Probabilistic cellular automata with memory two: invariant laws and multidirectional reversibility

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Abstract

We focus on a family of one-dimensional probabilistic cellular automata with memory two: the dynamics is such that the value of a given cell at time $t+1$ is drawn according to a distribution which is a function of the states of its two nearest neighbours at time t, and of its own state at time $t-1$. Such PCA naturally arise in the study of some models coming from statistical physics (8-vertex model, directed animals and gaz models, TASEP, etc.). We give conditions for which the invariant measure has a product form or a Markovian form, and we prove an ergodicity result holding in that context. The stationary space-time diagrams of these PCA present different forms of reversibility. We describe and study extensively this phenomenon, which provides families of Gibbs random fields on the square lattice having nice geometric and combinatorial properties.

Contents

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Probabilistic cellular automata (PCA) are a class of random discrete dynamical sytems. They can be seen both as the synchronous counterparts of finite-range interacting particle systems, and as a generalization of deterministic cellular automata: time is discrete and at each time step, all the cells are updated independently in a random fashion, according to a distribution depending only on the states of a finite number of their neighbours.

In this article, we focus on a family of one-dimensional probabilistic cellular automata with *memory two* (or *order two*): the value of a given cell at time $t+1$ is drawn according to a distribution which is a function of the states of its two nearest neighbours at time t, and of its own state at time $t-1$. The space-time diagrams describing the evolution of the states can thus be represented on a two-dimensional grid.

We study the invariant measures of these PCA with memory two. In particular, we give necessary and sufficient conditions for which the invariant measure has a product form or a Markovian form, and we prove an ergodicity result holding in that context. We also show that when the parameters of the PCA satisfy some conditions, the stationary space-time diagram presents some multidirectional (quasi)-reversibility property: the random field has the same distribution as if we had iterated a PCA with memory two in another direction (the same PCA in the reversible case, or another PCA in the quasi-reversible case). This can be seen has a probabilistic extension of the notion of *expansivity* for deterministic CA. For expansive CA, one can indeed reconstruct the whole space-time diagram from the knowledge of only one column. In the context of PCA with memory two, the criteria of quasi-reversibility that we obtain are reminiscent of the notion of *permutivity* for determistic CA. Stationary space-time diagrams of PCA are known to be Gibbs random fields [18, 24]. The family of PCA that we will describe thus provide examples of Gibbs fields with i.i.d. lines on many directions and nice combinatorial and geometric properties.

The first theoretical results on PCA and their invariant measures go back to the seventies [5, 22, 31], and were then gathered in a survey which is still today a reference book [12]. In particular, it contains a detailed study of binary PCA with memory one with only two neighbours, including a presentation of the necessary and sufficient conditions that the four parameters defining the PCA must satisfy for having an invariant measure with a product form or a Markovian form. Some extensions and alternative proofs were proposed by Mairesse and Marcovici in a later article [26], together with a study of some properties of the random fields given by stationary space-time diagrams of PCA having a product form invariant measure (see also the survey on PCA of the same authors [25]). The novelty was to highlight that these space-time diagrams are i.i.d. along many directions, and present a directional reversibility: they can also be seen as being obtained by iterating some PCA in another direction. Soon after, Casse and Marckert have proposed an indepth study of the Markovian case [10, 9]. Motivated by the study of the 8-vertex model, Casse was then led to introduce a class of one-dimensional PCA with memory two, called *triangular PCA* [8].

In the present article, we propose a comprehensive study of PCA with memory two having and invariant measure with a product form, and we show that their stationary space-time diagrams share some specificities. We first extend the notion of reversibility and quasi-reversibility to take into account other symmetries than the time reversal and, in a second time, we characterize PCA with an invariant product measure that are reversible or quasi-reversible. Even if most one-dimensional positive-rates PCA are usually expected to be ergodic, the ergodicity of PCA is known to be a difficult problem, algorithmically undecidable [12, 7]. In Section 2, after characterizing positive-rates PCA having a product invariant measure, we prove that these PCA are ergodic (Theorem 8). A novelty of our work is also to display some PCA for which the invariant measure has neither a product form nor a Markovian one, but for which the finite-dimensional marginals can be exactly computed (Theorems 19 and 36). In Section 4, we study PCA having Markov invariant measures. Section 5 is then devoted to the presentation of some applications of our models and results to statistical physics (8-vertex model, directed animals and gaz models, TASEP, etc.). In particular, we introduce an extension of the TASEP model, in which the probability for a particle to move depends on the distance of the previous particle and of its speed. It can also be seen as a traffic flow model, more realistic than the classical TASEP model. Finally, we give on one side a more explicit description of (quasi-)reversible binary PCA (Section 6), and on the other side, we provide some extensions to general sets of symbols (Section 7).

When describing the family of PCA presenting some given directional reversibility or quasi-reversibility property, for each family of PCA involved, we give the conditions that the parameters of the PCA must satisfy in order to present that behaviour, and we provide the dimension of the corresponding submanifold of the parameter space, see Table 1. Our purpose is to show that despite their specificity, these PCA build up rich classes, and we set out the detail of the computations in the last section.

1 Definitions and presentation of the results

1.1 Introductory example

In this paragraph, we give a first introduction to PCA with memory two, using an example motivated by the study of the 8-vertex model [8]. We present some properties of the stationary space-time diagram of this PCA: although it is a non-trivial random field, it is made of lines of i.i.d. random variables, and it is reversible. In the rest of the article, we will study exhaustively the families of PCA having an analogous behaviour.

Let us set $\mathbb{Z}^2_{\mathrm{e}} = \{(i,t) \in \mathbb{Z}^2 : i+t \equiv 0 \mod 2\}$, and introduce the notations: $\mathbb{Z}_t = 2\mathbb{Z}$ if $t \in 2\mathbb{Z}$, and $\mathbb{Z}_t = 2\mathbb{Z}+1$ if $t \in 2\mathbb{Z}+1$, so that the grid $\mathbb{Z}^2_{\mathrm{e}}$ can be seen as the union on $t \in \mathbb{Z}$ of the points $\{(i,t) : i \in \mathbb{Z}_t\}$, that will contain the information on the state of the system at time t. Note that one can scroll the positions corresponding to two consecutive steps of time along an horizontal zigzag line: $\dots (i,t),(i+1,t+1),(i+2,t),(i+3,t+1)\dots$ This will explain the terminology introduced later.

We now define a PCA dynamics on the *alphabet* $S=\{0,1\}$, which, through a recoding, can be shown to be closely related to the 8-vertex model (see Section 5 for details). The configuration η_t at a given time $t \in \mathbb{Z}$ is an element of $S^{\mathbb{Z}_t}$, and the evolution is as follows. Let us denote by $\mathcal{B}(q)$ the Bernoulli measure $q\delta_1 + (1-q)\delta_0$. Given the configurations η_t and η_{t-1} at times t and $t-1$, the configuration η_{t+1} at time $t+1$ is obtained by updating each site $i \in \mathbb{Z}_{t+1}$ simultaneously and independently, according to the distribution $T(\eta_t(i-1), \eta_{t-1}(i), \eta_t(i+1); \cdot)$, where

$$\begin{aligned}
T(0,0,1;\cdot) = T(1,0,0;\cdot) &= \mathcal{B}(q),\\
T(0,1,1;\cdot) = T(1,1,0;\cdot) &= \mathcal{B}(1-q)\\
T(0,1,0;\cdot) = T(1,1,1;\cdot) &= \mathcal{B}(r),\\
T(1,0,1;\cdot) = T(0,0,0;\cdot) &= \mathcal{B}(1-r).
\end{aligned}$$

As a special case, for $q=r$, we have: $T(a,b,c;\cdot) = q\,\delta_{a+b+c \mod 2} + (1-q)\,\delta_{a+b+c+1 \mod 2}$, so that the new state is equal to $a+b+c \mod 2$ with probability q, and to $a+b+c+1 \mod 2$ with probability $1-q$. Fig. 1 shows how η_{t+1} is computed from η_t and η_{t-1}, illustrating the progress of the Markov chain.

Let us assume that initially, (η_0, η_1) is distributed according to the uniform product measure $\lambda = \mathcal{B}(1/2)^{\otimes \mathbb{Z}_0} \otimes \mathcal{B}(1/2)^{\otimes \mathbb{Z}_1}$. Then, we can show that for any $t \in \mathbb{N}$, (η_t, η_{t+1}) is also distributed according to λ. We will say that the PCA has an invariant *Horizontal Zigzag Product Measure.* By stationarity, we can then extend the space-time diagram to a random field with values in $S^{\mathbb{Z}^2_{\mathrm{e}}}$. The study of the space-time diagram shows that it has some peculiar properties, which we will precise in the next sections. In particular, it is quasi-reversible: if we reverse the direction of time, the random field corresponds to the stationary space-time diagram of another PCA.

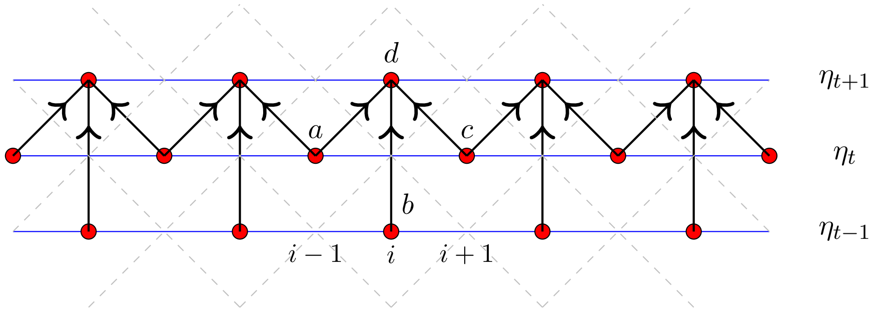
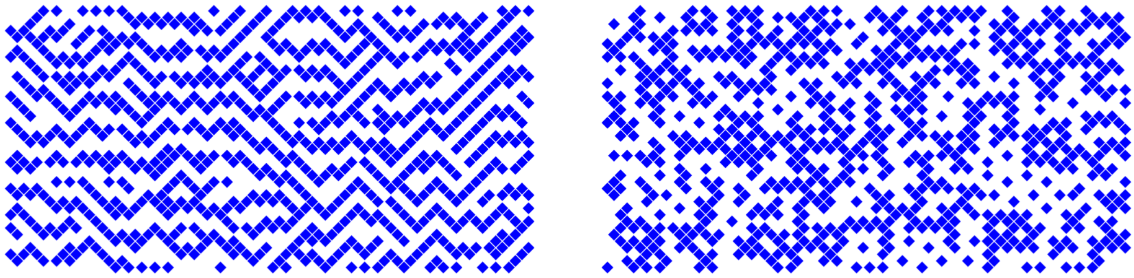


Figure 1: Illustration of the way η_{t+1} is obtained from η_t and η_{t-1}, using the transition kernel T. The value $\eta_{t+1}(i)$ is equal to d with probability $T(a,b,c;d)$, and conditionnally to η_t and η_{t-1}, the values $(\eta_{t+1}(i))_{i\in\mathbb{Z}_{t+1}}$ are independent.



$q = 0.9$ and $r = 0.2$ $\qquad$ $q = r = 0.2$

Figure 2: Examples of portions of stationary space-time diagrams of the 8-vertex PCA, for different values of the parameters. Cells in state 1 are represented in blue, and cells in state 0 are white.

Furthermore, the PCA is ergodic: whatever the distribution of (η_0, η_1), the distribution of (η_t, η_{t+1}) converges weakly to λ (meaning that for any $n \in \mathbb{N}$, the restriction of (η_t, η_{t+1}) to the cells of abscissa ranging between $-n$ and n converges to a uniform product measure). For $q = r$, the stationary space-time diagram presents even more symmetries and directional reversibilities: it has the same distribution as if we had iterated the PCA in any other of the four cardinal directions. In addition, any straight line drawn along the space-time diagram is made of i.i.d. random variables, see Fig. 2 for an illustration.

In the following of the article, we will show that this PCA belongs to a more general class of PCA that are all ergodic and for which the stationary space-time diagram share specific properties (independence, directional reversibility).

1.2 PCA with memory two and their invariant measures

In this article, we will only consider PCA with memory two for which the value of a given cell at time $t+1$ is drawn according to a distribution which is a function of the states of its two nearest neighbours at time t, and of its own state at time $t-1$. We thus introduce the following definition of transition kernel and of PCA with memory two.

Definition 1. Let S be a finite set, called the *alphabet*. A *transition kernel* is a function T that maps any $(a,b,c) \in S^3$ to a probability distribution on S. We denote by $T(a,b,c;\cdot)$ the distribution on S which is the image of the triplet $(a,b,c) \in S^3$, so that: $\forall d \in S$, $T(a,b,c;d) \in [0,1]$ and $\sum_{s\in S} T(a,b,c;d)$=1.

A *probabilistic cellular automaton (PCA) with memory two* of transition kernel T is a Markov chain of order two $(\eta_t)_{t\geq 0}$ such that η_t has values in $S^{\mathbb{Z}_t}$, and conditionnally to η_t and η_{t-1}, for any $i \in \mathbb{Z}_{t+1}$, $\eta_{t+1}(i)$ is distributed according to $T(\eta_t(i-1), \eta_{t-1}(i), \eta_t(i+1); \cdot)$, independently for different $i \in \mathbb{Z}_{t+1}$.

We say that a PCA has *positive rates* if its transition kernel T is such that $\forall a, b, c, d \in S$, $T(a, b, c; d) > 0$.

By definition, if $(\eta_t(i-1), \eta_{t-1}(i), \eta_t(i+1)) = (a, b, c)$, then $\eta_{t+1}(i)$ is equal to $d \in S$ with probability $T(a, b, c; d)$, see Fig. 1 for an illustration.

Let us introduce the two vectors $\mathrm{u} = (-1, 1)$ and $\mathrm{v} = (1, 1)$ of $\mathbb{Z}^2_{\mathrm{e}}$.

Let μ be a distribution on $S^{\mathbb{Z}_{t-1}} \times S^{\mathbb{Z}_t}$. We denote by $\sigma_v(\mu)$ the distribution on $S^{\mathbb{Z}_t} \times S^{\mathbb{Z}_{t+1}}$ which is the image of μ by the application:

$$\sigma_{\mathrm{v}} : ((x_k)_{k\in\mathbb{Z}_{t-1}}, (y_l)_{l\in\mathbb{Z}_t}) \to ((x_{k-1})_{k\in\mathbb{Z}_t}, (y_{l-1})_{l\in\mathbb{Z}_{t+1}}).$$

When considering the distribution μ as living on the two consecutive horizontal lines of the lattice $\mathbb{Z}^2_{\mathrm{e}}$, corresponding to times $t-1$ and t, the distribution $\sigma_{\mathrm{v}}(\mu)$ thus corresponds to shifting μ by a vector $\mathrm{v} = (1, 1)$. Similarly, we denote by $\sigma_{\mathrm{v-u}}(\mu)$ the distribution on $S^{\mathbb{Z}_{t-1}} \times S^{\mathbb{Z}_t}$ which is the image of μ by the application: $\sigma_{\mathrm{v-u}} : ((x_k)_{k\in\mathbb{Z}_{t-1}}, (y_l)_{l\in\mathbb{Z}_t}) \to ((x_{k-2})_{k\in\mathbb{Z}_{t-1}}, (y_l)_{l\in\mathbb{Z}_t})$.

In that context of PCA with memory two, we introduce the following definitions.

Definition 2. Let μ be a probability distribution on $S^{\mathbb{Z}_0} \times S^{\mathbb{Z}_1}$.

The distribution μ is said to be *shift-invariant* if $\sigma_{\mathrm{v-u}}(\mu) = \mu$.

The distribution μ on $S^{\mathbb{Z}_0} \times S^{\mathbb{Z}_1}$ is an *invariant distribution* of a PCA with memory two if the PCA dynamics is such that: $(\eta_0, \eta_1) \sim \mu \implies (\eta_1, \eta_2) \sim \sigma_{\mathrm{v}}(\mu)$.

By a standard compactness argument, one can prove that any PCA has at least one invariant distribution which is shift-invariant. In this article, we will focus on such invariant distributions. Note that if μ is both a shift-invariant measure and an invariant distribution of a PCA, then we also have $(\eta_0, \eta_1) \sim \mu \implies (\eta_1, \eta_2) \sim \sigma_{\mathrm{u}}(\mu)$.

Definition 3. Let p be a distribution on S. The p-HZPM (for *Horizontal Zigzag Product Measure*) on $S^{\mathbb{Z}_{t-1}} \times S^{\mathbb{Z}_t}$ is the distribution $\pi_p = \mathcal{B}(p)^{\otimes \mathbb{Z}_{t-1}} \otimes \mathcal{B}(p)^{\otimes \mathbb{Z}_t}$.

Observe that we do not specify t in the notation, since there will be no possible confusion. By definition, π_p is invariant for a PCA if:

$$(\eta_{t-1}, \eta_t) \sim \pi_p \implies (\eta_t, \eta_{t+1}) \sim \pi_p.$$

1.3 Stationary space-time diagrams and directional (quasi-)reversibility

Let A be a PCA and μ one of its invariant measures. Let $G_n = (\eta_t(i) : t \in \{-n, \dots, n\}, i \in \mathbb{Z}_t)$ be a space-time diagram of A under its invariant measure μ, from time $t = -n$ to $t = n$. Then $(G_n)_{n\geq 0}$ induces a sequence of compatible measures on $\mathbb{Z}^2_{\mathrm{e}}$ and, by Kolmogorov extension theorem, defines a unique measure on $\mathbb{Z}^2_{\mathrm{e}}$, that we denote by $G(A, \mu)$.

Definition 4. Let A be a PCA and μ one of its invariant distributions which is shift-invariant. A random field $(\eta_t(i) : t \in \mathbb{Z}, i \in \mathbb{Z}_t)$ which is distributed according to $G(A, \mu)$ is called a *stationary space-time diagram* of A taken under μ.

We denote by D_4 the dihedral group of order 8, that is, the group of symmetries of the square. We denote by r the rotation of angle $\pi/2$ and by h the horizontal reflection. We denote the vertical reflection by $v = r^2 \circ h$, and the identity by id. For a subset E of D_4, we denote by $< E >$ the subgroup of D_4 generated by the elements of E.

Definition 5. Let A be a positive-rates PCA, and let μ be an invariant measure of A which is shift-invariant. For $g \in D_4$, we say that (A, μ) is *g-quasi-reversible*, if there exists a PCA A_g and a measure μ_g such that $G(A, \mu) \overset{(d)}{=} g^{-1} \circ G(A_g, \mu_g)$. In this case, the pair (A_g, μ_g) is the g-reverse of (A, μ). If, moreover, $(A_g, \mu_g) = (A, \mu)$, then (A, μ) is said to be *g-reversible*.

For a subset E of D_4, we say that A is E-quasi-reversible (resp. E-reversible) if it is g-quasi-reversible (resp. g-reversible) for any $g \in E$.

Classical definitions of quasi-reversibility and reversibility of PCA correspond to time-reversal, that are, h-quasi-reversibility and h-reversibility. Geometrically, the stationary space-time diagram (A, μ) is g-quasi-reversible if after the action of the isometry g, the random field has the same distribution as if we had iterated another PCA A_g (or the same PCA A, in the reversible case). In particular, if (A, μ) is r-quasi-reversible (resp. r^2, r^3), it means that even if the space-time diagram is originally defined by an iteration of the PCA A towards the North, it can also be described as the stationary space-time diagram of another PCA directed to the East (resp. to the South, to the West).

Table 1 presents a summary of the results that will be proven in the next sections, concerning the stationary space-time diagrams of PCA having an invariant HZPM. For each possible (quasi-)reversibility behaviour, we give the conditions that the parameters of the PCA must satisfy (see Section 3 for details), and provide the number of degrees of freedom left by these equations, that is, the dimension of the corresponding submanifold of the parameter space (see Section 8).

2 Invariant product measures and ergodicity

To start with, next theorem gives a characterization of PCA with memory two having an HZPM invariant measure.

Theorem 6. *Let A be a positive-rates PCA with transition kernel T, and let p be a probability vector on S. The HZPM π_p is invariant for A if and only if*
Cond 1: *for any $a, c, d \in S$, $p(d) = \sum_{b \in S} p(b) T(a, b, c; d)$.*

Note that since A has positive rates, if π_p is invariant for A, then the vector p has to be positive.

Corollary 7. *Let A be a positive-rates PCA with transition kernel T. The PCA A has an invariant HZPM if and only if for any $a, c \in S$, the left eigenspace $E_{a,c}$ of matrices $(T(a, b, c; d))_{b,d \in S}$ related to the eigenvalue 1 is the same. In that case, the invariant HZPM is unique: it is the measure π_p defined by the unique vector p such that $E_{a,c} = Vect(p)$ for all $a, c \in S$ and $\sum_{b \in S} p(b) = 1$.*

Proof. Let p be a positive vector such that π_p is invariant by A and assume that $(\eta_{t-1}, \eta_t) \sim \pi_p$. Then, on the one hand, since π_p is invariant by A, we have:

$$\mathbb{P}\left(\eta_t(i-1) = a, \eta_{t+1}(i) = d, \eta_t(i+1) = c\right) = p(a)p(c)p(d).$$

And on the other hand, by definition of the PCA,

$$\mathbb{P}\left(\eta_t(i-1) = a, \eta_{t+1}(i) = d, \eta_t(i+1) = c\right) = \sum_{b \in S} p(a)p(b)p(c)T(a, b, c; d).$$

Cond. 1 follows.

Conversely, assume that Cond. 1 is satisfied, and that $(\eta_{t-1}, \eta_t) \sim \pi_p$. For some given choice of $n \in \mathbb{Z}_t$, let us denote: $X_i = \eta_{t-1}(n+1+2i), Y_i = \eta_t(n+2i), Z_i = \eta_{t+1}(n+1+2i)$, for $i \in \mathbb{Z}$,

Conditions on the parameters	Property of the PCA	Dimension of the submanifold (number of degrees of freedom)
Cond. 1: $\forall a,c,d \in S$, $p(d) = \sum_{b \in S} p(b)T(a,b,c;d)$	HZPM invariant $\{r^2, h\}$-quasi-reversible	$n^2(n-1)^2$
Cond. 1 + Cond. 2: $\forall a,b,d \in S$, $p(d) = \sum_{c \in S} p(c)T(a,b,c;d)$	r-quasi-reversible	$n(n-1)^3$
Cond. 1 + Cond. 3: $\forall b,c,d \in S$, $p(d) = \sum_{a \in S} p(a)T(a,b,c;d)$	r^{-1}-quasi-reversible	$n(n-1)^3$
Cond. 1 + Cond. 2 + Cond. 3	D_4-quasi-reversible	$(n-1)^4$
Cond. 1 + $\forall a,b,c,d \in S$, $T(a,b,c;d) = T(c,b,a;d)$	v-reversible	$\dfrac{(n-1)^2 n(n+1)}{2}$
Cond. 1 + $\forall a,b,c,d \in S$, $p(b)T(a,b,c;d) = p(d)T(c,d,a;b)$	r^2-reversible	$\dfrac{(n-1)^2 n(n+1)}{2}$
Cond. 1 + $\forall a,b,c,d \in S$, $p(b)T(a,b,c;d) = p(d)T(a,d,c;b)$	h-reversible	$\dfrac{n^3(n-1)}{2}$
Cond. 1 + $\forall a,b,c,d \in S$, $T(a,b,c;d) = T(c,b,a;d)$ and $p(b)T(a,b,c;d) = p(d)T(c,d,a;b)$	$< r^2, v >$-reversible	$\dfrac{(n-1)n^2(n+1)}{4}$
Cond. 1 + $\forall a,b,c,d \in S$, $p(a)T(a,b,c;d) = p(d)T(b,c,d;a)$	$< r >$-reversible	$\dfrac{n(n-1)(n^2-3n+4)}{4}$
Cond. 1 + $\forall a,b,c,d \in S$, $p(a)T(a,b,c;d) = p(d)T(d,c,b;a)$	$< r \circ v >$-reversible	$\dfrac{(n-1)^2(n^2-2n+2)}{2}$
Cond. 1 + $\forall a,b,c,d \in S$, $p(a)T(a,b,c;d) = p(d)T(b,c,d;a)$ and $T(a,b,c;d) = T(c,b,a;d)$	D_4-reversible	$\dfrac{n(n-1)(n^2-n+2)}{8}$

Table 1: Summary of the characterization of (quasi-)reversible PCA. We denote by n the cardinal of the alphabet S.

