Set-Valued Young Tableaux and Product-Coproduct Prographs

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Abstract

Standard set-valued Young tableaux are a generalization of standard Young tableaux where cells can contain unordered sets of integers, with the condition that every integer at position (i, j) must be smaller that every integer at both $(i+1, j)$ and $(i, j+1)$. In this paper, we explore properties of standard set-valued Young tableaux with three-rows and various row-constant cell densities. We show that standard set-valued Young tableaux of rectangular shape and row-constant density $\rho = (1, k-1, 1)$ are in bijection with closed k-ary prographs: planar graphs where all internal vertices may be interpreted as either a k-ary product or k-ary coproduct when read from bottom-to-top. That bijection is extended to three-row set-valued Young tableaux of non-rectangular and skew shape, and it is shown that a set-valued analogue of the Schẗzenberger involution on tableaux corresponds to 180-degree rotation of the associated prographs. As a set-valued analogue of the hook-length formula is currently lacking, we also present direct enumerations of three-row set-valued tableaux for a variety of row-constant densities and a small number of columns. We then argue why the resulting integers for density $\rho = (1, k-1, 1)$ should be interpreted as a one-parameter generalization of the three-dimensional Catalan numbers, mirroring the generalization of the (two-dimensional) Catalan numbers provided by the k-Catalan numbers.

1 Introduction: Standard Set-Valued Young Tableaux

For a non-increasing integer partition $\lambda = (\lambda_1, ..., \lambda_m)$ of the positive integer N, a Young diagram Y of shape λ is a left-justified array of N cells with λ_i cells in the i^{th} row. Given a Young diagram of shape λ, a Young tableau of the same shape is a bijection from the set of integers $[N] = \{1, ..., N\}$ to the cells of Y. In order for a Young tableau to be a standard Young tableau, the entries in the tableau must increase left-to-right across each row and down each column. We denote the set of standard Young tableaux of shape λ as $S(\lambda)$, and adopt the shorthand notation of $S(n^m)$ in the case of the m-row rectangular shape $\lambda = (n, \ldots, n)$. For a thorough introduction to Young tableaux, see Fulton [6].

The number of standard Young tableaux of arbitrary shape λ may be directly calculated using the hook-length formula, as originally given by Frame, Robinson and Thrall [5]. A quick application of the hook-length formula to the case of $\lambda = (n, n)$ yields the well-known identity that $|S(n^2)| = C_n = \frac{(2n)!}{n!(n+1)!}$, the n^{th} Catalan number. Generalizing to the d-row rectangular case of $\lambda = (n, \ldots, n)$ similarly yields $|S(n^d)| = C_{d,n}$, where $C_{d,n} = \frac{(d-1)!(dn)!}{n!(n+1)!\ldots(n+d-1)!}$ is the n^{th} d-dimensional Catalan number.

Given an integer partition $\lambda = (\lambda_1, \ldots, \lambda_{m_1})$ of N_1 and an integer partition $\mu = (\mu_1, \ldots, \mu_{m_2})$ of N_2, where $0 \leq \mu_i \leq \lambda_i$ for all i, one can also define a skew Young diagram of shape λ/μ by removing the μ_i leftmost cells in the i^{th} row of the Young diagram of shape λ, for all $1 \leq i \leq m$. A skew Young tableau of shape λ/μ is a bijection from $[N_1 - N_2]$ to the cells of the skew Young diagram of shape λ/μ. As above, such a tableau is said to be a skew standard Young tableau if its entries increase left-to-right across each row and down each column. We denote the set of skew standard Young tableaux of shape λ/μ by $S(\lambda/\mu)$.

This paper is focused on a generalization of standard Young tableaux known as standard set-valued Young tableaux. Consider a non-increasing integer partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ and a sequence of positive integers $\rho = (\rho_1, \ldots, \rho_m)$ such that $\sum_{i=1}^{m} \lambda_i \rho_i = M$. A **set-valued Young tableau** of shape λ and (row-constant) **density** is a function from $[M]$ to the cells of the Young diagram Y of shape λ such that every cell in the i^{th} row of Y receives precisely ρ_i integers. The resulting tableau qualifies as a **standard set-valued Young tableau** if, for each cell (i,j) of Y, every integer at position (i,j) is smaller than every integer in the cells at $(i+1,j)$ and $(i,j+1)$. In analogy with standard Young tableaux, we refer to these additional conditions as "column-standardness" and "row-standardness", respectively. We denote the set of standard set-valued Young tableaux of shape λ and density ρ as $\mathbb{S}(\lambda, \rho)$. See Figure 1 for a collection of standard set-valued Young tableaux with $\lambda = (3,3) = 3^2$ and $\rho = (2,1)$. Given a skew Young diagram of shape λ/μ, one may similarly define a **skew standard set-valued Young tableau** of shape λ/μ and (row-constant) density ρ. We denote the set of such skew tableau by $\mathbb{S}(\lambda/\mu, \rho)$.

1 2	4 5	6 7
3	8	9

1 2	3 5	7 8
4	6	9

1 2	3 4	5 8
6	7	9

1 2	3 4	5 6
7	8	9

1 2	4 5	7 8
3	6	9

Figure 1: Five of the twelve elements of $\mathbb{S}(3^2, \rho)$ with $\rho = (2,1)$.

Set-valued Young tableaux were originally introduced by Buch [2]. For later explorations into standard set-valued Young tableaux, see Heubach, Li and Mansour [7] or Drube [3].

It is important to emphasize the current lack of a set-valued analogue to the hook-length formula. This makes the enumeration of $\mathbb{S}(\lambda, \rho)$ for arbitrary λ and ρ an extremely challenging problem, and comprehensive attempts at counting standard set-valued Young tableaux of arbitrary density ρ have only been attempted for two-row shapes $\lambda = (a,b)$. See Drube [3] for calculations of $|S(\lambda, \rho)|$ in the two-row rectangular case, enumerations that corresponded to various generalizations of the (two-dimensional) Catalan numbers.

With the two-row case relatively well-understood, this paper presents the first thorough investigation of standard set-valued Young tableaux in the case of three-row shapes. Much as various choices of ρ allow $|\,\mathbb{S}(n^2, \rho)|$ to correspond to various generalizations of the two-dimensional Catalan numbers, $|\,\mathbb{S}(n^3, \rho)|$ and certain choices of ρ will correspond to various generalizations of the three-dimensional Catalan numbers. To our knowledge, all three-dimensional Catalan generalizations discussed in this paper have yet to appear anywhere in the literature.

