PARTITIONS OF UNITY IN $\mathrm{SL}(2,\mathbb{Z})$, NEGATIVE CONTINUED FRACTIONS, AND DISSECTIONS OF POLYGONS

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Abstract. We characterize sequences of positive integers $(a_1, a_2, \ldots, a_n)$ for which the 2×2 matrix $\begin{pmatrix} a_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & -1 \\ 1 & 0 \end{pmatrix}$ is either the identity matrix Id, its negative $-\mathrm{Id}$, or square root of $-\mathrm{Id}$. This extends a theorem of Conway and Coxeter that classifies such solutions subject to a total positivity restriction.

1. Introduction and main results

Let $M_n(a_1, \ldots, a_n) \in \mathrm{SL}(2,\mathbb{Z})$ be the matrix defined by the product

$$M_n(a_1, \ldots, a_n) := \begin{pmatrix} a_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.1}$$

where $(a_1, a_2, \ldots, a_n)$ are positive integers. The goal of this paper is to describe all solutions of the following three equations

$$\begin{array}{rclr} M_n(a_1, \ldots, a_n) & = & \mathrm{Id}, & \text{(Problem I)} \\ M_n(a_1, \ldots, a_n) & = & -\mathrm{Id}, & \text{(Problem II)} \\ M_n(a_1, \ldots, a_n)^2 & = & -\mathrm{Id}. & \text{(Problem III)} \end{array}$$

Problem II, with a certain total positivity restriction, was studied in [7, 6] under the name of "frieze patterns". The theorem of Conway and Coxeter [6] establishes a one-to-one correspondence between totally positive solutions of Problem II and triangulations of n-gons. Note also that Coxeter implicitly formulated Problem II in full generality, when he considered frieze patterns with zero and negative entries; see [8].

The following observations are obvious.

(a) Cyclic invariance: if $(a_1, a_2 \ldots, a_n)$ is a solution of one of the above problems, then $(a_2, \ldots, a_n, a_1)$ is also a solution of the same problem. It is thus often convenient to consider n-periodic infinite sequences $(a_i)_{i \in \mathbb{Z}}$ with the cyclic order convention $a_{i+n} = a_i$.

(b) The "doubling" $(a_1, \ldots, a_n, a_1, \ldots, a_n)$ of a solution of Problem II is a solution of Problem I, and the "doubling" of a solution of Problem III is a solution of Problem II.

A particular feature of Problem III (distinguishing it from Problems I and II) is that it is equivalent to a single Diophantine equation

$$\mathrm{tr}\ M_n(a_1, \ldots, a_n) = 0.$$

This equation was considered in [5], where the totally positive solutions were classified. For more details, see Section 4.

It turns out that the solutions of Problems I-III can be described combinatorially.

Definition 1.1. *We call a 3d-dissection a partition of a convex n-gon into sub-polygons by means of pairwise non-crossing diagonals, such that the number of vertices of every sub-polygon is a multiple of 3.*

In other words, a $3d$-dissection splits an n-gon into triangles, hexagons, nonagons, dodecagons, etc. Classical triangulations are a very particular case of a $3d$-dissection.

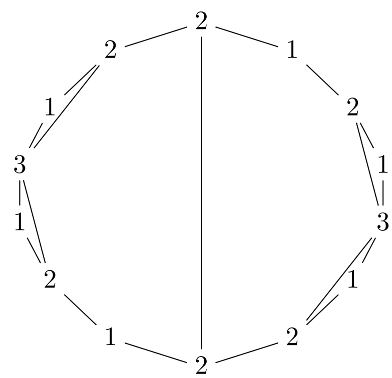
The following statement, proved in Section 3, is our main result.

Theorem 1. *(i) There is a one-to-one correspondence between the union of the set of solutions of Problems I and II and the set of $3d$-dissections of n-gons.*

(i) There is a one-to-one correspondence between the set of solutions of Problem III and the set of centrally symmetric $3d$-dissections of $2n$-gons.

In order to explain how to construct a solution of Problems I-III starting from $3d$-dissections we give here a simple example.

Example 1.2. Consider the following dissection of a tetradecagon ($n = 14$) into 4 triangles and 2 hexagons.



Label its vertices by integers. Every labeling integer counts the number of sub-polygons adjacent to the chosen vertex. Reading these numbers anti-clockwise along the border of the tetradecagon, one obtains a solution of Problem II $(a_1, \ldots, a_{14}) = (2, 2, 1, 3, 1, 2, 1, 2, 2, 1, 3, 1, 2, 1)$. Furthermore, the half-sequence $(a_1, \ldots, a_7) = (2, 2, 1, 3, 1, 2, 1)$ is a solution of Problem III, since the $3d$-dissection is centrally symmetric.

Theorem 1 reduces the problem of enumeration of solutions of Problems I-III to counting the number of $3d$-dissections of an n-gon. This is similar to the Conway and Coxeter theorem that implies that the totally positive solutions of Problem II are enumerated by triangulations of n-gons, so that the total number of solutions is given by the Catalan numbers. We refer to [12] for a general theorem on enumeration of polygon dissections.

We will also give an inductive procedure of construction of all the solutions of Problems I-III. Consider the following two families of "local surgery" operations on solutions of Problems I-III.

(a) The operations of the first type insert 1 into the sequence $(a_1, a_2, \ldots, a_n)$, increasing the two neighboring entries by 1:

$$(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) \mapsto (a_1, \ldots, a_i + 1,\ 1,\ a_{i+1} + 1, \ldots, a_n). \tag{1.2}$$

Within the cyclic ordering of a_i, the operation is defined for all $1 \leq i \leq n$. The operations (1.2) preserve the set of solutions of each of the above problems.

(b) The operations of the second type break one entry, a_i, replacing it by $a'_i, a''_i \in \mathbb{Z}_{>0}$ such that

$$a'_i + a''_i = a_i + 1,$$

and insert two consecutive 1's between them:

$$(a_1, \ldots, a_i, \ldots, a_n) \mapsto (a_1, \ldots, a'_i,\ 1,\ 1,\ a''_i, \ldots, a_n). \tag{1.3}$$

The operations (1.3) exchange solutions of Problems I and II, and preserve the set of solutions of Problem III.