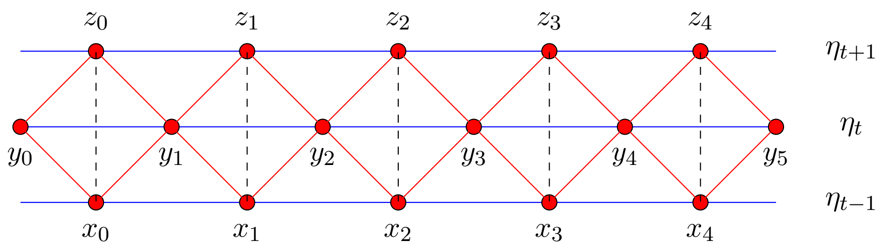


Figure 3: Illustration of the proof of Theorem 6.

see Fig. 3 for an illustration. Then, for any $k \geq 1$, we have:

$$\begin{aligned}
&\mathbb{P}\left((Y_i)_{0\leq i\leq k} = (y_i)_{0\leq i\leq k}, (Z_i)_{0\leq i\leq k-1} = (z_i)_{0\leq i\leq k-1}\right) \\
&= \sum_{(x_i:0\leq i\leq k-1)} \mathbb{P}\left((X_i)_{0\leq i\leq k} = (x_i)_{0\leq i\leq k-1}, (Y_i)_{0\leq i\leq k} = (y_i)_{0\leq i\leq k}\right) \prod_{i=0}^{k-1} T(y_i, x_i, y_{i+1}; z_i) \\
&= \sum_{(x_i:0\leq i\leq k-1)} \prod_{i=0}^{k-1} p(x_i) \prod_{i=0}^{k} p(y_i) \prod_{i=0}^{k-1} T(y_i, x_i, y_{i+1}; z_i) \\
&= \prod_{i=0}^{k} p(y_i) \prod_{i=0}^{k-1} \sum_{x_i\in S} p(x_i) T(y_i, x_i, y_{i+1}; z_i) \\
&= \prod_{i=0}^{k} p(y_i) \prod_{i=0}^{k-1} p(z_i) \text{ by Cond. 1,}
\end{aligned}$$

thus, π_p is invariant by A.

□

Theorem 8. *Let A be a PCA with transition kernel T and positive rates, satisfying Cond. 1. Then, A is ergodic. Precisely, whatever the distribution of (η_0, η_1) is, the distribution of (η_t, η_{t+1}) converges (weakly) to π_p.*

Proof. The proof we propose is inspired from [31], see also [12] and [27]. Let us fix some boundary conditions $(\ell, r) \in S^2$. Then, for any $k \geq 0$, the transition kernel T induces a Markov chain on S^{2k+1}, such that the probability of a transition from the sequence $(a_0, b_0, a_1, b_1, \ldots, b_{k-1}, a_k) \in S^{2k+1}$ to a sequence $(a'_0, b'_0, a'_1, b'_1, \ldots, b'_{k-1}, a'_k) \in S^{2k+1}$ is given by:

$$\begin{aligned}
&P_k^{(\ell,r)}((a_0, b_0, a_1, b_1, \ldots, b_{k-1}, a_k), (a'_0, b'_0, a'_1, b'_1, \ldots, b'_{k-1}, a'_k)) \\
&\quad = T(\ell, a_0, b_0; a'_0) T(a'_0, b_0, a_1; b'_0) T(b_0, a_1, b_1; a'_1) \cdots T(b_{k-1}, a_k, r; a'_k) \\
&\quad = T(\ell, a_0, b_0; a'_0) T(b_{k-1}, a_k, r; a'_k) \prod_{i=1}^{k-1} T(b_{i-1}, a_i, b_{i+1}; a'_i) \prod_{i=0}^{k-1} T(a'_i, b_i, a'_{i+1}; b'_i).
\end{aligned}$$

We refer to Fig. 4 for an illustration. Let us observe that the p-HZPM $\pi_p^k = \mathcal{B}(p)^{\otimes 2k+1}$ is left invariant by this Markov chain. This is an easy consequence from Cond. 1. For any $(\ell, r) \in S^2$, the transition kernel $P^{(\ell,r)}$ is positive. Therefore, there exists $\theta_{(\ell,r)} < 1$ such that for any probability distributions ν, ν' on S^{2k+1}, we have

$$||P_k^{(\ell,r)}\nu - P_k^{(\ell,r)}\nu'||_1 \leq \theta_k^{(\ell,r)} ||\nu - \nu'||_1,$$

the above inequality being true in particular for $\theta_k^{(\ell,r)} = 1 - \varepsilon_k^{(\ell,r)}$, where

$$\varepsilon_k^{(\ell,r)} = \min\{P_k^{(\ell,r)}(x, y) \,:\, x, y \in S^{2k+1}\}.$$

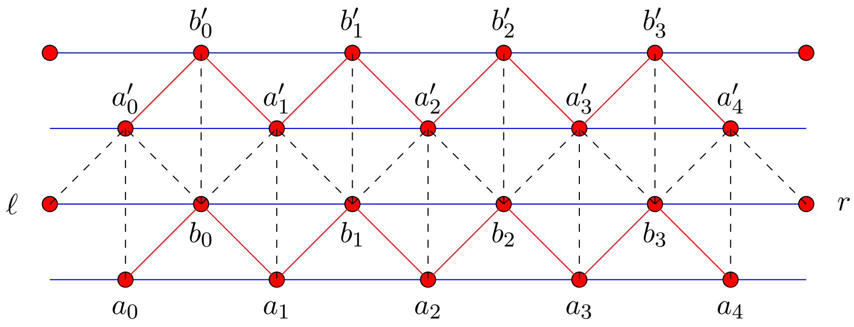


Figure 4: Illustration of the proof of Theorem 8.

Let us set $\theta_k = \max\{\theta_k^{(\ell,r)} : (\ell, r) \in S^2\}$. It follows that for any sequence $(\ell_t, r_t)_{t\geq 0}$ of elements of S^2, we have:

$$||P_k^{(\ell_{t-1},r_{t-1})} \cdots P_k^{(\ell_1,r_1)} P_k^{(\ell_0,r_0)}\nu - P_k^{(\ell_{t-1},r_{t-1})} \cdots P_k^{(\ell_1,r_1)} P_k^{(\ell_0,r_0)}\nu'||_1 \leq \theta^t ||\nu - \nu'||_1.$$

In particular, for $\nu' = \pi_p^k$, we obtain that for any distribution ν on S^{2k+1} and any sequence $(\ell_t, r_t)_{t\geq 0}$ of elements of S^2, we have:

$$||P_k^{(\ell_{t-1},r_{t-1})} \cdots P_k^{(\ell_1,r_1)} P_k^{(\ell_0,r_0)}\nu - \pi_p^k||_1 \leq 2\theta^t.$$

Let now μ be a distribution on $S^{\mathbb{Z}_0 \cup \mathbb{Z}_1}$, and let $k \geq 0$. When iterating A, the distribution μ induces a random sequence of symbols $\ell_t = \eta_{2t+1}(-(2k+1))$ and $r_t = \eta_{2t+1}(2k+1)$. Let us denote by ν_t the distribution of the sequence $(\eta_{2t}(-2k), \eta_{2t+1}(-2k+1), \eta_{2t}(-2k+2), \ldots, \eta_{2t}(2k-2), \eta_{2t+1}(2k-1), \eta_{2t}(2k))$, and let $\pi_p^{2k} = \mathcal{B}(p)^{\otimes 4k+1}$. We have:

$$\forall t \geq 0,\ ||\nu_t - \pi_p^{2k}||_1 \leq \max_{(\ell_0,r_0)\ldots(\ell_{t-1},r_{t-1})\in S^2} ||P_{2k}^{(\ell_{t-1},r_{t-1})} \ldots P_{2k}^{(\ell_1,r_1)} P_{2k}^{(\ell_0,r_0)}\nu_0 - \pi_p^{2k}||_1 \leq 2\theta^t.$$

This concludes the proof.

□

3 Directional (quasi-)reversibility of PCA having an invariant product measure

The stationary space-time diagram of a PCA (see Def. 4) is a random field indexed by $\mathbb{Z}_e^2$. For a point $x = (i, t) \in \mathbb{Z}_e^2$, we will also use the notation $\eta(x) = \eta(i, t) = \eta_t(i)$, and for a family $L \subset \mathbb{Z}_e^2$, we define $\eta(L) = (\eta(x))_{x\in L}$.

The following lemma proves that the space-time diagram of a positive-rate PCA characterizes its dynamics. Precisely, if two positive-rates PCA A and A' have the same space-time diagram taken under their respective invariant measures μ and μ', then $A = A'$ and $\mu = \mu'$.

Lemma 9. *Let (A, μ) and (A', μ') two positive-rates PCA with one of their invariant measure. Then,* $G(A, \mu) \overset{(d)}{=} G(A', \mu') \implies (A, \mu) = (A', \mu')$.

Proof. Let us set $G = G(A, \mu) = (\eta_t(i) : t \in \mathbb{Z}, i \in \mathbb{Z}_t)$ and $G' = G(A', \mu') = (\eta'_t(i) : t \in \mathbb{Z}, i \in \mathbb{Z}_t)$. By definition, $G|_{t=0,1} \sim \mu$ and $G'|_{t=0,1} \sim \mu'$. Since $G \overset{(d)}{=} G'$, we obtain $\mu = \mu'$.

Let us denote $\mu(a, b, c) = \mathbb{P}(\eta_1(-1) = a, \eta_0(0) = b, \eta_1(1) = c)$. As a consequence from the fact that A has positive rates, we have: $\forall (a, b, c) \in S^3, \mu(a, b, c) > 0$. Thus, for any $a, b, c, d \in S$, we have: $\mathbb{P}(\eta_1(-1) = a, \eta_0(0) = b, \eta_1(1) = c, \eta_2(0)) = d) = \mu(a, b, c)T(a, b, c; d) > 0$. The same relation holds for A' and as $G \overset{(d)}{=} G'$, we obtain: $\mu(a, b, c)T(a, b, c; d) = \mu'(a, b, c)T'(a, b, c; d)$. Since $\mu = \mu'$, we deduce that $T(a, b, c; d) = T'(a, b, c; d)$ for any $a, b, c, d \in S$. Hence, $A = A'$. □

By Lemma 9, if a PCA is g-quasi-reversible (see Def 5), its g-reverse is thus unique. Let's enumerate some easy results on quasi-reversible PCA and reversible PCA.

Proposition 10. *Let A be a positive-rates PCA and let μ be one of its invariant measures.*

1. *(A, μ) is id-reversible.*

2. *(A, μ) is v-quasi-reversible and the v-reverse PCA is defined by the transition kernel $T_v(c, b, a; d) = T(a, b, c; d)$.*

3. *For any $g \in D_4$, if (A, μ) is g-quasi-reversible, then its g-reverse (A_g, μ_g) is g^{-1}-quasi-reversible and (A, μ) is the g^{-1}-reverse of (A_g, μ_g).*

4. *If (A, μ) is g-quasi-reversible and (A_g, μ_g) is its g-reverse and if (A_g, μ_g) is g'-quasi-reversible and $(A_{g'g}, \mu_{g'g})$ is its g'-reverse, then (A, μ) is $g'g$-quasi-reversible and $(A_{g'g}, \mu_{g'g})$ is its $g'g$-reverse.*

5. *For any subset E of D_4, if (A, μ) is E-reversible, then (A, μ) is $< E >$-reversible.*

Remark 11. Since $< r, v >= D_4$, a consequence of the last point of Prop. 10 is that if (A, μ) is r and v-reversible, then it is D_4-reversible.

3.1 Quasi-reversible PCA with p-HZPM invariant

Let us denote by $\mathcal{T}_S$ the subset of positive-rates PCA with set of symbols S having an invariant HZPM. In addition, for a positive probability vector p on S, we define $\mathcal{T}_S\,(p)$ as the subset of $\mathcal{T}_S$ made of PCA for which the measure π_p is invariant. By Theorem 6, $\mathcal{T}_S\,(p)$ is thus the set of PCA satisfying Cond. 1.

In this section, we characterize PCA of $\mathcal{T}_S\,(p)$ that are g-quasi-reversible, for each possible $g \in D_4$. First of all, let us focus on the r^2-quasi-reversibility, or equivalently on the h-quasi-reversibility, which corresponds to time-reversal. For any stationary Markov chain, we can define a time-reversed chain, which has still the Markov property. But in general, the time-reversed chain of a PCA is no more a PCA. Next theorem shows that any PCA in $\mathcal{T}_S\,(p)$ is r^2-quasi-reversible, which means that the time-reversed chain of a PCA of $\mathcal{T}_S\,(p)$ is still a PCA with memory two, which furthermore belongs to $\mathcal{T}_S\,(p)$, since it preserves the measure π_p.

Theorem 12. *Any PCA $A \in \mathcal{T}_S\,(p)$ is r^2-quasi-reversible, and the transition kernel T_{r^2} of its r^2-reverse A_{r^2} is given by:*

$$\forall a, b, c, d \in S, \quad T_{r^2}(c, d, a; b) = \frac{p(b)}{p(d)} T(a, b, c; d).$$

Proof. For some given choice of $n \in \mathbb{Z}_t$, let us denote again: $X_i = \eta_{t-1}(n + 1 + 2i), Y_i = \eta_t(n + 2i), Z_i = \eta_{t+1}(n + 1 + 2i)$, for $i \in \mathbb{Z}$, see Fig. 3. The following computation proves the result wanted.

$$\begin{aligned}
&\mathbb{P}\left((X_i)_{0\le i\le k} = (x_i)_{0\le i\le k} | (Y_i)_{0\le i\le k+1} = (y_i)_{0\le i\le k+1}, (Z_i)_{0\le i\le k} = (z_i)_{0\le i\le k}\right) \\
&\quad = \frac{\mathbb{P}\left((X_i)_{0\le i\le k} = (x_i)_{0\le i\le k}, (Y_i)_{0\le i\le k+1} = (y_i)_{0\le i\le k+1}, (Z_i)_{0\le i\le k} = (z_i)_{0\le i\le k}\right)}{\mathbb{P}\left((Y_i)_{0\le i\le k+1} = (y_i)_{0\le i\le k+1}, (Z_i)_{0\le i\le k} = (z_i)_{0\le i\le k}\}\right)} \\
&\quad = \frac{p(y_0)\prod_{i=0}^{k} p(x_i)p(y_{i+1})T(y_i, x_i, y_{i+1}, z_i)}{p(y_0)\prod_{i=0}^{k} p(z_i)p(y_{i+1})} \\
&\quad = \prod_{i=0}^{k} \frac{p(x_i)}{p(z_i)} T(y_i, x_i, y_{i+1}, z_i).
\end{aligned}$$

□

With (2) and (4) of Prop. 10, we instantly obtain the following corollary.

Corollary 13. *Any PCA* $A \in \mathcal{T}_S$ *is* $\{h, r^2, v\}$*-quasi-reversible.*

Let us now focus on the space-time diagram $G(A, \pi_p)$ of a PCA $A \in \mathcal{T}_S\,(p)$, taken under its unique invariant measure π_p. By definition, any horizontal line of that space-time diagram is i.i.d. The following proposition extends that result to other types of lines.

Definition 14. A *zigzag polyline* is a sequence $(i, t_i)_{m \le i \le n} \in \mathbb{Z}^2_{\mathrm{e}}$ such that for any $i \in \{m, \dots, n\}$, $(t_{i+1} - t_i) \in \{-1, 1\}$.

Proposition 15. *Let* $A \in \mathcal{T}_S\,(p)$ *be a PCA of stationary space-time diagram* $G(A, \pi_p) = (\eta_t(i) : t \in \mathbb{Z}, i \in \mathbb{Z}_t)$. *For any zigzag polyline* $(i, t_i)_{m \le i \le n}$*, we have:* $(\eta_{t_i}(i) : i \in \{m, \dots, n\}) \sim \mathcal{B}(p)^{\otimes(n-m+1)}$.

Observe that Prop. 15 implies that (bi-)infinite zigzag polylines are also made of i.i.d. $\mathcal{B}(p)$ random variables.

Proof. The proof is done by induction on $T = \max(t_i) - \min(t_i)$. If $T = 1$, then the zigzag polyline is an horizontal zigzag, and since $A \in \mathcal{T}_S\,(p)$, the result is true.

Now, suppose that the result is true for any zigzag polyline such that $\max(t_i) - \min(t_i) = T$, and consider a zigzag polyline $(i, t_i)_{m \le i \le n}$ such that $\max(t_i) - \min(t_i) = T + 1$. Then, there exists t such that $\min(t_i) = t$ and $\max(t_i) = t + T + 1$. Let $M = \{i \in \{m, \dots, n\} : t_i = t + T + 1\}$. For any $i \in M$, we have $t_{i\pm1} = t + T$ (we assume that $0, n \notin M$, even if it means extending the line). So, by induction, we have that $(\eta(i, t_i - 2\ 1_{i \in M}) : i \in \{m, \dots, n\}) \sim \mathcal{B}(p)^{\otimes(m-n+1)}$. For any $(a_i)_{m \le i \le n} \in S^{m-n+1}$, we have:

$$
\begin{aligned}
&\mathbb{P}\left(\eta(x_i, t_i) = a_i : m \le i \le n\right) \\
&= \sum_{(b_i : i \in M) \in S^M} \mathbb{P}\left(\{\eta(i, t_i) = a_i : i \notin M\}, \{\eta(i, t_i - 2) = b_i : i \in M\}\right) \prod_{i \in M} T(a_{i-1}, b_i, a_{i+1}; a_i) \\
&= \sum_{(b_i : i \in M) \in S^M} \prod_{i \notin M} p(a_i) \prod_{i \in M} p(b_i) T(a_{i-1}, b_i, a_{i+1}; a_i) \\
&= \prod_{i \notin M} p(a_i) \prod_{i \in M} \sum_{b_i \in S} p(b_i) T(a_{i-1}, b_i, a_{i+1}; a_i) \\
&= \prod_{i=m}^{n} p(a_i).
\end{aligned}
$$

□

Now, we will characterize PCA in $\mathcal{T}_S$ that are r-quasi-reversible.

Proposition 16. *Let* $A \in \mathcal{T}_S\,(p)$*.* A *is* r*-quasi-reversible if and only if:*
Cond 2: *for any* $a, b, d \in S$, $\sum_{c \in S} p(c) T(a, b, c; d) = p(d)$.
In that case, the transition kernel T_r *of its* r*-reverse* A_r *is given by:*

$$
\forall a, b, c, d \in S, \quad T_r(d, a, b; c) = \frac{p(c)}{p(d)} T(a, b, c; d). \tag{1}
$$

Proposition 17. *Let* $A \in \mathcal{T}_S\,(p)$*.* A *is* r^{-1}*-quasi-reversible if and only if:*
Cond 3: *for any* $b, c, d \in S$, $\sum_{a \in S} p(a) T(a, b, c; d) = p(d)$.
In that case, the transition kernel $T_{r^{-1}}$ *of its* r^{-1}*-reverse* $A_{r^{-1}}$ *is given by:*

$$
\forall a, b, c, d \in S, \quad T_{r^{-1}}(b, c, d; a) = \frac{p(a)}{p(d)} T(a, b, c; d). \tag{2}
$$

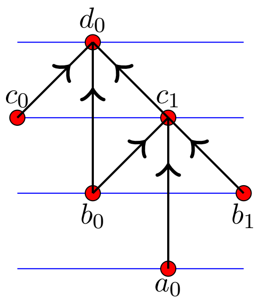


Figure 5: The pattern L.

We prove only Prop. 16, the proof of Prop. 17 being similar.

Proof. • Let us first prove that if A is r-quasi-reversible, then Cond. 2 holds, and that the r-reverse satisfies: $T_r(d,a,b;c) = \frac{p(c)}{p(d)} T(a,b,c;d)$. Let us recall the notations $\mathrm{u} = (-1,1)$ and $\mathrm{v} = (1,1)$. Since $A \in \mathcal{T}_S(p)$, for any $x \in \mathbb{Z}^2_{\mathrm{e}}$ and $a,b,c,d \in S$, we have:

$$\mathbb{P}\left(\eta(x+\mathrm{u}) = a, \eta(x) = b, \eta(x+\mathrm{v}) = c, \eta(x+\mathrm{u}+\mathrm{v}) = d\right) = p(a)p(b)p(c)T(a,b,c;d).$$

Hence,

$$\begin{aligned} T_r(d,a,b;c) &= \mathbb{P}\left(\eta(x+\mathrm{v}) = c | \eta(x+\mathrm{u}) = a, \eta(x) = b, \eta(x+\mathrm{u}+\mathrm{v}) = d\right) \\ &= \frac{p(a)p(b)p(c)T(a,b,c;d)}{\sum_{c' \in S} p(a)p(b)p(c')T(a,b,c';d)} \\ &= \frac{p(c)T(a,b,c;d)}{\sum_{c' \in S} p(c')T(a,b,c';d)} \end{aligned} \tag{3}$$

For some $x \in \mathbb{Z}^2_{\mathrm{e}}$, let us introduce the pattern $L = (x, x+\mathrm{u}, x+\mathrm{v}, x+2\mathrm{u}, x+\mathrm{u}+\mathrm{v}, x+2\mathrm{u}+\mathrm{v})$, see Fig. 5. For $a_0, b_0, b_1, c_0, c_1, d_0 \in S$, we are interested in the quantity:

$$Q(a_0, b_0, b_1, c_0, c_1, d_0) = \mathbb{P}\left(\eta(L) = (a_0, b_0, b_1, c_0, c_1, d_0)\right).$$

On the one hand, using the fact that we have a portion of the space-time diagram $G(A, \pi_p)$, Prop. 15 implies that:

$$\mathbb{P}\left(\eta(x+2\mathrm{u}) = c_0, \eta(x+\mathrm{u}) = b_0, \eta(x) = a_0, \eta(x+\mathrm{v}) = b_1\right) = p(c_0)p(b_0)p(a_0)p(b_1).$$

We thus obtain: $Q(a_0, b_0, b_1, c_0, c_1, d_0) = p(c_0)p(b_0)p(a_0)p(b_1)T(b_0, a_0, b_1; c_1)T(c_0, b_0, c_1; d_0)$. On the other hand, using the fact that A is r-quasi-reversible, we have:

$$Q(a_0, b_0, b_1, c_0, c_1, d_0) = \sum_{b'_1, c'_1 \in S} Q(a_0, b_0, b'_1, c_0, c'_1, d_0) T_r(d_0, c_0, b_0; c_1) T_r(c_1, b_0, a_0; b_1).$$

It follows that:

$$\begin{aligned} 1 &= \sum_{b'_1, c'_1 \in S} \frac{Q(a_0, b_0, b'_1, c_0, c'_1, d_0)}{Q(a_0, b_0, b_1, c_0, c_1, d_0)} T_r(d_0, c_0, b_0; c_1) T_r(c_1, b_0, a_0; b_1) \\ &= \sum_{b'_1, c'_1 \in S} \frac{p(b'_1)T(b_0, a_0, b'_1; c'_1)T(c_0, b_0, c'_1; d_0)}{p(b_1)T(b_0, a_0, b_1; c_1)T(c_0, b_0, c_1; d_0)} T_r(d_0, c_0, b_0; c_1) T_r(c_1, b_0, a_0; b_1) \end{aligned} \tag{4}$$

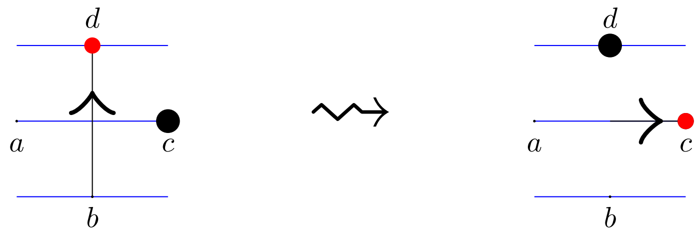


Figure 6: Elementary flip illustrating the relation $p(c)T(a,b,c;d) = p(d)T_r(d,a,b;c)$.

By (3), we have:

$$T_r(d_0,c_0,b_0;c_1) = \frac{p(c_1)T(c_0,b_0,c_1;d_0)}{\sum_{c\in S} p(c)T(c_0,b_0,c;d_0)},$$
$$T_r(c_1,b_0,a_0;b_1) = \frac{p(b_1)T(b_0,a_0,b_1;c_1)}{\sum_{b\in S} p(b)T(b_0,a_0,b;c_1)}.$$

After replacing in (4), we obtain:

$$\left(\sum_{b\in S} p(b)T(b_0,a_0,b;c_1)\right)\left(\sum_{c\in S} p(c)T(c_0,b_0,c;d_0)\right) = p(c_1)\sum_{b_1',c_1'\in S} p(b_1')T(b_0,a_0,b_1';c_1')T(c_0,b_0,c_1';d_0).$$

Summing over $d_0 \in S$ on both sides and simplifying gives: $\sum_{b\in S} p(b)T(b_0,a_0,b;c_1) = p(c_1)$. Hence, Cond. 2 is necessary. Together with (3), we deduce (1).