1.1 Outline of Paper

This paper proceeds as follows. In Section 2, we introduce k-ary product-coproduct prographs, a certain class of planar graphs that naturally extends existing combinatorial interpretations for both the two- and three-dimensional Catalan numbers. Much of Section 2 may be seen as a direct extension of the work of Borie [1]. This motivates our focus on the sets $\mathbb{S}(n^3, \rho)$ with $\rho = (1, k-1, 1)$, which are placed in bijection with k-ary product-coproduct prographs (Theorem 2.1). In Section 3, we focus upon the enumeration of these tableaux, yielding a family of integers $C^k_{3,n}$ that function as a three-dimensional analogue of the k-Catalan numbers $C^k_n = \frac{(kn)!}{(kn-n+1)!n!}$. Closed formulas for $C^k_{3,n}$ are derived in the cases of $n \leq 4$ (Propositions 3.1, 3.2, 3.3) and conjectures are made about the general case (Conjecture 3.4). In Section 4, we further explore the bijection between our standard set-valued Young tableaux and k-ary product-coproduct prographs. Appropriately generalized prographs are placed in bijection with various sets of (non-rectangular and skew) standard set-valued Young tableaux (Theorem 4.2), and a 180-degree rotation of prographs is shown to correspond to a set-valued analogue of the Schützenberger involution on standard Young tableaux (Theorem 4.3). Section 5 closes the paper with a pair of more cursory discussions, the first of which suggests additional combinatorial interpretations for $C^k_{d,n}$ and the second of which concerns three-row set-valued tableaux of entirely distinct densities ρ.

2 k-ary Product-Coproduct Prographs and Set-Valued Tableaux

It is well-known that the set $\mathcal{T}_n$ of binary trees with n non-terminal nodes is enumerated by the Catalan number $C_n = \frac{(2n)!}{n!(n+1)!}$. This places $\mathcal{T}_n$ in bijection with $S(n^2)$. One straightforward bijection $\phi : \mathcal{T}_n \to S(n^2)$ labels the edges $T \in \mathcal{T}_n$ according to a "leftward search" (see Step #1 below), places integers corresponding to left-children of T in the top row of $\phi(T)$, and places integers corresponding to right children of T in the bottom row of $\phi(T)$.

This bijection may be generalized to the set $\mathcal{T}_n^k$ of k-ary trees with n non-terminal nodes, which are enumerated by the k-Catalan numbers $C_n^k = \frac{(kn)!}{(kn-n+1)!n!}$. Heubach, Li and Mansour [7] proved that standard set-valued Young tableaux $\mathbb{S}(n^2, \rho)$ with $\rho = (k-1, 1)$ are also enumerated by k-Catalan numbers. See Figure 2 for an example of the bijection $\phi_k : \mathcal{T}_n^k \to \mathbb{S}(n^2, \rho)$, which works as follows:

1. Beginning at the root vertex, snake around the tree T in a clockwise direction and number edges according to the order in which they are first encountered.
2. Place all integers corresponding to rightmost-children of T in the bottom row of $\phi(T)$, in increasing order from left to right.
3. Place all remaining integers in the top row of $\phi(T)$, in increasing order from left to right and ensuring that each first row cell receives precisely $k-1$ integers.

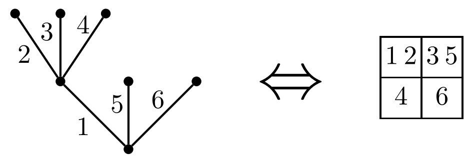

Figure 2: An example of the bijection $\phi_k : \mathcal{T}_n^k \to \mathbb{S}(n^2, \rho)$ between k-ary trees and two-row, standard set-valued Young tableaux of density $\rho = (k-1, 1)$.

Further generalizing this construction to three-row tableaux requires a class of planar graphs known as product-coproduct prographs. Binary product-coproduct prographs, as they relate to tableaux, were first considered by Borie [1]. Borie showed that the set $PC^2(n)$ of closed product-coproduct prographs with $2n$ internal vertices was enumerated by the three-dimensional Catalan numbers $C_{3,n} = \frac{2(3n)!}{n!(n+1)!(n+2)!}$. We begin by directly generalizing Borie's construction to the k-ary case.

A **k-ary (product-coproduct) prograph** is a planar graph that has been embedded in such a way that each of its internal vertices, when read from bottom to top, may be interpreted as either a k-ary product or a k-ary coproduct. Here a k-ary product takes in k edges and returns a single edge, whereas a k-ary coproduct takes in a single edge and returns k edges. See Figure 3 for examples of both components. If a prograph is not connected, it will also be necessary to require that the prograph possess no local extrema with respect to the y coordinate. This implies that every edge in a prograph has one input/initial end and one output/terminal end.

Figure 3: Products and coproducts for $k = 2$ and $k = 3$.

These k-ary product-coproduct prographs are best interpreted as a generalization of k-ary trees, with $\mathcal{T}_n^k$ corresponding to k-ary prographs that consist of n coproducts and zero products. In Section 4 we will consider a wide variety of k-ary prographs, but in this section we restrict our attention to closed prographs. A **closed k-ary (product-coproduct) prograph** is a k-ary product-coproduct prograph with a single input edge and a single output edge. Notice that a closed k-ary prograph must contain the same number of products as coproducts and must be connected. We denote the set of closed k-ary prographs with n products and n coproducts as $PC^k(n)$. See Figure 4 for an illustration of $PC^2(2)$.

We begin by introducing an algorithm for numbering the edges of any $G \in PC^k(n)$. Following Borie [1], we refer to this procedure as the "left-ascending search" on closed prographs:

1. Label the initial input edge with the integer 0.
2. Recursively consider the subgraph G_i' of G consisting only of the edges labelled $1, \ldots, i$, and "snake around" G_i' in the clockwise direction (beginning from the initial edge) until encountering a node with at least one output edge in $G - G_i'$.

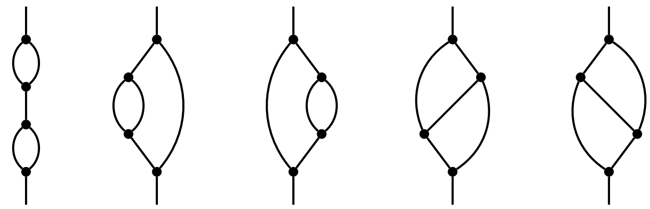

Figure 4: The set $PC^2(2)$ of closed 2-ary prographs with 2 products and 2 coproducts.

3. Label the leftmost unlabelled edge of that node with $i+1$.
4. Repeat Steps #2 and #3 until reaching the single output edge, which will receive the integer nk.

See Figure 5 for an example of the left-ascending search. Colloquially, the procedure may be described as "staying as leftward as possible, with the restriction that all input edges to a node must be labelled before any output edge from that node may be labelled".

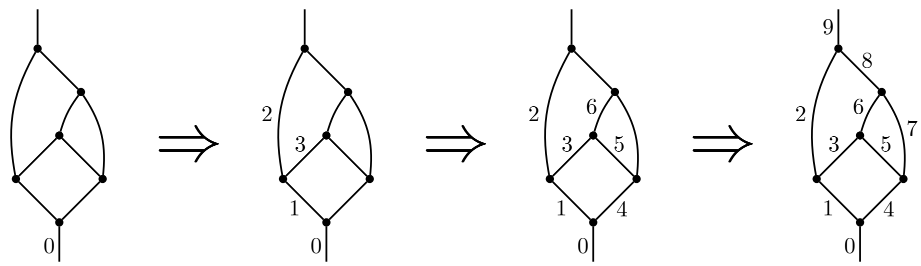

Figure 5: Example of the left-ascending search on an element of $PC^2(3)$.

