Divisors on overlapped intervals and multiplicative functions

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Abstract

Consider the real numbers

$$\ell_{n,k} = \ln\left(\frac{3}{2}\,k + \sqrt{\left(\frac{3}{2}\,k\right)^2 + 3\,n}\right)$$

and the intervals $\mathcal{L}_{n,k} = \left]\ell_{n,k} - \ln 3, \ell_{n,k}\right]$. For all $n \geq 1$, define

$$\frac{L_n(q)}{q^{n-1}} = \sum_{d|n} \sum_{k \in \mathbb{Z}} \mathbf{1}_{\mathcal{L}_{n,k}}\left(\ln d\right)\, q^k,$$

where $\mathbf{1}_A(x)$ is the characteristic function of the set A. Let $\sigma(n)$ be sum of divisors of n. We will prove that *A002324*$(n) = 4\,\sigma(n) - 3\,L_n(1)$ and *A096936*$(n) = L_n(-1)$, which are well-known multiplicative functions related to the number of representations of n by a given quadratic form.

1 Introduction

For a given integer $n \geq 1$, consider the two-sided sequence

$$p_{n,k} = \ln\left(k + \sqrt{k^2 + 2\,n}\right),$$

where $k \in \mathbb{Z}$ and define the intervals

$$\mathcal{P}_{n,k} = \left]p_{n,k} - \ln 2, p_{n,k}\right].$$

Kassel and Reutenauer [1] introduced the polynomials[1],

$$\frac{P_n(q)}{q^{n-1}} = \sum_{d|n} \sum_{k\in\mathbb{Z}} \mathbf{1}_{\mathcal{P}_{n,k}}\left(\ln d\right)\, q^k,$$

where $\mathbf{1}_A(x)$ is the characteristic function of the set A, i.e. $\mathbf{1}_A(x) = 1$ if $x \in A$, otherwise $\mathbf{1}_A(x) = 0$. Each polynomial $P_n(q)$ is monic of degree $2n-2$, its coefficients are non-negative integers and it is self-reciprocal (see [2]). The evaluations of $P_n(q)$ at some complex roots of 1 have number-theoretical interpretations (see [2]), e.g.

$$\begin{aligned}
\sigma(n) &= P_n(1), \\
\frac{r_{1,0,1}(n)}{4} &= P_n(-1), \\
\frac{r_{1,0,2}(n)}{2} &= \left|P_n\left(\sqrt{-1}\right)\right|, \\
\frac{r_{1,1,1}(n)}{6} &= \mathrm{Re}\, P_n\left(\frac{-1+\sqrt{-3}}{2}\right),
\end{aligned}$$

where $\sigma(n)$, $\frac{r_{1,0,1}(n)}{4}$, $\frac{r_{1,0,2}(n)}{2}$ and $\frac{r_{1,1,1}(n)}{6}$ are multiplicative functions[2] given by

$$\begin{aligned}
\sigma(n) &= \sum_{d|n} d, \\
r_{a,b,c}(n) &= \#\left\{(x,y)\in\mathbb{Z}^2 : \quad a\,x^2 + b\,x\,y + c\,y^2 = n\right\}.
\end{aligned}$$

Furthermore, for $q = \frac{1+\sqrt{-3}}{2}$, the same sequence $n \mapsto P_n(q)$ is related to $r_{1,0,1}(n)$ in three ways (see [3]), depending on the congruence class of n in $\mathbb{Z}/3\mathbb{Z}$,

$$\left|P_n\left(\frac{1+\sqrt{-3}}{2}\right)\right| = \begin{cases} r_{1,0,1}(n) & \text{if } n \equiv 0 \pmod 3, \\ \frac{1}{4}\, r_{1,0,1}(n) & \text{if } n \equiv 1 \pmod 3, \\ \frac{1}{2}\, r_{1,0,1}(n) & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

For any integer $n \geq 1$, consider the two-sided sequence

$$\ell_{n,k} = \ln\left(\tfrac{3}{2}\,k + \sqrt{\left(\tfrac{3}{2}\,k\right)^2 + 3\,n}\right)$$

and the intervals

$$\mathcal{L}_{n,k} = \left]\ell_{n,k} - \ln 3, \ell_{n,k}\right],$$

where k runs over the integers. Define the following variation of the polynomials $P_n(q)$,

[1] The original definition of $P_n(q)$, which we refer to as *Kassel-Reutenauer polynomials*, given in [1] is rather different, but equivalent, to the one presented here. We preferred to take the logarithm of the divisors in place of the divisors themselves in order to work with intervals $\mathcal{P}_{n,k}$ of constant length.