• Let us now assume that Cond. 2 holds, and let T_r be defined by (1). For any $d,a,b \in S$, we have:

$$\sum_{c\in S} T_r(d,a,b;c) = \frac{\sum_{c\in S} p(c)T(a,b,c;d)}{p(d)} = 1.$$

Hence, T_r is a transition kernel.

For some $x \in \mathbb{Z}^2_{\mathrm{e}}$, and $m \in \mathbb{N}$ let us define the pattern $M = (x + i\mathrm{u} + j\mathrm{v})_{0\le i,j\le m}$. Using Prop. 15, for any $(a_{i,j})_{0\le i,j\le m} \in S^{\{0,1,\dots,m\}^2}$, we have:

$$\mathbb{P}\left(\eta(M) = (a_{i,j})_{0\le i,j\le m}\right) = \prod_{i=0}^{m} p(a_{i,0}) \prod_{j=1}^{m} p(a_{0,j}) \prod_{i=1}^{m}\prod_{j=1}^{m} T(a_{i,j-1},a_{i-1,j-1},a_{i-1,j};a_{i,j}).$$

This computation is represented on Fig. 7 (a). The points for which $p(a_{i,j})$ appears in the product are marked by black dots, while the black vertical arrows represent the values that are computed through the transition kernel T. Now, by (1), we know that:

$$\forall a,b,c,d \in S, \quad p(c)T(a,b,c;d) = p(d)T_r(d,a,b;c). \tag{5}$$

It means that in the product above, we can perform flips as represented in Fig. 6, where an arrow to the right now represents a computation made with the transition kernel T_r. We say that such a use of (5) is a flip of (c,d). By flipping successively the cells from right to left and bottom to top: first $(a_{0,m},a_{1,m})$, then $(a_{0,m-1},a_{1,m-1})$, $(a_{1,m},a_{2,m})$, and $(a_{0,m-2},a_{1,m-2})$, $(a_{1,m-1},a_{2,m-1})$, $(a_{2,m},a_{3,m})$ etc., we finally obtain (see Fig. 7 for an illustration):

$$\mathbb{P}\left(\eta(M) = (a_{i,j})_{0\le i,j\le m}\right) = \prod_{i=0}^{m} p(a_{i,0}) \prod_{j=1}^{m} p(a_{m,j}) \prod_{i=0}^{m-1}\prod_{j=1}^{m} T_r(a_{i+1,j},a_{i+1,j-1},a_{i,j-1};a_{i,j}). \tag{6}$$

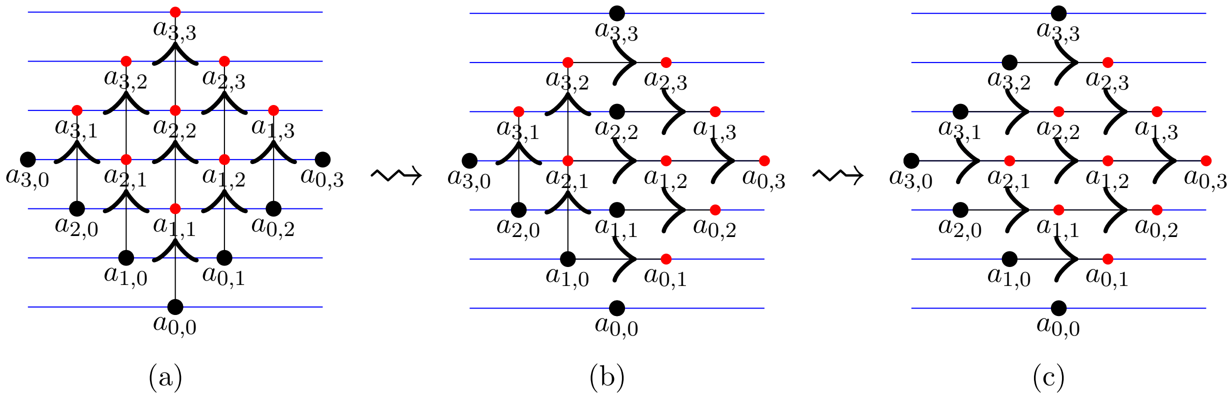


Figure 7: From T to T_r using flips.

Let us define the vertical lines: $V_{-1} = (x + (m-i)\mathrm{u} + (m-i-1)\mathrm{v})_{0\le i\le m-1}, V_0 = (x + (m-i)\mathrm{u} + (m-i)\mathrm{v})_{0\le i\le m}, V_1 = (x + (m-i-1)\mathrm{u} + (m-i)\mathrm{v})_{0\le i\le m-1}$. From (6), we deduce that:

$$\mathbb{P}\left(\eta(V_1) = (a_{m,m-1},\dots,a_{1,0}) \mid \eta(V_0) = (a_{m,m},\dots,a_{0,0}), \eta(V_{-1}) = (a_{m,m-1},\dots,a_{1,0})\right)$$
$$= \prod_{i=0}^{m-1} T_r(a_{m-i,m}, a_{m-i,m-i-1}, a_{m-i-1,m-i-1}; a_{m-i-1,m-i})$$

Since this is true for any $x \in \mathbb{Z}^2_{\mathrm{e}}$ and any $m \in \mathbb{N}$, it follows that the rotation of $G(A, \pi_p)$, the space-time diagram of A, by the rotation r is a space-time diagram of A_r, whose transition kernel is T_r, under one of its invariant measure that we denote by $\mu = (\pi_p)_r$ (observe that we do not specify the dependence on A in that last notation, although the measure depends on A). Furthermore, we can express explicitly the finite-dimensional of μ. For any $m \in \mathbb{N}$, we have:

$$\mu((a_{i+1,i})_{0\le i\le m-1}, (a_{i,i})_{0\le i\le m}) = \mathbb{P}\left(\eta(V_0) = (a_{m,m},\dots,a_{0,0}), \eta(V_1) = (a_{m,m-1},\dots,a_{1,0})\right)$$
$$= \sum_{(a_{i,j}\,:\, i,i+1\neq j)} \prod_{i=0}^{m} p(a_{i,0}) \prod_{j=1}^{m} p(a_{0,j}) \prod_{1\le i,j\le m} T(a_{i,j-1}, a_{i-1,j-1}, a_{i-1,j}; a_{i,j}) \quad (7)$$
$$= \sum_{(a_{i,j}\,:\, j<i)} \prod_{i=0}^{m} p(a_{i,0}) \prod_{j=1}^{m} p(a_{m,j}) \prod_{1\le j\le i+1\le m} T_r(a_{i+1,j}, a_{i+1,j-1}, a_{i,j-1}; a_{i,j}) \quad (8)$$

□

Note that in Prop. 16 and Prop. 17, the reverse PCA is not necessary an element of $\mathcal{T}_S(p)$. In the space-time diagram $G(A, \pi_p)$, the points $x, x+\mathrm{u}, x+2\mathrm{u}, \dots, x+m\mathrm{u}, x+m\mathrm{u}+\mathrm{v}, \dots, x+m\mathrm{u}+m\mathrm{v}$ consist in independent $\mathcal{B}(p)$ random variables. But if we now consider only the three points $x, x+\mathrm{v}, x+\mathrm{u}+\mathrm{v}$, they have no reason to be independent, so that μ can be different from π_p. Next theorem specifies the cases for which the reverse PCA A_r is an element of $\mathcal{T}_S(p)$, meaning that $\mu = \pi_p$.

But before, let us prove that in the space-time diagram $G(A, \pi)$, each vertical line V_i consists in independent variables. This means that even if the reverse PCA does not have necessarily an invariant p-HZPM, the measure μ is at least such that each horizontal (straight) line consists in independent $\mathcal{B}(p)$ random variables.

Proposition 18. *Let $A \in \mathcal{T}_S(p)$ be an r-quasi-reversible PCA (resp. an r^{-1}-quasi-reversible PCA). Then, for any $x \in \mathbb{Z}^2_{\mathrm{e}}$, the vertical line $V = \{x + k\mathrm{u} + k\mathrm{v} : k \in \mathbb{Z}\}$ consists in independent $\mathcal{B}(p)$ random variables.*

Proof. We assume that A is r-quasi-reversible, the case r^{-1}-quasi-reversible being similar. Let us consider again the Fig. 7. Precisely, let us do the succession of flips leading to Fig. 7 (b). Then, by summing over $(a_{i,j})_{i<j}$ and then over $(a_{i,j})_{i>j>0}$, we obtain :

$$\mu((a_{i,0})_{0<i\leq m},(a_{i,i})_{0\leq i\leq m}) = \prod_{0<i\leq m} p(a_{i,0}) \prod_{0\leq i\leq m} p(a_{i,i}).$$

This means that the points $x+m\mathrm{u}, x+(m-1)\mathrm{u},\ldots,x+\mathrm{u},x,x+(\mathrm{u}+\mathrm{v}),x+2(\mathrm{u}+\mathrm{v}),\ldots,x+m(\mathrm{u}+\mathrm{v})$ consist in independent $\mathcal{B}(p)$ random variables. As a consequence, the points of the vertical line V_0 are independent $\mathcal{B}(p)$ random variables. □

As already mentioned, in Prop. 16 and Prop. 17, if a PCA $A \in \mathcal{T}_S(p)$ satisfies Cond. 2 and not Cond. 3 (or the reverse), we get a PCA $C = A_r$ (or $A_{r^{-1}}$) for which we can compute exactly the marginals of an invariant measure, although it does not have a well-identified form. As a consequence of the previous results, we thus obtain next theorem, which gives conditions on the transitions of a PCA C for being of the form $C = A_r$, with A having an invariant p-HZMP. In that case, the measure $\mu = (\pi_p)_r$ is an invariant measure for C, and we have explicit formula for the computation of its marginals, see (7) and (8).

Theorem 19. *Let C be a PCA with transition kernel T. If there exists a probability distribution p such that Cond. 2 and Cond. 3 hold, then there exists a unique probability distribution μ on $S^{\mathbb{Z}_0} \times S^{\mathbb{Z}_1}$ such that*

- *μ is invariant by C,*
- *(C,μ) is $\{r^{-1},r\}$-quasi-reversible and its r^{-1}-reverse is $(C_{r^{-1}},\pi_p)$ with $C_{r^{-1}} \in \mathcal{T}_S(p)$, same hold for the r-reverse,*
- *$\mu_{|\mathbb{Z}_0} = \mathcal{B}(p)^{\otimes \mathbb{Z}_0}$ and $\mu_{|\mathbb{Z}_1} = \mathcal{B}(p)^{\otimes \mathbb{Z}_1}$.*

Moreover, we have explicit formula for the computation of the marginals of μ.

In Section 6, Example 52 provides an example of a PCA satisfying only Cond. 2, so that its r-reverse A_r satisfies the conditions of Theorem 19 above. In contrast, next theorem describes the family of PCA satisfying both Cond. 2 and Cond. 3.

Theorem 20. *Let $A \in \mathcal{T}_S(p)$. The following properties are equivalent:*

1. *A is $\{r,r^{-1}\}$-quasi-reversible.*
2. *A is r-quasi-reversible and $A_r \in \mathcal{T}_S(p)$,*
3. *A is r^{-1}-quasi-reversible and $A_{r^{-1}} \in \mathcal{T}_S(p)$,*
4. *Cond. 2 and Cond. 3 hold,*
5. *A is D_4-quasi-reversible.*

Proof.

$1 \Rightarrow 2$ If A is r-quasi-reversible, then its r-reverse A_r is defined by the transition kernel

$$T_r(d,a,b;c) = \frac{p(c)}{p(d)} T(a,b,c;d).$$

We thus have:

$$\sum_{a\in S} p(a) T_r(d,a,b;c) = p(c) \frac{\sum_{a\in S} p(a)T(a,b,c;d)}{p(d)} = p(c),$$

using Cond. 3, since A is r^{-1}-quasi-reversible. Thus, Cond. 1 holds for T_r and, by Theorem 6, $A_r \in \mathcal{T}_S(p)$.

$1 \Leftarrow 2$ Since $A_r \in \mathcal{T}_S(p)$, by Theorem 12, A_r is r^2-quasi-reversible. Then, by the property 4. of Prop. 10, A is $r^3 = r^{-1}$-quasi-reversible.

$1 \Leftrightarrow 3$ Same proof as $1 \Leftrightarrow 2$.

$1 \Leftrightarrow 4$ It is a consequence of Prop. 16 and Prop. 17.

$1 \Leftrightarrow 5$ It is a consequence of the points 2. and 4. of Prop 10, together with Theorem 12 (see also Corollary 13) .

□

Remark 21. It follows from the previous results that for $g \in D_4$, if A is $g-$quasi-reversible, then the transition kernel T_g of its g-reverse A_g is given by:

$$T_g(\sigma_g(a,b,c;d)) = \frac{p(\pi_4(\sigma_g(a,b,c;d)))}{p(d)} T(a,b,c;d),$$

where σ_g is the permutations of the four vertices a, b, c, d induced by the transformation $g \in D_4$, and where π_4 is the projection on the fourth letter, so that $\pi_4(a,b,c;d) = d$.

3.2 Reversible PCA with p-HZPM invariant

As a consequence of the previous results, we obtain the following characterization of reversible PCA.

Theorem 22. *Let $A \in \mathcal{T}_S(p)$.*

1. *A is v-reversible iff $T(a,b,c;d) = T(c,b,a;d)$ for any $a,b,c,d \in S$.*
2. *A is r^2-reversible iff $p(b)T(a,b,c;d) = p(d)T(c,d,a;b)$ for any $a,b,c,d \in S$.*
3. *A is h-reversible iff $p(b)T(a,b,c;d) = p(d)T(a,d,c;b)$ for any $a,b,c,d \in S$.*
4. *A is $< r^2, v >$-reversible iff $T(a,b,c;d) = T(c,b,a;d)$ and $p(b)T(a,b,c;d) = p(d)T(a,d,c;b)$ for any $a,b,c,d \in S$.*
5. *A is $< r >$-reversible iff $p(a)T(a,b,c;d) = p(d)T(b,c,d;a)$ for any $a,b,c,d \in S$.*
6. *A is $< r \circ v >$-reversible iff $p(a)T(a,b,c;d) = p(d)T(d,c,b;a)$ for any $a,b,c,d \in S$.*
7. *A is D_4-reversible iff $T(a,b,c;d) = T(c,b,a;d)$ and $p(a)T(a,b,c;d) = p(d)T(b,c,d;a)$ for any $a,b,c,d \in S$.*

Proof. Let $A \in \mathcal{T}_S(p)$.

1. This is an elementary property, true even if $A \notin \mathcal{T}_S(p)$.
2. A is r^2-reversible iff A is r^2-quasi-reversible and its r^2-reverse is A. Now, by Theorem 12, if $A \in \mathcal{T}_S(p)$, then A is r^2-quasi-reversible and the transition kernel T_{r^2} of its r^2-reverse is $T_{r^2}(c,d,a;b) = \frac{p(b)}{p(d)} T(a,b,c;d)$ for any $a,b,c,d \in S$.
3. A is h-reversible iff A is h-quasi-reversible and its h-reverse is A. By Corollary 13, if $A \in \mathcal{T}_S(p)$, then A is h-quasi-reversible and, as mentionned in Remark 21, we can show that $T_h(a,d,c;b) = \frac{p(b)}{p(d)} T(a,b,c;d)$ for any $a,b,c,d \in S$.
4. It is an easy consequence of the previous points.

5. A is r-reversible iff it is r-quasi-reversible and its r-reverse is A. Hence, by Prop. 16, if $A \in \mathcal{T}_S(p)$, then A is reversible iff Cond. 2 is satisfied and $T(b,c,d;a) = \frac{p(a)}{p(d)} T(a,b,c;d)$ for any $a,b,c,d \in S$. It is in fact sufficient to have $T(b,c,d;a) = \frac{p(a)}{p(d)} T(a,b,c;d)$ for any $a,b,c,d \in S$, since we then have:

$$\sum_{a \in S} p(a) T(a,b,c;d) = \sum_{a \in S} p(a) \frac{p(d)}{p(a)} T(b,c,d;a) = p(d),$$

   meaning that Cond. 2 is satisfied

6. A is $r \circ v$-reversible iff it is r-quasi-reversible and its $r \circ v$-reverse is A. Since $A \in \mathcal{T}_S(p)$, by Prop 16, A is r-quasi reversible iff Cond. 2, and we can prove that the transition kernel of its $r \circ v$-reverse is then given by $T_{r \circ v}(d,c,b;a) = \frac{p(a)}{p(d)} T(a,b,c;d)$ for any $a,b,c,d \in S$. As in the above point, it is sufficient to have $T(d,c,b;a) = \frac{p(a)}{p(d)} T(a,b,c;d)$ for any $a,b,c,d \in S$, since it implies Cond. 2.

7. It follows from points 1 and 5.

□

3.3 Independence properties of the space-time diagram

Theorem 23. *Let us consider a PCA $A \in \mathcal{T}_S(p)$ and its stationary space-time diagram $G = (A, \pi_p)$. Then for any $|a| \leq 1$, the points of G indexed by the discrete line $L_{a,b} = \{(x,y) \in \mathbb{Z}^2_{\mathrm{e}} : y = ax + b\}$ consist in i.i.d. random variables.*

Proof. This is a consequence of Prop. 15. We can assume without loss of generality that $b = 0$ and that $0 < a \leq 1$. Let $(x,y) \in \mathbb{Z}^2_{\mathrm{e}}$ be the first point with positive coordinates belonging to the integer line, so that we have in particular $0 < y \leq x$. Let us define the sequence $(t_i)_{\in \mathbb{Z}}$ by $t_{i+kx} = i + ky$ for $i \in \{0, \ldots, y-1\}$ and $t_{i+kx} = y + \frac{(-1)^{i-y}-1}{2} + ky$ for $i \in \{y, \ldots, x-1\}$, and any $k \in \mathbb{Z}$. This sequence satisfies the conditions of Prop. 15, so that $(\eta_{t_i}(i) : i \in \mathbb{Z}) \sim \mathcal{B}(p)^{\otimes \mathbb{Z}}$. Since $L_{a,b} \subset \{(i, t_i) : i \in \mathbb{Z}\}$, the result follows. □

Theorem 24. *Let us consider a PCA $A \in \mathcal{T}_S(p)$ satisfying Cond. 2 or Cond. 3. Then, for any line of its stationary space-time diagram $G = (A, \pi_p)$, nodes on that line are i.i.d.*

Proof. We prove the result for a PCA $A \in \mathcal{T}_S(p)$ satisfying Cond. 2. In that case, A is r-quasi-reversible. Now, take any line L in G.

If the equation of L is $y = ax + b$ with $|a| \leq 1$, then by Theorem 23, nodes on that line are i.i.d. By Prop. 18, the same property holds if the equation of L is $x = c$.

Let us now consider an equation of the form $y = ax + b$ with $|a| > 1$. We can assume without loss of generality that $b = 0$. Let $(x,y) \in \mathbb{Z}^2_{\mathrm{e}}$ be the first point with a positive coordinates belonging to the integer line, so that we have in particular $0 < |x| < y$. Then we can perform flips, similarly as the ones done in Prop. 18 (see Fig. 7), to get that, for any m, the points $m\mathrm{u}, (m-1)\mathrm{u}, \ldots, \mathrm{u}, (0,0), (x,y), (2x,2y), \ldots, (kx,ky)$ (with $k = \lfloor m/(x+y) \rfloor$) are i.i.d. In particular, $(0,0), (x,y), (2x,2y), \ldots, (kx,ky)$ are i.i.d. □

Remark 25. Observe that as a consequence of Theorem 19, the same result holds for a PCA that does not belong to $\mathcal{T}_S(p)$ but satisfies both Cond. 2 and Cond. 3.

PCA with strong independence. Let us recall that we denote $\mathrm{u} = (-1, 1), \mathrm{v} = (1, 1)$.

Definition 26. Let $G = (A, \mu)$ be a stationary space-time diagram of a PCA A under one of its invariant measure μ. We say that G is *top* (resp. *bottom*, *left*, *right*) *i.i.d.* if, for any $x \in \mathbb{Z}^2_{\mathrm{e}}$, $\{\eta(x), \eta(x-\mathrm{u}), \eta(x-\mathrm{v})\}$ (resp.$\{\eta(x), \eta(x+\mathrm{u}), \eta(x+\mathrm{v})\}$, $\{\eta(x), \eta(x-\mathrm{u}), \eta(x+\mathrm{v})\}$ $\{\eta(x), \eta(x+\mathrm{u}), \eta(x-\mathrm{v})\}$) are i.i.d. A PCA is said to be 3-to-3 i.i.d. if it is top, bottom, left and right i.i.d.

Proposition 27. *$G = (A, \mu)$ is both top and bottom i.i.d. if and only if $A \in \mathcal{T}_S$ and μ is its invariant HZPM.*

Proof. Let $G = (A, \mu)$ be a top and bottom i.i.d. PCA. We denote by p the one-dimensional marginal of μ. Then, we have, for any $a, b, c, d \in S$,

$$\begin{aligned}\mathbb{P}(\eta(x) = d, \eta(x-\mathrm{u}) = a, \eta(x-\mathrm{v}) = c) &= p(a)p(d)p(c) \text{ (top i.i.d.)} \\ &= \sum_{b \in S} p(a)p(b)p(c)T(a, b, c; d) \text{ (bottom i.i.d.).}\end{aligned}$$

Hence, Cond. 1 holds and, by Theorem 6, $A \in \mathcal{T}_S(p)$, and $\mu = \pi_p$. The reverse statement is trivial. □

Proposition 28. *$G = (A, \mu)$ is 3-to-3 i.i.d. if and only if A is a D_4-quasi-reversible PCA of $\mathcal{T}_S$ and μ is its invariant HZPM.*

Proof. Let (A, μ) be a 3-to-3 i.i.d. PCA, then $A \in \mathcal{T}_S$ because A is both top and bottom i.i.d. Moreover,

$$p(a)p(b)p(d) = \sum_{c \in S} p(a)p(b)p(c)T(a, b, c; d) \text{ and } p(c)p(b)p(d) = \sum_{a \in S} p(a)p(b)p(c)T(a, b, c; d),$$

using the fact that A is top and left (resp. right) i.i.d. But these are respectively Cond. 2 and Cond. 3 and, so, by Theorem 20, A is D_4-quasi-reversible. The reverse statement is trivial. □

Note that $G = (A, \mu)$ is top, bottom, and left (resp. right) i.i.d. if and only if A is a r-quasi-reversible PCA (resp. r^{-1}-quasi-reversible PCA) of $\mathcal{T}_S$ and μ is its invariant HZPM.

Connection with previous results for PCA with memory one. In the special case when the PCA has memory one, meaning that the probability transitions $T(a, b, c; d)$ do no depend on $b \in S$, Cond. 1 reduces to: $\forall a, c, d \in S$, $p(d) = T(a, \cdot, c; d)$. So, the only PCA having an invariant HZPM are trivial ones (no time dependence at all). In that context, it is in fact more relevant to study PCA having simply an invariant horizontal product measure, as done in [26]. Observe that when there is no dependence on $b \in S$, Cond. 2 et Cond. 3 become:

$$\forall a, d \in S, \sum_{c \in S} p(c)T(a, \cdot, c; d) = p(d) \qquad \text{and} \qquad \forall c, d \in S, \sum_{a \in S} p(a)T(a, \cdot, c; d) = p(d).$$

We recover the two sufficient conditions for having an horizontal product measure, as described in Theorem 5.6 of [26]. In that article, the space-time diagrams are represented on a regular triangular lattice, which is more adapted to the models that are considered. The authors show that under one or the other of these two conditions, there exists a transversal PCA, so that after an appropriate rotation of the triangular lattice, the stationary space-time diagram can also be described as the one of another PCA. With our terminology, this corresponds to a quasi-reversibility property.

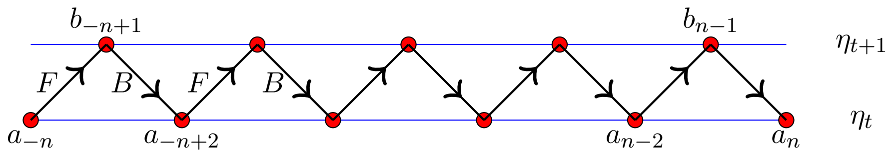


Figure 8: Illustration of Def. 30.

