Stieltjes moment sequences of polynomials

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This paper is dedicated to the memory of Professor Jeff Remmel, who recently passed away.

Abstract

A sequence $(a_n)_{n\geq 0}$ is Stieltjes moment sequence if it has the form $a_n = \int_0^\infty x^n d\mu(x)$ for μ is a nonnegative measure on $[0,\infty)$. It is known that $(a_n)_{n\geq 0}$ is a Stieltjes moment sequence if and only if the matrix $H = [a_{i+j}]_{i,j\geq 0}$ is totally positive, i.e., all its minors are nonnegative. We define a sequence of polynomials in $x_1, x_2, \ldots, x_n$ $(a_n(x_1, x_2, \ldots, x_n))_{n\geq 0}$ to be a Stieltjes moment sequence of polynomials if the matrix $H = [a_{i+j}(x_1, x_2, \ldots, x_n)]_{i,j\geq 0}$ is $x_1, x_2, \ldots, x_n$-totally positive, i.e., all its minors are polynomials in $x_1, x_2, \ldots, x_n$ with nonnegative coefficients. The main goal of this paper is to produce a large class of Stieltjes moment sequences of polynomials by finding multivariable analogues of Catalan-like numbers as defined by Aigner.

1 Introduction

A sequence $(a_n)_{n\geq 0}$ is a *Stieltjes moment sequence* if it has the form

$$a_n = \int_0^\infty x^n d\mu(x)$$

where μ is a nonnegative measure on $[0,\infty)$. There are several other characterizations of Stieltjes moment sequences. For example, in [16, Theorem 1.3], it is proved that a sequence $(a_n)_{n\geq 0}$ is a Stieltjes moment sequence if and only if both the matrices $[a_{i+j}]_{0\leq i,j\leq n}$ and $[a_{i+j+1}]_{0\leq i,j\leq n}$ are positive semidefinite for all $n \geq 0$ (see [13, 14]).

Another characterization Stieltjes moment sequences comes from the theory of total positivity. Let $A = [a_{n,k}]_{n,k\geq 0}$ be a finite or infinite matrix. We say that A is *totally positive of order* r if all its minors of order $1, 2, \ldots, r$ are nonnegative and we say that A is *totally positive* if it is totally positive of order r for all $r \geq 1$ (see [2, 3, 4, 5, 6, 7] for instance). Given a sequence $\alpha = (a_n)_{n\geq 0}$, we define the *Hankel matrix of* α, $H(\alpha)$, by

$$H(\alpha) = [a_{i+j}]_{i,j\geq 0} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then it is proved in [14] that α is a Stieltjes moment sequence if and only if $H(\alpha)$ is TP.

Let $\mathbb{R}$ denote the real numbers and $\mathbf{x} = x_1, \ldots, x_n$. In this paper, we may define when a sequence of polynomials $(a_n(\mathbf{x}))_{n\geq 0}$ in the polynomial ring $\mathbb{R}[\mathbf{x}]$ is a Stieltjes moment sequence of polynomials. For any polynomial $f(\mathbf{x}) = \sum c_{i_1,\ldots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ in $\mathbb{R}[\mathbf{x}]$, we let $f(\mathbf{x})|_{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}} = c_{i_1,\ldots,i_n}$ denote the coefficient of $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ in $f(\mathbf{x})$. We say that $f(\mathbf{x})$ is $\mathbf{x}$-nonnegative, written $f(\mathbf{x}) \geq_{\mathbf{x}} 0$, if

$$f(\mathbf{x})|_{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}} \geq 0 \text{ for all } i_1, \ldots, i_n.$$

Given a pair of polynomials in $f(\mathbf{x})$ and $g(\mathbf{x})$, we shall write

$$f(\mathbf{x}) \geq_{\mathbf{x}} g(\mathbf{x})$$

if $f(\mathbf{x}) - g(\mathbf{x}) \geq_{\mathbf{x}} 0$. Let $M = [m_{n,k}(\mathbf{x})]_{n,k\geq 0}$ be a finite or infinite matrix of polynomials in $\mathbb{R}[\mathbf{x}]$. We say that M is $\mathbf{x}$*-totally positive of order* r ($\mathbf{x}$-TP_r) if all its minors of order $1, 2, \ldots, r$ are polynomials in $\mathbf{x}$ with nonnegative coefficients and we say that M is $\mathbf{x}$*-totally positive* ($\mathbf{x}$-TP) if it is $\mathbf{x}$-totally positive of order r for all $r \geq 1$.

Given a sequence $\alpha = (a_k(\mathbf{x}))_{k\geq 0}$ of polynomials in $\mathbb{R}[\mathbf{x}]$, we define the *Hankel matrix of* α, $H(\alpha, \mathbf{x})$, by the following

$$H(\alpha, \mathbf{x}) = [a_{i+j}(\mathbf{x})]_{i,j\geq 0} = \begin{bmatrix} a_0(\mathbf{x}) & a_1(\mathbf{x}) & a_2(\mathbf{x}) & a_3(\mathbf{x}) & \cdots \\ a_1(\mathbf{x}) & a_2(\mathbf{x}) & a_3(\mathbf{x}) & a_4(\mathbf{x}) & \cdots \\ a_2(\mathbf{x}) & a_3(\mathbf{x}) & a_4(\mathbf{x}) & a_5(\mathbf{x}) & \cdots \\ a_3(\mathbf{x}) & a_4(\mathbf{x}) & a_5(\mathbf{x}) & a_6(\mathbf{x}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then if α is a *Stieltjes moment sequence of polynomials* if and only if $H(\alpha, \mathbf{x})$ is $\mathbf{x}$-TP. In the case where $n = 1$ so that we are considering polynomials in a single variable, our definition coincides with the definition of Stieltjes moment sequences of polynomials as defined by Wang and Zhu [20].

The main goal of this paper is to produce a number of combinatorially defined Stieltjes moment sequences of polynomials. We shall do this by finding appropriate multivariable analogues of Catalan-like numbers as defined by Aigner [1]. Aigner's idea is the following. Let $\sigma = (s_k)_{k\geq 0}$ and $\tau = (t_{k+1})_{k\geq 0}$ be two sequences of nonnegative numbers. Then define an infinite lower triangular matrix $A := A^{\sigma,\tau} = [a_{n,k}]_{n,k\geq 0}$ where the $a_{n,k}$s are defined by the recursions

$$a_{n+1,k} = a_{n,k-1} + s_k a_{n,k} + t_{k+1} a_{n,k+1} \tag{1}$$

subject to the initial conditions that $a_{0,0} = 1$ and $a_{n,k} = 0$ unless $n \geq k \geq 0$. Aigner called $A^{\sigma,\tau}$ the *recursive matrix* corresponding to (σ, τ) and he called the sequence $(a_{n,0})_{n\geq 0}$, the *Catalan-like numbers* corresponding to (σ, τ). Recently, Liang et al. [11] showed that many Catalan-like numbers are Stieltjes moment sequences by proving that the Hankel matrix of the sequence $(a_{n,0})_{n\geq 0}$ is totally positive. Such examples include the Catalan numbers, the Bell numbers, the central Delannoy numbers, the restricted hexagonal numbers, the central binomial coefficients, and the large Schröder numbers.

Liu and Wang [12] defined a sequence of polynomials $(f_n(q))_{n\geq 0}$ over $\mathbb{R}$ to be *q-log convex* (*q-LCX*) if for all $n \geq 1$,

$$(f_n(q))^2 \geq_q f_{n-1}(q)f_{n+1}(q) \tag{2}$$

and defined a sequence of polynomials $(f_n(q))_{n\geq 0}$ to be *strongly q-log convex* (*q-SLCX*) if for all $n \geq m \geq 1$,

$$f_n(q)f_m(q) \geq_q f_{n-1}(q)f_{m+1}(q). \tag{3}$$

Zhu [21] produced many examples of q-$SLCX$ sequences of polynomials by modifying Aigner's Catalan-like numbers. In such a situation, Zhu [21] showed that the sequence of polynomials $(m_{n,0}(q))_{n\geq 0}$ is a q-$SLCX$ sequence of polynomials if for all $k \geq 0$, $s_k(q)s_{k+1}(q) - t_{k+1}(q)r_{k+1}(q) \geq_q 0$. However, it is not the case that such a sequence of polynomials $(m_{n,0}(q))_{n\geq 0}$ is always a Stieltjes moment sequence of polynomials. For example, suppose that a and b are nonnegative real numbers and $r_k(q) = 1$ for $k \geq 1$, $s_0(q) = q^2$ and $s_k(q) = 1 + q^2 + a * q^b$ for $k \geq 1$, and $t_1(q) = q^4$ and $t_k(q) = q^2 + q^4$ for $k \geq 2$. It is easy to check that for all $k \geq 0$, $s_k(q)s_{k+1}(q) - t_{k+1}(q)r_{k+1}(q) \geq_q 0$. First one can compute that

$$\begin{aligned}
m_{0,0}(q) &= 1, \\
m_{1,0}(q) &= q^2, \\
m_{2,0}(q) &= q^4 + 4q^6 + aq^{4+b}, \\
m_{3,0}(q) &= q^4 + 5q^6 + 9q^8 + 2aq^{4+b} + 4aq^{6+b} + a^2q^{4+2b}, \text{ and} \\
m_{4,0}(q) &= q^4 + 8q^6 + 20q^8 + 21q^{10} + 3aq^{4+b} + 13aq^{6+b} + \\
&\quad 15aq^{8+b} + 3a^2q^{4+2b} + 5a^2q^{6+2b} + a^3q^{4+3b}.
\end{aligned}$$

Then one can compute that

$$\det\left(\begin{bmatrix} m_{0,0}(q) & m_{1,0}(q) & m_{2,0}(q) \\ m_{1,0}(q) & m_{2,0}(q) & m_{3,0}(q) \\ m_{2,0}(q) & m_{3,0}(q) & m_{4,0}(q) \end{bmatrix}\right) =$$
$$-q^8 - 4q^{10} + 6q^{12} + 36q^{14} + 27q^{16} - 64q^{18} - 3aq^{8+b} - 2aq^{10+b} + 27aq^{12+b} + 35aq^{14+b} -$$
$$48aq^{16+b} - 3a^2q^{8+2b} + 5a^2q^{10+2b} + 14a^2q^{12+2b} - 12a^2q^{14+2b} - a^3q^{8+3b} + 3a^3q^{10+3b} - a^3q^{12+3b}$$

which is not a polynomial in q with nonnegative coefficients for all integers $a, b \geq 0$.