[2] The proofs that the functions $\frac{r_{1,0,1}(n)}{4}$, $\frac{r_{1,0,2}(n)}{2}$ and $\frac{r_{1,1,1}(n)}{6}$ are multiplicative can be found in [4], pages 413, 417 and 431 respectively.

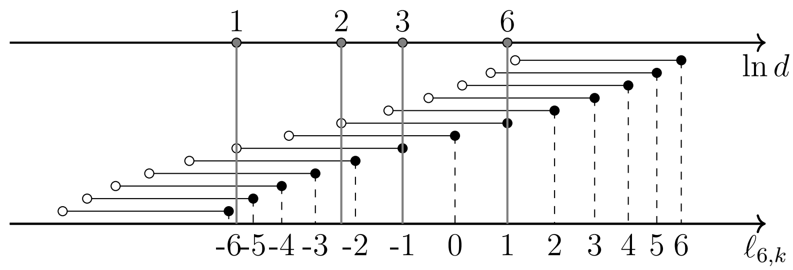

Figure 1: Representation of $L_6(q)$.

$$\frac{L_n(q)}{q^{n-1}} = \sum_{d|n} \sum_{k\in\mathbb{Z}} \mathbf{1}_{\mathcal{L}_{n,k}}(\ln d)\; q^k.$$

For example, in order to compute $L_6(q)$ from the definition, we need to consider the intervals $]\ell_{6,k} - \ln 3, \ell_{6,k}]$ on the real line and to count the number of values of $\ln d$ inside each interval, where d runs over the divisors of n. This data is shown in Fig 1, where the numbers $\ell_{6,k}$ are plotted on the line below (the corresponding values of k are labelled) whereas the numbers $\ln d$ are plotted on the line above (the corresponding values of d are labelled). Counting the number of intersections between the horizontal and the vertical lines, we obtain that the coefficients of $\frac{L_6(q)}{q^{6-1}}$ are as follows,

$$\frac{L_6(q)}{q^{6-1}} = q^5 + q^4 + q^3 + 2\,q^2 + 2\,q + 2\,q^0 + 2\,q^{-1} + 2\,q^{-2} + q^{-3} + q^{-4} + q^{-5}.$$

Like $P_n(q)$, the polynomial $L_n(q)$ is monic of degree $2\,n-2$, self-reciprocal and its coefficients are non-negative integers. The aim of this paper is to express the multiplicative functions[3] $\frac{r_{1,1,1}(n)}{6}$ and $\frac{r_{1,0,3}(n)}{2}$ in terms of the evaluations of $L_n(q)$ at roots of the unity. More precisely, we will prove the following result.

Theorem 1. *For each $n \geq 1$,*

$$A002324(n) \stackrel{\text{def}}{=} \frac{r_{1,1,1}(n)}{6} = 4\,\sigma(n) - 3\,L_n(1)\,, \tag{1}$$

$$A096936(n) \stackrel{\text{def}}{=} \frac{r_{1,0,3}(n)}{2} = L_n(-1)\,. \tag{2}$$

2 Auxiliary results for the identity (1)

For any $n \geq 1$, we will use the following notation,

$$d_{a,m}(n) := \#\{d|n : \quad d \equiv a \pmod{m}\}\,.$$

[3] The proof that the function $\frac{r_{1,0,3}(n)}{2}$ is multiplicative can be found in [4], page 421.