4 Horizontal zigzag Markov chains

4.1 Conditions for having an invariant HZMC

In this section, we recall some previous results obtained in [8] about PCA with memory two having an invariant measure which is a Horizontal Zigzag Markov Chain. Our purpose is to keep the present article as self-contained as possible.

First, let us recall what is a (F, B)-HZMC distribution. This is the same notion as (D, U)-HZMC in [8], but to be consistent with the orientation chosen here for the space-time diagrams, we prefer using the notations F for forward in time, and B for backward in time (rather than D for down and U for up). The definition we give below relies on the following lemma.

Lemma 29. *Let S be a finite set, and let $F = (F(a; b) : a, b \in S)$ and $B = (B(b; c) : b, c \in S)$ be two positive transition matrices from S to S. We denote by ρ_B (resp. ρ_F) the invariant probability distribution of B (resp. F), that is, the normalised left-eigenvector of B (resp. F) associated to the eigenvalue 1. If $FB = BF$, then $\rho_B = \rho_F$.*

Proof. Note that by Perron-Frobenius, B and F have a unique invariant probability distribution, satisfying respectively $\rho_B B = B$ and $\rho_F F = F$. Since $FB = BF$, we have $\rho_B FB = \rho_B BF = \rho_B F$, so that the vector $\rho_B F$ is an invariant probability distribution of B. By uniqueness, we obtain $\rho_B F = \rho_B$. Since the invariant probability distribution of F is also unique, we obtain $\rho_B = \rho_F$. □

Definition 30. Let S be a finite set, and let F and B be two transition matrices from S to S, such that $FB = BF$. We denote by ρ their (common) left-eigenvector associated to the eigenvalue 1. The (F, B)-HZMC (for *Horizontal Zigzag Markov Chain*) on $S^{\mathbb{Z}_t} \times S^{\mathbb{Z}_{t+1}}$ is the distribution $\zeta_{F,B}$ such that, for any $n \in \mathbb{Z}_t$, for any $a_{-n}, a_{-n+2}, \dots, a_n \in S, b_{-n+1}, n_{n+3}, \dots, b_{n-1} \in S$,

$$\mathbb{P}\left((\zeta_{F,B}(i,t) = a_i, \zeta_{F,B}(i,t+1) = b_i : -n \le i \le n)\right) = \rho(a_{-n}) \prod_{i=-n+1}^{n-1} F(a_{i-1}; b_i) B(b_i; a_{i+1}).$$

We give a simple necessary and sufficient condition that depends on both T and (F, B) for a (F, B)-HZMC to be an invariant measure of a PCA with transition kernel T.

Proposition 31 (Lemma 5.10 of [8])**.** *Let S be a finite set. Let A be a PCA with positive rates and let F and B be two transition matrices from S to S. The (F, B)-HZMC distribution is an invariant probability distribution of A iff*

Cond 4: *for any $a, c, d \in S$,*

$$F(a; d)B(d; c) = \sum_{b \in S} B(a; b)F(b; c)T(a, b, c; d).$$

In the context of PCA having an invariant (F, B)-HZMC, Prop. 15 can be extended as follows. The proof being similar, we omit it.

Proposition 32. *Let A be a PCA having a (F,B)-HZMC invariant measure, of stationary space-time diagram $G(A,\zeta_{F,B}) = (\eta_t(i) : t \in \mathbb{Z}, i \in \mathbb{Z}_t)$. For any zigzag polyline $(i,t_i)_{m\leq i\leq n}$, and any $(a_i)_{m\leq i\leq n} \in S^{n+1}$, we have:*

$$\mathbb{P}\left(\eta(i,t_i) = a_i : m \leq i \leq n\right) = \rho(a_0) \prod_{\substack{m\in\{0,\ldots,n-1\}\\ t_{i+1}=t_i+1}} F(a_i;a_{i+1}) \prod_{\substack{m\in\{0,\ldots,n-1\}\\ t_{i+1}=t_i-1}} B(a_i;a_{i+1}).$$

In general, the knowledge of the transition kernel T alone is not sufficient to be able to tell if the PCA A admits or not an invariant (F,B)-HZMC. Until now, the characterization of PCA having an invariant (F,B)-HZMC is known in only two cases: when $|S| = 2$ [8, Theorem 5.3], and when $F = B$ [8, Theorem 5.2]. In the other cases ($F \neq B$ and $|S| > 2$), it is an open problem.

4.2 Quasi-reversibility and reversibility

This section is devoted to PCA having an HZMC invariant measure, and that are (quasi-)reversible.

Proposition 33. *Let A be a PCA having a (F,B)-HZMC invariant distribution. Then, the stationary space-time diagram $(A,\zeta_{F,B})$ is $\{h,r^2,v\}$-quasi-reversible, and we have the following.*

- *The h-reverse is $(A_h,\zeta_{B,F})$ with, for any $a,b,c,d \in S$,*

$$T_h(a,d,c;b) = \frac{B(a;b)F(b;c)}{F(a;d)B(d;c)}T(a,b,c;d).$$

- *The v-reverse is (A_v,ζ_{B_h,F_h}) where, for any $a,b,c,d \in S$,*

$$T_v(c,b,a;d) = T(a,b,c;d),\ \ B_h(b;a) = \frac{\rho(a)}{\rho(b)}B(a;b) \text{ and } F_h(b;a) = \frac{\rho(a)}{\rho(b)}F(a;b).$$

- *The r^2-reverse is $(A_{r^2},\zeta_{F_h,B_h})$ with, for any $a,b,c,d \in S$,*

$$T_{r^2}(c,d,a;b) = \frac{B(a;b)F(b;c)}{F(a;d)B(d;c)}T(a,b,c;d).$$

Proof. The proof of the h-quasi-reversibility is similar to the one of Theorem 12, so we omit it, and the fact that $(A,\zeta_{F,B})$ is v-quasi-reversible is obvious (see Prop. 10). Let us denote by (A_v,μ_v) the v-reverse and let us prove that $\mu_v = \zeta_{B_h,F_h}$. For any $i,j \in \mathbb{Z}$, $x_i,y_i,\ldots,y_{j-1},x_j \in S$,

$$\begin{aligned}\mu_v(x_i,y_i,\ldots,y_{j-1},x_j) &= \zeta_{F,B}(x_j,y_{j-1},\ldots,y_i,x_i)\\ &= \rho(x_j)F(x_j;y_{j-1})B(y_{j-1};x_{j-1})\ldots F(y_i;x_i)\\ &= F_h(y_{j-1};x_j)\rho(y_{j-1})B(y_{j-1};x_{j-1})\ldots F(y_i;x_i)\\ &= \ldots\\ &= \rho(x_i)F_h(x_i;y_i)B_h(y_i,x_{i+1})\ldots B_h(y_{j-1};x_j)\end{aligned}$$

The fact that $(A,\zeta_{F,B})$ is r^2-quasi-reversible is due to the fact that $r^2 = h\circ v$. □

Proposition 34. *Let A be a PCA having a (F,B)-HZMC invariant distribution. $(A,\zeta_{F,B})$ is r-quasi-reversible iff*

Cond 5: *for any $a,c,d \in S$,*

$$F(a;d) = \sum_{c\in S} F(b;c)T(a,b,c;d).$$

In that case, the transition kernel of the reverse A_r is given, for any $a,b,c,d \in S$, by:

$$T_r(d,a,b;c) = \frac{F(b;c)}{F(a;d)}T(a,b,c;d). \tag{9}$$

Proof. The proof follows the same idea as the proof of Prop. 16, and uses Prop. 32, the analog of Prop. 15.

• Suppose that A is r-reversible. Then, for any a, b, c, d, any $x \in \mathbb{Z}^2_{\mathrm{e}}$,

$$\begin{aligned} T_r(d, a, b; c) &= \mathbb{P}\left(\eta(x + \mathrm{v}) = c | \eta(x + \mathrm{u} + \mathrm{v}) = d, \eta(x + \mathrm{u}) = a, \eta(x) = b\right) \\ &= \frac{\rho(a)B(a; b)F(b; c)T(a, b, c; d)}{\sum_{c' \in S} \rho(a)B(a; b)F(b; c')T(a, b, c'; d)} = \frac{F(b; c)T(a, b, c; d)}{\sum_{c' \in S} F(b; c')T(a, b, c'; d)}. \end{aligned} \tag{10}$$

For some $x \in \mathbb{Z}^2_{\mathrm{e}}$, let us reintroduce the pattern $L = (x, x+\mathrm{u}, x+\mathrm{v}, x+2\mathrm{u}, x+\mathrm{u}+\mathrm{v}, x+2\mathrm{u}+\mathrm{v})$, see Fig. 5. For $a_0, b_0, b_1, c_0, c_1, d_0 \in S$, we are interested in the quantity: $Q(a_0, b_0, b_1, c_0, c_1, d_0) = \mathbb{P}\left(\eta(L) = (a_0, b_0, b_1, c_0, c_1, d_0)\right)$.

On the one hand, we have:

$$Q(a_0, b_0, b_1, c_0, c_1, d_0) = \rho(c_0)B(c_0; b_0)B(b_0; a_0)F(a_0; b_1)T(b_0, a_0, b_1; c_1)T(c_0, b_0, c_1; d_0).$$

On the other hand, we have:

$$Q(a_0, b_0, b_1, c_0, c_1, d_0) = \sum_{b'_1, c'_1 \in S} Q(a_0, b_0, b'_1, c_0, c'_1, d_0)T_r(d_0, c_0, b_0; c_1)T_r(c_1, b_0, a_0; b_1).$$

Using the expressions of $T_r(d_0, c_0, b_0; c_1)$ and $T_r(c_1, b_0, a_0; b_1)$ given by (10) and simplifying, we get:

$$\begin{aligned} &\left(\sum_{b \in S} F(a_0; b)T(b_0, a_0, b; c_1)\right)\left(\sum_{c \in S} F(b_0; c)T(c_0, b_0, c; d_0)\right) \\ &= F(b_0; c_1) \sum_{b, c \in S} F(a_0; b)T(b_0, a_0, b; c)T(c_0, b_0, c; d_0). \end{aligned}$$

Now summing on $d_0 \in S$, we find, for any $b_0, c_1 \in S$, $F(b_0; c_1) = \sum_{b \in S} F(a_0; b)T(b_0, a_0, b; c_1)$.

• Conversely, suppose that A is a PCA having an invariant measures (F, B)-HZMC and that Cond. 5 holds. Then, we can perform flips thanks to (9) as in Fig. 6 and 7. □

Proposition 35. *Let A be a PCA whose an invariant probability distribution is a (F, B)-HZMC distribution. $(A, \zeta_{(F,B)})$ is r^{-1}-quasi-reversible iff*

Cond 6: *for any b, c, d,*

$$\frac{p(d)}{p(c)}B(d; c) = \sum_{a \in S} \frac{p(a)}{p(b)}B(a; b)T(a, b, c; d).$$

In that case, the transition kernel of the reverse $A_{r^{-1}}$ is given, for any $a, b, c, d \in S$, by:

$$T_{r^{-1}}(b, c, d; a) = \frac{B_h(b; a)}{B_h(c; d)}T(a, b, c; d). \tag{11}$$

4.3 PCA with an explicit invariant law that is not Markovian

As evocated in Section 3.1, there exist PCA A of $\mathcal{T}_S(p)$ that are r-quasi-reversible, and for which the r-reverse A_r does not belong to $\mathcal{T}_S(p)$. In that case, A_r has an invariant measure $\mu = (\pi_p)_r$ which is not a product measure, and for which we know formula allowing to compute exactly all the marginals, see equations (7) and (8). Let us point out that the measure μ can not be HZMC. Consider indeed the stationary space-time diagram $(A, \pi_p) = (\eta(i, t) : (i, t) \in \mathbb{Z}^2_{\mathrm{e}})$, and assume that μ is a (F, B)-HZMC measure. Then, the marginal of size one of μ is equal to $\rho = p$, and for any $x \in \mathbb{Z}^2_{\mathrm{e}}$, we have: $\mathbb{P}\left(\eta(x + \mathrm{u}) = a, \eta(x) = b\right) = p(a)p(b) = \rho(a)F(a; b)$ and

$\mathbb{P}(\eta(x) = b, \eta(x+\mathrm{v}) = c) = p(b)p(c) = \rho(c)B(c;b)$. Thus, we obtain $F(a;b) = B(c;b) = p(b)$ for any $a, b, c \in S$, meaning that the (F,B)-HZMC is in fact a p-HZMP, which is not possible since A_r does not belong to $\mathcal{T}_S(p)$.

So, the PCA A_r has an invariant measure that we can compute, and that has neither a product form nor a Markovian one. That was a real surprise of this work. We give an explicit example of such a PCA in Section 6, see Example 52.

Similarly, if a PCA A with a (F,B)-invariant HZMC is r-quasi-reversible, and is such that its r-reverse A_r does not have an invariant HZMC, then we can compute exactly the invariant measure of A_r, although it does not have a well-known form. This provides an analogous of Theorem 19, in the Markovian case. Precisely, next theorem gives conditions on the transitions of a PCA C for being of the form $C = A_r$, with A having a (F,B)-invariant HZMC. To the best knowledge of the authors, this is the first time that we can compute from the transition kernel an invariant law that is not Markovian.

Theorem 36. *Let C be a PCA of transition kernel T. For any $a \in S$, let $(F(a;b))_{b\in S}$ be the left eigenvector of $(T(b,a,a;c))_{b,c\in S}$ associated to the eigenvalue 1 and $(B(a;b))_{b\in S}$ be the left-eigenvector of $(T(a,a,b;c))_{b,c\in S}$. The two following conditions*

Cond 7: *for any $a, b, c \in S$, $F(b;c) = \sum_{d\in S} F(a;d)T(d,a,b;c)$;*

Cond 8: *for any $a, c, d \in S$, $B(d;c) = \sum_{b\in S} B(a;b)T(d,a,b;c)$;*

are equivalent to: there exists a probability measure μ on S such that

(i) μ is invariant by C,

(ii) (C,μ) is $\{r, r^{-1}\}$-quasi-reversible and

(iii) the r^{-1}-reverse is $(C_{r^{-1}}, \zeta_{(F,B)})$ with $T_{r^{-1}}(a,b,c;d) = \frac{F(a;d)}{F(b;c)}T(d,a,b;c)$.

(iv) the r-reverse is $(C_r, \zeta_{(B_h,F_h)})$ with $T_r(a,b,c;d) = \frac{B(c;d)}{B(b;a)}T(b,c,d;a)$ and B_h and F_h as defined in Prop 33..

Moreover, we have explicit formula for the computation of the marginals of μ.

Remark 37. In general, μ is not a Markovian law, nevertheless, sometimes it is. In that case, the PCA C is of the form $C = A_r$, with a PCA A that is not only r-quasi-reversible but also r^{-1}-quasi-reversible. Note also that in Theorem 36, we are able to find the expression of the invariant HZMC of $C_{r^{-1}}$ (resp. C_r) from the transition kernel T of C, whereas in all generality, given the values of the transition kernel $T_{r^{-1}}$ (resp. T_r), we are unable to say if the associated PCA has an invariant HZMC.

Proof. Let us assume that there exists a probability measure μ on S satisfying *(i)*, *(ii)*, and *(iii)*. Then, summing the equation of *(iii)* on $d \in S$, we find, for any $a, b, c \in S$,

$$\sum_{d\in S} F(a;d)T(d,a,b;c) = F(b;c)\sum_{d\in S} T_{r^{-1}}(a,b,c;d) = F(b;c).$$

For $a = b$, this equation shows that $(F(a;d))_{d\in S}$ is a left-eigenvector of $(T(d,a,a;c))_{d,c\in S}$ associated to 1.

Moreover, as $\zeta_{(F,B)}$ is invariant by $C_{r^{-1}}$, by Prop. 31, for any $a, c, d \in S$,

$$\begin{aligned} F(a;d)B(d;c) &= \sum_{b\in S} B(a;b)F(b;c)T_{r^{-1}}(a,b,c;d) \\ &= \sum_{b\in S} B(a;b)F(a;d)T(d,a,b;c). \end{aligned}$$

Dividing by $F(a,d)$ on both sides, we get Cond. 8, and for $d=a$ we obtaint that $(B(a;c))_{c\in S}$ is the left-eigenvector of $(T(a,a,b;c))_{b,c\in S}$.

Conversely, let us define

$$\tilde{T}(a,b,c;d) = \frac{F(a;d)}{F(b;c)} T(d,a,b;c).$$

Then, as Cond. 7 and Cond. 8 hold, we can check that $\tilde{T}$ is a transition kernel and satisfies Cond. 4, so $\zeta_{(F,B)}$ is an invariant measure of $\tilde{C}$, PCA whose transition kernel is $\tilde{T}$. Moreover,

$$\sum_{c\in S} F(b;c)\tilde{T}(a,b,c;d) = \sum_{c\in S} F(a;d)T(d,a,b;c) = F(a;d).$$

That is Cond. 5, and we conclude by application of Prop. 34 and by Lemma 9 (uniqueness of the r-reverse). Finally, multidimensional laws of μ are deduced from the space-time diagram (C,μ). Indeed, we know that rotated by $\pi/2$, it has the same distribution as $(C_r, \zeta_{F,B})$. So, we can compute all the finite-dimensional marginals of the space-time diagram, and in particular the multidimensional laws of μ. □

5 Applications to statistical physics

We now develop four examples of PCA with memory two, that are inspired from statistical physics. The first two ones are defined on a finite symbol set, while the third one is defined on the alphabet $\mathbb{Z}$, and the last one is defined on a continuous set of symbols. Formal definitions making rigorous these last two models will be given in Section 7, in a more general context.

5.1 The 8-vertex models

Let us recall the notations $\mathrm{u}=(1,1), \mathrm{v}=(-1,1)$. For some $n\in 2\mathbb{Z}$, we consider the graph G_n whose set of vertices is $V_n = \mathbb{Z}^2_{\mathrm{e}} \cap [-n,n]^2$, the restriction of the even lattice to a finite box, and whose set of edges is $E_n = \{(x,x+\mathrm{u}) : x, x+\mathrm{u} \in V_n\} \cup \{(x,x+\mathrm{v}) : x, x+\mathrm{v} \in V_n\}$. We define the boundary of V_n by $\partial V_n = \{(x_1,x_2)\in V_n : \max(|x_1|,|x_2|) = n\}$.

For each edge of G_n, we choose an orientation. This defines an orientation O of G_n, and we denote by O_n the set of orientations of G_n. For a given orientation $O\in O_n$, and an edge $e\in E_n$, we denote:

$$o(e) = \begin{cases} 0 & \text{if the edge } e \text{ is oriented from top to bottom in } O \ (\searrow \text{ or } \swarrow), \\ 1 & \text{if the edge } e \text{ is oriented from bottom to top in } O \ (\nwarrow \text{ or } \nearrow). \end{cases}$$

Hence, an orientation $O\in O_n$ can be seen as an element $(o(e))_{e\in E_n}$ of $\{0,1\}^{E_n}$.

Around each vertex $x\in V_n\setminus\partial V_n$, there are 4 oriented edges, giving a total of 16 possible local configurations, defining the *type* of the vertex x. In the 8-vertex model case, we consider only the orientations O such that around each vertex $x\in V_n\setminus\partial V_n$, there is an even number (0, 2 or 4) of incoming edges, so that only 8 local configurations remain, see Fig. 9. To each local configuration i among these 8 local configurations, we associate a local weight w_i. This allows to define a global weight W on the set $\tilde{O}_n$ of admissible orientations, by:

$$W(O) = \prod_{x\in V_n\setminus\partial V_n} w_{\mathrm{type}(x)}, \qquad \text{for } O\in\tilde{O}_n. \tag{12}$$

Thanks to these weights, we finally define a probability distribution $\mathbb{P}_W$ on $\tilde{O}_n$, by:

$$\mathbb{P}_W(O) = \frac{W(O)}{\sum_{O\in\tilde{O}_n} W(O)}. \tag{13}$$

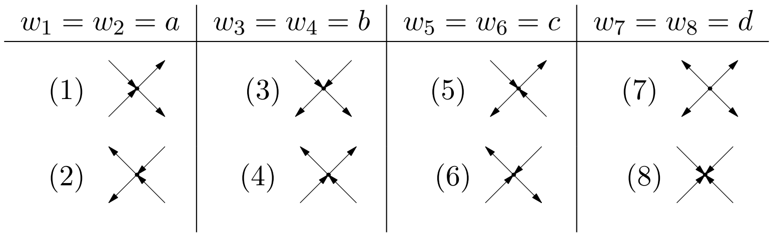


Figure 9: The 8 possible local configurations around any vertex.

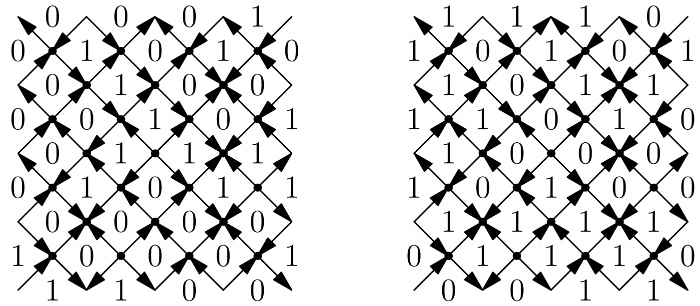


Figure 10: An orientation O and its two possible 2-colorings.

As usual in statistical physics, the partition function $\sum_{O\in\tilde{O}_n} W(O)$ is denoted by Z_n. In the following, we consider the more studied 8-vertex model, for which the parameters satisfy $w_1 = w_2 = a$, $w_3 = w_4 = b$, $w_5 = w_6 = c$ and $w_7 = w_8 = d$. We furthermore assume that $a + c = b + d$.

The 8-vertex model was introduced by Sutherland [30] and Fan and Wu [16] in 1970 as a generalization of the 6-vertex model (for which $d = 0$), which was introduced by Pauling in 1935 to study the ice in two dimension [29]. In [2], Baxter computes the partition function via Bethe's ansatz methods and deduces 5 asymptotic behaviours for the 8-vertex model [3, Section 10.11]. For the interested reader, we recommend [3, Chapter 8], [13] and reference therein for more information on 6-vertex model and [3, Chapter 10] and reference therein for more information on 8-vertex model.

In [3, Section 10.2], Baxter presents a "two-to-one" map $\mathcal{C}_8$ between 2-colorings of faces of G_n and admissible orientations of the 8-vertex model on G_n. Let F_n be the set of faces of G_n, that is, the set of quadruplet $(x, x + \mathrm{u}, x + \mathrm{u} + \mathrm{v}, x + \mathrm{v}) \in (\mathbb{Z}_\mathrm{e}^2)^4$ for which at least 3 of the 4 vertices belong to G_n. The map is the following. Let $C \in \{0, 1\}^{F_n}$ be a 2-coloring of faces of G_n, and take any edge $e \in E_n$. We denote by f_e and f'_e the two adjacent faces of e. Then, we define:

$$o(e) = \begin{cases} 1 & \text{if } C(f_e) = C(f'_e), \\ 0 & \text{otherwise } (i.e. \text{ if } C(f_e) \neq C(f'_e)). \end{cases} \tag{14}$$

It is a "two-to-one" map because from an admissible orientation O, we obtain two 2-colorings in $\mathcal{C}_8^{-1}(O) = \{C_0, C_1\}$. These two colorings have the following properties $C_0(f) = 1 - C_1(f)$ for any $f \in F_n$, see Fig. 10.

Let us set $q = a/(a+c)$ and $r = b/(b + d)$, and consider the PCA A_8 whose transition kernel

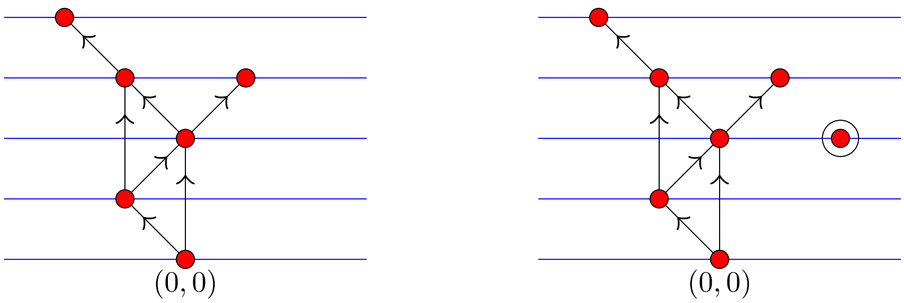


Figure 11: The set on the left is a directed animal on the triangular lattice, while the set on the right is not.