Wang and Zhu [20] showed that many of the special sequences considered by Zhu [21] are in fact Stieltjes moment sequences of polynomials over q. These include the following well-known polynomials which are q-analogues of Catalan-like numbers.

1. The Bell polynomials $B_n(q) = \sum_{k=0}^{n} S(n,k)q^k$ when $r_k(q) = 1$, $s_k(q) = k + q$, and $t_k(q) = kq$. Here $S(n,k)$ is the Stirling number of the second kind which counts the number of set partitions of $\{1,\ldots,n\}$ into k parts.

2. The Eulerian polynomials $A_n(q) = \sum_{k=0}^{n} A(n,k)q^k$ when $r_k(q) = 1$, $s_k(q) = (k+1)q + k$, and $t_k(q) = k^2q$. Here $A(n,k)$ is the number of permutations of n with k descents.

3. The q-Schröder numbers, $r_n(q) = \sum_{k=0}^{n} \frac{1}{k+1}\binom{2k}{k}\binom{n+k}{n-k} q^k$ when $r_k(q) = 1$, $s_0(q) = 1+q$, $s_k(q) = 1+2q$ for $k \geq 1$, and $t_k(q) = q(1+q)$.

4. The q-central Delannoy numbers $D_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k}\binom{2k}{k} q^k$ when $r_k(q) = 1$, $s_k(q) = 1+2q$, $t_1(q) = 2q(q+1)$, and $t_k(q) = q(1+q)$ for $k > 1$.

5. The Narayana polynomials $N_n(q) = \sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} q^k$ when $r_k(q) = 1$, $s_0(q) = q$, $s_k(q) = 1+q$ for $n \geq 1$, and $t_k(q) = q$.

6. The Narayana polynomials $W_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 q^k$ of type B when $r_k(q) = 1$, $s_k(q) = 1+q$, $t_1(q) = 2q$, and $t_k(q) = q$ for $k > 1$.

In this paper, we consider multivariable analogues Aigner's Catalan-like numbers. That is, suppose that we are given three sequences of polynomials over $\mathbb{R}$ with nonnegative coefficients

$$\pi = (r_k(\mathbf{x}))_{k\geq 1}, \ \sigma = (s_k(\mathbf{x}))_{k\geq 0}, \ \text{and } \tau = (t_{k+1}(\mathbf{x}))_{k\geq 0}.$$

Then we define a lower triangular matrix of polynomials

$$M(\mathbf{x}) := M^{\pi,\sigma,\tau}(\mathbf{x}) = [m_{n,k}(\mathbf{x})]_{0\leq k\leq n}$$

where the $m_{n,k}(\mathbf{x})$ are defined by the recursions

$$m_{n+1,k}(\mathbf{x}) = r_k(\mathbf{x})m_{n,k-1}(\mathbf{x}) + s_k(\mathbf{x})m_{n,k}(\mathbf{x}) + t_{k+1}(\mathbf{x})m_{n,k+1}(\mathbf{x}) \tag{4}$$

subject to the initial conditions that $m_{0,0}(\mathbf{x}) = 1$ and $m_{n,k}(\mathbf{x}) = 0$ unless $0 \leq k \leq n$.

We note that one can give simple combinatorial interpretations of the polynomial $m_{n,k}(\mathbf{x})$ defined by the recursions (4) in terms of Motzkin paths. A Motzkin path is path that starts at $(0,0)$ and consist of three types of steps, up-steps $(1,1)$, down-steps $(1,-1)$, and level-steps $(1,0)$. We let $\mathcal{M}_{n,k}$ denote the set all paths that start at $(0,0)$, end at (n,k), and stays on or above the x-axis. We weight an up-step at that ends at level k with $r_k(\mathbf{x})$, a level-step that ends at level k with $s_k(\mathbf{x})$, and a down-step that ends at level k with $t_{k+1}(\mathbf{x})$. See Figure 1. Given a path P in $\mathcal{M}_{n,k}$, we let the weight of P, $w(P)$, equal the product of all the weights of the steps in P. Then if we let

$$m_{n,k}(\mathbf{x}) = \sum_{P\in\mathcal{M}_{n,k}} w(P),$$

it is easy to see that the $m_{n,k}(\mathbf{x})$ satisfy the recursions (4).

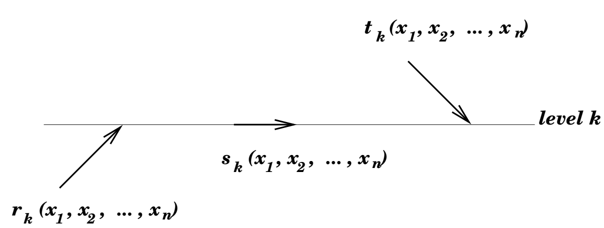

Figure 1: The weight of steps in Motzkin paths

Suppose that we are given three sequence of polynomials over $\mathbb{R}$ with nonnegative coefficients $\pi = (r_k(\mathbf{x}))_{k\geq 1}$, $\sigma = (s_k(\mathbf{x}))_{k\geq 0}$, and $\tau = (t_{k+1}(\mathbf{x}))_{k\geq 0}$. The main goal of this paper is to give some necessary conditions that will ensure that the sequence of polynomials $(m_{s,0}(\mathbf{x}))_{s\geq 0}$ is a Stieltjes sequence of polynomials where the polynomials $m_{n,k}(\mathbf{x})$ are defined by (4). First we will prove that $(m_{s,0}(\mathbf{x}))_{s\geq 0}$ a Stieltjes sequence of polynomials if the matrix

$$J(\mathbf{x}) := J^{(\pi,\sigma,\tau)}(\mathbf{x}) = \begin{bmatrix} s_0(\mathbf{x}) & r_1(\mathbf{x}) & & & \\ t_1(\mathbf{x}) & s_1(\mathbf{x}) & r_2(\mathbf{x}) & & \\ & t_2(\mathbf{x}) & s_2(\mathbf{x}) & r_3(\mathbf{x}) & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

is $\mathbf{x}$-TP and $r_k(\mathbf{x}) = 1$ for all $k \geq 0$. Following Wang and Zhu [20], we shall show that if there are sequences of polynomials with nonnegative coefficients in $\mathbb{R}[\mathbf{x}]$,

$$(b_1(\mathbf{x}), b_2(\mathbf{x}), \ldots) \text{ and } (c_1(\mathbf{x}), c_2(\mathbf{x}), \ldots)$$

such that

$$\begin{aligned} s_n(\mathbf{x}) &= b_{n+1}(\mathbf{x}) + c_{n+1}(\mathbf{x}) \text{ for } n \geq 0, \\ t_n(\mathbf{x}) &= c_n(\mathbf{x}) b_{n+1}(\mathbf{x}) \text{ for } n \geq 0, \text{ and} \\ r_n(\mathbf{x}) &= 1 \text{ for } n \geq 0, \end{aligned}$$

then $J^{(\pi,\sigma,\tau)}(\mathbf{x})$ is $\mathbf{x}$-TP.
We shall also show that $J^{(\pi,\sigma,\tau)}(\mathbf{x})$ is $\mathbf{x}$-TP if the following conditions hold:

(i) $s_0(\mathbf{x}) - 1 \geq_{\mathbf{x}} 0$,

(ii) $s_i(\mathbf{x}) s_{i+1}(\mathbf{x}) - t_{i+1}(\mathbf{x}) r_{i+1}(\mathbf{x}) \geq_{\mathbf{x}} 0$ for all $i \geq 0$,

(iii) $s_{i+1}(\mathbf{x}) - t_{i+1}(\mathbf{x}) r_{i+1}(\mathbf{x}) - 1 \geq_{\mathbf{x}} 0$ for all $i \geq 0$.

We will then use these facts to produce many examples of Stieltjes sequences of polynomials.