We are now ready to place $PC^k(n)$ in bijection with an appropriate collection of standard set-valued Young tableaux. Theorem 2.1 below directly recovers the result of Borie [1] in the case of $k = 2$.

Theorem 2.1. *Fix $n \geq 1$ and $k \geq 2$. Then $|PC^k(n)| = |\,\mathbb{S}(\lambda, \rho)|$ for $\lambda = (n, n, n)$ and $\rho = (1, k-1, 1)$.*

Proof. We construct a bijection $\phi : PC^k(n) \to \mathbb{S}(n^3, \rho)$. See Figure 6 for an example of this map.

1. Label the edges of $G \in PC^k(n)$ according to the left-ascending search defined above.
2. Place integers corresponding to leftmost coproduct children of G in the first row of $\phi(G) \in \mathbb{S}(n^3, \rho)$, in increasing order from left to right.
3. Place integers corresponding to all remaining coproduct children of G along the middle row of $\phi(G)$, in increasing order from left to right and ensuring that each second row cell receives precisely $k-1$ integers.
4. Place integers corresponding to product children of G along the third row of $\phi(G)$, in increasing order from left to right.

Notice that the initial input label of 0 is ignored in this procedure. The map ϕ is clearly injective. As $\phi(G)$ is row-standard by construction, to show that ϕ is a well-defined map into $\mathbb{S}(n^3, \rho)$ we merely need to argue that $\phi(G)$ is column-standard. This follows directly from the definition of our left-ascending search. As the leftmost child of a given coproduct node will always be labelled prior to the $k-1$ non-leftmost children of that same node, all second-row entries of $\phi(G)$ must be larger than the first-row entry in their column. Also notice that the addition of a coproduct node increases free edges by $k-1$ while the addition of a product node decreases free edges. Thus, jk available edges requires the addition of at least j coproduct nodes. Since each product node requires k distinct labelled inputs, this means that the j^{th} product edge to be labelled in our left-ascending search is proceeded by at least jk children of at least j different coproduct nodes. It follows that all third-row entries of $\phi(G)$ must be larger than all first- and second-row entries from the same column.

To show that ϕ is a bijection, we construct an injective map $\phi^{-1} : \mathbb{S}(n^3, \rho) \to PC^k(n)$. Starting with $T \in \mathbb{S}(\lambda, \rho)$, we place an initial input edge (labelled 0) and then recursively "build up" an edge-labelled version of $\phi^{-1}(T) \in PC^k(n)$ by working through the entries of T in numerical order as described below:

1. If i is in the top row of T, place a coproduct whose input is the edge labelled $i-1$. Then label the leftmost child of that coproduct with i.
2. If i is in the middle row of T, follow the left-ascending search through the partially-constructed graph from the edge labelled $i-1$. Then label the first unlabelled edge encountered (which will always be a non-leftmost coproduct child) with the integer i.
3. If i is in the bottom row of T, place a product whose rightmost input is the edge labelled $i-1$ and whose remaining inputs are the $k-1$ nearest free edges immediately to the left of the edge labelled $i-1$. Then label the output of that product i.

The map ϕ^{-1} is clearly injective, but its well-definedness depends upon whether the actions described above are possible at every step. In particular, there may not exist a rightward unlabelled edge when applying Option #2, or there not be enough leftward free edges (all labelled) when adding the product node in Option #3.

First note that, as leftmost coproduct children and product children are immediately labelled as soon as their source node is placed, unlabelled free edges at any intermediate step of the procedure must correspond to non-leftmost coproduct children. So assume that the entry i lies in the second-row cell $(2, j)$ of T, and that i is larger than precisely x other integers in that cell ($0 \leq x \leq k-2$). Row- and column-standardness of T guarantees that at least j coproducts have already been placed before we reach this point in the procedure, and that $(k-1)(j-1)+x$ of the non-leftmost children from those coproducts have already been labelled. This means there are at least $(k-1) * j - (k-1)(j-1) - x = k-1-x \geq 1$ unlabelled non-leftmost coproduct children at this step. The nature of the left-ascending search, as applied to previous integers in the second row of T, ensures that these unlabelled coproduct edges all lie to the right of the edge labelled $i-1$. Thus the operation of Option #2 is always possible.

Now assume that i lies in the third-row cell $(3, j)$ of T. Row- and column-standardness of T once again guarantee that at least j coproducts and precisely $j-1$ products have already been placed at this point in the procedure, with at least kj coproduct children and all $j-1$ product children already having been labelled. As $j-1$ labelled inputs are needed for the placement of each coproduct, this means that there are at least $kj-(k-1)(j-1)-(j-1) = k$ labelled free edges when i is the active integer. Due to the nature of the left-ascending search, the edge labelled $i-1$ is the rightmost of these free edges. Thus the operation described in Option #3 is always possible. $\square$

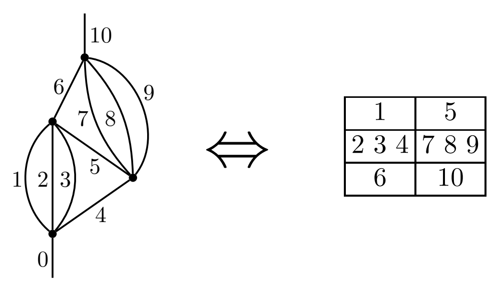

Figure 6: An example of the bijection $\phi : PC^k(n) \to \mathbb{S}(n^3, \rho)$ for $n = 2$ and $k = 4$.

3 Enumerating $\mathbb{S}(n^3, \rho)$ for $\rho = (1, k-1, 1)$

Theorem 2.1 suggests that $\mathbb{S}(n^3, \rho)$ with $\rho = (1, k-1, 1)$ generalizes $S(n^3)$ in a manner similar to how $\mathbb{S}(n^2, \widetilde{\rho})$ with $\widetilde{\rho} = (k-1, 1)$ generalizes $S(n^2)$. As $\mathbb{S}(n^2, \widetilde{\rho})$ is enumerated by the k-Catalan numbers C_n^k, we henceforth refer to the cardinalities $|\,\mathbb{S}(n^3, \rho)| = C_{3,n}^k$ as the **three-dimensional k-Catalan numbers**.

The purpose of this section is to develop closed formulas $C_{3,n}^k$. Sadly, developing such a formula or deriving a multivariate generating function for arbitrary $n \geq 1$, $k \geq 1$ do not appear to be tractable problems. As such, we restrict our attention to cases of small n. See Figure 1 of Appendix A for a table of known values of $C_{3,n}^k$, which combines the explicit results of this section with computer calculations performed in Java.

In all that follows, notice that the "degenerate" $k = 1$ case corresponds to three-row tableaux with empty cells across their middle row and hence that those enumerations predictably reduce to pre-existing results about two-row tableaux: that $C^1_{3,n} = |\mathbb{S}(n^3, \rho)| = |S(n^2)| = C_n$ for all $n \geq 1$ and $\rho = (1, 0, 1)$.