The crucial difference between these two classes of operations is that every operation (1.2) increases the number of sub-polygons of a $3d$-dissection by 1, while an operation (1.3) keeps this number unchanged. Indeed, an operation (1.2) consists in a gluing an extra "exterior" triangle, while an operation (1.3) selects one sub-polygon and increases the number of its vertices by 3. For more details, see Section 3. Note also that for a given n, there are exactly n different operations of type (1.2), while the total number of different operations of type (1.3) is equal to $a_1 + \cdots + a_n$. Every operation (1.2) transforms n into $n+1$, while every operation (1.3) transforms n into $n+3$.

The following statement, proved in Section 2, is an "algorithmic version" of Theorem 1.

Theorem 2. *(i) If $n = 2$, then Problem III has exactly two solutions:*

$$(a_1, a_2) = (1, 2), \text{ or } (2, 1), \tag{1.4}$$

and every solution of Problem III can be obtained from (1.4) by a sequence of operations (1.2) and (1.3).

(ii) If $n = 3$, then Problem II has a unique solution:

$$(a_1, a_2, a_3) = (1, 1, 1), \tag{1.5}$$

and every solution of Problem I (resp. II) can be obtained from (1.5) by a sequence of operations (1.2) and (1.3), such that the total number of operations (1.3) is odd (resp. even).

Note that the operations (1.2) are very well known. They were used by Conway and Coxeter [6]; see also [10, 3] and many other sources. In particular, the totally positive solutions of Problem II are precisely the solutions obtained by a sequence of operations (1.2); see Section 5. The operations (1.3) seem to be new. They change the combinatorial nature of solutions (from triangulations to $3d$-dissections), they also have a geometric meaning in terms of the homotopy class of a curve on the projective line; see Section 6.

The following topics are related to Problems I-III, and motivated our study.

a) Combinatorics of Coxeter's frieze patterns [7, 6]. Note that classical Farey sequences can be understood as very particular cases of Coxeter friezes; see [7] (and also [20]). In particular, the index of a Farey sequence defined in [13], is a totally positive solution of Problem II. Coxeter's friezes became an active area of research; see [18] and references therein. Although this is not the main subject of the paper, we outline in Section 7 the class of Coxeter's friezes corresponding to arbitrary solutions of Problems II and III.

b) The theory of negative continued fractions

$$[a_1, a_2, \ldots, a_n] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}$$

is relevant for the subject of this paper, although in this theory one usually considers $a_i \geq 2$ and the matrix $M_n(a_1, \ldots, a_n)$ is hyperbolic. Some ideas of the theory have found application to Farey sequences; see [22, 13].

c) The matrix $M_n(a_1, \ldots, a_n)$ is the monodromy matrix of the discrete Sturm-Liouville equation

$$V_{i-1} - a_i V_i + V_{i+1} = 0. \tag{1.6}$$

The condition $M_n(a_1, \ldots, a_n) = \mathrm{Id}$ (resp. $-\mathrm{Id}$) means that every solution of (1.6) is periodic (resp. antiperiodic): $V_{i+n} = V_i$ (resp. $-V_i$) for all i. Sturm oscillation theory (see, e.g., [21]), and in particular the notion of rotation number [11], can then be applied to give a geometric invariant separating the classes of solutions of Problems I-III, counting the number of operations (1.3); see Section 6.

2. Proof of Theorem 2

In this section we prove Theorem 2 and give some of its simplest corollaries.

2.1. **Induction basis.** Let us first consider the simplest cases.

a) If $n = 2$, then the matrix $M_2(a_1, a_2)$ is as follows:

$$\begin{pmatrix} a_2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_1a_2 - 1 & -a_2 \\ a_1 & -1 \end{pmatrix},$$

with $a_1, a_2 > 0$. Since this matrix cannot be $\pm\mathrm{Id}$, Problems I and II have no solutions.

Consider Problem III. The condition $M_2(a_1, a_2)^2 = -\mathrm{Id}$ implies that the trace of this matrix vanishes, so that the coefficients a_1, a_2 must satisfy the equation $a_1a_2 = 2$. The only positive integers satisfying this equation are $(a_1, a_2) = (2, 1)$ and $(1, 2)$.

b) Consider the case $n = 3$ and assume that the sequence (a_1, a_2, a_3) contains two consecutive 1's. Set $(a_1, a_2, a_3) = (a, 1, 1)$. The matrix $M_3(a_1, a_2, a_3)$ is then given by

$$\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1-a \\ 0 & -1 \end{pmatrix}.$$

Hence Problem II has one solution $(1, 1, 1)$, corresponding to $a = 1$, while Problems I and III have no solutions.

2.2. **The surgery operations.** Let us analyze how the operations (1.2) and (1.3) act on the matrix (1.1). This is just an elementary computation.

An operation (1.2) replaces the product of two elementary matrices

$$\begin{pmatrix} a_{i+1} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_i & -1 \\ 1 & 0 \end{pmatrix}$$

in the expression for $M_n(a_1, \ldots, a_n)$ by

$$\begin{aligned}\begin{pmatrix} a_{i+1}+1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_i+1 & -1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} a_ia_{i+1} - 1 & -a_i \\ a_{i+1} & -1 \end{pmatrix} \\ &= \begin{pmatrix} a_{i+1} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_i & -1 \\ 1 & 0 \end{pmatrix}.\end{aligned}$$

Therefore, an operation (1.2) does not change the matrix:

$$M_{n+1}(a_1, \ldots, a_i + 1, 1, a_{i+1} + 1, \ldots, a_n) = M_n(a_1, \ldots, a_n).$$

It follows that the operations (1.2) preserve the sets of solutions of Problems I-III.

Consider now an operation (1.3). Since

$$\begin{pmatrix} a_i'' & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_i' & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 - a_i' - a_i'' & 1 \\ -1 & 0 \end{pmatrix} = -\begin{pmatrix} a_i & -1 \\ 1 & 0 \end{pmatrix},$$

the matrix $M_n(a_1, \ldots, a_n)$ changes its sign. Hence the operations (1.3) preserve the set of solutions of Problem III. Furthermore, if the number of the operations (1.3) is even, then the sequence of operations also preserves the set of solutions of Problems I and II.

2.3. **Induction step.** We need the following lemma, which was essentially proved in [6].