Lemma 2. *For all integers $n \geq 1$,*

$$\frac{r_{1,1,1}(n)}{6} = d_{1,3}(n) - d_{2,3}(n).$$

Proof. This result can be found as equation (3) in [5]. $\square$ $\square$

Lemma 3. *For any integer $n \geq 1$,*

$$3\left\lceil 3^{-1}\, n\right\rceil - n = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3, \\ 2 & \text{if } n \equiv 1 \pmod 3, \\ 1 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

$$n - 3\left\lfloor 3^{-1}\, n\right\rfloor = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3, \\ 1 & \text{if } n \equiv 1 \pmod 3, \\ 2 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. It is enough to evaluate $3\left\lceil 3^{-1}\, n\right\rceil - n$ and $3\left\lceil 3^{-1}\, n\right\rceil - n$ at $n = 3k + r$, for $k \in \mathbb{Z}$ and $r \in \{0, 1, 2\}$. $\square$ $\square$

Lemma 4. *For any pair of integers $n \geq 1$ and k, the inequalities*

$$\ell_{n,k} - \ln 3 < \ln d \leq \ell_{n,k}$$

hold if and only if the inequalities

$$3^{-1}\, d - \frac{n}{d} \leq k < d - 3^{-1}\, \frac{n}{d}$$

hold.

Proof. The inequalities

$$\ell_{n,k} - \ln 3 < \ln d \leq \ell_{n,k}$$

are equivalent to

$$\ln d \leq \ell_{n,k} < \ln d + \ln 3.$$

Applying the strictly increasing function $x \mapsto \frac{e^x}{3} - n\, e^{-x}$ to the last inequalities we obtain the following equivalent inequalities

$$3^{-1}\, d - \frac{n}{d} \leq k < d - 3^{-1}\, \frac{n}{d}.$$

Indeed, $\frac{e^{\ln d}}{3} - n\, e^{-\ln d} = 3^{-1}\, d - \frac{n}{d}$, $\frac{e^{\ln d + \ln 3}}{3} - n\, e^{-(\ln d + \ln 3)} = d - 3^{-1} \frac{n}{d}$ and

$$\begin{aligned} \frac{e^{\ell_{n,k}}}{3} - n\, e^{-\ell_{n,k}} &= \frac{\frac{3}{2}\, k + \sqrt{\left(\frac{3}{2}\, k\right)^2 + 3\, n}}{3} - \frac{n}{\frac{3}{2}\, k + \sqrt{\left(\frac{3}{2}\, k\right)^2 + 3\, n}} \\ &= \frac{\frac{3}{2}\, k + \sqrt{\left(\frac{3}{2}\, k\right)^2 + 3\, n}}{3} + \frac{\frac{3}{2}\, k - \sqrt{\left(\frac{3}{2}\, k\right)^2 + 3\, n}}{3} \\ &= k. \end{aligned}$$

So, the lemma is proved. $\square$ $\square$

Lemma 5. *Let $n \geq 1$ be an integer. For all $d|n$,*

$$\sum_{k\in\mathbb{Z}} \mathbf{1}_{\mathcal{L}_{n,k}}(\ln d) = \left\lceil d - 3^{-1}\,\frac{n}{d}\right\rceil - \left\lceil 3^{-1}\,d - \frac{n}{d}\right\rceil.$$

Proof. For all integers $n \geq 1$ and k,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \mathbf{1}_{\mathcal{L}_{n,k}}(\ln d) \\
=\ &\#\{k \in \mathbb{Z} : \quad \ell_{n,k} - \ln 3 < \ln d \leq \ell_{n,k}\} \\
=\ &\#\left\{k \in \mathbb{Z} : \quad 3^{-1}\,d - \frac{n}{d} \leq k < d - 3^{-1}\,\frac{n}{d}\right\} \quad \text{(Lemma 4)} \\
=\ &\#\left\{k \in \mathbb{Z} : \quad \left\lceil 3^{-1}\,d - \frac{n}{d}\right\rceil \leq k < \left\lceil d - 3^{-1}\,\frac{n}{d}\right\rceil\right\}, \\
=\ &\left\lceil d - 3^{-1}\,\frac{n}{d}\right\rceil - \left\lceil 3^{-1}\,d - \frac{n}{d}\right\rceil
\end{aligned}$$

So, the lemma is proved. □ □

3 Auxiliary results for the identity (2)

Lemma 6. *The function $\frac{r_{1,0,3}(n)}{2}$ is multiplicative.*

Proof. See page 421 in [4]. □

Lemma 7. *For all integers $n \geq 1$,*

$$\frac{r_{1,0,3}(n)}{2} = d_{1,3}(n) - d_{2,3}(n) + 2\,(d_{4,12}(n) - d_{8,12}(n)).$$

Proof. This result can be found as equation (1) in [5]. □ □

Recall that the nonprincipal Dirichlet character mod 3 is the 3-periodic arithmetic function $\chi_3(n)$ given by $\chi_3(0) = 0$, $\chi_3(1) = 1$ and $\chi_3(2) = -1$.