For all of our enumerations we recursively place $\mathbb{S}(\lambda, \rho)$ in bijection with a collection of sets $\bigcup \mathbb{S}(\lambda_i, \rho)$ of strictly smaller shape yet equivalent density. Our technique is similar to pre-existing proofs for non-set-valued tableaux where the sub-shapes λ_i are determined via the removal of lower-right corners, corresponding to possible locations of the largest possible entry in a tableaux of shape λ. The difference here is that we never remove entries from a cell without eliminating all entries in that cell. If the removed cell contains entries other than the largest entry in the tableau, this necessitates that we account for the ordering of those smaller entries relative to integers appearing elsewhere in the tableau.

Before proceeding, observe that $|S(\lambda, \rho)|$ is easily calculable whenever $\lambda = (n, 1, \ldots, 1)$ is "hook-shaped". In this case, one merely needs to count the ways of partitioning entries between the rightward and downward "legs" of the tableau, giving an enumeration in terms of a single binomial coefficient $|S(\lambda, \rho)| = \binom{a}{b}$. See Figure 7 for an example of this phenomenon.

For the rest of this section, an unfilled Young diagram of shape λ is used to denote the cardinality $|\mathbb{S}(\lambda, \rho)|$, assuming ρ is well-understood.

$$\ydiagram{2,1,1} = \binom{k+1}{1} \qquad \ydiagram{4,1,1} = \binom{k+3}{3}$$

Figure 7: Cardinalities $|S(\lambda, \rho)|$ for $\rho = (1, k-1, 1)$ and several hook-shapes λ. As 1 must lie at position $(1, 1)$, one merely needs to determine which of $\{2, 3, \ldots\}$ lie in the remaining first-row cells.

Proposition 3.1. *Let $\rho = (1, k-1, 1)$. For any $k \geq 1$, $C^k_{3,2} = |\mathbb{S}(2^3, \rho)| = k^2 + 1$.*

Proof. As the largest entry of any $T \in \mathbb{S}(2^3, \rho)$ must lie at position $(3, 2)$, we investigate the integers $a_1 < \ldots < a_{k-1}$ lying at $(2, 2)$ in an arbitrary set-valued tableaux $T_1 \in \mathbb{S}(\lambda_1, \rho)$ of shape $\lambda_1 = (2, 2, 1)$. The only other entry in T_1 that may be larger than any of the a_i is the entry b at position $(3, 1)$. The subset of $\mathbb{S}(\lambda_1, \rho)$ satisfying $b \leq a_i$ for all i is then in bijection with $\mathbb{S}(\lambda_2, \rho)$ for $\lambda_2 = (2, 1, 1)$. If $b > a_1$, one must specify the ordering of b relative to $a_2, \ldots, a_{k-1}$. So assume that j is the largest index such that $a_j < b$ (where $1 \leq j \leq k-1$). Each choice of j defines a subset of $\mathbb{S}(\lambda_2, \rho)$ that is in bijection with $\mathbb{S}(\lambda_3, \rho)$ for $\lambda_3 = (2, 1)$, since for any choice of j the k largest entries of such a tableau $T_1 \in \mathbb{S}(\lambda_1, \rho)$ is split between positions $(2, 2)$ and $(3, 1)$. Combining these observations gives the string of equalities below.

$$\ydiagram{2,2,2} = \ydiagram{2,2,1} = \ydiagram{2,1,1} + (k-1)\,\ydiagram{2,1} = \binom{k+1}{1} + (k-1)\binom{k}{1} = k^2 + 1$$

$\square$

Proposition 3.2. *Let $\rho = (1, k-1, 1)$. For any $k \geq 1$,*

$$C^k_{3,3} = |\mathbb{S}(3^3, \rho)| = \frac{9k^4 - 2k^3 + 9k^2}{4} + 1$$

Proof. We begin by enumerating $\mathbb{S}(\lambda', \rho)$ for $\lambda' = (3, 2, 1)$. For arbitrary $T' \in \mathbb{S}(\lambda', \rho)$, let $a_1 < \ldots < a_{k-1}$ denote the entries at $(2, 2)$, b denote the entry at $(3, 1)$, and c denote the entry at $(1, 3)$. Proceeding as in the proof of Proposition 3.1, we subdivide $\mathbb{S}(\lambda', \rho)$ based on the relationship of b and c to the a_i and then delete all entries $x \geq a_1$ to place each subset in bijection with tableaux of some smaller shape. The equalities below synopsize our results, with the first summand corresponding to $b, c < a_1$, the second summand corresponding to the $k-1$ placements of b relative to $a_2 < \ldots < a_{k-1}$ when $b > a_1$ yet $c < a_1$, the third summand corresponding to the $k-1$ placements of c relative to $a_2 < \ldots < a_{k-1}$ when $c > a_1$ yet $b < a_1$, and the fourth summand corresponding to the $\binom{k}{k-2,1,1}$ placements of b, c relative to $a_2 < \ldots < a_{k-1}$ when $b, c > a_1$.

$$\ydiagram{3,2,1} = \ydiagram{3,1,1} + \binom{k-1}{1}\ydiagram{3,1} + \binom{k-1}{1}\ydiagram{2,1,1} + \binom{k}{k-2,1,1}\ydiagram{2,1}$$

$$= \binom{k+2}{2} + \binom{k-1}{1}\binom{k+1}{2} + \binom{k-1}{1}\binom{k+1}{1} + \binom{k}{k-2,1,1}\binom{k}{1} \;=\; \frac{3k^3+k^2+2k}{2}$$

For the full theorem, we once again proceed as in the proof to Proposition 3.1. After reducing to arbitrary $T \in \mathbb{S}(\lambda, \rho)$ with $\lambda = (3,3,2)$, we divide $\mathbb{S}(\lambda, \rho)$ into subsets depending upon how the entries $a_1 < \ldots < a_{k-1}$ at position $(2,3)$ relate to the entry b_1 at $(3,1)$ and the entry b_2 at $(3,2)$. The three summands in the first line of the equalities below corresponds to the cases of $b_1 < b_2 < a_1$, $b_1 < a_1 < b_2$, and $a_1 < b_1 < c_2$, respectively. In the second line of eqaulities, the first of those subsets is further subdivided based upon the relationship of the entry c at $(1,3)$ to the entry b_2 at $(3,2)$, with the two new summands corresponding to $b_2 < c$ and $c < b_2$, respectively. This leaves a sum of cardinalities $|\,\mathbb{S}(\lambda_i, \rho)|$ that are computable via Proposition 3.1, our informal lemma for shape $\lambda' = (3,2,1)$, and the result of Heubach, Li and Mansour [7] giving $|\,\mathbb{S}(n^2, \rho)| = C_n^k$.

$$\ydiagram{3,3,3} = \ydiagram{3,3,2} = \ydiagram{3,2,2} + \binom{k-1}{1}\ydiagram{3,2,1} + \binom{k}{2}\ydiagram{3,2}$$

$$= \ydiagram{2,2,2} + \ydiagram{3,2,1} + \binom{k-1}{1}\ydiagram{3,2,1} + \binom{k}{2}\ydiagram{3,2}$$

$$= (k^2+1) + k\left(\frac{3k^3+k^2+2k}{2}\right) + \binom{k}{2}C_3^k \;=\; \frac{9k^4-2k^3+9k^2}{4} + 1$$