We should note that one of the advantages of producing Stieltjes moment sequences of polynomials $(a_k(\mathbf{x}))_{k\geq 0}$ is that we automatically obtain infinitely many other examples of the Stieltjes moment sequences of polynomials by replacing x_i by $\phi_i(\mathbf{x})$ where $\phi_i(\mathbf{x})$ is a polynomial with nonnegative coefficients. That is,

$$(a_k(\phi_1(\mathbf{x}), \ldots, \phi_n(\mathbf{x})))_{k\geq 0}$$

will be another Stieltjes moment sequence of polynomials. In addition, we can produce infinitely many example of Stieltjes moment sequences by replacing x_i by any nonnegative real number r_i. That is, if $r_i \geq 0$ for $i = 1, \ldots, n$, then $(a_k(r_1, \ldots, r_n))_{k\geq 0}$ is a Stieltjes moment sequence.

The outline of this paper is as follows. In Section 2, we shall give several sufficient conditions which ensure that the $H(\alpha, \mathbf{x})$ is $\mathbf{x}$-totally positive for a sequence of polynomials $\alpha = (a_k(\mathbf{x}))_{k\geq 0}$. Then in Section 3, we shall uses these results to produce many combinatorial sequences defined Stieltjes moment sequences of polynomials $(a_k(\mathbf{x}))_{k\geq 0}$.

2 Premliminaries

We start with two lemmas about $n \times n$ tridiagonal matrices of polynomials in $\mathbf{x}$. Suppose that $J := J(\mathbf{x}) = [a_{i,j}(\mathbf{x})]_{i,j=1,\ldots,n}$ is tridiagonal matrix of nonnegative polynomials in $\mathbf{x}$ over $\mathbb{R}$. That is, $a_{i,j}(\mathbf{x}) \geq_{\mathbf{x}} 0$ for all i,j and $a_{i,j}(\mathbf{x}) = 0$ if $|i-j| > 1$. Let $J[\{i_1,\ldots,i_k\},\{j_1,\ldots,j_k\}]$ denote the $k \times k$ matrix which arises from J by taking the elements that lie in the intersection of the rows $i_1,\ldots,i_k$ in J and the columns $j_1,\ldots,j_k$ in J where $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$. We say that a minor $\det J[\{i_1,\ldots,i_k\},\{j_1,\ldots,j_k\}]$ is a consecutive principal minor if there exists an s such that $(i_1,i_2,\ldots,i_k) = (j_1,j_2,\ldots,j_k) = (s,s+1,\ldots,s+k-1)$.

Lemma 1. *Suppose that $J = [a_{i,j}(\mathbf{x})]_{i,j=1,\ldots,n}$ is tridiagonal matrix of nonnegative polynomials in $\mathbb{R}[\mathbf{x}]$. Then J is $\mathbf{x}$-TP if and only if all of its consecutive principle minors are polynomials in $\mathbf{x}$ with nonnegative coefficients.*

Proof. Consider a minor $\det J[\{i_1,\ldots,i_k\},\{j_1,\ldots,j_k\}]$. First we observe that $\det J[\{i_1,\ldots,i_k\},\{j_1,\ldots,j_k\}] = 0$ if there is an s such that $|i_s - j_s| > 1$. That is, suppose that $i_s < j_s$. Clearly $a_{i_s,j_t}(\mathbf{x}) = 0$ for $t = s, s+1, \ldots, k$. But then

$$\det J[\{i_1,\ldots,i_k\},\{j_1,\ldots,j_k\}] = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) a_{i_1,j_{\sigma_1}}(\mathbf{x}) \cdots a_{i_k,j_{\sigma_k}}(\mathbf{x})$$

where S_k is the symmetric group. It follows that if $a_{i_s,j_{\sigma_s}}(\mathbf{x}) \neq 0$, then $\sigma_s \leq s-1$. But in such a situation, there must be an $r < s$ such that $\sigma_r \geq s$ which would imply that $a_{i_r,j_{\sigma_r}}(\mathbf{x}) = 0$. Thus $\det J[\{i_1,\ldots,i_k\},\{j_1,\ldots,j_k\}] = 0$. A similar argument can be used to show that if $j_s < i_s$, then $\det J[\{i_1,\ldots,i_k\},\{j_1,\ldots,j_k\}] = 0$.

Thus we only have to consider minors of the form $\det J[\{i_1,\ldots,i_k\},\{j_1,\ldots,j_k\}]$ where $|i_s - j_s| \leq 1$. In such a situation, let $1 \leq t_1 < \cdots < t_r \leq k$ be the indices t such that such that $|i_t - j_t| = 1$. Then it is easy to see that $\det J[\{i_1,\ldots,i_k\},\{j_1,\ldots,j_k\}]$ equals $\prod_{s=1}^{r} a_{i_{t_s},j_{t_s}}(\mathbf{x})$ times a product of consecutive principle minors involving consecutive indices. For example,

$$\det J[\{1,2,3,5,6,9,11,12\},\{2,3,4,5,6,8,11,12\}] = \\ a_{1,2}(\mathbf{x})a_{2,3}(\mathbf{x})a_{3,4}(\mathbf{x})\det J[\{5,6\},\{5,6\}]a_{9,8}(\mathbf{x})\det J[\{11,12\},\{11,12\}].$$

Thus it follows that if all the consecutive principal minors are polynomials in $\mathbf{x}$ with nonnegative coefficients, then J is $\mathbf{x}$-TP. □

Lemma 2. *Suppose that*

$$J(\mathbf{x}) = \begin{bmatrix} s_0(\mathbf{x}) & r_1(\mathbf{x}) & & & \\ t_1(\mathbf{x}) & s_1(\mathbf{x}) & r_2(\mathbf{x}) & & \\ & t_2(\mathbf{x}) & s_2(\mathbf{x}) & r_3(\mathbf{x}) & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

is tridiagonal matrix of nonnegative polynomials in $\mathbb{R}[\mathbf{x}]$, where $\sigma = (s_i(\mathbf{x}))_{i\geq 1}$, $\pi = (r_i(\mathbf{x}))_{i\geq 0}$, and $\tau = (t_{i+1}(\mathbf{x}))_{i\geq 0}$ are sequences of non-zero polynomials over $\mathbb{R}$ with nonnegative coefficients such that

(i) $s_0(\mathbf{x}) - 1 \geq_{\mathbf{x}} 0$,

(ii) $s_i(\mathbf{x})s_{i+1}(\mathbf{x}) - t_{i+1}(\mathbf{x})r_{i+1}(\mathbf{x}) \geq_{\mathbf{x}} 0$ *for all* $i \geq 0$,

(iii) $s_{i+1}(\mathbf{x}) - t_{i+1}(\mathbf{x})r_{i+1}(\mathbf{x}) - 1 \geq_{\mathbf{x}} 0$ *for all* $i \geq 0$.

Then $J(\mathbf{x})$ is $\mathbf{x}$-TP.

Proof. By Lemma 1, we need only show that all the consecutive principal minors of $J(\mathbf{x})$ are nonnegative polynomials in $\mathbf{x}$.

Let $M_k(\mathbf{x}) = \det J[\{1,\ldots,k\},\{1,\ldots,k\}]$. First we shall prove by induction that $M_k(\mathbf{x}) \geq_{\mathbf{x}} 0$ for all $1 \leq k \leq n$ and $M_k(\mathbf{x}) - M_{k-1}(\mathbf{x}) \geq_{\mathbf{x}} 0$ for all $2 \leq k \leq n$. Note that $M_1(\mathbf{x}) = s_0(\mathbf{x}) \geq_{\mathbf{x}} 0$ and $M_2(\mathbf{x}) = s_0(\mathbf{x})s_1(\mathbf{x}) - t_1(\mathbf{x})r_1(\mathbf{x}) \geq_{\mathbf{x}} 0$ by assumption. Then note that

$$\begin{aligned} M_2(\mathbf{x}) - M_1(\mathbf{x}) &= s_0(\mathbf{x})s_1(\mathbf{x}) - t_1(\mathbf{x})r_1(\mathbf{x}) - s_0(\mathbf{x}) \\ &= s_0(\mathbf{x})(s_1(\mathbf{x}) - t_1(\mathbf{x})r_1(\mathbf{x}) - 1) + t_1(\mathbf{x})r_1(\mathbf{x})(s_0(\mathbf{x}) - 1) \geq_{\mathbf{x}} 0 \end{aligned}$$

since we are assuming that that $s_0(\mathbf{x}) - 1 \geq_{\mathbf{x}} 0$ and $s_1(\mathbf{x}) - t_1(\mathbf{x})r_1(\mathbf{x}) - 1 \geq_{\mathbf{x}} 0$.

Next assume $k \geq 3$ and by induction that $M_{k-1}(\mathbf{x}) \geq_{\mathbf{x}} 0$ and $M_{k-1}(\mathbf{x}) - M_{k-2}(\mathbf{x}) \geq_{\mathbf{x}} 0$. Then expanding M_k about the last row, we see that

$$\begin{aligned} M_k(\mathbf{x}) &= s_k(\mathbf{x})M_{k-1}(\mathbf{x}) - r_k(\mathbf{x})t_k(\mathbf{x})M_{k-2}(\mathbf{x}) \\ &= (s_k(\mathbf{x}) - r_k(\mathbf{x})t_k(\mathbf{x}))M_{k-1}(\mathbf{x}) + r_k(\mathbf{x})t_k(\mathbf{x})(M_{k-1}(\mathbf{x}) - M_{k-2}(\mathbf{x})) \geq_{\mathbf{x}} 0. \end{aligned}$$

Similarly,

$$M_k(\mathbf{x}) - M_{k-1}(\mathbf{x}) = (s_k(\mathbf{x}) - r_k(\mathbf{x})t_k(\mathbf{x}) - 1)M_{k-1}(\mathbf{x}) + r_k(\mathbf{x})t_k(\mathbf{x})(M_{k-1}(\mathbf{x}) - M_{k-2}(\mathbf{x})) \geq_{\mathbf{x}} 0.$$

Thus it follows that $M_k(\mathbf{x}) \geq_{\mathbf{x}} 0$ for all k.