Lemma 2.1. *Given a solution $(a_1, \ldots, a_n)$ of Problem I, II, or III, there exists at least one value of $1 \leq i \leq n$, such that $a_i = 1$.*

Proof. Assume that $a_i \geq 2$ for all i, and consider the solution $(V_i)_{i\in\mathbb{Z}}$ of the equation (1.6) with initial conditions $(V_0, V_1) = (0, 1)$. Since $V_{i+1} = a_i V_i - V_{i-1}$, we see by induction that $V_{i+1} > V_i$ for all i. Therefore, the solution $(V_i)_{i\in\mathbb{Z}}$ grows and cannot be periodic.

The matrix $M_n(a_1, \ldots, a_n)$ is the monodromy matrix of (1.6). More precisely, let $(V_i)_{i\in\mathbb{Z}}$ be a solution of the equation (1.6). Then for the vector $(V_{i+1}, V_i)^t$, we have

$$\begin{bmatrix} V_{i+1} \\ V_i \end{bmatrix} = \begin{pmatrix} a_i & -1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} V_i \\ V_{i-1} \end{bmatrix}, \qquad \ldots, \qquad \begin{bmatrix} V_{i+n} \\ V_{i+n-1} \end{bmatrix} = M_n(a_1, \ldots, a_n) \begin{bmatrix} V_i \\ V_{i-1} \end{bmatrix}.$$

Suppose first that $M_n(a_1, \ldots, a_n) = \mathrm{Id}$. Then every solution of (1.6) must be periodic, which is a contradiction.

If now $M_n(a_1, \ldots, a_n) = -\mathrm{Id}$ or $M_n(a_1, \ldots, a_n)^2 = -\mathrm{Id}$, then we can use the doubling argument to conclude that every solution of (1.6) must be $2n$-periodic or $4n$-periodic, respectively. □

We are ready to prove that every solution of Problems I-III can be obtained from the elementary solutions (1.5) and (1.4) by a sequence of the operations (1.2) and (1.3). Given a solution $(a_1, \ldots, a_n)$, by Lemma 2.1 there exists at least one coefficient a_i which is equal to 1. There are then two possibilities: (a) both $a_{i-1}, a_{i+1} \geq 2$; and (b) there are two consecutive 1's, say $a_i = a_{i+1} = 1$, i.e., the chosen solution has the following "fragment": $(\ldots, a_{i-1}, 1, 1, a_{i+2}, \ldots)$.

In the case (a), consider the $(n-1)$-tuple

$$(a_1, \ldots, a_{i-2},\, a_{i-1} - 1,\, a_{i+1} - 1,\, a_{i+2}, \ldots, a_n).$$

Clearly, the solution $(a_1, \ldots, a_n)$ can be obtained from this $(n-1)$-tuple by an operation (1.2). In the case (b), take the $(n-3)$-tuple

$$(a_1, \ldots, a_{i-2},\, a_{i-1} + a_{i+2} - 1,\, a_{i+3}, \ldots, a_n).$$

The chosen solution is a then result of the operation (1.3) applied to the coefficient $a_{i-1} + a_{i+2} - 1$.

The above inverse operations (1.2) and (1.3) can always be applied, unless $n = 2$, or unless $n = 3$ and there are at least two consecutive 1's. Theorem 2 is proved. □

2.4. **Simple corollaries.** An immediate consequence of Theorem 2 is the following upper bound for the coefficients.

Corollary 2.2. *If $(a_1, a_2, \ldots, a_n)$ is a solution to one of Problems I, II, or III, then*
(i) $a_i \leq n - 5$ (Problem I);
(ii) $a_i \leq n - 2$ (Problem II);
(iii) $a_i \leq n$ (Problem III).

Proof. The operations (1.3) cannot increase the values of the coefficients a_i, while the operations (1.2) simultaneously increase n and two coefficients by 1. □

The next corollary gives expressions for the total sum of the coefficients.

Corollary 2.3. *(i) If $(a_1, a_2, \ldots, a_n)$ is a solution of one of Problems I or II obtained from the initial solution $(a_1, a_2, a_3) = (1, 1, 1)$ by applying a sequence of S operations (1.2) and R operations (1.3), then*

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &= 3S + 3R + 3 \\ &= 3n - 6R - 6. \end{aligned} \tag{2.7}$$

(ii) If $(a_1, a_2, \ldots, a_n)$ is a solution of Problem III obtained from one of the initial solutions $(a_1, a_2) = (2, 1)$ or $(1, 2)$ by applying a sequence of S operations (1.2) and R operations (1.3), then

$$\begin{array}{rcl} a_1 + a_2 + \cdots + a_n & = & 3S + 3R + 3 \\ & = & 3n - 6R - 3. \end{array} \tag{2.8}$$

Proof. Both the operations (1.3) and (1.2) add 3 to the total sum of the coefficients. Furthermore, the operations (1.2) (resp. (1.3)) increase n by 1 (resp. by 3). □

Note that the numbers S and R depend only on the solution $(a_1, a_2, \ldots, a_n)$ (and independent of the choice of the sequence of operations producing the solution). The simplest case $R = 0$ is considered in Section 5 below.

2.5. **Solutions of Problem I for small n.** Let us give several examples constructed using the inductive procedure provided by Theorem 2. We start with the list of solutions of Problem I for $n \leq 8$.

(a) Part (i) of Corollary 2.2 implies that Problem I has no solutions for $n \leq 5$.

(b) For $n = 6$, Problem I has the unique solution $(a_1, a_2, a_3, a_4, a_5, a_6) = (1, 1, 1, 1, 1, 1)$ obtained from (1.5) by one operation (1.3).

(c) For $n = 7$, one has 7 different solutions:

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (2, 1, 2, 1, 1, 1, 1) \tag{2.9}$$

and its cyclic permutations.

(d) For $n = 8$, one has 34 different solutions, namely

$$(a_1, \ldots, a_8) = (3, 1, 1, 1, 1, 2, 2, 1), \quad (3, 1, 2, 1, 1, 1, 2, 1), \quad (2, 2, 1, 2, 1, 1, 2, 1), \quad (2, 1, 2, 1, 2, 1, 2, 1), \tag{2.10}$$

and their reflections and cyclic permutations.

2.6. **Solutions of Problem II for small n.** Below is the list of solutions of Problem II for $n \leq 10$.