$\square$

Proposition 3.3. *Let $\rho = (1, k-1, 1)$. For any $k \geq 1$,*

$$C_{3,4}^k = |\,\mathbb{S}(4^3, \rho)| = \frac{256k^6 - 114k^5 + 217k^4 - 12k^3 + 121k^2}{36} + 1$$

Sketch of Proof. As this enumeration differs from Propositions 3.1 and 3.2 only in the number of sub-cases involved, we immediately jump to the string of equalities illustrating our results. Before the primary decomposition, we directly present the result of lemmas for $\lambda_1' = (4,3,1)$ and $\lambda_2' = (4,3,2)$.

$$\ydiagram{4,3,1} \;=\; \frac{16}{3}k^4 - \frac{8}{3}k^3 + \frac{5}{3}k^2 - \frac{1}{3}k \qquad\qquad \ydiagram{4,3,2} \;=\; 4k^5 + \frac{7}{6}k^4 + 2k^3 + \frac{5}{6}k^2 + k$$

$$\ydiagram{4,4,4} = \ydiagram{4,4,3} = \ydiagram{4,3,3} + \binom{k-1}{1}\ydiagram{4,3,2} + \binom{k}{2}\ydiagram{4,3,1} + \binom{k+1}{3}\ydiagram{4,3}$$

$$= \ydiagram{3,3,3} + \ydiagram{4,3,2} + \binom{k-1}{1}\ydiagram{4,3,2} + \binom{k}{2}\ydiagram{4,3,1} + \binom{k+1}{3}\ydiagram{4,3}$$

$$= \left(\frac{9k^4-2k^3+9k^2}{4} + 1\right) + k\left(4k^5 + \frac{7}{6}k^4 + 2k^3 + \frac{5}{6}k^2 + k\right)$$

$$+ \binom{k}{2}\left(\frac{16}{3}k^4 - \frac{8}{3}k^3 + \frac{5}{3}k^2 - \frac{1}{3}k\right) + \binom{k+1}{3}C_4^k \;=\; \frac{256k^6 - 114k^5 + 217k^4 - 12k^3 + 121k^2}{36} + 1$$

$\square$

Although a general formula for $|\,\mathbb{S}(n^3, \rho)|$ when $n \geq 5$ does not appear tractable, similarities in our results for $n = 2,3,4$ allow us to make several conjectures about the general case. Notice that the expressions from Propositions 3.1, 3.2 and 3.3 each have degree $2n-2$ and a leading coefficient of $\frac{n^{2n-4}}{(n-1)!(n-1)!}$, with all non-constant coefficients being divisible by $(n-1)!(n-1)!$. Enforcing these conditions, in the case of $n = 5$ we may apply a linear regression to the experimentally-determined data points of Table 1 from Appendix A to obtain the following enumeration

Conjecture 3.4. *Let $\rho = (1, k-1, 1)$. For any $k \geq 1$,*

$$C_{3,5}^k = |\,\mathbb{S}(5^3, \rho)| = \frac{15626k^8 - 10092k^7 + 10258k^6 - 72k^5 + 5473k^4 - 204k^3 + 2628k^2}{576} + 1$$

4 Properties of k-ary Prographs and Set-Valued Tableaux

In this section we prove a generalization of Theorem 2.1 that applies to non-closed prographs satisfying certain basic properties. We then explore one significant application of our bijection that generalizes an unproven proposition of Borie [1], showing that rotation of prographs corresponds to a set-valued analogue of the Schützenberger involution on standard Young tableaux.

4.1 Non-Closed k-ary Prographs and Set-Valued Tableaux

We begin with a characterization of what sorts of non-closed k-ary prographs are possible for fixed k. Denote the set of k-ary product-coproduct prographs with precisely n coproducts, m products, and x initial input strands by $PC_x^k(n,m)$. Notice that these three parameters are sufficient to determine the number of output strands in any $G \in PC_x^k(n,m)$. More explicitly,

Proposition 4.1. *Take any k-ary product-coproduct prograph $G \in PC_x^k(n,m)$. Then G has precisely $y = (n-m)(k-1)+x$ output strands. In particular, $y \equiv x \mod (k-1)$.*

Proof. Observe that each k-ary coproduct increases the number of free edges by $k-1$, while each k-ary product decreases the number of free edges by $k-1$. If we begin with x free edges, after n coproducts and m products we have $y = x + n(k-1) - n(k-1)$ outgoing free edges. □

For any set of non-closed k-ary prographs $PC_x^k(n,m)$ where $x \equiv 1 \mod (k-1)$, there exists an injection $j : PC_x^k(n,m) \to PC^k(n + \frac{x-1}{k-1})$ that is formed by recursively joining incoming strands with coproducts, from left-to-right in sets of k, while recursively joining outgoing strands with products, from right-to-left in sets of k. See Figure 8 for an illustration of this operation. For any $G \in PC_x^k(n,m)$, we call the image $j(G) \in PC^k(n+\frac{x-1}{k-1})$ the **justified prograph (justification)** of G.

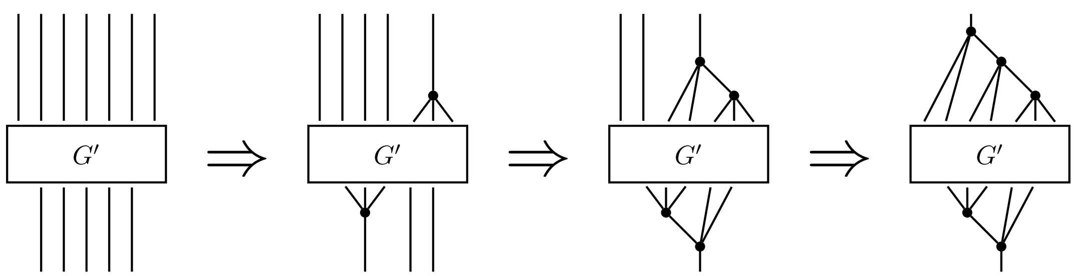

Figure 8: Closing $G \in PC_5^3(n,n-1)$ to obtain the justified prograph $j(G) \in PC^3(n + \frac{5-1}{3-1})$.

For any $G \in PC_x^k(n,m)$ with $x \equiv 1 \mod (k-1)$, applying our left-ascending search to $j(G)$ always results in the first $\frac{x-1}{k-1}$ labels being applied to the "new" coproduct edges on the bottom of $j(G)$ and the last $\frac{y-1}{k-1}$ labels being applied to the "new" product edges on the top of $j(G)$. This suggests a natural way to extend our left-ascending search to non-closed prographs when $x \equiv 1 \mod (k-1)$. To label $G \in PC_x^k(n,m)$, apply the left-ascending search to $j(G)$, delete all edges of $j(G)$ not in G, and subtract $\frac{x-1}{k-1}$ from all remaining labels. See Figure 9 for an example of this generalized search.