For consecutive principle minors of the form $\det J[\{m, m+1, \ldots, m+r\}, \{m, m+1, \ldots, m+r\}]$ where $m > 1$, we note $s_m(\mathbf{x}) - 1 \geq_{\mathbf{x}} 0$ since $s_m(\mathbf{x}) - t_m(\mathbf{x})r_m(\mathbf{x}) - 1 \geq_{\mathbf{x}} 0$. It then follows that the matrix $J[\{s, s+1, \ldots, s+r\}, \{s, s+1, \ldots, s+r\}]$ satisfies the hypothesis of the theorem so that the proof that $M_n(\mathbf{x}) \geq_{\mathbf{x}} 0$ for all $n \geq 1$ can be applied to show that

$$\det J[\{s, s+1, \ldots, s+r\}, \{s, s+1, \ldots, s+r\}] \geq_{\mathbf{x}} 0.$$

□

Wang and Zhu [20, Lemma 3.3], showed that if $b_n(q)$ and $c_n(q)$ be all q-nonnegative, then the corresponding tridiagonal matrix is q-TP. The method of the proof used in [20, Lemma 3.3] can be carried over verbatim to its $\mathbf{x}$-analogue. Here we omit the details for brevity.

Lemma 3. *Let $(b_1(\mathbf{x}), b_2(\mathbf{x}), \ldots)$ and $(c_1(\mathbf{x}), c_2(\mathbf{x}), \ldots)$ be sequences of polynomials in $\mathrm{R}[\mathbf{x}]$ with nonnegative coefficients. Then the tridiagonal matrix*

$$J^{b,c} = \begin{bmatrix} b_1(\mathbf{x}) + c_1(\mathbf{x}) & 1 & & \\ b_2(\mathbf{x})c_1(\mathbf{x}) & b_2(\mathbf{x}) + c_2(\mathbf{x}) & 1 & \\ & b_3(\mathbf{x})c_2(\mathbf{x}) & b_3(\mathbf{x}) + c_3(\mathbf{x}) & \ddots \\ & \ddots & \ddots & \end{bmatrix}$$

is $\mathbf{x}$-TP.

Theorem 4. *Let $J = J^{(\pi,\sigma,\tau)}(\mathbf{x})$ be the tridiagonal matrix*

$$J = \begin{bmatrix} s_0(\mathbf{x}) & r_1(\mathbf{x}) & & & \\ t_1(\mathbf{x}) & s_1(\mathbf{x}) & r_2(\mathbf{x}) & & \\ & t_2(\mathbf{x}) & s_2(\mathbf{x}) & r_3(\mathbf{x}) & \\ & \ddots & \ddots & \ddots & \\ & & t_{n-1}(\mathbf{x}) & s_{n-1}(\mathbf{x}) & r_n(\mathbf{x}) \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

where $\sigma = (s_i(\mathbf{x}))_{i\geq 1}$, and $\tau = (t_{i+1}(\mathbf{x}))_{i\geq 0}$ are sequences of non-zero polynomials over $\mathbb{R}$ with non-negative coefficients and $r_k(\mathbf{x}) = 1$ for all $k \geq 0$. Let $M(\mathbf{x})$ be the a lower triangular matrix of polynomials

$$M(\mathbf{x}) := M^{\pi,\sigma,\tau}(\mathbf{x}) = [m_{n,k}(\mathbf{x})]_{0\leq k\leq n}$$

where the $m_{n,k}(\mathbf{x})$ are defined by (4). Then if the coefficient matrix J is $\mathbf{x}$-totally positive, the sequence $(m_{n,0}(\mathbf{x}))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials.

Proof. Let $M(\mathbf{x}) = [m_{i,j}(\mathbf{x})]_{i,j\geq 0}$ and $H(\mathbf{x}) = [m_{i+j,0}(\mathbf{x})]_{i,j\geq 0}$ be the Hankel matrix of the $\mathbf{x}$-Catalan-like numbers $m_{n,0}(\mathbf{x})$. We need to show that $H(\mathbf{x})$ is $\mathbf{x}$-totally positive. Let $T_0(\mathbf{x}) = 1, T_k(\mathbf{x}) = t_1(\mathbf{x})\cdots t_k(\mathbf{x})$ and $T = \operatorname{diag}(T_0(\mathbf{x}), T_1(\mathbf{x}), T_2(\mathbf{x}), \ldots)$. Then it is not difficult to verify that

$$H(\mathbf{x}) = M(\mathbf{x})T(\mathbf{x})M(\mathbf{x})^t,$$

see [2, (2.5)]. By the Cauchy-Binet Theorem, any minor of $H(\mathbf{x})$ can be expressed of sums of products of minors of $M(\mathbf{x})$, $T(\mathbf{x})$, and $M(\mathbf{x})^t$. It follows that if $M(\mathbf{x})$, $T(\mathbf{x})$, and $M(\mathbf{x})^t$ are $\mathbf{x}$-TP matrices, then $H(\mathbf{x})$ is $\mathbf{x}$-TP. It is clear that $T(\mathbf{x})$ is $\mathbf{x}$-TP and $M(\mathbf{x})$ is $\mathbf{x}$-TP if and only if $M(\mathbf{x})^t$ is $\mathbf{x}$-TP. Thus to show that $H(\mathbf{x})$ is $\mathbf{x}$-TP, we need only show that $M(\mathbf{x})$ is $\mathbf{x}$-TP. Thus we need only show that $J(x)$ being $\mathbf{x}$-TP implies $M(\mathbf{x})$ is $\mathbf{x}$-TP. Let $M_n(\mathbf{x}) = [m_{i,j}(\mathbf{x})]_{\mathbf{0\leq i,j\leq n}}$ be the n-th leading principal submatrix of $M(\mathbf{x})$. Clearly, to show that $M(\mathbf{x})$ is $\mathbf{x}$-TP, it suffices to show that $M_n(\mathbf{x})$ are $\mathbf{x}$-TP for $n \geq 0$. We do this by induction on n. Obviously, $M_0(\mathbf{x})$ is $\mathbf{x}$-TP. Assume that $M_n(\mathbf{x})$ is $\mathbf{x}$-TP. Then one can easily show that (4) implies that

$$M_{n+1}(\mathbf{x}) = \begin{bmatrix} e_{n+1}(\mathbf{x}) & \\ 0_n & M_n(\mathbf{x}) \end{bmatrix} \begin{bmatrix} e_{n+1}(\mathbf{x}) \\ J_n(\mathbf{x}) \end{bmatrix},$$

where $e_{n+1}(\mathbf{x}) = [1, 0, \ldots, 0]$, 0_n is column of n 0s, and $J_n(\mathbf{x})$ is the $n\times(n+1)$ principal submatrix of $J(\mathbf{x})$. By the induction hypothesis, $M_n(\mathbf{x})$ is $\mathbf{x}$-TP so that $\begin{bmatrix} 1 & 0 \\ 0 & M_n(\mathbf{x}) \end{bmatrix}$ is $\mathbf{x}$-TP. On the other hand, $J_n(\mathbf{x})$ is $\mathbf{x}$-TP since it is a submatrix of the $\mathbf{x}$-TP matrix $J(\mathbf{x})$, so that $\begin{bmatrix} e_{n+1}(\mathbf{x}) \\ J_n(\mathbf{x}) \end{bmatrix}$ is $\mathbf{x}$-TP. Applying the Cauchy-Binet Theorem again, we see that $M_{n+1}(\mathbf{x})$ is $\mathbf{x}$-TP. $\square$

Given a polynomial $a(\mathbf{x}) = \sum_{(i_1,\ldots,i_n)\in I} c_{i_1,\ldots,i_n} x_1^{i_1}\cdots x_n^{i_n}$ where I is finite index set and $c_{i_1,\ldots,i_n} \neq 0$ for all $(i_1,\ldots,i_n) \in I$, we let the degree of $a(\mathbf{x})$, $\deg(a(\mathbf{x}))$, equal $\max(\{i_1+\cdots+i_n : (i_1,\ldots,i_n) \in I\})$. We say that $a(\mathbf{x})$ is homogeneous of degree n if $i_1+\cdots+i_n = n$ for all $(i_1,\ldots,i_n) \in I$ and is inhomogeneous otherwise. If $a(\mathbf{x})$ had degree n, then we let

$$H_{x_0}(a(\mathbf{x})) = x_0^n a\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

For example, if $a(x_1, x_2) = 1 + x_1 + x_1x_2 + x_1^3$, then

$$H_{x_0}(a(x_1,x_2)) = x_0^3\left(1 + \frac{x_1}{x_0} + \frac{x_1}{x_0}\frac{x_2}{x_0} + \frac{x_1}{x_0}\frac{x_1}{x_0}\frac{x_1}{x_0}\right) = x_0^3 + x_0^2x_1 + x_0x_1x_2 + x_1^3.$$

Clearly if $\deg(a(\mathbf{x})) = n$, then $H_{x_0}(a(\mathbf{x}))$ is a homogeneous polynomial of degree n.