T is defined, by

$$\begin{aligned} T(0,0,1;\cdot) &= T(1,0,0;\cdot) = \mathcal{B}(q), \\ T(0,1,1;\cdot) &= T(1,1,0;\cdot) = \mathcal{B}(1-q), \\ T(0,1,0;\cdot) &= T(1,1,1;\cdot) = \mathcal{B}(r), \\ T(1,0,1;\cdot) &= T(0,0,0;\cdot) = \mathcal{B}(1-r). \end{aligned}$$

This is the PCA presented as an introductory example in Section 1. With this PCA, we define a random 2-coloring of G_n in the following way. First, we color the faces centered on points of ordinate $-n$ and $-n+1$ (first two lines) and the faces centered on points of abscisse $-n$ and n (left and right boundary conditions), independently, with common law $\mathcal{B}(1/2)$. Then, we color the other faces by applying successively the PCA A_8, from bottom to top. We denote by $\mathcal{F}_n$ the law of the random 2-coloring of G_n obtained.

Proposition 38 ([8])**.** *If $C \sim \mathcal{F}_n$, then $\mathcal{C}_8(C) \sim \mathbb{P}_W$.*

This proposition is the first application of PCA with memory two in the literature. One can check that the PCA A_8 satisfies Cond. 1 with $p(0) = p(1) = 1/2$. The proof of Theorem 8 implies that this PCA is ergodic. When $n \to \infty$, the center of the square has the same behaviour whatever are the boundary conditions [8, Proposition 1.6].

Note that in what precedes, we have assumed that the weights satisfy the relation $a+c = b+d$. If we now assume that they rather satisfy $a + d = b + c$, we can design a PCA that, when iterated from left to right (or equivalently, from right to left), generates configurations distributed according to the required distrbution $\mathbb{P}_W$. When $a = b$ and $c = d$, so that both relations are satisfied, we obtain $q = r$, and the dynamics is D_4-reversible.

5.2 Directed animals and gaz models

A directed animal on the square lattice (resp. on the triangular lattice) is a set $A \subset \mathbb{Z}_e^2$ such that $(0,0) \in A$ and, for any $z \in A$ there exists a directed path $w = ((0,0) = x_0, x_1, \ldots, x_{m-1}, x_m = z)$ such that, for any $1 \le k \le m$,

$$x_k - x_{k-1} \in \{u, v\} \text{ (resp. } \{u, v, u+v\}).$$

Let us denote by $\mathcal{A}_S$ (resp. $\mathcal{A}_T$) the set of directed animals on the square (resp. triangular) lattice.

The *area* of an animal A is the cardinal of A and the *perimeter* of an animal A is the cardinal of $P(A) = \{x : x \notin A, \{x\} \cup A \text{ is a directed animal}\}$. Let us introduce the generating functions of directed animals enumerated according to their area, on the square lattice and on the triangular lattice:

$$G_S(z) = \sum_{A \in \mathcal{A}_S} z^{|A|} \qquad G_T(z) = \sum_{A \in \mathcal{A}_T} z^{|A|}. \tag{15}$$

The computation of G_S was done by Dhar in 1982 via the study of hard-particles model [11]. Here, we will present this work using PCA, see also [25] for details. Let B_S be the binary state PCA with memory one whose transition kernel T_S is given, for any $a, b \in \{0,1\}$, by

$$T_S(a,b;1) = \begin{cases} p_S & \text{if } a = b = 0, \\ 0 & \text{else.} \end{cases}$$

Theorem 39 ([11, 6, 23])**.** *For any p_S, let $(\eta(i,t) : (i,t) \in \mathbb{Z}^2_{\mathrm{e}})$ be the space-time diagram of B_S under its invariant probability measure, then*

$$\mathbb{P}\left(\eta(0,0) = 1\right) = -G_S(-p_S) \tag{16}$$

Note that the uniqueness of the invariant measure of B_S, for any choice of $p_S \in (0,1)$, was proven in [21]. Theorem 39 was generalized by [23] for directed animals on any "admissible" graph. In the case of directed animal on the square lattice, the invariant measure of B_S has a simple Markovian form (see [11, 6, 23, 10]), so that we can recover the following result.

Theorem 40 ([11])**.** *The area generating function of directed animals on the square lattice is*

$$G_S(z) = \frac{1}{2}\left(\left(1 - \frac{4z}{1+z}\right)^{-1/2} - 1\right) \tag{17}$$

In [6], the enumeration of directed animals on the square and triangular lattices was done according to others statistics.

Theorem 41 ([11, 6])**.** *The area generating function of directed animals on the triangular lattice is*

$$G_T(z) = \frac{1}{2}\left((1-4z)^{-1/2} - 1\right) \tag{18}$$

Observe that the following property holds.

Lemma 42.

$$G_T\left(\frac{z}{1+z}\right) = G_S(z). \tag{19}$$

Here, we will give a proof of this lemma using only results of [23] that generalize Theorem 39, and [8, Theorem 5.3] on PCA.

Proof. Let B_T be the binary state PCA with memory two of transition kernel T_T given, for any $a, b, c \in \{0,1\}$ by

$$T_T(a,b,c;1) = \begin{cases} p_T & \text{if } a = b = c = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then by Theorem 2.7 of [23] applied to the triangular lattice, we get that: if $(\eta(i,t) : (i,t) \in \mathbb{Z}^2_{\mathrm{e}})$ is the space-time diagram of B_T taken under its invariant measure, then

$$G_T(-p_T) = \mathbb{P}\left(\eta(i,t) = 1\right). \tag{20}$$

Now, let us prepare to apply [8, Theorem 5.3] to B_T. For any $a, c \in \{0,1\}$, the left eigenvector of $(T_T(a,b,c;d))_{b,d \in \{0,1\}}$ is

$$T(a,c;1) = \begin{cases} \dfrac{p_T}{1+p_T} & \text{if } a = c = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the associated PCA with memory one is B_S with $p_S = \dfrac{p_T}{1+p_T}$. As B_S satisfy conditions of [8, Theorem 5.3], we obtain that the invariant measure of B_T and of B_S with $p_S = \dfrac{p_T}{1+p_T}$ is the same. So, by Theorem 39 and 20,

$$-G_T(-p_T) = -G_S\left(-\frac{p_T}{1+p_T}\right). \tag{21}$$

Taking $x = -p_T$, we obtain $G_T(x) = G_S\left(\dfrac{x}{1-x}\right)$, which is equivalent to (19) when $x = \dfrac{z}{1+z}$.

An attentive reader would have seen that we have used [8, Theorem 5.3] for a PCA with non-positive rates. This is possible under some conditions on T_T and for this transition it works well. Nevertheless, the necessary and sufficient condition are not known in general. We refer the interested reader to [6, Section 4.4] and [10, Section 2.2] for some sufficient conditions and remarks about PCA with non-positive rates. □

Some words about the enumeration of directed animal by area and perimeter. Let

$$\tilde{G}_S(x,y) = \sum_{A\in\mathcal{A}_S} x^{|A|}y^{|P(A)|} \quad \text{and} \quad \tilde{G}_T(x,y) = \sum_{A\in\mathcal{A}_T} x^{|A|}y^{|P(A)|} \tag{22}$$

be the generating function of directed animal enumerated according to their area and perimeter on, respectively, square and triangular lattice. Let us introduce two PCA $\tilde{B}_S$ and $\tilde{B}_T$ of alphabet $S = \{0,1\}$. The PCA $\tilde{B}_S$ has memory one and transition kernel $\tilde{T}_S$, and the PCA $\tilde{B}_T$ has memory two and transition kernel $\tilde{T}_T$, with:

$$\tilde{T}_S(a,b;1) = \begin{cases} p+q & \text{if } a=b=1, \\ p & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{T}_T(a,b,c;1) = \begin{cases} p+q & \text{if } a=b=c=1, \\ p & \text{otherwise.} \end{cases}$$

For p sufficiently close to 0, these PCA can be proven to be ergodic, and we have the following result.

Theorem 43 ([23, Theorem 4.3])**.** *For p sufficiently close to 0, let $\tilde{\eta}_S$ (resp. $\tilde{\eta}_T$) be the space-time diagram of $\tilde{B}_S$ (resp. $\tilde{B}_T$) taken under its invariant measure. Then*

$$\mathbb{P}(\tilde{\eta}_S(0,0)=1) = q + \tilde{G}_S(p,q) \qquad (\textit{resp. } \mathbb{P}(\tilde{\eta}_T(0,0)=1) = q + \tilde{G}_T(p,q)).$$

Unfortunately, we have no explicit description of the invariant measures of these PCA.

5.3 Synchronous TASEP of order two

The TASEP (Totally ASymmetric Exclusion Process) describes the evolution of some particles that go from the left to the right on a line without overtaking. There are various kinds of models of TASEP models, with discrete or continuous time and space, and one or more types of particles. We refer the interested readers to the following articles [4, 19] for the description of some models with discrete time and space. Here, we present a new (to the best knowledge of the authors) generalization of TASEP called TASEP of order two on real line and discrete time.

The TASEP presented here models the behaviour of an infinite number of particles (indexed by $\mathbb{Z}$) on the real line, that move to the right, that do not bypass and that do not overlap. For $i,t \in \mathbb{Z}$, we denote by $x_i(t) \in \mathbb{R}$ the position of particle i at time t. Time is discrete, and at time t, each particle $i \in \mathbb{Z}$ moves with a random speed $v_i(t)$, independently of the others. The random speed $v_i(t)$ depends on the distance $x_{i+1}(t) - x_i(t)$ between the particle i and the particle $i+1$ in front of it, and of the speed $v_{i+1}(t-1) = x_{i+1}(t) - x_{i+1}(t-1)$ of the particle $i+1$ at time $t-1$. Formally, the evolution of $(x_i(t))_{i\in\mathbb{Z}}$ is defined by:

$$\forall i \in \mathbb{Z}, \quad x_i(t+1) = x_i(t) + v_i(t), \tag{23}$$

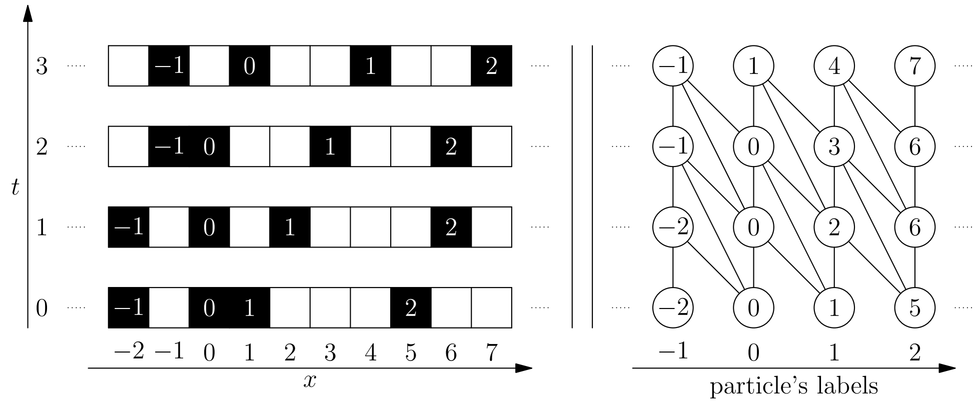


Figure 12: On the left, the classical representation of a TASEP: a white square is an empty square; a black square is a square that contains a particle, the white number is the label of this particle. On the right, the PCA that represents this TASEP; each column represent the trajectory of a particle.

where $v_i(t)$ is random and distributed following $\mu_{(x_{i+1}(t)-x_i(t),v_{i+1}(t-1))}$, a probability distribution on $\mathbb{R}^+$, and $(v_i(t))_{i\in\mathbb{Z}}$ are independent, knowing $(x_i(t))_{i\in\mathbb{Z}}$ and $(x_i(t-1))_{i\in\mathbb{Z}}$.

It is known that TASEP with discrete time can be represented by PCA [25, 9]. We adopt the sight presented in [9] to show that the TASEP of order two can be represented by a PCA with memory two: take $\eta(i,t) = x_i(t)$, then $(\eta(i,t) : i \in \mathbb{Z}, t \in \mathbb{N})$ is the space-time diagram of a PCA with memory two whose transition kernel T is, for any $a \in \mathbb{R}, x, y, v \in \mathbb{R}^+$,

$$T(a, a+x, a+x+y; a+v) = \mu_{(x+y,y)}(v). \tag{24}$$

Now, we will focus as an example on the simplest case where $v \in \{0,1\}$ a.s. and particles move on the integer line (for any $i \in \mathbb{Z}, t \in \mathbb{N}$, $x_i(t) \in \mathbb{Z}$). The constraints we have on T are the following:

- $T(a, a+1, a+1; a) = 1$ for any $a \in \mathbb{Z}$ (and $T(a, b, b+i; c)$ does not matter if $i \neq 0, 1$ or $b \leq a$),
- $T(a, a+k, a+k+i; c) = 0$ for any a, $k \geq 0$, $i \in \{0,1\}$ and $c \notin \{a, a+1\}$,
- $T(a, a+k, a+k+i; a) = T(b, b+k, b+k+i; b)$ for any $k \geq 1$, $i \in \{0,1\}$, $a, b \in \mathbb{Z}$.

The first two points signify that the next position has to be empty for a particle to move, and that a particle can only move of one unit forward. The last point is an hypothesis of translation invariance. Hence, the PCA can be described by the transitions $(T(0,k,k;0))_{k\geq 2}$ and $(T(0,k,k+1;0))_{k\geq 1}$.

The first result of this section is about the fact that there exists a family of (F,B)-HZMC that is stable by this PCA. First, let us define a q,p-HZMC for any probability q and p on $\mathbb{Z}$: a q,p-HZMC is a (F,B)-HZMC such that, for any $a, k \in \mathbb{Z}$, $F(a; a+k) = q(k)$ and $B(a; a+k) = p(k)$.

Lemma 44. *For any transition kernel T, if their exist p a probability on $\mathbb{N}^*$ and q on $\{0,1\}$ such that, for any $k \geq 1$,*

$$p(k)q(1)T(0,k,k+1;0) + p(k+1)q(0)T(0,k+1,k+1;0) = p(k+1)q(0) \tag{25}$$

then if we start under the law such that (η_0, η_1) is a q,p-HZMC with $\eta_0(0) = 0$ a.s., then any double line (η_t, η_{t+1}) is also distributed as a p,q-HZMC but the starting point is now $\eta_t(0)$ with

$$\mathbb{P}(\eta_t(0) = k) = \binom{t}{k} q(1)^k q(0)^{t-k}.$$

Note that (25) also implies

$$p(k)q(1)T(0,k,k+1;1)+p(k+1)q(0)T(0,k+1,k+1;1)=p(k)q(1), \tag{26}$$

both equations being equivalent to:

$$p(k)q(1)T(0,k,k+1;0)=p(k+1)q(0)T(0,k+1,k+1;1). \tag{27}$$

These two conditions (25) and (26) are similar to Cond. 4 of Prop. 31.

Proof. The proof is done by induction on $t\in\mathbb{N}$. For $t=0$, we assume that (η_0,η_1) is a q,p-HZMC with $\eta_0(0)=0$ a.s. Now, let us suppose that (η_t,η_{t+1}) is a q,p-HZMC with $\mathbb{P}(\eta_t(0)=k)=\binom{t}{k}q(1)^kq(0)^{t-k}$. Then, by conditionning by the possible values $(a_i)_{0\le i\le k+1}$ for $(\eta_t(i))_{0\le i\le k+1}$, we obtain that the finite dimensional laws of (η_{t+1},η_{t+2}) are given by:

$$\begin{aligned}
&\mathbb{P}\left((\eta_{t+1}(i)=b_i)_{0\le i\le k},(\eta_{t+2}(i)=c_i)_{0\le i\le k-1}\right)\\
&=\sum_{a_0,\dots,a_{k+1}}\binom{t}{a_0}q(1)^{a_0}q(0)^{t-a_0}q(b_0-a_0)\prod_{i=0}^{k-1}p(a_{i+1}-b_i)q(b_{i+1}-a_{i+1})T(b_i,a_{i+1},b_{i+1};c_i)\\
&=\left(\sum_{a_0}\binom{t}{a_0}q(1)^{a_0}q(0)^{t-a_0}q(b_0-a_0)\right)\left(\prod_{i=0}^{k-1}\sum_{a_{i+1}}p(a_{i+1}-b_i)q(b_{i+1}-a_{i+1})T(b_i,a_{i+1},b_{i+1};c_i)\right).
\end{aligned}$$

Since the only non-zero terms correspond to $a_i\in\{b_i,b_i-1\}$, the left parenthesis is equal to:

$$\sum_{a_0\in\{b_0,b_0-1\}}\binom{t}{a_0}q(1)^{a_0}q(0)^{t-a_0}q(b_0-a_0)=\binom{t+1}{b_0}q(1)^{b_0}q(0)^{t+1-b_0},$$

and the right one to:

$$\begin{aligned}
&\prod_{i=0}^{k-1}\sum_{a_{i+1}\in\{b_{i+1},b_{i+1}-1\}}p(a_{i+1}-b_i)q(b_{i+1}-a_{i+1})T(b_i,a_{i+1},b_{i+1};c_i)\\
&=\prod_{i=0}^{k-1}\sum_{a_{i+1}\in\{b_{i+1},b_{i+1}-1\}}p(a_{i+1}-b_i)q(b_{i+1}-a_{i+1})T(0,a_{i+1}-b_i,b_{i+1}-b_i;c_i-b_i)\\
&=\prod_{i=0}^{k-1}p(b_{i+1}-c_i)q(c_i-b_i),\quad\text{using (25) and (26).}
\end{aligned}$$

□

We can remark that q is the speed law of a particle under the stationary regime and p the distance law between two successive particles (to be precise the left one at current time t and the right one at previous time $t-1$).

Theorem 45. *For any T, for any distribution q on $\{0,1\}$ such that*

$$Z=\sum_{k=0}^{\infty}\left(\frac{q(1)}{q(0)}\right)^k\prod_{m=1}^{k}\frac{T(0,m,m+1;0)}{T(0,m+1,m+1;1)}<\infty, \tag{28}$$

there exists a unique distribution p on $\mathbb{N}^*$ such that (25) hold.

Moreover, this distribution p is, for any $k\ge 1$,

$$p(k)=\frac{\left(\frac{q(1)}{q(0)}\right)^{k-1}\prod_{m=1}^{k-1}\frac{T(0,m,m+1;0)}{T(0,m+1,m+1;1)}}{Z}. \tag{29}$$

Proof. Let q be a probability measure on $\{0,1\}$. By (27), we have:

$$\forall k \geq 1, \quad p(k+1) = p(k) \frac{T(0,k,k+1;0)}{T(0,k+1,k+1;1)} \frac{q(1)}{q(0)}. \tag{30}$$

By induction, we obtain:

$$\forall k \geq 1, \quad p(k+1) = \left(\frac{q(1)}{q(0)}\right)^k \prod_{m=1}^{k} \frac{T(0,m,m+1;0)}{T(0,m+1,m+1;1)} p(1) \tag{31}$$

As $\sum_{k \in \mathbb{N}^*} p(k) = 1$, we need (28). In that case, (29) follows. □

In the classical case of synchronous TASEP (presented in [25, Sections 2.3 & 4.3], [9, Section 3.3]), we have

$$T(0,k,k+1;1) = T(0,k+1,k+1;1) = p. \tag{32}$$

With Theorem 45, we recover the invariant measures of the classical synchronous TASEP.

This example is interesting because it does not enter in our previous framework for many reasons. First, it is easy to see that studying an invariant measure for this PCA is not interesting because it corresponds to the overloaded state where nobody move ($q(0) = 1$). That's why we focused here on a family of distributions that is stable by the PCA and not only on one distribution.

Moreover, the PCA has not positive rates for any μ, because we cannot get any configuration starting from any configuration. Nevertheless, studying carefully their eigenvectors on the good subspace, we solve the algebraic issues to find interesting results.

In addition, we find some results about a (F,B)-HZMC family (with $F \neq B$) and a PCA with an infinite alphabet, whereas our main result on PCA is about PCA with invariant (F,B)-HZMC but alphabet of size 2 or PCA with a general alphabet but with invariant (F,F)-HZMC (see end of Section 4.1).

5.4 Eden model on the triangular lattice

The Eden model is an aggregation model that was defined by Murray Eden in 1961 [15]. It describes a growth model on $\mathbb{Z}^2$ which growths by perimeter starting from a point. Here, we develop an Eden model on the triangular lattice as the one of [1] on the square lattice.

Let μ be a probability measure on $[0,\infty)$. The graph G we consider is the one with set of nodes $\{(x,y) \in \mathbb{Z}_{\mathrm{e}}^2 : y \geq 0\}$ and set E of edges that are the ones supported by vectors u, v and u + v. To each edge e of G, we associate a positive random variable $\omega(e)$ with distribution μ. For every directed path γ in G, we denote $\lambda(\gamma) = \sum_{e \in \gamma} \omega(e)$ the passage time of γ. The passage time between two connected nodes x and y is

$$d(x;y) = \inf_{\gamma \in \Gamma(x;y)} \lambda(\gamma)$$

where $\Gamma(x;y)$ is the set of directed paths starting from x and finishing in y. Finally, we define the passage time $\eta(x)$ on a node $x = (x_1, x_2) \in \mathbb{Z}_{\mathrm{e}}^2$ as

$$\eta(x) = \begin{cases} 0 & \text{if } x_2 \in \{0,1\}, \\ \displaystyle\min_{y \in \mathbb{Z}_0 \times \{0\} \cup \mathbb{Z}_1 \times \{1\}} d(y;x) & \text{otherwise.} \end{cases}$$

This is a stationary version of the first-passage percolation on a directed triangular lattice with μ as the law of the time to travel an edge. In the special case, where μ is distributed as an exponential random variable we obtain a stationary version of the Eden model on a directed triangular lattice.

This model can be seen as a PCA A of order two with alphabet $E = [0, \infty)$ where the state of a node $x \in \mathbb{Z}^2_{\mathrm{e}}$ is $\eta(x)$. In that case, for any $a, b, c \in E$, the law $T(a, b, c; .)$ is the one of $\min(a + \omega_1, b + \omega_2, c + \omega_3)$, where ω_1, ω_2 and ω_3 are i.i.d. of common law μ. Unfortunately, the theorems presented in this article do not apply to this PCA.

6 The binary case

In this section, we specify the conditions obtained in Section 3 to the case of a binary symbol set: $S = \{0, 1\}$.

Proposition 46. *For a binary symbol set $S = \{0, 1\}$,*

1. *Cond. 1 is equivalent to*

   Cond 9: $\forall a, c \in S, \quad p(0)T(a, 0, c; 1) = p(1)T(a, 1, c; 0)$.

2. *Cond. 2 is equivalent to*

   Cond 10: $\forall a, b \in S, \quad p(0)T(a, b, 0; 1) = p(1)T(a, b, 1; 0)$.

3. *Cond. 3 is equivalent to*

   Cond 11: $\forall b, c \in S, \quad p(0)T(0, b, c; 1) = p(1)T(1, b, c; 0)$.

Proof. In the binary case, Cond. 1 reduces to: $\forall a, c \in S,\ p(1) = p(0)T(a, 0, c; 1) + p(1)T(a, 1, c; 1)$, which is itself equivalent to Cond. 9. The proof is analogous for Cond. 10 and Cond. 11. □

Let p be a probability measure on $S = \{0, 1\}$. If we specify some results of Table 1 to the case $|S| = 2$, we obtain:

$$\dim(\mathcal{T}_S(p)) = \dim(\{A \in \mathcal{T}_S(p) : A \text{ is } h\text{-reversible}\}) = 4,$$
$$\dim(\{A \in \mathcal{T}_S(p) : A \text{ is } v\text{-reversible}\}) = \dim\big(\{A \in \mathcal{T}_S(p) : A \text{ is } r^2\text{-reversible}\}\big) = 3,$$
$$\dim(\{A \in \mathcal{T}_S(p) : A \text{ is } D_4\text{-quasi-reversible}\}) = \dim(\{A \in \mathcal{T}_S(p) : A \text{ is } D_4\text{-reversible}\}) = 1.$$

In this section, we will describe more precisely these different sets, which will give an alternative proof of the value of their dimension, in the binary case. First, next result shows that in the binary case, the sets above having the same dimension are equal.

Proposition 47. *Let p be any positive probability on $S = \{0, 1\}$, and let $A \in \mathcal{T}_S(p)$. Then, we have the following properties.*

1. *A is h-reversible.*
2. *A is v-reversible iff A is r^2-reversible.*
3. *A is D_4-quasi-reversible iff A is D_4-reversible.*

Proof.