Lemma 8. *For all $n \geq 1$,*

$$\frac{(-1)^{\lfloor 3^{-1}\,n\rfloor} - (-1)^{\lceil 3^{-1}\,n\rceil}}{2} = (-1)^{n-1}\,\chi_3(n).$$

Proof. It is enough to substitute $n = 3k + r$, with $k \in \mathbb{Z}$ and $r \in \{0, 1, 2\}$, in both sides in order to check that they are equal. □ □

Lemma 9. *For all $n \geq 1$,*

$$\sum_{d|n} (-1)^{\frac{n}{d}-1}\,(-1)^{d-1}\,\chi_3(d) = (-1)^{n-1}\,\frac{r_{1,0,3}(n)}{2}.$$

Proof. By Lemma 6, the function $\frac{r_{1,0,3}(n)}{2}$ is multiplicative. Also, it is easy to check that the functions $(-1)^{n-1}$ and $\chi_3(n)$ are multiplicative. So, the functions $f(n) = (-1)^{n-1}\ \frac{r_{1,0,3}(n)}{2}$ and $(-1)^{n-1}\ \chi_3(n)$ are multiplicative, because the multiplicative property is preserved by ordinary product. The function $g(n) = \sum_{d|n} (-1)^{\frac{n}{d}-1}\ (-1)^{d-1}\ \chi_3(d)$ is multiplicative, because Dirichlet convolution preserves the multiplicative property. So, it is enough to prove that $f\left(p^k\right) = g\left(p^k\right)$ for each prime power p^k.

Consider the case $p = 2$. The following elementary equivalences hold for any integer $m \geq 0$,

$$\begin{aligned}
2^m \equiv 1 \pmod 3 &\iff m \equiv 0 \pmod 2, \\
2^m \equiv 2 \pmod 3 &\iff m \equiv 1 \pmod 2, \\
2^m \equiv 4 \pmod{12} &\iff m \equiv 0 \pmod 2 \text{ and } m \neq 0, \\
2^m \equiv 8 \pmod{12} &\iff m \equiv 1 \pmod 2 \text{ and } m \neq 1.
\end{aligned}$$

So, for each integer $k \geq 1$,

$$\begin{aligned}
d_{1,3}\left(2^k\right) &= \#[0,k] \cap 2\mathbb{Z} = \left\lfloor \frac{k}{2} \right\rfloor + 1, \\
d_{2,3}\left(2^k\right) &= \#[1,k] \cap (2\mathbb{Z}+1) = \left\lceil \frac{k}{2} \right\rceil, \\
d_{4,12}\left(2^k\right) &= \#[2,k] \cap 2\mathbb{Z} = \left\lfloor \frac{k}{2} \right\rfloor, \\
d_{8,12}\left(2^k\right) &= \#[3,k] \cap (2\mathbb{Z}+1) = \left\lceil \frac{k}{2} \right\rceil - 1.
\end{aligned}$$

For any $k \geq 1$, it follows that

$$
\begin{aligned}
g\left(2^k\right) &= \sum_{j=0}^{k} (-1)^{2^{k-j}-1} \, (-1)^{2^j-1} \, \chi_3\left(2^j\right) \\
&= \sum_{j=0}^{k} (-1)^{2^{k-j}-1} \, (-1)^{2^j-1} \, (-1)^j \\
&= -1-(-1)^k+\sum_{j=1}^{k-1}(-1)^j \\
&= -1-(-1)^k+\frac{-1-(-1)^k}{2} \\
&= -3\,\frac{1+(-1)^k}{2} \\
&= -3\left(1+\left\lfloor\frac{k}{2}\right\rfloor-\left\lceil\frac{k}{2}\right\rceil\right) \\
&= -\left(\left(\left\lfloor\frac{k}{2}\right\rfloor+1\right)-\left\lceil\frac{k}{2}\right\rceil+2\left(\left\lfloor\frac{k}{2}\right\rfloor-\left(\left\lceil\frac{k}{2}\right\rceil-1\right)\right)\right) \\
&= (-1)^{2^k-1}\left(d_{1,3}\left(2^k\right)-d_{2,3}\left(2^k\right)+2\left(d_{4,12}\left(2^k\right)-d_{8,12}\left(2^k\right)\right)\right) \\
&= f\left(2^k\right) \quad \text{(Lemma 7).}
\end{aligned}
$$