Assuming $x \equiv 1 \mod (k-1)$, justification also suggests a natural way to directly generalize Theorem 2.1:

Theorem 4.2. *Fix $n, m \geq 1$, $k \geq 2$, and take any $x \geq 1$ such that $x \equiv 1 \mod (k-1)$. Then $|PC_x^k(n,m)| = |\mathbb{S}(\lambda/\mu, \rho)|$, where $\lambda = (n + \frac{x-1}{k-1}, n + \frac{x-1}{k-1}, m)$, $\mu = (\frac{x-1}{k-1}, 0, 0)$, and $\rho = (1, k-1, 1)$.*

Proof. Let $j : PC_x^k(n,m) \to PC^k(n + \frac{x-1}{k-1})$ be justification and let $\phi : PC^k(n + \frac{x-1}{k-1} \to \mathbb{S}((n + \frac{x-1}{k-1})^3, \rho)$ be the bijection from Theorem 2.1. Then define $\chi : \mathbb{S}((n + \frac{x-1}{k-1})^3, \rho) \to \mathbb{S}(\lambda/\mu, \rho)$ as the map that deletes the first $\frac{x-1}{k-1}$ cells in the first row of $T \in \mathbb{S}((n + \frac{x-1}{k-1})^3, \rho)$, deletes the last $\frac{y-1}{k-1}$ cells in the third row of T, and reindexes all remaining entries by $a \mapsto a - \frac{x-1}{k-1}$. We look to define a bijection $\psi : PC_x^k(n,m) \to \mathbb{S}(\lambda/\mu, \rho)$ by $\psi \equiv \chi \circ \phi \circ j$. See Figure 9 for an example of this map ψ.

The definition of j, ϕ, and χ ensure that ψ is well-defined. We have also already argued that j and ϕ are injective, so to show that ψ is injective we merely need to demonstrate that χ is injective when its domain is restricted to $\text{im}(\phi \circ j)$.

So take any $G \in PC_x^k(n,m)$. Using Proposition 4.1, the number of output strands in G is $y = (k-1)(n-m)+x$. Thus $n + \frac{x-1}{k-1} = m + \frac{y-1}{k-1}$, and $j(G) \in PC^k(n + \frac{x-1}{k-1})$ is obtained from G by recursively adding $\frac{x-1}{k-1}$ left-aligned coproducts to the bottom of G and $\frac{y-1}{k-1}$ right-aligned products to the top of G. The definition of our generalized left-ascending search guarantees that every $T \in \mathrm{im}(\phi \circ j)$ features $1, \ldots, \frac{x-1}{k-1}$ to begin its first row and $k(n+\frac{x-1}{k-1})+m+1, \ldots, (k+1)(n+\frac{x-1}{k-1})$ to end its third row. This implies that $\chi|_{\mathrm{im}(\phi\circ j)}$ is actually a bijection onto $\mathbb{S}(\lambda/\mu, \rho)$.

To show that ψ is bijective we provide an injective function $\psi^{-1} : \mathbb{S}(\lambda/\mu,\rho) \to PC_x^k(n,m)$. Define $\chi^{-1} : \mathbb{S}(\lambda/\mu,\rho) \to \mathbb{S}((n+\frac{x+1}{k-1})^3, \rho)$ as the map that reindexes entries of $T \in \mathbb{S}(\lambda/\mu,\rho)$ by $a \mapsto a + \frac{x-1}{k-1}$, appends $1, \ldots, \frac{x-1}{k-1}$ to the first row of the resulting tableau, and appends $k(n+\frac{x-1}{k-1})+m+1, \ldots, (k+1)(n+\frac{x-1}{k-1})$ to the third row of that same tableau. χ^{-1} is clearly bijective onto its image. As Theorem 2.1 guarantees that ϕ^{-1} is bijective, we only need to define an injective map $j^{-1} : \mathrm{im}(\phi^{-1} \circ \xi^{-1}) \to PC_x^k(n,m)$ that reverses the process of justification. Yet our definition of χ^{-1} ensures that $\mathrm{im}(\phi^{-1}\circ\chi^{-1})$ are precisely those prographs $G \in PC^k(n+\frac{x-1}{k-1})$ that feature $\frac{x-1}{k-1}$ consecutive left-aligned coproducts at their bottom and $\frac{y-1}{k-1}$ consecutive right-aligned products at their top. This means that every $G \in \mathrm{im}(\phi^{-1}\circ\chi^{-1})$ admits an "unjustification" where one simply deletes these $\frac{x-1}{k-1}$ initial products (along with their inputs) and $\frac{y-1}{k-1}$ terminal coproducts (along with their outputs). This unjustification map $j^{-1}|_{\mathrm{im}(\phi^{-1}\circ\chi^{-1})}$ is clearly a bijection. $\square$

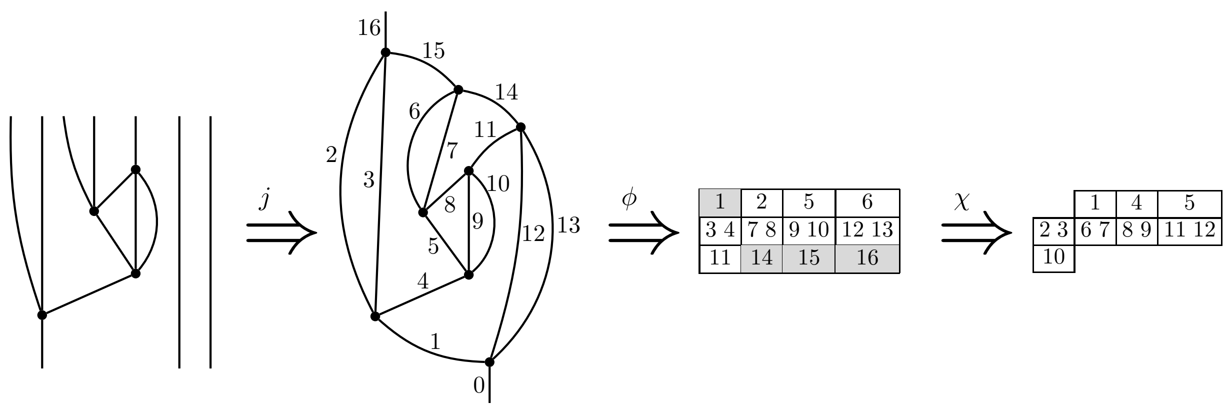

Figure 9: An example of the bijection $\psi : PC_x^k(n,m) \to \mathbb{S}(\lambda,\mu,\rho)$ for $x=3$, $n=3$, and $m=1$.

4.2 The Schütezenberger Involution

For any rectangular shape $\lambda \vdash N$, the Schützenberger involution is a map $f : S(\lambda) \to S(\lambda)$ that rotates $T \in S(\lambda)$ by 180-degrees and then renumbers entries via $a \mapsto N - a + 1$. This map may be extended to non-rectangular shapes, although in this case $f(T)$ is a standard Young tableaux of a (distinct) skew shape. As described by Drube [3], one may also extend the Schützenberger involution to standard set-valued Young tableaux. For any rectangular shape λ and row-constant density ρ, similarly define $f : \mathbb{S}(\lambda,\rho) \to \mathbb{S}(\lambda,\rho')$ via 180-degree rotation of $T \in \mathbb{S}(\lambda,\rho)$ and then reversing the order of entries in the resulting tableaux. Here $\rho' = (\rho_m, \ldots, \rho_1)$ if $\rho = (\rho_1, \ldots, \rho_m)$, meaning that only "vertically symmetric" densities are preserved by the map.