(a) For $n \leq 8$ all solutions of Problem II are given by Conway-Coxeter's "quiddities", and correspond to triangulations of n-gons; see Section 5 below. The number of solutions for a given n is thus equal to the Catalan number C_{n-2}, where $C_n = \frac{1}{n+1}\binom{2n}{n}$.

(b) For $n = 9$, in addition to 429 Conway-Coxeter quiddities, there is one extra solution:

$$(a_1, \ldots, a_9) = (1, 1, 1, 1, 1, 1, 1, 1, 1). \tag{2.11}$$

(c) For $n = 10$, in addition to 1430 Conway-Coxeter quiddities, there are 15 solutions:

$$(a_1, \ldots, a_{10}) = (2, 1, 2, 1, 1, 1, 1, 1, 1, 1), \quad (2, 1, 1, 1, 1, 2, 1, 1, 1, 1), \tag{2.12}$$

and their cyclic permutations.

3. The combinatorial model: $3d$-dissections

In this section we prove Theorem 1 deducing it from Theorem 2. Using the combinatorics of $3d$-dissections, we then obtain the formulas for the numbers of surgery operations for a given solution of Problems I or II. Finally, we revisit the examples from Sections 2.5 and 2.6 and give their combinatorial realizations.

3.1. **Quiddity of a dissection.** The correspondence between solutions of Problems I or II and $3d$-dissections is similar to the classical Conway and Coxeter construction [6] (see also Section 5).

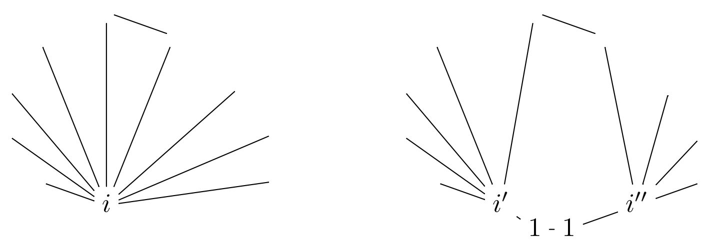
Definition 3.1. *The quiddity of a $3d$-dissection of an n-gon is the n-tuple of numbers $(a_1, \ldots, a_n)$ such that a_i is the number of sub-polygons adjacent to ith vertex of the n-gon.*

3.2. **Proof of Theorem 1.** Part (i). Consider a solution $(a_1, \ldots, a_n)$ of Problem I or II. We want to prove that there exists a $3d$-dissection of an n-gon such that its quiddity is precisely the chosen solution.

We proceed by induction in n. By Theorem 2, the chosen solution can be obtained from the initial solution (1.5) by a series of operations (1.2) and (1.3). Consider the solution (of length $n-1$ or $n-3$) obtained by the same sequence but without the last operation. By induction assumption, this solution corresponds to some $3d$-dissection, say D, (of an $(n-1)$-gon or an $(n-3)$-gon). There are then two possibilities.

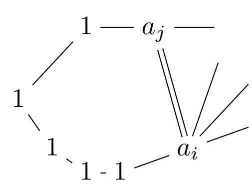
(a) If the last operation in the series is that of type (1.2), then the solution corresponds to the angulation D with extra exterior triangle glued to the edge $(i, i+1)$.

(b) Suppose that the last operation is that of type (1.3). Consider the new $3d$-dissection obtained from D by the following local surgery at vertex i, along a chosen sub-polygon:



that inserts two new vertices $1, 1$ between two copies of the vertex i. This leads to a $3d$-dissection of an $(n+3)$-gon which is exactly as in the right-hand-side of (1.3).

Conversely, given a $3d$-dissection of an n-gon, we need to show that its quiddity is a solution of Problem I or II. This follows from the obvious fact that any $3d$-dissection of an n-gon by pairwise non-crossing diagonals has an *exterior* sub-polygon. By "exterior" we mean a sub-polygon without diagonals which is glued to the rest of the $3d$-dissection along one edge



Such a $3d$-dissection can be reduced by applying the inverse of one of the operations (1.2) or (1.3). We then proceed by induction.

Part (i) of the theorem follows, the proof of Part (ii) is similar.

Theorem 1 is proved. □

Remark 3.2. Relationship between Problems I-III and combinatorics of dissections of n-gons is not very surprising. Since the work of Conway and Coxeter triangulations of various geometric objects play important role in the subject; see, e.g., [1, 2]. Higher angulations of n-gons have also been considered; see [16]. However, to the best of our knowledge, $3d$-dissections have not been considered in the literature on the subject.

3.3. **Counting the surgery operations.** Consider a solution of Problem I or II and the corresponding $3d$-dissection. Denote by N_d is the number of $3d$-gons in the $3d$-dissection.

Proposition 3.3. *Given a solution of Problem I or II constructed from (1.5) by a sequence of S operations (1.2) and R operations (1.3),* *(i) The number S counts the total number of sub-polygons except for the initial one:*

$$S = \sum_{d \leq \left[\frac{n}{3}\right]} N_d - 1. \tag{3.13}$$

(ii) The number of operations of the second type is the weighted sum

$$R = \sum_{d \leq \left[\frac{n}{3}\right]} (d-1)\, N_d. \tag{3.14}$$

In other words, to calculate R, one ignores the triangles, counts hexagons, counts nonagons 2 times, dodecagons 3 times, etc.

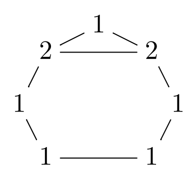
Proof. An operation (1.2) consists in a gluing a triangle. It increases the total number of sub-polygons by 1. This implies (3.13).

We have proved (see the proof of Theorem 1) that an operation (1.3) does not change the total number of sub-polygons of a $3d$-dissection, but adds 3 new vertices to one of the existing sub-polygons. Hence (3.14). □

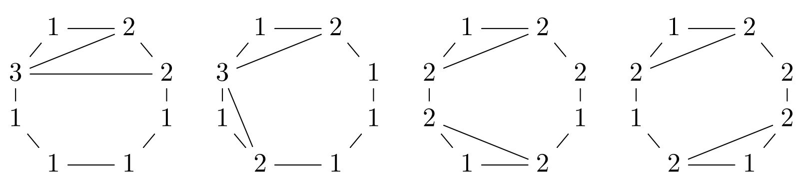
3.4. **Examples.** Let us give combinatorial entities of the examples from Section 2.