Theorem 5. *Suppose that $\alpha = (a_0(\mathbf{x}), a_1(\mathbf{x}), a_2(\mathbf{x}), \ldots)$ is a Stieltjes moment sequence of polynomials such that for all $n \geq 0$, $\deg(a_n(\mathbf{x})) = n$. Then $H_{x_0}(\alpha) = (H_{x_0}(a_0(\mathbf{x})), H_{x_0}(a_1(\mathbf{x})), \ldots)$ is a Stieltjes moment sequence of polynomials.*

Proof. Let

$$\mathbf{H} = [H_{x_0}(a_{i+j}(\mathbf{x}))]_{i,j\geq 0} = \begin{bmatrix} H_{x_0}(a_0(\mathbf{x})) & H_{x_0}(a_1(\mathbf{x})) & H_{x_0}(a_2(\mathbf{x})) & H_{x_0}(a_3(\mathbf{x})) & \dots \\ H_{x_0}(a_1(\mathbf{x})) & H_{x_0}(a_2(\mathbf{x})) & H_{x_0}(a_3(\mathbf{x})) & H_{x_0}(a_4(\mathbf{x})) & \dots \\ H_{x_0}(a_2(\mathbf{x})) & H_{x_0}(a_3(\mathbf{x})) & H_{x_0}(a_4(\mathbf{x})) & H_{x_0}(a_5(\mathbf{x})) & \dots \\ H_{x_0}(a_3(\mathbf{x})) & H_{x_0}(a_4(\mathbf{x})) & H_{x_0}(a_5(\mathbf{x})) & H_{x_0}(a_6(\mathbf{x})) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$H = H(\alpha, \mathbf{x}) = [a_{i+j}(\mathbf{x})]_{i,j\geq 0} = \begin{bmatrix} a_0(\mathbf{x}) & a_1(\mathbf{x}) & a_2(\mathbf{x}) & a_3(\mathbf{x}) & \dots \\ a_1(\mathbf{x}) & a_2(\mathbf{x}) & a_3(\mathbf{x}) & a_4(\mathbf{x}) & \dots \\ a_2(\mathbf{x}) & a_3(\mathbf{x}) & a_4(\mathbf{x}) & a_5(\mathbf{x}) & \dots \\ a_3(\mathbf{x}) & a_4(\mathbf{x}) & a_5(\mathbf{x}) & a_6(\mathbf{x}) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since α is a Stieltjes moment sequence of polynomials, we know that if $1 \leq i_1 < \cdots < i_k$ and $1 \leq j_1 < \cdots < j_k$, then the minor $\det(H[\{i_1, \dots, i_k\}, \{j_1, \dots, j_k\}])$ equals

$$\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{s=1}^{k} a_{i_s - 1 + j_{\sigma(s)} - 1}(\mathbf{x}) = \sum_{(r_1, \dots, r_n) \in I(i_1, \dots, i_n; j_1, \dots, j_n)} c_{r_1, \dots, r_n} x_1^{r_1} \cdots x_n^{r_n}$$

for some finite index set $I(i_1, \dots, i_n; j_1, \dots, j_n)$ where $c_{r_1, \dots, r_n} \geq 0$ for all $(r_1, \dots, r_n) \in I(i_1, \dots, i_n; j_1, \dots, j_n)$. Since $a_n(\mathbf{x}) \geq_{\mathbf{x}} 0$ and $\deg(a_n(\mathbf{x})) = n$, the degree of any term of the form $\prod_{s=1}^{k} a_{i_s - 1 + j_{\sigma(s)} - 1}(\mathbf{x})$ is

$$\sum_{s=0}^{k} i_s - 1 + j_{\sigma(s)} - 1 = \sum_{s=0}^{k} i_s - 1 + j_s - 1.$$

Thus the degree of $\det(H[\{i_1, \dots, i_k\}, \{j_1, \dots, j_k\}])$ less than or equal to $\sum_{s=0}^{k} i_s - 1 + j_s - 1$. In particular, $r_1 + \cdots + r_n \leq \sum_{s=0}^{k} i_s - 1 + j_s - 1$ for all $(r_1, \dots, r_n) \in I(i_1, \dots, i_n; j_1, \dots, j_n)$. But then $\det(\mathbf{H}[\{i_1, \dots, i_k\}, \{j_1, \dots, j_k\}])$ equals

$$\begin{aligned} &\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{s=1}^{k} x_0^{i_s - 1 + j_{\sigma(s)} - 1} a_{i_s - 1 + j_{\sigma(s)} - 1}\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \\ &= x_0^{\sum_{s=0}^{k} i_s - 1 + j_s - 1} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{s=1}^{k} a_{i_s - 1 + j_{\sigma(s)} - 1}\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \\ &= x_0^{\sum_{s=0}^{k} i_s - 1 + j_s - 1} \sum_{(r_1, \dots, r_n) \in I(i_1, \dots, i_n; j_1, \dots, j_n)} c_{r_1, \dots, r_n} \left(\frac{x_1}{x_0}\right)^{r_1} \cdots \left(\frac{x_n}{x_0}\right)^{r_n}. \end{aligned}$$

By our remarks above, $x_0^{\sum_{s=0}^{k} i_s - 1 + j_s - 1} \sum_{(r_1, \dots, r_n) \in I(i_1, \dots, i_n; j_1, \dots, j_n)} c_{r_1, \dots, r_n} \left(\frac{x_1}{x_0}\right)^{r_1} \cdots \left(\frac{x_n}{x_0}\right)^{r_n}$ is polynomial in $x_0, \mathbf{x}$ with nonnegative coefficients. Thus $\mathbf{H}(\alpha)$ is $(x_0, \mathbf{x})$-TP so that $H_{x_0}(\alpha)$ is a Stieltjes moment sequence of polynomials. $\square$

3 Applications

In this section, we shall use the results of the previous section to produce many combinatorially interesting examples of Stieltjes moment sequences of polynomials.

Example 3.1. Let $\pi = (r_1(q), r_2(q), r_3(q), \ldots) = (1, 1, 1, \ldots)$, $\sigma = (s_0(q), s_1(q), s_2(q), \ldots) = (1, 1+q, 1+q, \ldots)$ and $\tau = (t_1(q), t_2(q), t_3(q), \ldots) = (q, q, q, \ldots)$. It is easy to check that these sequences satisfy the hypothesis of Lemma 2. In this case, we are considering the polynomials defined by

$$\begin{aligned} a_{0,0}(q) &= 1, \\ a_{n+1,0}(q) &= a_{n,0}(q) + q a_{n,1}(q) \text{ for } n \geq 1, \text{ and} \\ a_{n+1,k}(q) &= a_{n,k-1}(q) + (1+q)a_{n,k}(q) + q a_{n,k+1}(q) \text{ for } 1 \leq k \leq n. \end{aligned}$$

where $a_{n,k}(q) = 0$ unless $n \geq k \geq 0$. In this case, the underlying combinatorial objects are Motzkin paths where the weights of the up-steps are 1, the weights of the down-steps are q and the weights of the level-steps are 1 at level 0 and $1+q$ at levels $k > 0$. Thus we can interpret $a_{n,k}(q)$ as the sum of the weights of Motzkin paths that start at $(0,0)$ and end at (n,k). For example, if $A(q) = [a_{n,k}(q)]$, then

$$A(q) = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1+q & 2+q & 1 & & \\ 1+3q+q^2 & 3+5q+q^2 & 3+2q & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Proposition 6. *The sequence $(a_{n,0}(q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials.*

A *Riordan array*, denoted by $(d(x), h(x))$, is an infinite lower triangular matrix whose generating function of the kth column is $x^k h^k(x) d(x)$ for $k = 0, 1, 2, \ldots$, where $d(0) = 1$ and $h(0) \neq 0$ [15]. A Riordan array $R = [r_{n,k}]_{n,k\geq 0}$ can be characterized by two sequences $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ such that

$$r_{0,0} = 1, \quad r_{n+1,0} = \sum_{j\geq 0} z_j r_{n,j}, \quad r_{n+1,k+1} = \sum_{j\geq 0} a_j r_{n,k+j} \tag{5}$$

for $n, k \geq 0$ (see [10] for instance). Call $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ the A- and Z-sequences of R respectively. Let $Z(x) = \sum_{n\geq 0} z_n x^n$ and $A(x) = \sum_{n\geq 0} a_n x^n$ be the generating functions of $(z_n)_{n\geq 0}$ and $(a_n)_{n\geq 0}$ respectively. Then it follows from (5) that

$$d(x) = \frac{1}{1 - xZ(xh(x))} \text{ and } h(x) = A(xh(x)). \tag{6}$$

The matrix $R(a, b; c, e) = [r_{n,k}]_{n,k\geq 0}$, where

$$\begin{cases} r_{0,0} = 1, \quad r_{n+1,0} = a r_{n,0} + b r_{n,1}, \\ r_{n+1,k+1} = r_{n,k} + c r_{n,k+1} + e r_{n,k+2}, \end{cases} \tag{7}$$

is called the recursive matrix. The coefficient matrix of the recursive matrix (7) is defined to be

$$J(p,q;s,t)=\begin{bmatrix} a & 1 & & & \\ b & c & 1 & & \\ & e & c & 1 & \\ & & e & c & \ddots \\ & & & \ddots & \ddots \end{bmatrix}. \tag{8}$$