Now define an involution $r : PC^k(n) \to PC^k(n)$ on closed k-ary prographs that corresponds to 180-degree rotation. For any $k \geq 2$, this rotation operator on G proves to be compatible with the Schützenberger involution on $\phi(G)$, where ϕ is the bijection of Theorem 2.1. See Figure 11 for an example of this correspondence

Theorem 4.3. *Fix $k \geq 2$ and $n \geq 1$, and let $\rho = (1, k-1, 1)$. For $\phi : PC^k(n) \to \mathbb{S}(\lambda,\rho)$ defined as in Theorem 2.1 and $r : PC^k(n) \to PC^k(n)$, $f : \mathbb{S}(\lambda,\rho) \to \mathbb{S}(\lambda,\rho)$ defined as above, $\phi \circ r \equiv f \circ \phi$.*

Proof. Take arbitrary $G \in PC^k(n)$ and set $N = (k+1)n$, so that G contains $N+1$ total edges and the cells of $\phi(G) \in \mathbb{S}(\lambda,\rho)$ are filled with $\{1,\ldots,N\}$. We show that leftmost coproduct children in G correspond to bottom row entries in both $\phi \circ r(G)$ and $f \circ \phi(G)$, while product outputs in G correspond to top row entries in both $\phi \circ r(G)$ and $f \circ \phi(G)$. This implies that $\phi \circ r(G)$ and $f \circ \phi(G)$ feature identical sequences of integers across their top and bottom rows, and hence must be the same tableau.

So assume G has been labelled according to our left ascending search, utilizing the labels $0, \ldots, N$. Observe that, if an edge in G is labelled with the integer a, the corresponding edge in the rotated prograph $r(G)$ is labelled with $N - a$. This is because performing a left-ascending search on $r(G)$ is equivalent to performing a "right-descending search" on G, and these two labellings proceed through the edges of G in opposite orders.

First consider the case where a labels a leftmost coproduct output in G. It is necessarily the case that $a - 1$ labels the input to the same coproduct node for which this a labels the leftmost child. As demonstrated on the left side of Figure 10, this means that $N - (a - 1)$ labels a product output in the rotated prograph $r(G)$. It follows that $N - a + 1$ appears in the bottom row of $\phi \circ r(G)$. On the other hand, a being a leftmost coproduct child implies that a appears in the top row of $\phi(G)$, and hence that $N - a + 1$ appears in the bottom row of $f \circ \phi(G)$.

Similarly consider the case where a labels a product output in G. It is necessarily the case that $a - 1$ labels the rightmost input of the same product node for which a labels the output. As seen on the right side of Figure 10, this means that $N - (a - 1)$ labels a leftmost coproduct child in the rotated prograph $r(G)$ and hence that $N - a + 1$ appears in the top row of $\phi \circ r(G)$. On the other hand, a appears in the bottom row of $\phi(G)$ and consequently $N - a + 1$ appears in the top row of $f \circ \phi(G)$. ☐

Figure 10: A demonstration of how the edge labels of leftmost coproduct outputs (left) and product outputs (right) behave under 180-rotation of the underlying prograph.

The result of Theorem 4.3 may be directly extended to non-closed prographs, with rotation of prographs now corresponding to rotation and reindexing of skew, non-rectangular tableaux. This phenomenon is consistent with Theorem 4.2: 180-degree rotation of prographs transposes the number of input and output strands and hence flips the number of "missing boxes" in the first and third rows of the corresponding skew tableaux.

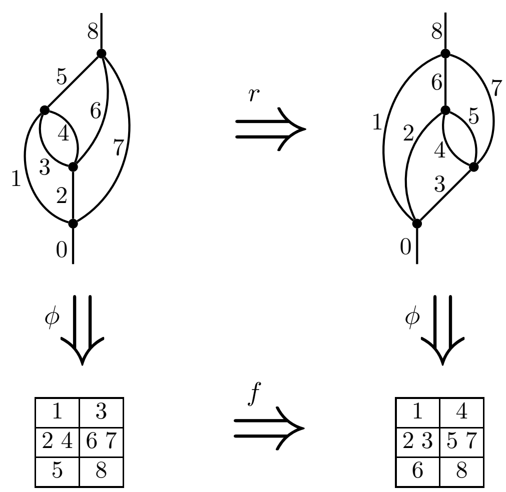

Figure 11: An example of the relationship between rotation r of k-ary product-coproduct prographs and the generalized Schützenberger involution f on standard set-valued Young tableaux.

5 Future Directions

5.1 Additional Combinatorial Interpretations for $\mathbb{S}(n^3, \rho)$ with $\rho = (1, k-1, 1)$

It is natural to suppose that all combinatorial interpretations of the three-dimensional Catalan numbers admit one-parameter generalizations that lie in bijection with $\mathbb{S}(n^3, \rho)$ for $\rho = (1, k-1, 1)$. Below we briefly conjecture as to how several more of those interpretations may be k-generalized. See sequence A005789 of OEIS [9] for a full list of candidates. The authors are especially interested in how the pattern-avoiding permutations of Lewis [8] may be generalized using standard set-valued Young tableaux.

1. The three-dimensional Catalan number $C_{3,n}$ is known to count the number of walks in the first quadrant of $\mathbb{Z}^2$ that start and end at $(0,0)$ and use $3n$ total steps from $\{(-1,0),(0,1),(1,-1)\}$. The authors conjecture that $C_{3,n}^k$ counts the number of walks in the first quadrant of $\mathbb{Z}^2$ that start and end at $(0,0)$ and use $(k+1)n$ total steps from $\{(-(k-1),0),(0,k-1),(1,-1)\}$.

2. $C_{3,n}$ is also known to count three-dimensional integer lattice paths from $(0,0,0)$ to (n,n,n) that use steps from $\{(1,0,0),(0,1,0),(0,0,1)\}$ and that satisfy $x \geq y \geq z$ at every lattice point (x,y,z) along the path. The authors conjecture that $C_{3,n}^k$ counts the number of integer lattice paths from $(0,0,0)$ to $((k-1)n, n, (k-1)n)$ that use steps from $\{(1,0,0),(0,1,0),(1,0,0)\}$ and which satisfy $(k-1)x \geq y \geq (k-1)z$ at every point (x,y,z).

For a somewhat different application of set-valued tableaux with $\rho = (1, k-1, 1)$, we refer the reader to the work of Eu [4]. Eu places all standard Young tableaux with at most three rows and any shape $\lambda \vdash N$ in bijection with Motzkin paths of length n: lattice paths from $(0,0)$ to $(n,0)$ that use steps from $\{1,1,(1,-1),(1,0)\}$ and never fall below the x-axis.