(a) Consider again the solutions of Problem I for small n; see Section 2.5. For $n = 6$ the unique solution $(a_1, \ldots, a_6) = (1, \ldots, 1)$ is given by the hexagon without interior diagonals.

For $n = 7$ the unique modulo cyclic permutations solution (2.9) corresponds to a triangle glued to an hexagon

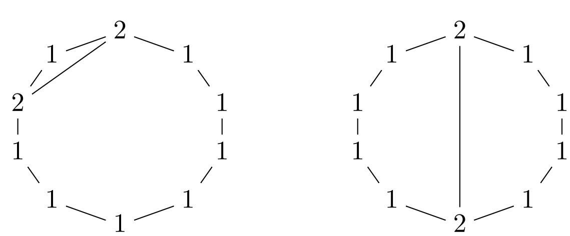


For $n = 8$ the solutions of Problem I correspond to dissections of the octagon into hexagon and two triangles. There are exactly 4 such dissections (modulo reflections and rotations):



in full accordance with (2.10).

(b) Consider now the solutions of Problem II discussed in Section 2.6. For $n = 9$ the solution (2.11) obviously corresponds to the nonagon with no dissection. For $n = 10$ there are two possibilities: a triangle glued to a nonagon and two glued hexagons



This corresponds (modulo cyclic permutations) to the solutions (2.12).

4. Problem III and the zero-trace equation

An elementary observation shows that Problem III is equivalent to a single Diophantine equation. Theorems 1 and 2 imply that the number of solutions of this equation is finite and give an algorithm to construct them all.

4.1. **The "Rotundus" polynomial.** The trace of the matrix (1.1), that we denote by $R_n(a_1, \ldots, a_n)$, is a beautiful cyclically invariant polynomial in $a_1, \ldots, a_n$. The first examples are:

$$\begin{array}{rcl} R_1(a) & = & a, \\ R_2(a_1, a_2) & = & a_1 a_2 - 2, \\ R_3(a_1, a_2, a_3) & = & a_1 a_2 a_3 - a_1 - a_2 - a_3, \\ R_4(a_1, a_2, a_3, a_4) & = & a_1 a_2 a_3 a_4 - a_1 a_2 - a_2 a_3 - a_3 a_4 - a_1 a_4 + 2, \\ R_5(a_1, a_2, a_3, a_4, a_5) & = & a_1 a_2 a_3 a_4 a_5 \\ & & -a_1 a_2 a_3 - a_2 a_3 a_4 - a_3 a_4 a_5 - a_1 a_4 a_5 - a_1 a_2 a_5 \\ & & +a_1 + a_2 + a_3 + a_4 + a_5. \end{array}$$

The polynomial $R_n(a_1, \ldots, a_n)$ was called the "Rotundus" in [5], where it is proved that $R_n(a_1, \ldots, a_n)$ can also be calculated as the Pfaffian of a certain skew-symmetric matrix. Note that $R_n(a_1, \ldots, a_n)$ is the polynomial part of the rational function

$$a_1 a_2 \cdots a_n \left(1 - \frac{1}{a_1 a_2}\right) \left(1 - \frac{1}{a_2 a_3}\right) \cdots \left(1 - \frac{1}{a_n a_1}\right).$$

4.2. **The "Rotundus equation".**

Proposition 4.1. *Every solution of Problem III is a solution of the equation*

$$R_n(a_1, \ldots, a_n) = 0, \tag{4.15}$$

and vice-versa.

Proof. A trace zero element of SL(2, $\mathbb{Z}$) has eigenvalues i and $-i$. This is equivalent to the fact that it squares to $-\mathrm{Id}$. □

Remark 4.2. Note also that every solution of Problem I or II satisfies the equation $R_n(a_1, \ldots, a_n) = 2$ or -2, respectively. However, the converse is false: a solution of one of these equations is not necessarily a solution of Problem I or II. It is also easy to see that the equation $R_n(a_1, \ldots, a_n) = \pm 2$ has infinitely many positive integer solutions for sufficiently large n. For instance, one has $R_n(a, 1, 1) = -2$ for any a.

4.3. **The list of solutions of Problem III for small** n**.** Let us give a complete list of solutions of Problem III for $n \leq 6$.

(a) For $n = 2, 3$, and 4, all the solutions are given by centrally symmetric triangulations of a quadrilateral (2), hexagon (6), and octagon (20), respectively. Note also that the cases $n = 2$ and 3 are well-known.

(b) For $n = 5$, besides 70 solutions corresponding to centrally symmetric triangulations of the decagon (see Example 5.9), one obtains 5 additional solutions: $(a_1, a_2, a_3, a_4, a_5) = (2, 1, 1, 1, 1)$ and its cyclic permutations. These 5 new solutions are obtained from (1.4) by applying one operation (1.3).

(c) For $n = 6$, besides 252 solutions corresponding to centrally symmetric triangulations of the dodecagon, one gets 26 additional solutions:

$$(a_1, a_2, a_3, a_4, a_5, a_6) = (3, 1, 2, 1, 1, 1), \quad (2, 2, 1, 2, 1, 1), \quad (2, 1, 2, 1, 2, 1),$$

their cyclic permutations and reflections.

We mention that the sequence $2, 6, 20, 75, 278, \ldots$ corresponding to the total number of solutions of Problem III is not in the OEIS.

5. Conway-Coxeter quiddities and Farey sequences

In the seminal paper [6], Conway and Coxeter classified solutions of Problem II[1] that satisfy a certain condition of total positivity. These are precisely the solutions obtained from the initial solution $(a_1, a_2, a_3) = (1, 1, 1)$ by a sequence of operations (1.2). Their classification of totally positive solutions beautifully relates Problem II to such classical subjects as triangulations of n-gons. Furthermore, the close relation of the topic to Farey sequences was already mentioned in [7]. It turns out that the Conway-Coxeter theorem implies some results of [13] about the index of a Farey sequence.

This section is an overview and does not contain new results. We describe the Conway-Coxeter theorem, formulated in terms of matrices $M_n(a_1, \ldots, a_n)$, and a similar result in the case of Problem III, obtained in [5]. We also briefly discuss the relation to Farey sequences.

5.1. **Total positivity.** The class of totally positive solutions of Problem II can be defined in several equivalent ways. The simplest definition is based on the properties of solutions of the Sturm-Liouville equation.