Now $R(a,b;c,e)$ is a Riordan array with $Z(x)=a+bx$ and $A(x)=1+cx+ex^2$. Let $R(a,b;c,e)=(d(x),h(x))$. Then by (6), we have

$$d(x)=\frac{1}{1-x(a+bxh(x))} \text{ and } h(x)=1+cxh(x)+ex^2h^2(x).$$

It follows that

$$h(x)=\frac{1-cx-\sqrt{1-2cx+(c^2-4e)x^2}}{2ex^2}$$

and

$$d(x)=\frac{2e}{2e-b+(bc-2ae)x+b\sqrt{1-2cx+(c^2-4e)x^2}}$$

(see [19] for details). From this formula we can now derive a number of interesting examples. For example, taking $a=1$, $b=q$, $c=1+q$ and $e=q$ in (8), we obtain the generating function of the $(a_{n,0}(q))$ is

$$d_A(x,q)=\sum_{n\geq 0}a_{n,0}(q)x^n=\frac{2}{1+(q-1)x+\sqrt{1-2(1+q)x+(1-q)^2x^2}}.$$

Remark 7. 1. When we set $q=1$ in $A(q)$, we obtain the Catalan triangle of Aigner [1]. See also sequence [17, A039599] in the On-line Encyclopedia of Integer Sequences. It follows that $a_{n,0}(1)=C_n$ where $C_n=\frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number [17, A000108]. Hence $a_{n,0}(q)$ is a q-analogue of the Catalan number C_n.

2. When we set $q=2$ in $A(q)$, we obtain the triangle [17, A172094] and $a_{n,0}(2)$ are the little Schröder numbers S_n [17, A001003]. It follows that $a_{n,0}(2q)$ is a q-analogue of n-th little Schröder number S_n.

3. When we set $q=3$, the sequence $(a_{n,0}(3))_{n\geq 0}$ is sequence [17, A007564]. It follows that $a_{n,0}(3q)$ is a q-analogue of the sequence [17, A007564].

4. When we set $q=4$, the sequence $(a_{n,0}(4))_{n\geq 0}$ is sequence [17, A059231]. It follows that $a_{n,0}(4q)$ is a q-analogue of the sequence [17, A059231].

We know that for any $m\geq 1$, the sequence $(a_{n,0}(m))_{n\geq 0}$ is a Stieltjes moment sequence. In particular, the Catalan numbers C_n and the little Schröder numbers S_n are a Stieltjes moment sequences.

Example 3.2. Let

$$\begin{aligned}
\pi &= (r_1(q),r_2(q),r_3(q),\ldots)=(1,1,1,\ldots),\\
\sigma &= (s_0(q),s_1(q),s_2(q),\ldots)=(1+q+q^2,1+q+q^2,1+q+q^2,\ldots), \text{ and}\\
\tau &= (t_1(q),t_2(q),t_3(q),\ldots)=(q,q,q,\ldots).
\end{aligned}$$

It is easy to check that these sequences satisfy the hypothesis of Lemma 2. In this case, we are considering the polynomials defined by

$$\begin{aligned}
b_{0,0}(q) &= 1, \\
b_{n+1,0}(q) &= (1+q+q^2)b_{n,0}(q) + qb_{n,1}(q) \text{ for } n \geq 1, \text{and} \\
b_{n+1,k}(q) &= b_{n,k-1}(q) + (1+q+q^2)b_{n,k}(q) + qb_{n,k+1}(q) \text{ for } 1 \leq k \leq n
\end{aligned}$$

where $b_{n,k}(q) = 0$ unless $n \geq k \geq 0$. In this case, the underlying combinatorial objects are Motzkin paths where the weights of the up-steps are 1, the weights of the down-steps are q, and the weights of the level-steps $1+q+q^2$. Thus we can interpret $b_{n,k}(q)$ as the sum of the weights of Motzkin paths that start at $(0,0)$ and end at (n,k). For example, if $B(q) = [b_{n,k}(q)]$, then

$$B(q) = \begin{bmatrix}
1 & & & & \\
1+q+q^2 & 1 & & & \\
1+3q+3q^2+2q^3+q^4 & 2+2q+2q^2 & 1 & & \\
\begin{array}{c}\big(1+6q+9q^2+10q^3+ \\ 6q^4+3q^5+q^6\big)\end{array} & 3+8q+9q^2+6q^3+3q^4 & 3+3q+3q^2 & 1 & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.$$

Proposition 8. *The sequence $(b_{n,0}(q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials.*

Taking $a = 1+q+q^2$, $b = q$, $c = 1+q+q^2$ and $e = q$ in (8), we obtain the generating function of the $(b_{n,0}(q))$ is

$$d_B(x,q) = \sum_{n\geq 0} b_{n,0}(q)x^n = \frac{2}{1-(1+q+q^2)x+\sqrt{1-2(1+q+q^2)x+((1+q+q^2)^2-4q)x^2}}.$$

Remark 9. In this case, the triangle $B(1)$ is [17, A091965] and the first column $(b_{n,0}(1))_{n\geq 0}$ is sequence [17, A002212]. Clearly $b_{n,0}(1)$ counts the number of 3-colored Motzkin paths of length n and the number of restricted hexagonal polyominoes with n cells.

It follows that for any $m \geq 1$, the sequence $(d_{n,0}(m))_{n\geq 0}$ is a Stieltjes moment sequence. In particular, the sequence which counts the number of restricted hexagaonal polynominoes is a Stieltjes momoment sequence.

Example 3.3. Let

$$\begin{aligned}
\pi &= (r_1(p,q), r_2(p,q), r_3(p,q), \ldots) = (1,1,1,\ldots), \\
\sigma &= (s_0(p,q), s_1(p,q), s_2(p,q), \ldots) = (1+p+q, 1+p+q, 1+p+q, \ldots), \text{ and} \\
\tau &= (t_1(p,q), t_2(p,q), t_3(p,q), \ldots) = (q,q,q,\ldots).
\end{aligned}$$

It is easy to check that these sequences satisfy the hypothesis of Lemma 2. In this case, we are considering the polynomials defined by

$$\begin{aligned}
c_{0,0}(p,q) &= 1, \\
c_{n+1,0}(p,q) &= (1+p+q)c_{n,0}(p,q) + qc_{n,1}(p,q) \text{ for } n \geq 1, \text{ and} \\
c_{n+1,k}(p,q) &= c_{n,k-1}(p,q) + (1+p+q)c_{n,k}(p,q) + qc_{n,k+1}(p,q) \text{ for } 1 \leq k \leq n
\end{aligned}$$

where $c_{n,k}(p,q) = 0$ unless $n \geq k \geq 0$. In this case, the underlying combinatorial objects are Motzkin paths where the weights of the up-steps are 1, the weights of the down-steps are q, and the weights of the level-steps $1+p+q$. Thus we can interpret $c_{n,k}(p,q)$ as the sum of the weights of Motzkin paths that start at $(0,0)$ and end at (n,k). In particular, we can interpret $c_{n,}(p,q)$ as weighted sum over three colored Motzkin paths. That is, the levels of the Motzkin path can be colored with one of three colors, namely, color 0 which has weight 1, color 1 which has weight q, and color 2 which has weight p, and the down-steps have weight q. For example, if $C(p,q) = [c_{n,k}(p,q)]$, then

$$C(p,q) = \begin{bmatrix} 1 & & & & \\ 1+p+q & 1 & & & \\ 1+3p+2q+2pq+p^2+q^2 & 2+2p+2q & 1 & & \\ \begin{pmatrix}1+6p+3q+6p^2+9pq+3q^2+\\ p^3+3p^2q+3pq^2+q^3\end{pmatrix} & \begin{pmatrix}3+8p+6q+\\ 3p^2+6pq+3q^2\end{pmatrix} & 3+3p+3q & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Proposition 10. *The sequence $(c_{n,0}(p,q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials.*

Taking $a = 1+p+q$, $b = q$, $c = 1+p+q$ and $e = q$ in (8), we obtain the generating function of the $c_{n,0}(p,q)$s is

$$d_C(x,p,q) = \sum_{n\geq 0} c_{n,0}(p,q)x^n = \frac{2}{1-(1+p+q)x+\sqrt{1-2(1+p+q)x+((1+p+q)^2-4q)x^2}}.$$

Remark 11.

1. When we set $p = q = 1$ in $(c_{n,0}(1,1))_{n\geq 0}$, we obtain the $1, 3, 10, 36, 137, \ldots$ which is sequence [17, A002212]. Besides counting 3-colored Motzkin path, it also the number of restricted hexagonal polyominoes with n-cells.