Direct computations for small n reveal that a similar result may hold for standard set-valued Young tableaux with at most three rows, precisely $n(k-1)$ entries, and densities (determined by the number of rows) of either $\rho_1 = (1)$, $\rho_2 = (1, k-1)$, or $\rho = (1, k-1, 1)$. Such tableaux appear to lie in bijection with what we refer to as k-Motzkin paths of length n: lattice paths from $(0,0)$ to $(n,0)$ that use steps from $\{(k-1,1),(1,-1),(1,0)\}$ and which never fall below the x-axis. The only caveat here is that one cannot include tableaux with "partially filled" cells: every cell must have the full complement of entries determined by ρ_i. See Figure 12 for evidence of our proposed bijection.

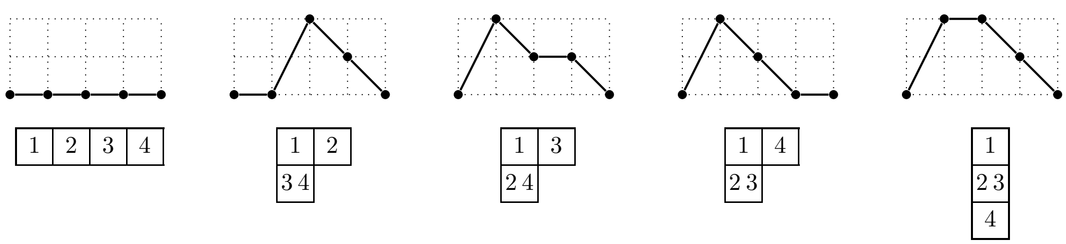

Figure 12: 3-Motzkin paths of length 4 and standard set-valued Young tableaux with 4 entries across at most three-rows and densities of either $\rho_1 = (1)$, $\rho_2 = (1,2)$, or $\rho_3 = (1,2,1)$.

5.2 $\mathbb{S}(\lambda, \rho)$ for Distinct Three- and Four-Row Densities

We close this paper by briefly exploring several additional densities for standard set-valued Young tableaux of shapes $\lambda = n^3$ and $\lambda = n^4$. The cardinalities of the resulting sets $\mathbb{S}(\lambda, \rho)$ correspond to one-parameter generalizations of the three- and four-dimensional Catalan numbers that are distinct from the three-dimensional k-Catalan numbers $C_{3,n}^k$ of previous sections. It is our hope that combinatorial interpretations as interesting as those for $C_{3,n}^k$ will eventually be found for each of these generalizations.

First consider the case of $\lambda = n^3$ and $\widetilde{\rho} = (k-1, 1, 1)$, where $k \geq 1$. We informally refer to the resulting integers $\widetilde{C}_{3,n}^k = |\mathbb{S}(n^3, \widetilde{\rho})|$ as the non-involutory three-dimensional k-Catalan numbers, since our set-valued Schützenberger involution is no longer an automorphism of $\mathbb{S}(n^3, \widetilde{\rho})$ but a bijection onto the distinct set $\mathbb{S}(n^3, \widetilde{\rho}')$ with $\widetilde{\rho}' = (1, 1, k-1)$. Notice that $\widetilde{C}_{3,n}^k \leq C_{3,n}^k$ for all choices of n, k where both values are known.

Applying the methods of Section 3 to $\mathbb{S}(\lambda, \widetilde{\rho})$ yields the enumerations below. See Table 2 of Appendix A for all experimentally-determined values of $\widetilde{C}^k_{3,n} = |\,\mathbb{S}(n^3, \widetilde{\rho})|$.

Proposition 5.1. *Let $\widetilde{\rho} = (k-1, 1, 1)$. For any $k \geq 1$,*

$$\widetilde{C}^k_{3,2} = |\,\mathbb{S}(2^3, \widetilde{\rho})| = \frac{1}{2}k^2 + \frac{3}{2}k$$

$$\widetilde{C}^k_{3,3} = |\,\mathbb{S}(3^3, \widetilde{\rho})| = \frac{2}{3}k^4 + 3k^3 + \frac{7}{3}k^2 - k$$

$$\widetilde{C}^k_{3,4} = |\,\mathbb{S}(4^3, \widetilde{\rho})| = \frac{25}{18}k^6 + \frac{61}{8}k^5 + \frac{175}{18}k^4 - \frac{35}{24}k^3 - \frac{37}{9}k^2 + \frac{5}{6}k$$

In the case of $\lambda = n^4$, we recognize the densities $\xi_i = (1, k-1, k-1, 1)$ and $\xi_2 = (k-1, 1, 1, 1)$ as prime candidates to obtain what should be referred to as the (involutory) four-dimensional k-Catalan numbers $C^k_{4,n} = |\,\mathbb{S}(4^n, \xi_1)|$ and the non-involutory four-dimensional k-Catalan numbers $\widetilde{C}^k_{4,n} = |\,\mathbb{S}(4^n, \xi_2)|$.

As the addition of a fourth row makes the techniques of Section 3 significantly harder to apply, we simply direct the reader to Tables 3 and Table 4 of Appendix A for all known values of $C^k_{4,n} = |\,\mathbb{S}(4^n, \xi_1)|$ and $\widetilde{C}^k_{4,n} = |\,\mathbb{S}(4^n, \xi_2)|$. Notice that we once again appear to have $\widetilde{C}^k_{4,n} \leq C^k_{4,n}$ for all $n \geq 1$ and $k \geq 1$.

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A Tables of Values

All enumerations below were performed in Java by Benjamin Levandowski of Valparaiso University. Java code is available upon request.

Table 1: Known values of $C^k_{3,n} = |\mathbb{S}(n^3, \rho)|$ for $\rho = (1, k-1, 1)$

$k\backslash n$	1	2	3	4	5	6
1	1	2	5	14	42	132
2	1	5	42	462	6006	87516
3	1	10	190	4295	153415	5396601
4	1	17	581	27461	1566018	100950800
5	1	26	1401	105026	9511451	
6	1	37	2890	315014	41500117	
7	1	50	5342	797917		

Table 2: Known values of $\widetilde{C}^k_{3,n} = |\mathbb{S}(n^3, \widetilde{\rho})|$ for $\widetilde{\rho} = (k-1, 1, 1)$

$k\backslash n$	1	2	3	4	5	6
1	1	2	5	14	42	132
2	1	5	42	462	6006	87516
3	1	9	153	3579	101630	3288871
4	1	14	396	15830	779063	44072801
5	1	20	845	51325	3872370	
6	1	27	1590	136234	14589623	
7	1	35	2737	314202		

Table 3: Known values of $|\mathbb{S}(n^4, \xi_1)|$ for $\xi_1 = (1, k-1, k-1, 1)$

$k\backslash n$	1	2	3	4	5	6
1	1	5	42	462	6006	87516
2	1	14	462	24024	1662804	140229804
3	1	84	24521	13074832		
4	1	460	960875	3959335892		
5	1	2380	31378194			
6	1	11814				
7	1	57288				

Table 4: Known values of $|\mathbb{S}(n^4, \xi_2)|$ for $\xi_2 = (k-1, 1, 1, 1)$

$k\backslash n$	1	2	3	4	5	6
1	1	5	42	462	6006	87516
2	1	14	462	24024	1662804	140229804
3	1	28	2158	281571	50972547	
4	1	48	6990	1798860	658138000	
5	1	75	18275	8103935		
6	1	110	41382	28950168		
7	1	154	84427			