Definition 5.1. *A solution of Problem II is called totally positive if there exists a solution $(V_i)_{i\in\mathbb{Z}}$ of the equation (1.6) that does not change its sign on the interval $[1, \ldots, n]$, i.e., the sequence $(V_1, V_2, \ldots, V_n)$ is positive or negative.*

In the context of Sturm oscillation theory, this case should be called "non-oscillant", or "non-osculating".

Remark 5.2. Note that since $M(a_1, \ldots, a_n) = -\mathrm{Id}$, every solution is n-anti-periodic, so that it must change sign on the interval $[1, n + 1]$.

Let us give an equivalent combinatorial definition. Consider the following tridiagonal $i\times i$-determinant

$$K(a_1, \ldots, a_i) = \begin{vmatrix} a_1 & 1 & & & \\ 1 & a_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & a_{i-1} & 1 \\ & & & 1 & a_i \end{vmatrix}.$$

It is well-known (and can be easily checked directly) that the matrix (1.1) can be written in terms of these determinants as follows:

$$M(a_1, \ldots, a_n) = \begin{pmatrix} K(a_1, \ldots, a_n) & -K(a_2, \ldots, a_n) \\ K(a_1, \ldots, a_{n-1}) & -K(a_2, \ldots, a_{n-1}) \end{pmatrix}.$$

Definition 5.3. *A solution $(a_1, \ldots, a_n)$ of Problem II is totally positive if*

$$K(a_i, \ldots, a_{i+j}) > 0$$

for all $j \leq n - 2$ and all i. Note that we use the cyclic ordering of the a_i.

Remark 5.4. (a) The condition $M(a_1, \ldots, a_n) = -\mathrm{Id}$ implies that $K(a_i, \ldots, a_{i+n-1}) = 0$, for all i.

[1] Conway and Coxeter worked with so-called frieze patterns (see Section 7 below), but the equivalence of their result to the classification of solutions of Problem II is a simple observation; see [3, 19].

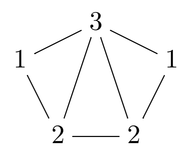
5.2. **Triangulated n-gons: "quiddities".** The Conway-Coxeter result states that totally positive solutions of Problem II are in one-to-one correspondence with triangulations of n-gons.

Given a triangulation of an n-gon, let a_i be the number of triangles adjacent to the i^{th} vertex. This yields an n-tuple of positive integers, $(a_1, \ldots, a_n)$. Conway and Coxeter call an n-tuple obtained from such a triangulation a *quiddity*.

Theorem. [6] *Any quiddity is a totally positive solution of Problem II, and every totally positive solution arises in this way.*

A direct proof of the Conway-Coxeter theorem in terms of 2×2-matrices is given in [10, 3]. For a simple proof, see also [15].

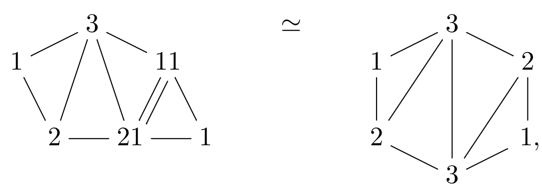
Example 5.5. For $n = 5$, the triangulation of the pentagon



generates a solution $(a_1, a_2, a_3, a_4, a_5) = (1, 3, 1, 2, 2)$ of Problem II. All other solutions for $n = 5$ are obtained by cyclic permutations of this one.

5.3. **Gluing triangles.** Obviously, every triangulation of an n-gon can be obtained from a triangle by adding new exterior triangles.

Example 5.6. Gluing a triangle to the above triangulated pentagon



one obtains the solution $(1, 3, 2, 1, 3, 2) = (1, 3, 1 + 1,\ 1,\ 2 + 1, 2)$ of Problem II, for $n = 6$.

An operation (1.2) applied to a quiddity consists in gluing a triangle to a triangulated n-gon, so that the Conway-Coxeter theorem implies the following statement (see also [10], Theorem 5.5).

Corollary 5.7. *Every totally positive solution of Problem II can be obtained from the initial solution* $(a_1, a_2, a_3) = (1, 1, 1)$ *by a sequence of operations (1.2).*

For a clear and detailed discussion; see [3].

5.4. **Indices of Farey sequences as Conway-Coxeters quiddities.** Relation to Farey sequences and negative continued fractions was already mentioned by Coxeter [7] (see also [20]).

Rational numbers in $[0, 1]$ whose denominator does not exceed N written in a form of irreducible fractions, whose denominators do not exceed N form the *Farey sequence* of order N. Elements of the Farey sequence form the *Farey graph*: two rationals, $v_1 = \frac{a_1}{b_1}$ and $v_2 = \frac{a_2}{b_2}$, are joined by an edge if and only if

$$|a_1 b_2 - a_2 b_1| = 1.$$

The Farey graph is often embedded into the hyperbolic plane, the edges being realized as geodesics joining rational points on the ideal boundary.

The main properties of Farey sequences can be found in [14]. A simple but important property is that every Farey sequence forms a triangulated polygon in the Farey graph. A Conway-Coxeters quiddity is then precisely the index of a Farey sequence, defined in [13].

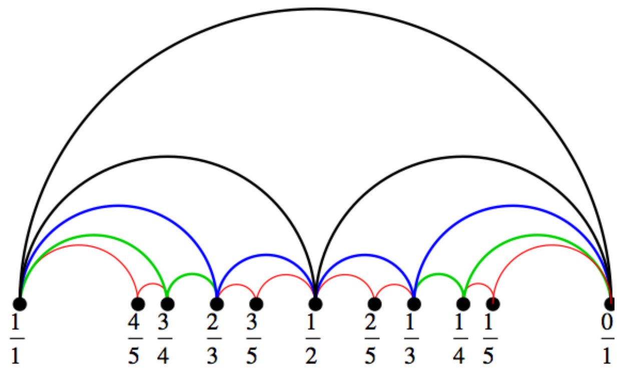


FIGURE 1. The Farey sequence of order 5 and the triangulated hendecagon.