Let p and $k \geq 1$ be an odd prime and an integer respectively. Notice that $(-1)^{p^j-1}=1$ for all $0 \leq j \leq k$. Also, $d_{4,12}\left(p^k\right)=d_{8,12}\left(p^k\right)=0$, because p^k has no even divisor. So, for any $k \geq 1$,

$$
\begin{aligned}
g\left(p^k\right) &= \sum_{j=0}^{k} (-1)^{p^{k-j}-1} \, (-1)^{p^j-1} \, \chi_3\left(p^j\right) \\
&= \sum_{j=0}^{k} \chi_3\left(p^j\right) \\
&= d_{1,3}\left(p^k\right)-d_{2,3}\left(p^k\right) \\
&= (-1)^{p^k-1}\left(d_{1,3}\left(p^k\right)-d_{2,3}\left(p^k\right)+2\left(d_{4,12}\left(p^k\right)-d_{8,12}\left(p^k\right)\right)\right) \\
&= f\left(p^k\right) \quad \text{(Lemma 7).}
\end{aligned}
$$

Therefore, $f(n)=g(n)$ for all $n \geq 1$. □ □

Lemma 10. *For each $d|n$,*

$$
\sum_{k\in\mathbb{Z}} \mathbf{1}_{\mathcal{L}_{n,k}}(\ln d)\,(-1)^k=\frac{1}{2}\left((-1)^{\left\lceil 3^{-1}d-\frac{n}{d}\right\rceil}-(-1)^{\left\lceil d-3^{-1}\frac{n}{d}\right\rceil}\right).
$$

Proof. For any integer $n \geq 1$ and any $d|n$,

$$\sum_{k\in\mathbb{Z}} \mathbf{1}_{\mathcal{L}_{n,k}}(\ln d)\,(-1)^k = \sum_{3^{-1}\,d-\frac{n}{d}\leq k<d-3^{-1}\,\frac{n}{d}} (-1)^k \quad \text{(Lemma 4)}$$
$$= \sum_{\lceil 3^{-1}\,d-\frac{n}{d}\rceil\leq k<\lceil d-3^{-1}\,\frac{n}{d}\rceil} (-1)^k.$$

Substituting $a = \lceil 3^{-1}\,d - \frac{n}{d}\rceil$, $b = \lceil d - 3^{-1}\,\frac{n}{d}\rceil$ and $q = -1$ in the geometric sum

$$\sum_{a\leq k<b} q^k = \frac{q^a - q^b}{1-q}$$

we obtain

$$\sum_{k\in\mathbb{Z}} \mathbf{1}_{\mathcal{L}_{n,k}}(\ln d)\,(-1)^k = \frac{(-1)^{\lceil 3^{-1}\,d-\frac{n}{d}\rceil} - (-1)^{\lceil d-3^{-1}\,\frac{n}{d}\rceil}}{1-(-1)}$$
$$= \frac{1}{2}\left((-1)^{\lceil 3^{-1}\,d-\frac{n}{d}\rceil} - (-1)^{\lceil d-3^{-1}\,\frac{n}{d}\rceil}\right).$$

So, the lemma is proved. □ □

4 Proof of the main result

We proceed now with the proof of the main result of this paper.