1. Since A is in $\mathcal{T}_S(p)$, A is h-quasi-reversible, and the transition kernel T_h of its h-reverse satisfies, for any $a, b, c, d \in S$,

   $$T_h(a, d, c; b) = \frac{p(b)}{p(d)} T(a, b, c; d).$$

   For $b = d$, this gives $T_h(a, b, c; d) = T(a, b, c; d)$, and for $b \neq d$, Cond. 9 provides the result.

2. It is a corollary of 1. Indeed, if A is in $\mathcal{T}_S(p)$ and v-reversible, then it is h and v-reversible, and so also $r^2 = v \circ h$-reversible. And conversely, if it is r^2-reversible, then it is $v = r^2 \circ h$-reversible.

3. This will be a consequence of Theorem 50.

□

As a consequence of Prop. 46, we obtain the following descriptions of binary PCA having an invariant HZPM.

Theorem 48. *Let A be a PCA with transition kernel T (with positive rates). Then A has an invariant HZPM iff*

Cond 12: *there exists $k \in]0,\infty[$ such that for any $a, c \in S$,*

$$\frac{T(a,1,c;0)}{T(a,0,c;1)} = k.$$

More explicitly, this is equivalent to the following condition.

Cond 13: *there exists $k \in]0,\infty[$ and $\begin{cases} q_{0,0}, q_{0,1}, q_{1,0}, q_{1,1} \in (0,1) & \text{if } k \in (0,1] \\ q_{0,0}, q_{0,1}, q_{1,0}, q_{1,1} \in (1-k^{-1},1) & \text{if } k \in [1,\infty) \end{cases}$ such that, for any $a, c \in S$,*

$$\begin{aligned} T(a,0,c;0) &= q_{a,c}, \\ T(a,1,c;0) &= k(1-q_{a,c}) = k - kq_{a,c}. \end{aligned}$$

In that case, the p-HZPM invariant is $(p(0), p(1))$ where $p(1) = 1 - p(0) = \dfrac{1}{1+k}$.

Proof. The PCA A has an invariant HZPM iff there exists a probability p on S such that Cond. 9 is satisfied, which can easily be shown to be equivalent to the above conditions. □

Proposition 49. *Let p be a positive probability on S, and let $k = p(0)/p(1)$.*

Then, the PCA A is a r-quasi-reversible of $\mathcal{T}_S(p)$ iff

Cond 14: *there exists $\begin{cases} q_0, q_1 \in (0,1) & \text{if } k \in (0,1] \\ q_0, q_1 \in (1-k^{-1}, 1-k^{-1}+k^{-2}) & \text{if } k \in [1,\infty) \end{cases}$ such that, for any $a \in S$,*

$$\begin{aligned} T(a,0,0;0) &= q_a, \\ T(a,0,1;0) = T(a,1,0;0) &= k(1-q_c) = k - kq_a \\ T(a,1,1;0) &= k(1-k(1-q_c)) = k - k^2 + k^2 q_a \end{aligned}$$

Similarly, the PCA A is a r^{-1}-quasi-reversible of $\mathcal{T}_S(p)$ iff

Cond 15: *there exists $\begin{cases} q_0, q_1 \in (0,1) & \text{if } k \in (0,1] \\ q_0, q_1 \in (1-k^{-1}, 1-k^{-1}+k^{-2}) & \text{if } k \in [1,\infty) \end{cases}$ such that, for any $c \in S$,*

$$\begin{aligned} T(0,0,c;0) &= q_c, \\ T(0,1,c;0) = T(1,0,c;0) &= k(1-q_c) = k - kq_c \\ T(1,1,c;0) &= k(1-k(1-q_c)) = k - k^2 + k^2 q_c \end{aligned}$$

Proof. We prove the first statement. Let A be a r-quasi-reversible PCA in $\mathcal{T}_S(p)$. Then T satisfies Cond. 9 and 10, meaning that for any $a, b, c \in S$,

$$\frac{T(a,1,c;0)}{T(a,0,c;1)} = \frac{T(a,b,1;0)}{T(a,b,0;1)} = k = \frac{p(0)}{p(1)}.$$

Taking $b = c = 0$, we find, for any $a \in S$,

$$T(a,1,0;0) = T(a,0,1;0) = kT(a,0,0;1)$$

and taking $b = c = 1$, we find that for any $a \in S$,

$$T(a, 0, 1; 1) = T(a, 1, 0; 1) = k^{-1}T(a, 1, 1; 0).$$

Hence, for any $a \in S$, we get

$$T(a, 1, 1; 0) = k(1 - T(a, 0, 1; 0)) = k(1 - k(1 - T(a, 0, 0; 0))).$$

Then, every $T(a, b, c; d)$ can be express in terms of $T(0, 0, 0; 0) = q_0$, $T(0, 0, 1; 0) = q_1$ and k, which gives Cond. 14, and the range of q_0, q_1 is deduced from the fact that, for any $a, b, c, d \in S$, $T(a, b, c; d) \in (0, 1)$.

Conversely, let A be such that Cond. 14 holds. Then Cond. 9 and 10 hold, so $A \in \mathcal{T}_S(p)$ and A is r-quasi-reversible. □

Theorem 50. *Let p be a positive probability on S, and let $k = p(0)/p(1)$.*
Then, the PCA A is a $\{r, r^{-1}\}$-quasi-reversible of $\mathcal{T}_S(p)$ iff

Cond 16: *there exists* $\begin{cases} q_0 \in (0,1) & \text{if } k \in (0,1] \\ q_0 \in (1-k^{-1}+k^{-2}-k^{-3}, 1-k^{-1}+k^{-2}) & \text{if } k \in [1,\infty) \end{cases}$ *such that*

$$\begin{aligned}
&T(0,0,0;0) = q_0,\\
&T(0,0,1;0) = T(0,1,0;0) = T(1,0,0;0) = k(1-q_0) = k - kq_0\\
&T(0,1,1;0) = T(1,1,0;0) = T(1,0,1;0) = k(1-k(1-q_0)) = k - k^2 + k^2q_0\\
&T(1,1,1;0) = k(1-k(1-k(1-q_0))) = k - k^2 + k^3 - k^3q_0.
\end{aligned}$$

Moreover, in that case, A is D_4-reversible.

Proof. The PCA A is a $\{r, r^{-1}\}$-quasi-reversible PCA of $\mathcal{T}_S(p)$ iff Cond. 9, 10 and 11 are satisfied, which can easily be shown to be equivalent to the above condition.

Now, we prove the D_4-reversibility of A. First, A is symmetric, so A is v-reversible. Second, the r-reverse of A is the PCA A_r with transition kernel T_r given by (1) on p.11. In particular, (1) provides: $T_r(0,0,0;0) = T(0,0,0;0)$ and $T_r(0,0,1;0) = T(0,1,0;0) = T(0,0,1;0)$. Furthermore, A_r is r^{-1}-quasi-reversible, and by Theorem 20, we have: $A_r \in \mathcal{T}_S(p)$. So, A_r must satisfy Cond. 15, which allows to express all the transitions of T_r from the values of $T_r(0,0,0;0)$ and $T_r(0,0,1;0)$. These values being the same as for A, which is also r^{-1}-quasi-reversible, it follows that $T_r = T$. Since A is r and v-reversible, by (5) of Prop. 10, A is D_4-reversible. □

Example 51. Let us consider the special case when p is the uniform distribution on S, meaning that $p(0) = p(1) = 1/2$. Then, $k = 1$, and the family of PCA above corresponds to:

$$\forall a,b,c,d \in S, \quad T(a,b,c;d) = \begin{cases} q_0 & \text{if } d = a+b+c \mod 2 \\ 1-q_0 & \text{otherwise.} \end{cases}$$

In the deterministic case ($q_0 = 1$), we get a linear CA. Such CA have been intensively studied. Here, in the probabilstic setting, the PCA we obtain can be seen as noisy versions of that linear CA (with a probability $1 - q_0$ of doing an error, independently for different cells). This is a special case of the 8-vertex PCA, with $p = r$.

Example 52. Let us consider the probability distribution on S given by $p(0) = 1/3$ and $p(1) = 2/3$, so that $k = 2$. When specifying Cond. 14 to $q_0 = 3/4$ and $q_1 = 4/5$, we obtain:

$$\begin{aligned}
&T(0,0,0;0) = 3/4, && T(1,0,0;0) = 4/5,\\
&T(0,0,1;0) = T(0,1,0;0) = 1/8, && T(1,0,1;0) = T(1,1,0;0) = 1/10,\\
&T(0,1,1;0) = 7/16, && T(1,1,1;0) = 9/20.
\end{aligned}$$

The PCA A of transition kernel T is r-quasi-reversible, but one can check that it does not satisfy Cond. 11, so that it is not r^{-1}-quasi-reversible. So, T_r does not belong to $\mathcal{T}_S(p)$, and following the argument developed in Section 4.3, T_r does not have an invariant HZMC either. Nevertheless, one can compute exactly the marginals of its invariant measure μ, see (7) and (8). The transtions of T_r are the following one:

$$\begin{array}{ll} T_r(0,0,0;0) = 3/4, & T_r(1,0,0;0) = 1/8, \\ T_r(0,0,1;0) = 1/8, & T_r(1,0,1;0) = 7/16, \\ T_r(0,1,0;0) = 4/5, & T_r(1,1,0;0) = 1/10, \\ T_r(0,1,1;0) = 1/10, & T_r(1,1,1;0) = 9/20. \end{array}$$

7 Extension to general alphabet

We now present some extensions of our methods and results to general set of symbols. First of all, we extend the definition of PCA to any Polish space S, as it has been done in [9] for PCA with memory one. The transition kernel T of a PCA with memory two must now satisfies:

- for any Borel set $D \in \mathcal{B}(S)$, the map $\begin{array}{llll} T_D : & S^3 & \longrightarrow & \mathbb{R} \\ & (a,b,c) & \longmapsto & T(a,b,c;D) \end{array}$ is $\mathcal{B}(S^3)$-mesurable;
- for any $a,b,c \in S$, the function $\begin{array}{llll} T_{a,b,c} : & \mathcal{B}(S) & \longrightarrow & \mathbb{R} \\ & D & \longmapsto & T(a,b,c;D) \end{array}$ is a probability measure on S.

For any σ-finite measure μ on S, the transition kernel T is said to be μ-positive if, for μ^3-almost every $(a,b,c) \in S^3$, $T(a,b,c;.)$ is absolutely continuous according to μ and μ is absolutely continuous according to $T(a,b,c;.)$. In that case, thanks to Radon-Nikodym theorem, we can define the density of T according to μ, that is a μ^4-measurable positive function where, for μ^3-almost every $(a,b,c) \in S^3$,

$$t(a,b,c;d) = \frac{\mathrm{d}T(a,b,c;.)}{\mathrm{d}\mu}(d) \tag{33}$$

where $\dfrac{\mathrm{d}T(a,b,c;.)}{\mathrm{d}\mu}$ is the Radon-Nikodym derivative of $T(a,b,c;.)$ according to μ.

Theorem 53. *Let μ be any σ-finite measure on a Polish space S. Let A be a PCA with a μ-positive transition kernel T on S. Then, A has an invariant μ-positive HZPM iff*

Cond 17: *there exist μ-measurable positive function p on S such that, for μ^3-almost every $(a,c,d) \in S^3$,*

$$p(d) = \int_E p(b)t(a,b,c;d)\mathrm{d}\mu(b) \tag{34}$$

and

$$\mu(p) = \int_E p(b)\mathrm{d}\mu(b) < \infty, \tag{35}$$

where t is the μ-density of T.

Then, the P-HZPM is invariant by A where $p(.)/\mu(p)$ is the μ-density of P.

Proof. The proof follows the same idea that the one of Theorem 6, except that we are now on a Polish space S. Let A be a μ-positive triangular PCA with alphabet S.

• Suppose that A has an invariant μ-positive P-HZPM and that (η_t, η_{t+1}) follows a P-HZPM distribution. Then, for any $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \in \mathcal{B}(S)$,

$$\begin{aligned}
&\mathbb{P}\left(\eta_t(i-1) \in \tilde{A}, \eta_{t+1}(i) \in \tilde{D}, \eta_t(i+1) \in \tilde{C}\right) \\
&\quad = \int_{\tilde{A}\times\tilde{C}\times\tilde{D}} p(a)p(c)p(d)\mathrm{d}\mu^3(a,c,d) && \text{on the one hand,} \\
&\quad = \int_{\tilde{A}\times\tilde{C}\times\tilde{D}} \left(\int_S p(a)p(b)p(c)t(a,b,c;d)\mathrm{d}\mu(b)\right) \mathrm{d}\mu^3(a,c,d) && \text{on the other hand.}
\end{aligned}$$

Hence, for μ-almost $a, c, d \in S$,

$$p(a)p(c) \int_S p(b)t(a,b,c;d)\mathrm{d}\mu(b) = p(a)p(d)p(c)$$

and so, as $p(a), p(c) > 0$ for μ-almost $a, c \in S$, Cond. 17 holds.

• Conversely, assume that Cond. 17 is satisfied, and that (η_{t-1}, η_t) follows a μ-positive P-HZPM distribution. For some given choice of $n \in \mathbb{Z}_t$, let us denote: $X_i = \eta_{t-1}(n+1+2i), Y_i = \eta_t(n+2i), Z_i = \eta_{t+1}(n+1+2i)$, for $i \in \mathbb{Z}$, see Fig. 3 on p. 8 for an illustration. Then, for any $k \geq 1$, for any μ-measurable Borel sets $B_0, B_1, \ldots, B_k, C_0, \ldots, C_{k-1}$,

$$\begin{aligned}
&\mathbb{P}\left((Y_i)_{0\leq i\leq k} \in B_0 \times \cdots \times B_k, (Z_i)_{0\leq i\leq k-1} \in C_0 \times \cdots \times C_{k-1}\right) \\
&= \int_{C_0\times\cdots\times C_{k-1}} \int_{B_0\times\cdots\times B_k} \left(\prod_{i=0}^{k-1} \int_S t(y_i, x_i, y_{i+1}; z_i)p(x_i)\mathrm{d}\mu(x_i)\right) \\
&\qquad\qquad p(y_0)\ldots p(y_k)\mathrm{d}\mu(y_0,\ldots,y_k)\mathrm{d}\mu(z_0,\ldots,z_{k-1}) \\
&= \int_{C_0\times\cdots\times C_{k-1}} \int_{B_0\times\cdots\times B_k} \left(\prod_{i=0}^{k-1} p(z_i)\right) p(y_0)\ldots p(y_k)\mathrm{d}\mu(y_0,\ldots,y_k)\mathrm{d}\mu(z_0,\ldots,z_{k-1})
\end{aligned}$$

thus, the P-HZPM distribution is invariant by A. □

Now, the problem is reduced to find eigenfunction associated to the eigenvalue 1 of some integral operator. If this problem is solved by Gauss elimination in the case of a finite space, this is more complicated in the general case. Indeed, such a function does not always exist, but, when it is the case, the solution is unique (up to a multiplicative constant), see the following lemma.

Lemma 54 (Durrett [14, Theorem 6.8.7])**.** *Let $\mathcal{A}$ be an integral operator of kernel m:*

$$\mathcal{A} : f \to \left(\mathcal{A}(f) : y \to \int_S f(x)m(x;y)\mathrm{d}\mu(x)\right).$$

If m is the μ-density of a μ-positive t. k. M from S to S, then $\mathcal{A}$ possesses at most one positive eigenfunction in $L^1(\mu)$ (up to a multiplicative constant).

Moreover, the previous results concerning the characterization of reversible and quasi-reversible PCA extend for PCA with general alphabet. The difference is that we are considering μ-positive PCA and that Cond. 2 and 3 become respectively

Cond 18: for μ^3-almost every $(a, b, d) \in S^3$, $\int_S p(c)t(a,b,c;d)\mathrm{d}\mu(c) = p(d)$

and

Cond 19: for μ^3-almost every $(b, c, d) \in S^3$, $\int_S p(a)t(a,b,c;d)\mathrm{d}\mu(a) = p(d)$.

Following the same idea as in [9], many results on PCA with invariant (F, B)-HZMC can also be generalized to PCA on general alphabets.

8 Dimension of the manifolds

In this section, we give the dimensions of $\mathcal{T}_S(p)$ and of its subsets of (quasi)-reversible PCA (see Table 1). But first, we need some results about dimensions of sets of matrices.

8.1 Preliminaries: dimensions of sets of matrices with a given eigenvector

Let S be a finite set and u, v be two probabilities on S. We denote $\mathcal{M}_S(u,v)$ the set of positive matrices $M = (m_{ij})_{i,j\in S}$ such that M is a stochastic matrix and $uM = v$, i.e.

$$\begin{aligned}\mathcal{M}_S(u,v) = \quad & \{M = (m_{ij})_{i,j\in S} : \text{ for any } i,j \in S,\ 0 < m_{ij} < 1; \\ & \text{for any } i \in S,\ \sum_{j\in S} m_{ij} = 1; \\ & \text{for any } j \in S,\ \sum_{i\in S} u(i) m_{ij} = v(j)\}.\end{aligned}$$

A particular case is when $u = v = p$, in that case, p is a left-eigenvector of M associated to the eigenvalue 1 and the set is denoted $\mathcal{M}_S(p)$. Moreover, we will need to know the dimension of the subset $\mathcal{M}_S^{\mathrm{sym}}(p)$ of $\mathcal{M}_S(p)$ defined by

$$\mathcal{M}_S^{\mathrm{sym}}(p) = \{M \in \mathcal{M}_S(p) : \forall i,j \in S,\ p(i)m_{ij} = p(j)m_{ji}\}. \tag{36}$$

Our first lemma is about the dimension of $\mathcal{M}_S(u,v)$.

Lemma 55. *Let S be a finite set of size n. Then,*

$$\dim \mathcal{M}_S(u,v) = (n-1)^2. \tag{37}$$

Proof. First, we prove $\dim \mathcal{M}_S(u,v) \le (n-1)^2$. $\mathcal{M}_S(u,v)$ is defined by the $2n$ linear equations $\forall i \in S$, $\sum_{j\in S} m_{ij} = 1$ and $\forall j \in S$, $\sum_{i\in S} u(i)m_{ij} = v(j)$. This gives $2n-1$ independent linear equations on the n^2 variables $(m_{ij})_{i,j\in S}$. So $\dim \mathcal{M}_S(p) \le n^2 - (2n-1) = (n-1)^2$.

We do not have the equality yet because we have the additional condition: $\forall i,j \in S$, $m_{ij} > 0$. Hence, we have to ensure that $\mathcal{M}_S(u,v)$ is not empty, and that we are not in any other degenerate for which the dimension would be strictly smaller than $(n-1)^2$. For that, we first exhibit a solution of the system such that $m_{ij} > 0$ and then find a neighbourhood around this solution having the dimension we want.

First, the matrix $M = (v(j))_{i,j\in S}$ is in $M_S(u,v)$. Now, let $s \in S$ be a distinguished element of S. Let us set: $S^\star = S\backslash\{s\}$. One can check that there exists a neighbourhood V_0 of 0 in $\mathbb{R}^{(S^\star)^2}$ such that for any $(\epsilon_{ij} : i,j \in S^\star) \in V_0$, the matrix $M_\epsilon = (m_{ij})_{i,j\in S}$ defined by:

$$\begin{aligned} m_{ij} &= v(j) + \epsilon_{ij} \text{ for any } i,j \in S^\star, \\ m_{is} &= v(s) - \sum_{j'\in S^\star} \epsilon_{ij'}, \\ m_{sj} &= v(j) - \frac{\sum_{i'\in S^\star} u(i')\epsilon_{i'j}}{u(s)}, \\ m_{ss} &= v(s) + \frac{\sum_{i'\in S^\star}\sum_{j'\in S^\star} u(i')\epsilon_{i'j'}}{u(s)}, \end{aligned}$$

is positive, stochastic, and satisfies $uM = v$. So $\dim \mathcal{M}_S(p) \ge \dim(S^\star)^2 = (n-1)^2$. □

In the particular case when $u = v = p$, we get

Corollary 56. *Let S be a finite set of size n. Then,*

$$\dim \mathcal{M}_S(p) = (n-1)^2. \tag{38}$$

Now we give two properties about families of matrices in $\mathcal{M}_S(p)$.

Lemma 57. *Let S be a finite set and p be a probability on S. Let $(M_k = (m_{k,ij})_{i,j\in S} : k \in S)$ be a collection of positive matrices indexed by S such that, for any $i,j \in S$,*

$$\sum_{k\in S} p(k) m_{k,ij} = p(j). \tag{39}$$

Let $s \in S$ and define $S^\star = S\backslash\{s\}$. If, for any $k \in S^\star$, $M_k \in \mathcal{M}_S(p)$, then $M_s \in \mathcal{M}_S(p)$.

Proof. By (39), coefficients of the matrix M_s according to the ones of the other matrices is, for any $i,j \in S$,

$$m_{s,ij} = \frac{p(j) - \sum_{k\in S^\star} p(k) m_{k,ij}}{p(s)}.$$

First, let us prove that M_s is stochastic: for any i,

$$\sum_{j\in S} m_{s,ij} = \frac{1 - \sum_{k\in S^\star} p(k)}{p(s)} = \frac{1-(1-p(s))}{p(s)} = 1;$$

then, that p is a left-eigenvector of M_s: for any j,

$$\begin{aligned}\sum_{i\in S} p(i) m_{s,ij} &= \frac{p(j) - \sum_{k\in S^\star} p(k) \sum_{i\in S} p(i) m_{k,ij}}{p(s)} \\ &= \frac{p(j) - \sum_{k\in S^\star} p(k)p(j)}{p(s)} \\ &= p(j)\frac{1-(1-p(s))}{p(s)} = p(j).\end{aligned}$$

□

Lemma 58. *Let S be a finite set and let p be a probability on S. Let $M = (m_{ij})_{i,j\in S}$ be a matrix in $\mathcal{M}_S(p)$. Then $\tilde{M} = \left(\frac{p(j)}{p(i)} m_{ji}\right)_{i,j\in S} \in \mathcal{M}_S(p)$.*

Proof. First, let us prove that $\tilde{M}$ is stochastic: for any $i \in S$, $\sum_{j\in S} \tilde{m}_{ij} = \sum_{j\in S} \frac{p(j)}{p(i)} m_{ji} = 1$; then, that p is a left-eigenvector of $\tilde{M}$: for any $j \in S$, $\sum_{i\in S} p(i)\tilde{m}_{ij} = \sum_{i\in S} p(j) m_{ji} = p(j)$. □

Finally, we get the dimension of $\mathcal{M}_S^{\mathrm{sym}}(p)$.

Lemma 59. *Let S be a finite set of size n and p be a probability on S,*

$$\dim \mathcal{M}_S^{sym}(p) = \frac{(n-1)n}{2}.$$

Proof. First, $\dim \mathcal{M}_S^{\mathrm{sym}}(p) \leq \frac{(n-1)n}{2}$ because we know by proof of Lemma 55 that we can describe a matrix in the manifold $\mathcal{M}_S(p)$ by knowing $(m_{ij} : i,j \in S^*)$. But, with the new contrain $p(i)m_{ij} = p(j)m_{ji}$ for any $i,j \in S$, it is sufficient to know only $(m_{ij} : i \leq j,\ i,j \in S^*)$.

Conversely, let us take $(m_{ij} : i \leq j,\ i,j \in S^*)$ in a neighbourhood V of $(m_{ij} = p(j) : i \leq j,\ i,j \in S^*)$ in $\mathbb{R}^{n(n-1)/2}$. Let us take:

- for any $i,j \in S^*$, $i > j$, $m_{ij} = \dfrac{p(j)}{p(i)} m_{ji}$;

- for any $i \in S^*$, $m_{is} = 1 - \sum_{j \in S^*} m_{ij}$;
- for any $j \in S^*$, $m_{sj} = \dfrac{p(j) - \sum_{i \in S^*} p(i) m_{ij}}{p(s)}$;
- $m_{ss} = 1 - \sum_{j \in S} m_{sj}$.

By the same argument as in the proof of Lemma 55, there exists a neighboorhood V of dimension $\frac{(n-1)n}{2}$ such that for any point on it, $M = (m_{ij})_{ij \in S} \in \mathcal{M}_S^{\text{sym}}(p)$. And, so,

$$\dim \mathcal{M}_S^{\text{sym}}(p) \geq \frac{(n-1)n}{2}.$$

□

These preliminary results will be useful to prove dimensions of sets of (quasi-)reversible PCA.