2. When we set $p = 1$ and $q = 2$ in $(c_{n,0}(p,q))_{n\geq 0}$, we obtain the $1, 4, 18, 88, 456, 2464, \ldots$ which is sequence [17, A024175].

3. When we set $p = 2$ and $q = 2$ in $(c_{n,0}(p,q))_{n\geq 0}$, we obtain the $1, 4, 20, 112, 672, 4224, \ldots$ which is sequence [17, A003645] whose n-th term is 2^nC_{n+1}.

There are many variations of the $t_k(p,q)$-sequence that will also produce Steiltjes moment sequence of polynomials. For example, suppose we define $t_k^{(s)}(p,q)$ to be q if $k \leq s$ and p if $k > s$ and let $\tau^{(s)} = (t_1^{(s)}(p,q), t_2^{(s)}(p,q), t_3^{(s)}(p,q), \ldots)$. It is easy to see the sequences $\pi = (r_1(p,q), r_2(p,q), r_3(p,q), \ldots) = (1,1,1,\ldots)$, $\sigma = (s_0(p,q), s_1(p,q), s_2(p,q), \ldots) = (1+p+q, 1+p+q, 1+p+q, \ldots)$ and $\tau^{(s)}$ satisfy the hypothesis of Lemma 2 for all s. Then we can define the polynomials $c_{n,k}^{(s)}(p,q)$ by

$$\begin{aligned} c_{0,0}^{(s)}(p,q) &= 1, \\ c_{n+1,0}^{(s)}(p,q) &= (1+p+q)c_{n,0}^{(s)}(p,q) + t_1^{(s)}(p,q)c_{n,1}^{(s)}(p,q) \text{ for } n \geq 1, \text{ and} \\ c_{n+1,k}^{(s)}(p,q) &= c_{n,k-1}^{(s)}(p,q) + (1+p+q)c_{n,k}^{(s)}(p,q) + t_{k+1}^{(s)}(p,q)c_{n,k+1}^{(s)}(p,q) \text{ for } 1 \leq k \leq n \end{aligned}$$

where $c_{n,k}^{(s)}(p,q) = 0$ unless $n \geq k \geq 0$.

Proposition 12. *For all $s \geq 0$, $(c_{n,0}^{(s)}(p,q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials.*

One of the advantages of this set up is that we can set $p = 0$ in such sequences. In particular, $(c_{n,0}^{(s)}(0,q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials. In such a situation, $c_{n,0}^{(s)}(0,q)$ is the sum over the weights of 2-colored Motzkin paths of height $\leq s$. That is, the level steps can be colored with color 0 which has weight 1 or colored with color 1 which has weight q. The down-steps all have weight q and the up-steps all have weight 1.

We can also generalize this example by adding more variables. That is, let $\mathbf{x} = (\mathbf{x})$ where $n \geq 3$ and let $1 \leq s_1 < \cdots < s_{n-1}$. Then let $r_i(\mathbf{x}) = 1$ for all $i \geq 1$, $s_i(\mathbf{x}) = 1 + x_1 + \cdots + x_n$ for all $i \geq 1$, and $t_i^{(s_1,\ldots,s_{n-1})}(\mathbf{x})$ equal x_1 if $i \leq s_1$, x_j if $s_{j-1} < i \leq s_j$, and x_n if $i > s_{n-1}$. Then let $\pi = (r_1(\mathbf{x}), r_2(\mathbf{x}), r_3(\mathbf{x}), \ldots) = (1,1,1,\ldots)$, $\sigma = (s_0(\mathbf{x}), s_1(\mathbf{x}), s_2(\mathbf{x}), \ldots)$ and $\tau^{(s_1,\ldots,s_{n-1})} = (t_1^{(s_1,\ldots,s_{n-1})}(\mathbf{x}), t_2^{(s_1,\ldots,s_{n-1})}(\mathbf{x}), t_3^{(s_1,\ldots,s_{n-1})}(\mathbf{x}), \ldots)$. It is easy to check that for any $1 \leq s_1 < \cdots < s_{n-1}$, these sequences satisfy the hypothesis of Lemma 2. In this case, we are considering the polynomials defined by

$c_{0,0}^{(s_1,\ldots,s_{n-1})}(\mathbf{x}) = 1,$
$c_{n+1,0}^{(s_1,\ldots,s_{n-1})}(\mathbf{x}) = (1 + x_1 + \cdots + x_n)c_{n,0}^{(s_1,\ldots,s_{n-1})}(\mathbf{x}) + t_1^{(s_1,\ldots,s_{n-1})}(\mathbf{x})c_{n,1}^{(s_1,\ldots,s_{n-1})}(\mathbf{x})$ for $n \geq 1$, and
$c_{n+1,k}^{(s_1,\ldots,s_{n-1})}(\mathbf{x}) = c_{n,k-1}^{(s_1,\ldots,s_{n-1})}(\mathbf{x}) + (1 + x_1 + \cdots + x_n)c_{n,k}^{(s_1,\ldots,s_{n-1})}(\mathbf{x}) + t_{k+1}^{(s_1,\ldots,s_{n-1})}c_{n,k+1}^{(s_1,\ldots,s_{n-1})}(\mathbf{x})$
for $1 \leq k \leq n$,

where $c_{n,k}^{(s_1,\ldots,s_{n-1})}(\mathbf{x}) = 0$ unless $n \geq k \geq 0$. In this case, the underlying combinatorial objects are Motzkin paths where the weights of up-steps are 1, the weights of down-steps ending at level k are $t_{k+1}^{(s_1,\ldots,s_{n-1})}(\mathbf{x})$, and the weights of level-steps $1 + x_1 + \cdots + x_n$. Thus we can interpret $c_{n,k}(\mathbf{x})$ as the sum of the weights of Motzkin paths that start at $(0,0)$ and end at (n,k). In particular, we can interpret $c_{n,0}(\mathbf{x})$ as weighted sum over $(n+1)$-colored Motzkin paths. That is, the levels of the Motzkin path can be colored with one of $(n+1)$-colors, namely, color 0 which has weight 1, color i which has weight x_i for $i = 1, \ldots, n$, and the down-steps that end at level k have weight $t_{k+1}^{(s_1,\ldots,s_{n-1})}(\mathbf{x})$.

Proposition 13. *For all $1 \leq s_1 < \cdots < s_{n-1}$, $(c_{n,0}^{(s_1,\ldots,s_{n-1})}(\mathbf{x}))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials.*

Once again for any $2 \leq j \leq x_n$, the polynomials $(c_{n,0}^{(s_1,\ldots,s_{n-1})}(x_1, \ldots, x_{j-1}, 0, \ldots, 0))_{n\geq 0}$ Stieltjes moment sequence of polynomials and $c_{n,0}^{(s_1,\ldots,s_{n-1})}(x_1, \ldots, x_{j-1}, 0, \ldots, 0)$ equals to the sum of j-colored Motzkin paths of length n and height less that s_j.

Example 3.4. Let $[n]_{p,q} = \frac{p^n - q^n}{p-q} = p^{n-1} + qp^{n-2} + \cdots + q^{n-2}p + q^{n-1}$. Let

$$\begin{aligned}
\pi^{(u)} &= (r_1(p,q), r_2(p,q), r_3(p,q), \ldots) = (1,1,1,\ldots), \\
\sigma^{(u)} &= (s_0(p,q), s_1(p,q), s_2(p,q), \ldots) = ([u]_{p,q}, p+q, p+q, \ldots), \text{ and} \\
\tau^{(u)} &= (t_1(p,q), t_2(p,q), t_3(p,q), \ldots) = (pq[u-1]_{p,q}, pq, pq, \ldots),
\end{aligned}$$

where $u \geq 3$. In this case, we are considering the polynomials defined by

$$\begin{aligned} d_{0,0}^{(u)}(p,q) &= 1, \\ d_{n+1,0}^{(u)}(p,q) &= [u]_{p,q} d_{n,0}^{(u)}(p,q) + pq[u-1]_{p,q} d_{n,1}^{(u)}(p,q) \text{ for } n \geq 1, \text{ and} \\ d_{n+1,k}^{(u)}(p,q) &= d_{n,k-1}^{(u)}(p,q) + (p+q) d_{n,k}^{(u)}(p,q) + pq d_{n,k+1}^{(u)}(p,q) \text{ for } 1 \leq k \leq n \end{aligned}$$

where $d_{n,k}^{(u)}(p,q) = 0$ unless $n \geq k \geq 0$. In this case, the underlying combinatorial objects are Motzkin paths where the weights of the up-steps are 1, the weights of the down-steps that ends a level 0 are $pq[u-1]_{p,q}$ and the weights of the down-steps that end at level $k > 0$ are pq, and the weights of the level-steps is $[u]_{p,q}$ if the step is at level 0 and $p+q$ if the step is at level $k \geq 1$. Thus we can interpret $d_{n,k}^{(u)}(p,q)$ as the sum of the weights of Motzkin paths that start at $(0,0)$ and end at (n,k). For example, if $D^{(3)}(p,q) = [e_{n,k}(p,q)]$, then

$$D^{(3)}(p,q) = \begin{bmatrix} 1 & & & \\ p^2+pq+q^2 & 1 & & \\ (p^2+pq+q^2)^2 + pq(p+q) & p^2+pq+q^2+p+q & 1 & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In this case, the sequences do not satisfy the hypothesis of both Lemma 2 and Lemma 3. Nevertheless, we can prove directly that the tridiagonal matrix $J^{(u)} := J^{(\pi,\sigma,\tau)}(p,q)$ where

$$J^{(u)} = \begin{bmatrix} [u]_{p,q} & 1 & & & & \\ pq[u-1]_{p,q} & p+q & 1 & & & \\ & pq & p+q & 1 & & \\ & & pq & p+q & 1 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

is p,q-TP. By Lemma 1, we need only show that principal minors of the form $\det J^{(u)}[\{n,n+1,\ldots,n+k-1\},\{n,n+1,\ldots,n+k-1\}]$ are polynomials with nonnegative coefficients.