Proof of Theorem 1. Identity (1) follows from the following transformations,

$$
\begin{aligned}
L_n(1) &= \sum_{d|n} \sum_{k\in\mathbb{Z}} \mathbf{1}_{\mathcal{L}_{n,k}}(\ln d) \\
&= \sum_{d|n} \left(\left\lceil d - 3^{-1}\frac{n}{d} \right\rceil - \left\lceil 3^{-1} d - \frac{n}{d} \right\rceil \right) \quad \text{(Lemma 5)} \\
&= \sum_{d|n} \left(d + \frac{n}{d} \right) + \sum_{d|n} \left\lceil -3^{-1}\frac{n}{d} \right\rceil - \sum_{d|n} \left\lceil 3^{-1} d \right\rceil \\
&= \sum_{d|n} \left(d + \frac{n}{d} \right) - \sum_{d|n} \left\lfloor 3^{-1}\frac{n}{d} \right\rfloor - \sum_{d|n} \left\lceil 3^{-1} d \right\rceil \\
&= \frac{2}{3}\sum_{d|n} \left(d + \frac{n}{d} \right) + \frac{1}{3}\sum_{d|n} \left(\frac{n}{d} - 3\left\lfloor 3^{-1}\frac{n}{d} \right\rfloor \right) - \frac{1}{3}\sum_{d|n} \left(3\left\lceil 3^{-1} d \right\rceil - d \right) \\
&= \frac{4\sigma(n)}{3} + \frac{d_{1,3}(n) + 2d_{2,3}(n)}{3} - \frac{2d_{1,3}(n) + d_{2,3}(n)}{3} \quad \text{(Lemma 3)} \\
&= \frac{4\sigma(n)}{3} - \frac{d_{1,3}(n) - d_{2,3}(n)}{3} \\
&= \frac{4}{3}\sigma(n) - \frac{1}{3}\frac{r_{1,1,1}(n)}{6} \quad \text{(Lemma 2)}.
\end{aligned}
$$

Identity (2) follows from the following transformations,

$$
\begin{aligned}
\frac{L_n(-1)}{(-1)^{n-1}} &= \sum_{d|n} \sum_{k\in\mathbb{Z}} \mathbf{1}_{\mathcal{L}_{n,k}}(\ln d)\,(-1)^k \\
&= \sum_{d|n} \frac{1}{2} \left((-1)^{\left\lceil 3^{-1} d - \frac{n}{d} \right\rceil} - (-1)^{\left\lceil \frac{n}{d} - 3^{-1} d \right\rceil} \right) \quad \text{(Lemma 10)} \\
&= \sum_{d|n} \frac{1}{2} \left((-1)^{\left\lceil 3^{-1} d \right\rceil - \frac{n}{d}} - (-1)^{\frac{n}{d} - \left\lfloor 3^{-1} d \right\rfloor} \right) \\
&= \sum_{d|n} (-1)^{\frac{n}{d}-1} \frac{(-1)^{\left\lfloor 3^{-1} d \right\rfloor} - (-1)^{\left\lceil 3^{-1} d \right\rceil}}{2} \\
&= \sum_{d|n} (-1)^{\frac{n}{d}-1} (-1)^{d-1} \chi_3(d) \quad \text{(Lemma 8)} \\
&= (-1)^{n-1} \frac{r_{1,0,3}(n)}{2} \quad \text{(Lemma 9)}.
\end{aligned}
$$

So, the theorem is proved. □ □

5 Final remarks

1. Let k be a field and $\mathcal{R}$ be a k-algebra. The *codimension* of an ideal I of $\mathcal{R}$ is the dimension of the quotient $\mathcal{R}/I$ as a vector space over k.

Consider the free abelian group of rank 2, denoted $\mathbb{Z} \oplus \mathbb{Z}$. Let $k = \mathbb{F}_q$ be the finite field with q elements and $\mathcal{R} = \mathbb{F}_q[\mathbb{Z} \oplus \mathbb{Z}]$ its group algebra. Kassel and Reutenauer [1] proved that, for any prime power q, the number of ideals of codimension $n \geq 1$ of $\mathbb{F}_q[\mathbb{Z} \oplus \mathbb{Z}]$ is $(q-1)^2 \, P_n(q)$. So, it is natural to look for connections between the values of $L_n(q)$, when q is a prime power, and the algebraic structures related to $\mathbb{F}_q$.

2. The polynomials $P_n(q)$ are generated by the product (see [2])

$$\prod_{m \geq 1} \frac{(1-t^m)^2}{(1-q\,t^m)\,(1-q^{-1}\,t^m)} = 1 + \left(q + q^{-1} - 2\right) \sum_{n=1}^{\infty} \frac{P_n(q)}{q^{n-1}}\, t^n.$$

It would be interesting to find a similar generating function for $L_n(q)$.

Acknowledge

The author thanks S. Brlek and C. Reutenauer for they valuable comments and suggestions concerning this research.

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2010 *Mathematics Subject Classification*: Primary 11A25; Secondary 11A07, 11A05.
Keywords: multiplicative function, representation by quadratic form, polynomial, divisor.

(Concerned with sequence A002324, A096936.)