8.2 Dimensions of $\mathcal{T}_S(p)$ and its subsets

Theorem 60. *Let S be a finite set of size n and p be a positive probability on S.*

1. $\dim(\mathcal{T}_S(p)) = n^2(n-1)^2$.
2. $\dim(\{A \in \mathcal{T}_S(p) : A \text{ is } r\text{-quasi-reversible}\}) = n(n-1)^3$,
3. $\dim\left(\{A \in \mathcal{T}_S(p) : A \text{ is } r^{-1}\text{-quasi-reversible}\}\right) = n(n-1)^3$,
4. $\dim(\{A \in \mathcal{T}_S(p) : A \text{ is } D_4\text{-quasi-reversible}\}) = (n-1)^4$.
5. $\dim(\{A \in \mathcal{T}_S(p) : A \text{ is } v\text{-reversible}\}) = \dfrac{(n-1)^2 n(n+1)}{2}$.
6. $\dim\left(\{A \in \mathcal{T}_S(p) : A \text{ is } r^2\text{-reversible}\}\right) = \dfrac{(n-1)^2 n(n+1)}{2}$.
7. $\dim(\{A \in \mathcal{T}_S(p) : A \text{ is } h\text{-reversible}\}) = \dfrac{n^3(n-1)}{2}$.
8. $\dim\left(\{A \in \mathcal{T}_S(p) : A \text{ is } < r^2, v >\text{-reversible}\}\right) = \dfrac{(n-1)n^2(n+1)}{4}$.
9. $\dim(\{A \in \mathcal{T}_S(p) : A \text{ is } < r >\text{-reversible}\}) = \dfrac{n(n-1)(n^2-3n+4)}{4}$.[a]
10. $\dim(\{A \in \mathcal{T}_S(p) : A \text{ is } < r \circ v >\text{-reversible}\}) = \dfrac{(n-1)^2(n^2-2n+2)}{2}$.[b]
11. $\dim(\{A \in \mathcal{T}_S(p) : A \text{ is } D_4\text{-reversible}\}) = \dfrac{n(n-1)(n^2-n+2)}{8}$.[c]

Proof. Let $s \in S$, $S^\star = S \backslash \{s\}$ and $|S| = n$.

1. By Theorem 6, a PCA A is in $\mathcal{T}_S(p)$ if for all $a, c \in S$, $(T(a,b,c;d))_{b,d \in S} \in \mathcal{M}_S(p)$. It follows, by Corollary 56, that: $\dim \mathcal{T}_S(p) = |S|^2 \dim \mathcal{M}_S(p) = n^2(n-1)^2$.

[a] OEIS A006528
[b] OEIS A037270
[c] OEIS A002817

2. By Theorem 6, as $A \in \mathcal{T}_S(p)$, for any $a, c \in S$, $(T(a,b,c;d))_{b,d\in S} \in \mathcal{M}_S(p)$. Moreover, A is r-reversible so, by Prop. 16,

$$\sum_{c\in S} p(c)T(a,b,c;d) = p(d).$$

By Lemma 57, for any $a \in S$, we can choose freely $(T(a,b,c;d))_{b,d\in S} : c \in S^\star) \in \mathcal{M}_S(p)$ and $(T(a,b,s;d))_{b,d\in S}$ is then uniquely obtained from them and in $\mathcal{M}_S(p)$. That's why

$$\dim\left(\{A \in \mathcal{T}_S(p) : A \text{ is } r\text{-quasi-reversible}\}\right) = |S||S^*| \dim \mathcal{M}_S(p) = n(n-1)^3.$$

3. The proof is similar to the previous one.

4. As before, for any $a, c \in S$, $(T(a,b,c;d))_{b,d\in S} \in \mathcal{M}_S(p)$. By Theorem 20, we need in addition that, for any $b, d \in S$,

$$\sum_{a\in S} p(a)T(a,b,c;d) = p(d) \text{ for any } c \in S, \text{ and } \sum_{c\in S} p(c)T(a,b,c;d) = p(d) \text{ for any } a \in S.$$

Hence, we can choose freely a collection of $|S^*|^2$ matrices $((T(a,b,c;d))_{b,d\in S} \in \mathcal{M}_S(p) : a, c \in S^\star)$. Then, by Lemma 57, matrices $((T(s,b,c;d))_{b,d\in S} : c \in S^\star)$ and $(T(a,b,s;d)_{b,d\in S} : a \in S^\star)$ are uniquely defined and in $\mathcal{M}_S(p)$. Finally, the last matrix $(T(s,b,s;d))_{b,d\in S}$ can be obtained from two various methods but define the same matrix at the end (the proof is similar to the one of Lemma 55). Hence,

$$\dim\left(\{A \in \mathcal{T}_S(p) : A \text{ is } D_4\text{-quasi-reversible}\}\right) = |S^*|^2 \dim \mathcal{M}_S(p) = (n-1)^4.$$

5. If $A \in \mathcal{T}_S(p)$ is v-reversible, then $(T(a,b,c;d))_{b,d\in S} \in \mathcal{M}_S(p)$ and $T(a,b,c;d) = T(c,b,a;d)$. So, matrices $\{(T(a,b,a;d))_{b,d\in S} : a \in S\}$ can be chosen freely in $\mathcal{M}_S(p)$, but as $T(a,b,c;d) = T(c,b,a;d)$ when $a \neq c$, hence only $\{(T(a,b,c;d))_{b,d\in S} : a < c\}$ can be choosen freely in $\mathcal{M}_S(p)$, $\{(T(a,b,c;d))_{b,d\in S} : a > c\}$ are imposed by $\{(T(c,b,a;d))_{b,d\in S} : c < a\}$. Hence

$$\begin{aligned}\dim\left(\{A \in \mathcal{T}_S(p) : A \text{ is } v\text{-reversible}\}\right) &= \left(|S| + \binom{|S|}{2}\right) \dim \mathcal{M}_S(p)\\ &= \left(n + \frac{n(n-1)}{2}\right)(n-1)^2\\ &= \frac{(n-1)^2 n(n+1)}{2}.\end{aligned}$$

6. If $A \in \mathcal{T}_S(p)$ is r^2-reversible, then $(T(a,b,c;d))_{b,d\in S} \in \mathcal{M}_S(p)$ and $T(c,d,a;b) = \frac{p(b)}{p(d)} T(a,b,c;d)$. Hence, if we take a matrix $(T(a,b,c;d))_{b,d\in S} \in \mathcal{M}_S(p)$ with $a < c$, then $(T(c,b,a;d))_{b,d\in S}$ is known and $\in \mathcal{M}_S(p)$ by Lemma 58. So, we can just choose freely matrices $(T(a,b,c;d))_{b,d\in S} \in \mathcal{M}_S(p)$ with $a \leq c$. That is why the dimension is the same as for v-reversible matrices.

7. If $A \in \mathcal{T}_S(p)$ is h-reversible, then $(T(a,b,c;d))_{b,d\in S} \in \mathcal{M}_S(p)$ and $T(a,d,c;b) = \frac{p(b)}{p(d)} T(a,b,c;d)$. Then, for any $a, c \in S$, $(T(a,b,c;d))_{b,d\in S} \in \mathcal{M}_S^{\text{sym}}(p)$ and, moreover, they can be choosen freely. So, by Lemma 59,

$$\dim\left(\{A \in \mathcal{T}_S(p) : A \text{ is } h\text{-reversible}\}\right) = |S|^2 \dim \mathcal{M}_S^{\text{sym}}(p) = \frac{(n-1)n^3}{2}.$$

8. If $A \in \mathcal{T}_S(p)$ is $< r^2, v >$-reversible, then $(T(a,b,c;d))_{b,d\in S} \in \mathcal{M}_S(p)$, $T(a,b,c;d) = T(c,b,a;d)$ and $T(a,d,c;b) = \frac{p(b)}{p(d)} T(a,b,c;d)$. Then, it is equivalent to choose freely $\{(T(a,b,c;d))_{b,d\in S} : a \leq c\}$ in $\mathcal{M}_S^{\text{sym}}(p)$. That's why,

$$\begin{aligned}\dim\left(\{A \in \mathcal{T}_S(p) : A \text{ is } < r^2, v > \text{-reversible}\}\right) &= \left(|S| + \binom{|S|}{2}\right) \dim \mathcal{M}_S^{\text{sym}}(p) \\ &= \frac{n(n+1)}{2} \frac{(n-1)n}{2} \\ &= \frac{(n-1)n^2(n+1)}{4}.\end{aligned}$$

9,10,11. Proofs are long and relatively similar. They are done in Section 8.3.

□

Corollary 61. *Let S be a finite set of size n*

$$\dim\left(\cup_p \mathcal{T}_S(p)\right) = (n^3 - n^2 + 1)(n-1).$$

Proof. We just add to the previous result the dimension of the set of positive probability measures on S that is $n-1$. □

Remark 62. If one prefer to know the dimension of the set of D-(quasi-)reversible PCA, for any $D \subset D_4$, it is sufficient to add $n-1$ to the result of Theorem 60 corresponding to this set to find the dimension as we have done in Corollary 61, .

A word about the dimension of the set of PCA having a (F, B)-HZMC invariant distribution. Let us denote $\mathcal{T}_S((F,B))$ this set.

Proposition 63. *Let S be a finite set of size n. For any (F, B) such that $FB = BF$,*

$$\dim \mathcal{T}_S((F,B)) = n^2(n-1)^2. \tag{40}$$

Proof. For any (F, B), A is in $\mathcal{T}_S((F,B))$ iff the two following conditions hold (see Prop. 31)

1. for any $a,b,c \in S$, $\sum_{d\in S} T(a,b,c;d) = 1$,
2. for any $a,c,d \in S$,

$$\frac{F(a;d)B(d;c)}{(FB)(a;c)} = \sum_{b\in S} \frac{B(a;b)F(b;c)}{(FB)(a;c)} T(a,b,c;d).$$

Now, by Lemma 55, for any u, v,

$$\dim \mathcal{M}_S(u,v) = (n-1)^2.$$

To conclude, we just have to say that, for any a, c, we can take freely $(T(a,b,c;d))_{b,d\in S} \in \mathcal{M}_S(u,v)$ with $u = \left(\frac{B(a;b)F(b;c)}{(FB)(a;c)}\right)_{b\in S}$ and $v = \left(\frac{F(a;d)B(d;c)}{(FB)(a;c)}\right)_{d\in S}$. □

But getting the dimension of $\cup_{\{(F,B):FB=BF\}} \mathcal{T}_S((F,B))$ is complicated due to the fact that the set $\{(F,B) : FB = BF\}$ is not really well known yet even by algebraists, see [28, 17, 20] for references on this subject.

8.3 Annex: proofs of points 9, 10 and 11 of Theorem 60

In this annex, let S be any finite set, s be any point on S and let p be any probability measure on S. We denote $S^* = S\backslash\{s\}$ and $n = |S|$.

The proofs of the last three points of Theorem 60 are long because they consist in reducing an affine system with $|S|^4$ equations and $|S|^4$ variables (containing some redundant equations) into one with only free equations describing the same manifold. Furthermore, we must ensure that there exists a solution with positive coefficients and that we are not in a degenerate case (see the discussion in the middle of the proof of Lemma 55). Since the proofs of the three points are similar, but not exactly the same, we first define some conditions that are useful for the three cases, then we detail the proof of point 9 and finally, we focus on the differences for the two other cases in comparison with the point 9.

8.3.1 Preliminary results

This section is technical and must be seen as a reference for the sections that are following, so it can be omitted in a first lecture.

First, we define some conditions on the transition kernel T.

Cond 20: For any $a, b, c, d \in S^*$, we have

$$T(a,b,c;s) = 1 - \sum_{d\in S^*} T(a,b,c;d); \tag{41}$$

$$T(s,b,c;d) = \frac{p(d)}{p(s)} - \sum_{a\in S^*} \frac{p(a)}{p(s)} T(a,b,c;d); \tag{42}$$

$$T(a,s,c;d) = \frac{p(d)}{p(s)} - \sum_{b\in S^*} \frac{p(b)}{p(s)} T(a,b,c;d); \tag{43}$$

$$T(a,b,s;d) = \frac{p(d)}{p(s)} - \sum_{c\in S^*} \frac{p(c)}{p(s)} T(a,b,c;d); \tag{44}$$

$$T(s,b,c;s) = p(s)\left(1 - \left(\frac{1-p(s)}{p(s)}\right)^2\right) + \sum_{a\in S^*}\sum_{d\in S^*} \frac{p(a)}{p(s)} T(a,b,c;d); \tag{45}$$

$$T(a,s,c;s) = p(s)\left(1 - \left(\frac{1-p(s)}{p(s)}\right)^2\right) + \sum_{b\in S^*}\sum_{d\in S^*} \frac{p(b)}{p(s)} T(a,b,c;d); \tag{46}$$

$$T(a,b,s;s) = p(s)\left(1 - \left(\frac{1-p(s)}{p(s)}\right)^2\right) + \sum_{c\in S^*}\sum_{d\in S^*} \frac{p(c)}{p(s)} T(a,b,c;d); \tag{47}$$

$$T(s,s,c;d) = p(d)\left(1 - \left(\frac{1-p(s)}{p(s)}\right)^2\right) + \sum_{a\in S^*}\sum_{b\in S^*} \frac{p(a)p(b)}{p(s)^2} T(a,b,c;d); \tag{48}$$

$$T(s,b,s;d) = p(d)\left(1 - \left(\frac{1-p(s)}{p(s)}\right)^2\right) + \sum_{a\in S^*}\sum_{c\in S^*} \frac{p(a)p(c)}{p(s)^2} T(a,b,c;d); \tag{49}$$

$$T(a,s,s;d) = p(d)\left(1 - \left(\frac{1-p(s)}{p(s)}\right)^2\right) + \sum_{b\in S^*}\sum_{c\in S^*} \frac{p(b)p(c)}{p(s)^2} T(a,b,c;d); \tag{50}$$

$$T(s,s,c;s) = p(s)\left(1 + \left(\frac{1-p(s)}{p(s)}\right)^3\right) - \sum_{a\in S^*}\sum_{b\in S^*}\sum_{d\in S^*} \frac{p(a)p(b)}{p(s)^2} T(a,b,c;d); \tag{51}$$

$$T(s,b,s;s) = p(s)\left(1+\left(\frac{1-p(s)}{p(s)}\right)^3\right) - \sum_{a\in S^*}\sum_{c\in S^*}\sum_{d\in S^*}\frac{p(a)p(c)}{p(s)^2}T(a,b,c;d); \tag{52}$$

$$T(a,s,s;s) = p(s)\left(1+\left(\frac{1-p(s)}{p(s)}\right)^3\right) - \sum_{b\in S^*}\sum_{c\in S^*}\sum_{d\in S^*}\frac{p(b)p(c)}{p(s)^2}T(a,b,c;d); \tag{53}$$

$$T(s,s,s;d) = p(d)\left(1+\left(\frac{1-p(s)}{p(s)}\right)^3\right) - \sum_{a\in S^*}\sum_{b\in S^*}\sum_{c\in S^*}\frac{p(a)p(b)p(c)}{p(s)^3}T(a,b,c;d); \tag{54}$$

$$T(s,s,s;s) = p(s)\left(1-\left(\frac{1-p(s)}{p(s)}\right)^4\right) + \sum_{a\in S^*}\sum_{b\in S^*}\sum_{c\in S^*}\sum_{d\in S^*}\frac{p(a)p(b)p(c)}{p(s)^3}T(a,b,c;d). \tag{55}$$

Cond 21: For any $a, b, c \in S$, $\sum_{d\in S} T(a,b,c;d) = 1$.
Cond 22: For any $a, c, d \in S$, $\sum_{b\in S} p(b)T(a,b,c;d) = p(d)$.
Cond 23: For any $a, b, c, d \in S$, $0 < T(a,b,c;d) < 1$.

Lemma 64. *Cond. 20 ⇒ Cond. 21*

Proof.

- For any $a, b, c \in S^*$,

$$\begin{aligned}\sum_{d\in S} T(a,b,c;d) &= T(a,b,c;s) + \sum_{d\in S^*} T(a,b,c;d)\\ &= 1 - \sum_{d\in S^*} T(a,b,c;d) + \sum_{d\in S^*} T(a,b,c;d)\\ &= 1.\end{aligned}$$

- For any $a, b \in S^*$,

$$\begin{aligned}\sum_{d\in S} T(a,b,s;d) &= T(a,b,s;s) + \sum_{d\in S^*} T(a,b,s;d)\\ &= p(s)\left(1-\left(\frac{1-p(s)}{p(s)}\right)^2\right) + \sum_{c\in S^*}\sum_{d\in S^*}\frac{p(c)}{p(s)}T(a,b,c;d)\\ &\quad + \sum_{d\in S^*}\left(\frac{p(d)}{p(s)} - \sum_{c\in S^*}\frac{p(c)}{p(s)}T(a,b,c;d)\right)\\ &= p(s)\left(1-\left(\frac{1-p(s)}{p(s)}\right)^2\right) + \frac{1-p(s)}{p(s)}\\ &= p(s) - (1-p(s))\frac{1-p(s)}{p(s)} + \frac{1-p(s)}{p(s)}\\ &= p(s) + (1-p(s)) = 1.\end{aligned}$$

- Similarly, for any $a, b, c \in S^*$,

$$\sum_{d\in S} T(a,s,c;d) = 1 \text{ and } \sum_{d\in S} T(s,b,c;d) = 1.$$

- For any $a \in S^*$,

$$\begin{aligned}
\sum_{d\in S} T(a,s,s;d) &= T(a,s,s;s) + \sum_{d\in S^*} T(a,s,s;d) \\
&= p(s)\left(1+\left(\frac{1-p(s)}{p(s)}\right)^3\right) - \sum_{b\in S^*}\sum_{c\in S^*}\sum_{d\in S^*}\frac{p(b)p(c)}{p(s)^2}T(a,b,c;d) \\
&\quad + \sum_{d\in S^*}\left(p(d)\left(1-\left(\frac{1-p(s)}{p(s)}\right)^2\right) + \sum_{b\in S^*}\sum_{c\in S^*}\frac{p(b)p(c)}{p(s)^2}T(a,b,c;d)\right) \\
&= p(s)\left(1+\left(\frac{1-p(s)}{p(s)}\right)^3\right) + (1-p(s))\left(1+\left(\frac{1-p(s)}{p(s)}\right)^2\right) \\
&= p(s) + \frac{(1-p(s))^3}{p(s)^2} + 1 - p(s) - \frac{(1-p(s))^3}{p(s)^2} \\
&= 1.
\end{aligned}$$

- Similarly, for any $b, c \in S^*$,

$$\sum_{d\in S} T(s,b,s;d) = 1 \text{ and } \sum_{d\in S} T(s,s,c;d) = 1.$$

- Finally,

$$\begin{aligned}
\sum_{d\in S} T(s,s,s;d) &= T(s,s,s;s) + \sum_{d\in S^*} T(s,s,s;d) \\
&= p(s)\left(1-\left(\frac{1-p(s)}{p(s)}\right)^4\right) + \sum_{a\in S^*}\sum_{b\in S^*}\sum_{c\in S^*}\sum_{d\in S^*}\frac{p(a)p(b)p(c)}{p(s)^3}T(a,b,c;d) \\
&\quad + \sum_{d\in S^*}\left(p(d)\left(1+\left(\frac{1-p(s)}{p(s)}\right)^3\right) - \sum_{a\in S^*}\sum_{b\in S^*}\sum_{c\in S^*}\frac{p(a)p(b)p(c)}{p(s)^3}T(a,b,c;d)\right) \\
&= p(s) - \frac{(1-p(s))^4}{p(s)^3} + 1 - p(s) + \frac{(1-p(s))^4}{p(s)^3} \\
&= 1.
\end{aligned}$$

□

8.3.2 Proof of 9 of Theorem 60 (r-reversible)

To prove point 9, we define now the following condition:

Cond 24: For any $a, b, c, d \in S$, $p(a)T(a,b,c;d) = p(d)T(b,c,d;a)$.

Hence, by Theorem 6 and 22,

$$\begin{aligned}
&\dim(\{A \in \mathcal{T}_S(p) : A \text{ is } < r >\text{-reversible}\}) \\
&= \dim\{(T(a,b,c;d) : a,b,c,d \in S) : Cond.\ 21 + Cond.\ 22 + Cond.\ 23 + Cond.\ 24\} \qquad (56)
\end{aligned}$$

Now, we define a condition similar to Cond. 24 but only on S^*:

Cond 24*: For any $a, b, c, d \in S^*$, $p(a)T(a,b,c;d) = p(d)T(b,c,d;a)$.

We have some properties that links all these previous conditions by the two following lemmas.

Lemma 65. *(Cond. 21 + Cond. 22 + Cond. 24) ⇔ (Cond. 21 + Cond. 24)*

Proof. $\Rightarrow$ is obvious. Now to prove $\Leftarrow$ we do the following computation: for any $a, c, d \in S$,

$$\sum_{b\in S} p(b)T(a,b,c;d) = \sum_{b\in S} p(b)\frac{p(d)}{p(a)}T(b,c,d;a)$$
$$= \sum_{b\in S} p(b)\frac{p(d)}{p(a)}\frac{p(a)}{p(b)}T(c,d,a;b) = p(d).$$

□

Lemma 66. *(Cond. 21 + Cond. 24) $\Leftrightarrow$ (Cond. 20 + Cond. 24*)*

Proof. This proof is algebraic.

$\Rightarrow$: Let suppose that T satisfy Cond. 21 + Cond. 24. Then Cond. 24* obviously holds. Now, we will prove that Cond. 20 holds too.

- For any $a, b, c \in S^*$,
$$T(a,b,c;s) = 1 - \sum_{d\in S^*} T(a,b,c;d).$$

- For any $a, b, d \in S^*$,
$$T(a,b,s;d) = \frac{p(d)}{p(s)}T(d,a,b;s) = \frac{p(d)}{p(s)}\left(1 - \sum_{c\in S^*} T(d,a,b;c)\right) = \frac{p(d)}{p(s)} - \sum_{c\in S^*}\frac{p(c)}{p(s)}T(a,b,c;d).$$

- Similarly, we get (42) and (43).

- For any $a, b \in S^*$,
$$T(a,b,s;s) = 1 - \sum_{d\in S^*} T(a,b,s;d) = 1 - \sum_{d\in S^*}\left(\frac{p(d)}{p(s)} - \sum_{c\in S^*}\frac{p(c)}{p(s)}T(a,b,c;d)\right)$$
$$= \frac{2p(s)-1}{p(s)} + \sum_{c\in S^*}\sum_{d\in S^*}\frac{p(c)}{p(s)}T(a,b,c;d)$$
$$= p(s) - \frac{(1-p(s))^2}{p(s)} + \sum_{c\in S^*}\sum_{d\in S^*}\frac{p(c)}{p(s)}T(a,b,c;d)$$
$$= p(s)\left(1 - \left(\frac{1-p(s)}{p(s)}\right)^2\right) + \sum_{c\in S^*}\sum_{d\in S^*}\frac{p(c)}{p(s)}T(a,b,c;d).$$

- Similarly, we get (45) and (46).

- For any $a, d \in S^*$,
$$T(a,s,s;d) = \frac{p(d)}{p(s)}T(d,a,s;s) = \frac{p(d)}{p(s)}\left(2 - \frac{1}{p(s)} + \sum_{b\in S^*}\sum_{c\in S^*}\frac{p(b)}{p(s)}T(d,a,b;c)\right)$$
$$= \frac{p(d)}{p(s)}\left(2 - \frac{1}{p(s)}\right) + \sum_{b\in S^*}\sum_{c\in S^*}\frac{p(b)p(c)}{p(s)^2}T(a,b,c;d)$$
$$= p(d)\left(1 - \left(\frac{1-p(s)}{p(s)}\right)^2\right) + \sum_{b\in S^*}\sum_{c\in S^*}\frac{p(b)p(c)}{p(s)^2}T(a,b,c;d);$$

- Similarly, we get (49) and (48).