Remark 5.8. The Conway-Coxeter theorem implies that

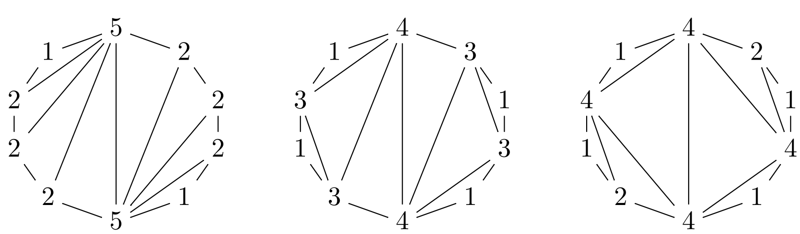
$$a_1 + a_2 + \cdots + a_n = 3n - 6.$$

Indeed, the total number of triangles in a triangulation is $n-2$, and each triangle has three angles that contribute to a quiddity. The above formula is equivalent to Theorem 1 of [13]. Moreover, it holds not only for the complete Farey sequence, but also for an arbitrary *path in the Farey graph.* Consider the Farey sequence of order 5 presented in Figure 1. It has many different shorter paths, for instance, $\left\{\frac{1}{1}, \frac{2}{3}, \frac{3}{5}, \frac{1}{2}, \frac{1}{3}, \frac{0}{1}\right\}$.

5.5. **Totally positive solutions of Problem III.** A solution $(a_1, \ldots, a_n)$ of Problem III is totally positive if its double $(a_1, \ldots, a_n, a_1, \ldots, a_n)$ is a totally positive solution of Problem II. Every totally positive solution can be obtained from one of the solutions $(a_1, a_2) = (1, 2)$, or $(2, 1)$ by a sequence of operations (1.2).

The Conway-Coxeter theorem implies that there is a one-to-one correspondence between totally positive solutions of Problem III and *centrally symmetric* triangulations of $2n$-gons; see [5].

Example 5.9. There exist 70 different centrally symmetric triangulations of the decagon, for instance



The corresponding sequences $(a_1, a_2, a_3, a_4, a_5) = (5, 2, 2, 2, 1), (4, 3, 1, 3, 1), (4, 2, 1, 4, 1), \ldots$ are totally positive solutions of Problem III.

The total number of totally positive solutions of Problem III is given by the central binomial coefficient $\binom{2n}{n} = 1, 2, 6, 20, 70, 252, 924, \ldots$

6. THE ROTATION INDEX

In this section we define a geometric invariant of solutions of Problem I, II, and III. It is given by the index of a broken line in $\mathbb{R}^2$ (i.e., the homotopy class of an n-gon in the projective line), or equivalently as the rotation number of the equation (1.6). This index actually counts the number of operations (1.3).

6.1. **The star-shaped broken line.** Recall that the index of a smooth closed plane curve is the number of rotations of its tangent vector. For a *star-shaped* curve, the index can be equivalently calculated as the homotopy class of the projection of the curve to $\mathbb{RP}^1$.

Given a solution $(a_1, \ldots, a_n)$ of Problem I, II, or III, let us construct a star-shaped (broken) line in $\mathbb{R}^2$. Consider the corresponding discrete Sturm-Liouville equation

$$V_{i+1} = a_i V_i - V_{i-1},$$

where the set of coefficients a_i is understood as an infinite n-periodic sequence $(a_i)_{i\in\mathbb{Z}}$. Chose two linearly independent solutions, $V^{(1)} = (V_i^{(1)})_{i\in\mathbb{Z}}$ and $V^{(2)} = (V_i^{(2)})_{i\in\mathbb{Z}}$. One then has a sequence of points in $\mathbb{R}^2$:

$$V_i = \left(V_i^{(1)}, V_i^{(2)}\right),$$

These points form a broken star-shaped line. Indeed, the determinant

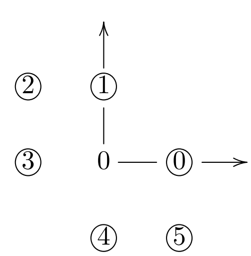
$$W(V^{(1)}, V^{(2)}) := \begin{vmatrix} V_{i+1}^{(1)} & V_i^{(1)} \\ V_{i+1}^{(2)} & V_i^{(2)} \end{vmatrix},$$

usually called the *Wronski determinant*, is constant, i.e., does not depend on i. Therefore, the sequence of points $(V_i)_{i\in\mathbb{Z}}$ in $\mathbb{R}^2$ always rotates around the origin in the same (positive or negative, depending on the choice of the two solutions) direction. Note that a different choice of the solutions $V^{(1)}$ and $V^{(2)}$ gives the same broken line, modulo a linear coordinate transformation in $\mathbb{R}^2$.

If $M_n(a_1, \ldots, a_n) = \mathrm{Id}$ (resp. $-\mathrm{Id}$), then the broken line thus constructed is periodic, i.e., $V_{i+n} = V_i$ (resp. anti-periodic, $V_{i+n} = -V_i$). We will be interested in the *index* of this broken line.

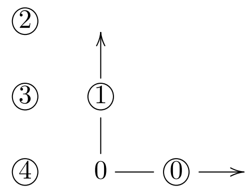
Remark 6.1. Note that the index of an antiperiodic star-shaped broken line is a well-defined half-integer. If $M_n(a_1, \ldots, a_n)^2 = -\mathrm{Id}$, then, using the doubling procedure, we can still define the index of the corresponding star-shaped broken line as a multiple of $\frac{1}{2}$.

Example 6.2. (a) Consider the sequence $(a_1, \ldots, a_6) = (1, 1, 1, 1, 1, 1)$, which is the solution of Problem I obtained from $(1, 1, 1)$ by applying one operation (1.3). Choosing the solutions with the initial conditions $(V_0^{(1)}, V_1^{(1)}) = (1, 0)$ and $(V_0^{(2)}, V_1^{(2)}) = (0, 1)$, one obtains the following hexagon in $\mathbb{R}^2$: $\{(1, 0), (0, 1), (-1, 1), (-1, 0), (0, -1), (1, -1)\}$.



The index of this hexagon is 1.

(b) Consider the solution of Problem II $(a_1, a_2, a_3, a_4) = (2, 1, 2, 1)$ obtained from $(1, 1, 1)$ by applying one operation (1.2). Choosing the solutions with the same initial conditions as above, one obtains the following antiperiodic quadrelateral in $\mathbb{R}^2$: $\{(1, 0), (0, 1), (-1, 2), (-1, 1)\}$.



whose index is $\frac{1}{2}$.

