.1)

$$q(n) = \sum_{k=1}^n \sum_{\substack{n_1 + \dots + n_k = n \\ n_1, \dots, n_k \geq 1}} \binom{n}{n_1, \dots, n_k} \prod_{\substack{i=1 \\ n_i \geq 2}}^k 2.$$

Setting $m_i = n_i - 1$ for $i = 1, \dots, k$ in the latter formula, we obtain

$$\begin{aligned} q(n) &= \sum_{k=1}^n \sum_{\substack{m_1 + \dots + m_k = n-k \\ m_1, \dots, m_k \geq 0}} \frac{n!}{(m_1 + 1)! \dots (m_k + 1)!} \prod_{i=1}^k s(m_i) \\ &= \sum_{k=1}^n \frac{n!}{(n-k)!} \sum_{\substack{m_1 + \dots + m_k = n-k \\ m_1, \dots, m_k \geq 0}} \frac{(n-k)!}{m_1! \dots m_k!} \prod_{i=1}^k \frac{s(m_i)}{m_i + 1} \end{aligned}$$

Using (8), the linearity of T_k , the multinomial theorem, and then (9), we obtain

$$\begin{aligned} q(n) &= \sum_{k=1}^n \frac{n!}{(n-k)!} \sum_{\substack{m_1 + \dots + m_k = n-k \\ m_1, \dots, m_k \geq 0}} \binom{n-k}{m_1, \dots, m_k} T_k \left(\prod_{i=1}^k x_i^{m_i} \right) \\ &= \sum_{k=1}^n \frac{n!}{(n-k)!} T_k \left(\sum_{\substack{m_1 + \dots + m_k = n-k \\ m_1, \dots, m_k \geq 0}} \binom{n-k}{m_1, \dots, m_k} \prod_{i=1}^k x_i^{m_i} \right) \\ &= \sum_{k=1}^n \frac{n!}{(n-k)!} T_k \left(\left(\sum_{i=1}^k x_i \right)^{n-k} \right) = \sum_{k=1}^n k! \sum_{i=0}^k (-1)^{i+k} 2^i \binom{n}{k-i} \begin{Bmatrix} n-k+i \\ i \end{Bmatrix}. \end{aligned}$$

Permuting the sums in the last expression and observing that $\begin{Bmatrix} n \\ 0 \end{Bmatrix} = 0$, we finally obtain

$$q(n) = \sum_{i=0}^n (-2)^i \sum_{k=i}^n (-1)^k \binom{n}{k-i} \begin{Bmatrix} n-k+i \\ i \end{Bmatrix} k!,$$

which is exactly the claimed expression of $q(n)$. \square

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