- For any $a \in S^*$,

$$
\begin{aligned}
T(a,s,s;s) &= 1 - \sum_{d \in S^*} T(a,s,s;d) = 1 - \sum_{d \in S^*} \left(\frac{p(d)}{p(s)} \left(2 - \frac{1}{p(s)} \right) + \sum_{b \in S^*} \sum_{c \in S^*} \frac{p(b)p(c)}{p(s)^2} T(a,b,c;d) \right) \\
&= \frac{3p(s)^2 - 3p(s) + 1}{p(s)^2} - \sum_{b \in S^*} \sum_{c \in S^*} \sum_{d \in S^*} \frac{p(b)p(c)}{p(s)^2} T(a,b,c;d) \\
&= p(s) \left(1 + \left(\frac{1 - p(s)}{p(s)} \right)^3 \right) - \sum_{b \in S^*} \sum_{c \in S^*} \sum_{d \in S^*} \frac{p(b)p(c)}{p(s)^2} T(a,b,c;d);
\end{aligned}
$$

- Similarly, we get (52) and (51).
- For any $d \in S^*$,

$$
\begin{aligned}
T(s,s,s;d) &= \frac{p(d)}{p(s)} T(s,s,d;s) = p(d) \left(1 - \frac{(1-p(s))^3}{p(s)^3} \right) - \sum_{a \in S^*} \sum_{b \in S^*} \sum_{c \in S^*} \frac{p(b)p(c)p(d)}{p(s)^3} T(b,c,d;a) \\
&= p(d) \left(1 + \left(\frac{1 - p(s)}{p(s)} \right)^3 \right) - \sum_{a \in S^*} \sum_{b \in S^*} \sum_{c \in S^*} \frac{p(a)p(b)p(c)}{p(s)^3} T(a,b,c;d)
\end{aligned}
$$

and, finally,

$$
\begin{aligned}
T(s,s,s;s) &= 1 - \sum_{d \in S^*} T(s,s,s;d) \\
&= 1 - \sum_{d \in S^*} \left(p(d) \left(1 + \frac{(1-p(s))^3}{p(s)^3} \right) - \sum_{a \in S^*} \sum_{b \in S^*} \sum_{c \in S^*} \frac{p(a)p(b)p(c)}{p(s)^3} T(a,b,c;d) \right) \\
&= 1 - (1 - p(s)) \left(1 + \frac{(1-p(s))^3}{p(s)^3} \right) + \sum_{a \in S^*} \sum_{b \in S^*} \sum_{c \in S^*} \sum_{d \in S^*} \frac{p(a)p(b)p(c)}{p(s)^3} T(a,b,c;d) \\
&= p(s) \left(1 - \left(\frac{1 - p(s)}{p(s)} \right)^4 \right) + \sum_{a \in S^*} \sum_{b \in S^*} \sum_{c \in S^*} \sum_{d \in S^*} \frac{p(a)p(b)p(c)}{p(s)^3} T(a,b,c;d).
\end{aligned}
$$

So, Cond. 20 holds.

$\Leftarrow$: Now, suppose that Cond. 20 and Cond. 24* hold. Cond. 21 hold by Lemma 64. We will prove that Cond. 24 holds. It is obvious when $a, b, c, d \in S^*$ by Cond. 24*. Furthermore, we have the following properties.

- For any $a, b, c \in S^*$,

$$
\begin{aligned}
p(a)T(a,b,c;s) &= p(a) \left(1 - \sum_{d \in S^*} T(a,b,c;d) \right) \\
&= p(a) - \sum_{d \in S^*} p(a) T(a,b,c;d) \\
&= p(s) \left(\frac{p(a)}{p(s)} - \sum_{d \in S^*} \frac{p(d)}{p(s)} T(b,c,d;a) \right) \\
&= p(s) T(b,c,s,a).
\end{aligned}
$$

- For any $a, b, d \in S^*$,

$$\begin{aligned}
p(a)T(a,b,s;d) &= \frac{p(a)p(d)}{p(s)} - \sum_{c\in S^*} \frac{p(c)}{p(s)} p(a)T(a,b,c;d) \\
&= \frac{p(a)p(d)}{p(s)} - \sum_{c\in S^*} \frac{p(c)}{p(s)} p(d)T(b,c,d;a) \\
&= p(d)\left(\frac{p(a)}{p(s)} - \sum_{c\in S^*} \frac{p(c)}{p(s)} T(b,c,d;a)\right) \\
&= p(d)T(b,s,d;a).
\end{aligned}$$

- Similarly, for any $a, c, d \in S^*$, $p(a)T(a,s,c;d) = p(d)T(s,c,d;a)$.
- For any $b, c, d \in S^*$,

$$\begin{aligned}
p(s)T(s,b,c;d) &= p(d) - \sum_{a\in S^*} p(a)T(a,b,c;d) \\
&= p(d)(1 - \sum_{a\in S^*} T(b,c,d;a) \\
&= p(d)T(b,c,d;s).
\end{aligned}$$

- For any $a, b \in S^*$,

$$\begin{aligned}
p(a)T(a,b,s;s) &= p(a)p(s)\left(1 - \left(\frac{1-p(s)}{p(s)}\right)^2\right) + \sum_{c\in S^*}\sum_{d\in S^*} \frac{p(c)}{p(s)} p(a)T(a,b,c;d) \\
&= p(a)p(s)\left(1 - \left(\frac{1-p(s)}{p(s)}\right)^2\right) + \sum_{c\in S^*}\sum_{d\in S^*} \frac{p(c)}{p(s)} p(d)T(b,c,d;a) \\
&= p(s)\left(p(a)\left(1 - \left(\frac{1-p(s)}{p(s)}\right)^2\right) + \sum_{c\in S^*}\sum_{d\in S^*} \frac{p(c)p(d)}{p(s)^2} T(b,c,d;a)\right) \\
&= p(s)T(b,s,s;a).
\end{aligned}$$

- The other cases are similar and left to the readers.

That ends the proof. $\square$

Due to Eq 56 and the two preceding lemmas, we obtain:

$$\dim\left(\{A \in \mathcal{T}_S\left(p\right) : A \text{ is } < r >\text{-reversible}\}\right) \leq \dim\{(T(a,b,c;d) : a,b,c,d \in S^*) : Cond.\ 24^*\}. \tag{57}$$

Now, we compute $\dim\{(T(a,b,c;d) : a,b,c,d \in S^*) : Cond.\ 24^*\}$ to find the upper bound.

Lemma 67. *For any finite set S,*

$$\dim\{(T(a,b,c;d) : a,b,c,d \in S^*) : Cond.\ 24^*\} = \frac{n(n-1)(n^2-3n+4)}{4}. \tag{58}$$

Proof. This proof is half-algebraic and half-combinatorics. The goal is to use Cond. 24* to split the set $\{T(a,b,c;d) : a,b,c,d \in S^*\}$ in some subsets such that variables in each subset depend of only one free parameter. The partition is the following one:

$$
\begin{aligned}
&\{\{T(i,i,i;i)\} : i \in S^*\} \\
&\bigcup \{\{T(i,i,i;j), T(i,i,j;i), T(i,j,i;i), T(j,i,i;i)\} : i,j \in S^*, i \neq j\} \\
&\bigcup \{\{T(i,j,i;j), T(j,i,j;i)\} : i,j \in S^*, i < j\} \\
&\bigcup \{\{T(i,i,j;j), T(i,j,j;i), T(j,j,i;i), T(j,i,i;j)\} : i,j \in S^*, i \neq j\} \\
&\bigcup \{\{T(i,k,i;j), T(k,i,j;i), T(i,j,i;k), T(j,i,k;i)\} : i,j,k \in S^*, i \neq j,k, j < k\} \\
&\bigcup \{\{T(i,i,j;k), T(i,j,k;i), T(j,k,i;i), T(k,i,i;j)\} : i,j,k \in S^*, i \neq j \neq k \neq i\} \\
&\bigcup \{\{T(a,b,c;d), T(b,c,d;a), T(c,d,a;b), T(d,a,c;b)\} : a,b,c,d \in S^*, a \neq b \neq c \neq d \neq a \neq c, d \neq b\}
\end{aligned}
$$

One can check that, in each subset of this partition, there is exactly only one free variable according to Cond. 24*, see Table 2 to find the equations that link them. Now, the dimension is just the size of this partition. Enumeration is done in Table 2. By adding the fourth column, we find

$$\dim\{(T(a,b,c;d) : a,b,c,d \in S^*) : Cond.\ 24^*\} = \frac{n(n-1)(n^2-3n+4)}{4}.$$

□

To get the lower bound for $\dim\left(\{A \in \mathcal{T}_S\left(p\right) : A \text{ is } < r >\text{-reversible}\}\right)$, we use a similar trick that we have done in the proof of Lemma 55. We first remark that $T(a,b,c;d) = p(d)$ is a solution and, then by all the previous equations, it is not difficult to construct a neighboorhood whose dimension is $\dim\{(T(a,b,c;d) : a,b,c,d \in S^*) : Cond.\ 24^*\}$ and for which we do not lose positivities of $T(a,b,c;d)$ for any $(a,b,c,d) \in S^4$. Then, we get that

$$\dim\left(\{A \in \mathcal{T}_S\left(p\right) : A \text{ is } < r >\text{-reversible}\}\right) \geq \dim\{(T(a,b,c;d) : a,b,c,d \in S^*) : Cond.\ 24^*\}. \tag{59}$$

That ends the proof of point 9 of Theorem 60. □

8.3.3 Proof of 10 of Theorem 60 ($r \circ v$-reversible)

The proof of 10 is similar to the one of 9. Hence, we will omit some parts of the proof that are the same. We only detail the partition in Lemma 69, because it differs from the one of Lemma 67.

The conditions we will need here are the two following ones:

Cond 25: For any $a,b,c,d \in S$, $p(a)T(a,b,c;d) = p(d)T(d,c,b;a)$.

Cond 25*: For any $a,b,c,d \in S^*$, $p(a)T(a,b,c;d) = p(d)T(d,c,b;a)$.

That are linked by the following lemma:

Lemma 68. *(Cond. 21 + Cond. 22 + Cond. 25) ⇔ (Cond. 20 + Cond. 25*)*

Proof. Proof is similar to the one of Lemma 66. □

Hence, by Theorems 6 and 22 and Lemma 68, we get

$$\dim\left(\{A \in \mathcal{T}_S\left(p\right) : A \text{ is } < r \circ v >\text{-reversible}\}\right) \leq \dim\{(T(a,b,c;d) : a,b,c,d \in S^*) : Cond.\ 25^*\}.$$

Now, we compute the upper bound:

Lemma 69. *For any finite set S:*

$$\dim\{(T(a,b,c;d) : a,b,c,d \in S^*) : Cond.\ 25^*\} = \frac{(n-1)^2(n^2-2n+2)}{2}. \tag{60}$$

Subset type	Involved equations	Conditions on the arguments	Number of subsets of this type
$\{T(i,i,i;i)\}$	$T(i,i,i;i)$	$i \in S^*$	$\lvert S^*\rvert = n-1$
$\{T(i,i,i;j),$ $T(i,i,j;i),$ $T(i,j,i;i),$ $T(j,i,i;i)\}$	$p(i)T(i,i,i;j)$ $= p(j)T(i,i,j;i)$ $= p(j)T(i,j,i;i)$ $= p(j)T(j,i,i;i)$	$i,j \in S^*$ $i \neq j$	$\binom{\lvert S^*\rvert}{1}\binom{\lvert S^*\rvert-1}{1}$ $= (n-1)(n-2)$
$\{T(i,j,i;j),$ $T(j,i,j;i)\}$	$p(i)T(i,j,i;j)$ $= p(j)T(j,i,j;i)$	$i,j \in S^*$ $i < j$	$\binom{\lvert S^*\rvert}{2} = \frac{(n-1)(n-2)}{2}$
$\{T(i,i,j;j),$ $T(i,j,j;i),$ $T(j,j,i;i),$ $T(j,i,i;j)\}$	$p(i)T(i,i,j;j)$ $= p(j)T(i,j,j;i)$ $= p(j)T(j,j,i;i)$ $= p(i)T(j,i,i;j)$	$i,j \in S^*$ $i \neq j$	$\binom{\lvert S^*\rvert}{2} = \frac{(n-1)(n-2)}{2}$
$\{T(i,k,i;j),$ $T(k,i,j;i),$ $T(i,j,i;k),$ $T(j,i,k;i)\}$	$p(i)p(k)T(i,k,i;j)$ $= p(j)p(k)T(k,i,j;i)$ $= p(i)p(j)T(i,j,i;k)$ $= p(j)p(k)T(j,i,k;i)$	$i,j,k \in S^*$ $i \neq j,k$ $j < k$	$\binom{\lvert S^*\rvert}{1}\binom{\lvert S^*\rvert-1}{2}$ $= \frac{(n-1)(n-2)(n-3)}{2}$
$\{T(i,i,j;k),$ $T(i,j,k;i),$ $T(k,j,i;i),$ $T(j,i,i;k)\}$	$p(i)p(j)T(i,i,j;k)$ $= p(j)p(k)T(i,j,k;i)$ $= p(j)p(k)T(j,k,i;i)$ $= p(i)p(k)T(k,i,i;j)$	$i,j,k \in S^*$ $i \neq j \neq k \neq i$	$\binom{\lvert S^*\rvert}{1}\binom{\lvert S^*-1\rvert}{1}\binom{\lvert S^*\rvert-2}{1}$ $= (n-1)(n-2)(n-3)$
$\{T(a,b,c;d),$ $T(b,c,d;a),$ $T(c,d,a;b),$ $T(d,a,b;c)\}$	$p(a)p(b)p(c)T(a,b,c;d)$ $= p(b)p(c)p(d)T(b,c,d;a)$ $= p(a)p(c)p(d)T(c,d,a;b)$ $= p(a)p(b)p(d)T(d,a,b;c)$	$a,b,c,d \in S^*$ $a < b,c,d$ $b \neq c \neq d \neq b$	$\frac{1}{4}\lvert S^*\rvert(\lvert S^*\rvert-1)(\lvert S^*\rvert-2)(\lvert S^*\rvert-3)$ $= \frac{(n-1)(n-2)(n-3)(n-4)}{4}$

Table 2: Partition of $\{T(a,b,c;d) : a,b,c,d \in S^*\}$ according to Cond. 24*. On each line, we detail one of the type of the subset involved in the partition. The first column is the subset type. The second gives the equations that link the variables in the subset; these equations are obtained by specifications of Cond. 24*. The third column gives conditions on the arguments to get independent sets when we enumerate them. The fourth column is the enumeration of subsets of that type.

Subset type	Involved equations	Conditions on the arguments	Number of subsets of this type
$\{T(i,i,i;i)\}$	$T(i,i,i;i)$	$i \in S^*$	$\lvert S^*\rvert = (n-1)$
$\{T(i,i,i;j),$ $T(j,i,i;i)\}$	$p(i)T(i,i,i;j)$ $= p(j)T(j,i,i;i)$	$i,j \in S^*$ $i \neq j$	$\binom{\lvert S^*\rvert}{1}\binom{\lvert S^*\rvert-1}{1}$ $= (n-1)(n-2)$
$\{T(i,i,j;i),$ $T(i,j,i;i)\}$	$p(i)T(i,i,j;i)$ $= p(j)T(i,j,i;i)$	$i,j \in S^*$ $i \neq j$	$\binom{\lvert S^*\rvert}{1}\binom{\lvert S^*\rvert-1}{1}$ $= (n-1)(n-2)$
$\{T(i,j,i;j),$ $T(j,i,j;i)\}$	$p(i)T(i,j,i;j)$ $= p(j)T(j,i,j;i)$	$i,j \in S^*$ $i < j$	$\binom{\lvert S^*\rvert}{2} = \frac{(n-1)(n-2)}{2}$
$\{T(i,i,j;j),$ $T(j,j,i;i)\}$	$p(i)T(i,i,j;j)$ $= p(j)T(j,j,i;i)$	$i,j \in S^*$ $i < j$	$\binom{\lvert S^*\rvert}{2} = \frac{(n-1)(n-2)}{2}$
$\{T(i,j,j;i)\}$	$T(i,j,j;i)$	$i,j \in S^*$ $i \neq j$	$\binom{\lvert S^*\rvert}{1}\binom{\lvert S^*\rvert-1}{1}$ $= (n-1)(n-2)$
$\{T(i,i,j;k),$ $T(k,j,i;i)\}$	$p(i)T(i,i,j;k)$ $= p(k)T(k,j,i;i)$	$i,j,k \in S^*$ $i \neq j \neq k \neq i$	$\binom{\lvert S^*\rvert}{1}\binom{\lvert S^*\rvert-1}{1}\binom{\lvert S^*\rvert-2}{1}$ $= (n-1)(n-2)(n-3)$
$\{T(i,j,i;k),$ $T(k,i,j;i)\}$	$p(i)T(i,j,i;k)$ $= p(k)T(k,i,j;i)$	$i,j,k \in S^*$ $i \neq j \neq k \neq i$	$\binom{\lvert S^*\rvert}{1}\binom{\lvert S^*\rvert-1}{1}\binom{\lvert S^*\rvert-2}{1}$ $= (n-1)(n-2)(n-3)$
$\{T(i,j,k;i),$ $T(i,k,j;i)\}$	$T(i,j,k;i)$ $= T(i,k,j;i)$	$i,j,k \in S^*$ $i \neq j,k$ $j < k$	$\binom{\lvert S^*\rvert}{1}\binom{\lvert S^*\rvert-1}{2}$ $= \frac{(n-1)(n-2)(n-3)}{2}$
$\{T(j,i,i;k),$ $T(k,i,i;j)\}$	$p(j)T(j,i,i;k)$ $= p(k)T(k,i,i;j)$	$i,j,k \in S^*$ $i \neq j,k$ $j < k$	$\binom{\lvert S^*\rvert}{1}\binom{\lvert S^*\rvert-1}{2}$ $= \frac{(n-1)(n-2)(n-3)}{2}$
$\{T(a,b,c;d),$ $T(d,c,b;a)\}$	$p(a)T(a,b,c;d)$ $= p(d)T(d,c,b;a)$	$a,b,c,d \in S^*$ $a < d$ $a \neq b \neq c \neq a$ $d \neq b,c$	$\frac{1}{2}\lvert S^*\rvert(\lvert S^*\rvert-1)(\lvert S^*\rvert-2)(\lvert S^*\rvert-3)$ $= \frac{(n-1)(n-2)(n-3)(n-4)}{2}$

Table 3: Partition of $\{T(a,b,c;d) : a,b,c,d \in S^*\}$ according to Cond. 25*.

Proof. Proof is similar to the one of Lemma 67, except that the variable space is not partitioned in the same way. The new partition (based on Cond. 25*) and its enumeration is given in the Table 3. Thus, the size of this partition is $\frac{(n-1)^2(n^2-2n+2)}{2}$.

The end of the proof is like the ones of Lemma 55 and 67. It consists in checking that there exists a neighbourhood of the point $(T(a,b,c;d) = p(d) : a,b,c,d \in S)$ with the good dimension such that any point of this neighbourhood satisfies the required conditions. □

8.3.4 Proof of 11 of Theorem 60 (D_4-reversible)

The proof of point 11 is similar to the two previous ones. We begin by introducing the two new following conditions.

Cond 26: For any $a,b,c,d \in S$, $T(a,b,c;d) = T(c,b,a;d)$.

Cond 26*: For any $a,b,c,d \in S^*$, $T(a,b,c;d) = T(c,b,a;d)$.

We have then the following relation.

Lemma 70. *(Cond. 21 + Cond. 24 + Cond. 26) ⇔ (Cond. 20 + Cond. 24* + Cond. 26*)*

By Theorem 6 and 22 and Lemma 70, we have

$$\begin{aligned}&\dim\left(\{A \in \mathcal{T}_S\,(p) : A \text{ is } D_4\text{-reversible}\}\right)\\ &\leq \dim\{(T(a,b,c;d) : a,b,c,d \in S^*) : \mathit{Cond.}\ 24^* + \mathit{Cond.}\ 26^*\}.\end{aligned}$$

Now, we compute the dimension.

Lemma 71.

$$\dim\{(T(a,b,c;d) : a,b,c,d \in S^*) : \mathit{Cond.}\ 24^* + \mathit{Cond.}\ 26^*\} = \frac{(n-1)^2(n^2-2n+2)}{2}. \quad (61)$$

Proof. As before, the main argument is to find the partition of T based on Cond. 24* and Cond. 26*. This partition and its enumeration is given in Table 4. Thus, the size of this partition is $\frac{n(n-1)(n^2-n+2)}{8}$.

To prove equality between $\dim\left(\{A \in \mathcal{T}_S\,(p) : A \text{ is } D_4\text{-reversible}\}\right)$ and $\dim\{(T(a,b,c;d) : a,b,c,d \in S^*) : \mathit{Cond.}\ 24^* + \mathit{Cond.}\ 26^*\}$, we use the same trick as developed in the end of the proof of Lemma 55. □

Subset type	Involved equations	Conditions on the arguments	Number of subsets of this type
$\{T(i,i,i;i)\}$	$T(i,i,i;i)$	$i \in S^*$	$\lvert S^*\rvert$
$\{T(i,i,i;j)$, $T(i,i,j;i)$, $T(i,j,i;i)$, $T(j,i,i;i)\}$	$p(i)T(i,i,i;j)$ $= p(j)T(i,i,j;i)$ $= p(j)T(i,j,i;i)$ $= p(j)T(j,i,i;i)$	$i,j \in S^*$ $i \neq j$	$\binom{\lvert S^*\rvert}{1}\binom{\lvert S^*\rvert-1}{1} = (n-1)(n-2)$
$\{T(i,j,i;j)$, $T(j,i,j;i)\}$	$p(i)T(i,j,i;j)$ $= p(j)T(j,i,j;i)$	$i,j \in S^*$ $i < j$	$\binom{\lvert S^*\rvert}{2} = \frac{(n-1)(n-2)}{2}$
$\{T(i,i,j;j)$, $T(i,j,j;i)$, $T(j,j,i;i)$, $T(j,i,i;j)\}$	$p(i)T(i,i,j;j)$ $= p(j)T(i,j,j;i)$ $= p(j)T(j,j,i;i)$ $= p(i)T(j,i,i;j)$	$i,j \in S^*$ $i < j$	$\binom{\lvert S^*\rvert}{2} = \frac{(n-1)(n-2)}{2}$
$\{T(i,i,j;k)$, $T(k,i,i;j)$, $T(j,k,i,i)$, $T(i,j,k;i)$, $T(j,i,i;k)$, $T(i,i,k;j)$, $T(i,k,j;i)$, $T(k,j,i;i)\}$	$p(i)p(j)T(i,i,j;k)$ $= p(j)p(k)T(i,j,k;i)$ $= p(j)p(k)T(j,k,i;i)$ $= p(i)p(k)T(k,i,i;j)$ $= p(i)p(j)T(j,i,i;k)$ $= p(j)p(k)T(k,j,i;i)$ $= p(j)p(k)T(i,k,j;i)$ $= p(i)p(k)T(i,i,k;j)$	$i,j,k \in S^*$ $i \neq j,k$ $j < k$	$\binom{\lvert S^*\rvert}{1}\binom{\lvert S^*\rvert-1}{2} = \frac{(n-1)(n-2)(n-3)}{2}$
$\{T(i,j,i;k)$, $T(k,i,j;i)$, $T(i,k,i;j)$, $T(j,i,k;i)\}$	$p(i)p(k)T(i,k,i;j)$ $= p(j)p(k)T(k,i,j;i)$ $= p(i)p(j)T(i,j,i;k)$ $= p(j)p(k)T(j,i,k;i)$	$i,j,k \in S^*$ $i \neq j,k$ $j < k$	$\binom{\lvert S^*\rvert}{1}\binom{\lvert S^*-1\rvert}{2} = \frac{(n-1)(n-2)(n-3)}{2}$
$\{T(a,b,c;d)$, $T(d,a,b;c)$, $T(c,d,a;b)$, $T(b,c,d;a)$, $T(c,b,a;d)$, $T(b,a,d;c)$, $T(a,d,c;b)$, $T(d,c,b;a)\}$	$p(a)p(b)p(c)T(a,b,c;d)$ $= p(b)p(c)p(d)T(b,c,d;a)$ $= p(a)p(c)p(d)T(c,d,a;b)$ $= p(a)p(b)p(d)T(d,a,b;c)$ $= p(a)p(b)p(c)T(c,b,a;d)$ $= p(b)p(c)p(d)T(d,c,b;a)$ $= p(a)p(c)p(d)T(a,d,c;b)$ $= p(a)p(b)p(d)T(b,a,d;c)$	$a,b,c,d \in S^*$ $a < b,c,d$ $b < c,d$ $c \neq d$	$\frac{1}{8}\lvert S^*\rvert(\lvert S^*\rvert-1)(\lvert S^*\rvert-2)(\lvert S^*\rvert-3) = \frac{(n-1)(n-2)(n-3)(n-4)}{8}$

Table 4: Partition of $\{T(a,b,c;d) : a,b,c,d \in S^*\}$ according to Cond. 24* and Cond. 26*.

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