6.2. **The index.**

Proposition 6.3. *For a solution of Problem I or II obtained from (1.5) by a sequence of S operations (1.2) and R operations (1.3), the index of the corresponding broken line is equal to $\frac{1}{2}(R+1)$.*

Proof. We need to show that the operations of the first type applied to solution of Problems I and II do not change the index of the corresponding broken line, while the operations of the second type increase this index by $\frac{1}{2}$.

An operation (1.2) adds one additional point, $V_i + V_{i+1}$, between the points V_i and V_{i+1} in the sequence of points $(V_i)_{i\in\mathbb{Z}}$. The resulting sequence is $(\ldots, V_i, V_i + V_{i+1}, V_{i+1}, \ldots)$, which has the same index as the initial one.

An easy computation shows that the operation (1.3) transforms the sequence of points $(V_i)_{i\in\mathbb{Z}}$ as follows:

$$(\ldots, V_{i-1}, V_i, V_{i+1}, \ldots) \mapsto (\ldots, V_{i-1}, V_i,\ a'_i V_i - V_{i-1},\ (a'_i - 1)V_i - V_{i-1},\ -V_i, -V_{i+1}, \ldots).$$

Indeed, the sequence on the right-hand-side is a solution of the equation (1.6) with coefficients

$$(a_1, \ldots, a'_i,\ 1,\ 1,\ a''_i, \ldots, a_n).$$

Therefore, the operation (1.3) rotates the picture by 180° and increases the index by $\frac{1}{2}$. □

If, for a solution of Problem II, the index is equal to $\frac{1}{2}$, i.e., the number R of operations (1.3) equals zero, then the solution is called non-osculating. The class of non-osculating solutions of Problem II is precisely that of Conway and Coxeter, considered in Section 5. Similarly, the class of non-osculating solutions of Problem III is that corresponding to symmetric triangulations of a $2n$-gon. Solutions of Problem I cannot be non-osculating because R is odd in this case.

7. Oscillating tame friezes

A frieze pattern can be viewed as a "matrix" of the Sturm-Liouville operator (1.6) acting on the infinite-dimensional space of sequences of numbers. This point of view allows one to apply the tools of linear algebra; see [19], and is useful for the spectral theory of linear difference operators; see [17].

We briefly introduce the notion of tame oscillating Coxeter friezes. We show that this notion is equivalent to solutions of Problem II. Theorems 2 and 1 then provide a classification of tame oscillating friezes. It is easy to see that oscillating Coxeter friezes satisfy the main properties of the classical friezes, such as Coxeter's glide symmetry.

7.1. **Classical Coxeter friezes.**

Coxeter's frieze [7] is an array of $(n+1)$ infinite rows of integers, with the first and the last rows consisting of 0's, the second and nth rows of 1's, and all the other entries are *positive integers*. Consecutive rows are shifted, and the so-called Coxeter *unimodular rule*:

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}, \qquad\qquad ad - bc = 1,$$

is satisfied for every elementary "diamond".

The Conway-Coxeter theorem [6] provides a classification of the Coxeter friezes. Every frieze corresponds to a triangulated n-gon, the rows 3 and $n-1$ being the Conway-Coxeter quiddity.

Example 7.1. For example, the frieze (here and below we omit the first and the last rows of 0's):

$$
\begin{array}{cccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 1 & & 3 & & 1 & & 2 & & 2 & & \cdots \\
\cdots & & 2 & & 2 & & 1 & & 3 & & 1 & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots
\end{array}
$$

is the unique (up to a cyclic permutation) Coxeter frieze for $n=5$. It corresponds to the quiddity $(a_1,a_2,a_3,a_4,a_5)=(1,3,1,2,2)$.

Therefore, there is one-to-one correspondence between Coxeter friezes and totally positive solutions of Problem II; see Section 5.

We refer to [18] for a survey on friezes and their connection to various topics.

7.2. **Tameness.** Let us relax the positivity assumption. Then frieze patterns may become very "wild", and the classification of such friezes is out of reach; cf. [9]. An important property that we keep is that of tameness, first introduced in [3].

Definition 7.2. *A frieze is tame if every* 3×3*-diamond vanishes.*

Remark 7.3. Note that every classical Coxeter frieze is tame. This follows easily from the positivity assumption.

Solutions of Problems II and III correspond to tame friezes with $(a_i)_{i\in\mathbb{Z}}$ in the 3rd row. More precisely, we have the following

Proposition 7.4. *There is a one-to-one correspondence between*

(i) Solutions of Problem II and tame friezes with the 3*rd row all positive integers;*

(ii) Solutions of Problem III and tame friezes with even n *and the* 3*rd row of positive integers which are invariant under reflection in the middle row.*

Proof. The following fact was noticed in [6] for classical Coxeter friezes, and proved in [19] for tame friezes.

Lemma 7.5. *Every diagonal of a tame frieze is a solution of the equation (1.6) with coefficients* $(a_i)_{i\in\mathbb{Z}}$ *in the* 3*rd row of the frieze.*

Part (i) readily follows from the lemma, while Part (ii) is then a consequence of Coxeter's glide symmetry. $\square$

Example 7.6. The solution of Problem III with $(a_1,a_2,a_3,a_4,a_5)=(2,1,1,1,1)$ generates the following tame frieze with $n=10$:

$$
\begin{array}{cccccccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 2 & & 1 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\cdots & & 1 & & 0 & & 0 & & 0 & & 1 & & 1 & & 0 & & 0 & & 0 & & 1 & \\
 & 0 & & -1 & & -1 & & -1 & & -1 & & 0 & & -1 & & -1 & & -1 & & -1 & & \cdots \\
\cdots & & -1 & & -2 & & -1 & & -2 & & -1 & & -1 & & -2 & & -1 & & -2 & & -1 & \\
 & 0 & & -1 & & -1 & & -1 & & -1 & & 0 & & -1 & & -1 & & -1 & & -1 & & \cdots \\
\cdots & & 1 & & 0 & & 0 & & 0 & & 1 & & 1 & & 0 & & 0 & & 0 & & 1 & \\
 & 2 & & 1 & & 1 & & 1 & & 1 & & 2 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$

Every row is 5-periodic, and the frieze is symmetric under the reflection.

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