For minors of the form $\det J^{(u)}[\{n,n+1,\ldots,n+k-1\},\{n,n+1,\ldots,n+k-1\}]$ where $n > 1$, we are dealing with the tridiagonal matrix

$$L = \begin{bmatrix} p+q & 1 & & & & \\ pq & p+q & 1 & & & \\ & pq & p+q & 1 & & \\ & & pq & p+q & 1 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

which does satisfy the hypothesis of Lemma 3 where $b_i(p,q) = p$ and $c_i(p,q) = q$ for all $i \geq 1$. Thus we need only consider minors of the form $N_k^{(u)}(p,q) = \det J^{(u)}[\{1,2,\ldots,k\},\{1,2,\ldots,k\}]$. We shall prove by induction that $N_k^{(u)}(p,q) = [k+u-1]_{p,q}$. Note that $p[n]_{p,q} = p^n + pq[n-1]_{p,q}$ and $q[n]_{p,q} = pq[n-1]_{p,q} + q^n$. Hence $N_1^{(u)} = [u]_{p,q}$ and

$$N_2^{(u)} = (p+q)[u]_{p,q} - pq[u-1]_{p,q} = p^u + pq[u-1]_{p,q} + pq[u-1]_{p,q} + q^u - pq[u-1]_{p,q} = [u+1]_{p,q}.$$

Now assume that $k \geq 3$. Then by expanding the determinant about the last row of $J^{(u)}[\{1,2,\ldots,k\},\{1,2,\ldots,k\}]$, we see that

$$\begin{aligned} N_k^{(u)} &= (p+q)N_{k-1} - pqN_{k-2}^{(u)} = (p+q)[k-1+u-1]_{p,q} - pq[k-2+u-1]_{p,q} \\ &= p^{k+u-2} + pq[k+u-3]_{p,q} + pq[k+u-3]_{p,q} + q^{k+u-2} - pq[k+u-3]_{p,q} = [k+u-1]_{p,q}. \end{aligned}$$

Thus $J^{(u)}$ is q-TP for all $u \geq 3$.

Thus we can apply Theorem 4 to obtain the following result.

Proposition 14. *For all $u \geq 3$, $(d_{n,0}^{(u)}(p,q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials.*

Taking $a = [u]_{p,q}$, $b = pq[u-1]_{p,q}$, $c = p+q$ and $e = pq$ in (8), we obtain the generating function of the $(d_{n,0}^{(u)})$ is

$$d_{D,u}(x,p,q) = \sum_{n\geq 0} d_{n,0}^{(u)} x^n =$$

$$\frac{2}{2-[u-1]_{p,q} + ([u-1]_{p,q}(p+q) - 2[u]_{p,q})x + [u-1]_{p,q}\sqrt{1-2(p+q)x+(p-q)^2x^2}}.$$

Remark 15. Setting $p = q = 1$ in $(d_{n,0}^{(3)}(p,q))_{n\geq 0}$ gives the sequence $1, 3, 11, 43, 173, 707, 2917, \ldots$ which is sequence [17, A026671]. The combinatorial interpretation for the sequence is the number of lattice paths from $(0,0)$ to (n,n) using steps $(1,0)$, $(0,1)$, and $(1,1)$ when the step is on the diagonal.

Example 3.5. Let

$$\begin{aligned} \pi &= (r_1(p,q,r), r_2(p,q,r), r_3(p,q,r), \ldots) = (1,1,1,\ldots), \\ \sigma &= (s_0(p,q,r), s_1(p,q,r), s_2(p,q,r), \ldots) = (q+r, p+q+r, p+q+r, \ldots), \text{ and} \\ \tau &= (t_1(p,q,r), t_2(p,q,r), t_3(p,q,r), \ldots) = (q(p+r), q(p+r), q(p+r), \ldots). \end{aligned}$$

In this case, we are considering the polynomials defined by

$$\begin{aligned} i_{0,0}(p,q,r) &= 1, \\ i_{n+1,0}(p,q,r) &= (q+r)i_{n,0}(p,q,r) + q(p+r)i_{n,1}(p,q,r) \text{ for } n \geq 1, \text{and} \\ i_{n+1,k}(p,q,r) &= i_{n,k-1}(p,q,r) + (p+q+r)i_{n,k}(p,q,r) + q(p+r)i_{n,k+1}(p,q,r) \text{ for } 1 \leq k \leq n \end{aligned}$$

where $i_{n,k}(p,q,r) = 0$ unless $n \geq k \geq 0$. In this case, the underlying combinatorial objects are Motzkin paths where the weights of the up-steps are 1, the weights of the down-steps are $q(p+r)$, and the weights of the level steps at level 0 is $q+r$ and the weights of the level steps at level $k \geq 1$ are $p+q+r$. Thus we can interpret $i_{n,k}(p,q,r)$ as the sum of the weights of Motzkin paths that start at $(0,0)$ and end at (n,k). For example, if $I(p,q,r) = [i_{n,k}(q)]$, then

$$I(p,q,r) =$$
$$\begin{bmatrix} 1 & & & & \\ q+r & 1 & & & \\ (pq+q^2)+3qr+r^2 & (p+2q)+2r & 1 & & \\ \begin{matrix}(p^2q+3pq^2+q^3)+ \\ (4pq+6q^2)r+6qr^2+r^3\end{matrix} & (p^2+5pq+3q^2)+(3p+8q)r+3r^2 & (2p+3q)+3r & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In this case, the sequences do not satisfy the hypothesis of both Lemma 2 and Lemma 3. However, we can prove directly that the tridiagonal matrix $J := J^{(\pi,\sigma,\tau)}$ where

$$J = \begin{bmatrix} q+r & 1 & & & \\ q(p+r) & p+q+r & 1 & & \\ & q(p+r) & p+q+r & 1 & \\ & & q(p+r) & p+q+r & 1 \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

is (p,q,r)-TP. By Lemma 1, we need only show that principal minors of the form $\det J[\{n, n+1, \ldots, n+k-1\}, \{n, n+1, \ldots, n+k-1\}]$ are polynomials with nonnegative coefficients.

For consecutive minors of the form $\det J[\{n, n+1, \ldots, n+k-1\}, \{n, n+1, \ldots, n+k-1\}]$ where $n \geq 2$, are considering the matrix

$$\overline{J} = \begin{bmatrix} p+r+q & 1 & & & \\ q(p+r) & p+r+q & 1 & & \\ & q(p+r) & p+r+q & 1 & \\ & & q(p+r) & p+r+q & 1 \\ & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

This matrix satisfies the hypothesis of Lemma 3 where $(b_1(p,q,r), b_2(p,q,r), b_3(p,q,r), \ldots) = (p+r, p+r, p+r, \ldots)$ and $(c_1(p,q,r), c_2(p,q,r), c_3(p,q,r), \ldots) = (q,q,q\ldots)$.

Thus we need only consider minors of the form $P_k = \det J[\{1,2,\ldots,k\},\{1,2,\ldots,k\}]$. We shall prove by induction that $P_k = q^n + \sum_{k=1}^{n} r^k \left(\sum_{j=0}^{k} \binom{n-k-j+k-1}{k-1} q^j p^{n-k-j} \right)$. Clearly $P_1 = q+r$ and $P_2 = (q+r)(p+q+r) - q(p+r) = q^2 + (p+q)r + r^2$. Now assume that $k \geq 3$. Then by expanding the determinant about the last row of $J[\{1,2,\ldots,k\},\{1,2,\ldots,k\}]$, we see that

$$P_k = (p+q+r)P_{k-1} - q(p+r)P_{k-2}.$$

In particular,

$$P_k|_{r^0} = (p+q)P_{k-1}|_{r^0} - qpP_{k-2}|_{r^0} = (p+q)q^{k-1} - qpq^{k-2} = q^k.$$

Similarly,

$$P_k|_{r^n} = r * P_{k-1}|_{r^n} - q(p+r)P_{k-2}|_{r^n} = r * r^{n-1} - 0 = r^k.$$

Now suppose that $1 \leq r \leq n-1$. Then we must show that

$$P_k|_{r^k q^j} = (p+q)P_{k-1}|_{r^k q^j} + P_{k-1}|_{r^{k-1} q^j} - pqP_{k-2}|_{r^k q^j} - qP_{k-2}|_{r^{k-1} q^j}. \tag{9}$$

One can show that this yields a identity among binomial coefficients which can be directly checked with Mathematica. Thus J is p,q,r-TP.

Thus we can apply Theorem 4 to obtain the following result.

Proposition 16. *The sequence $(i_{n,0}(p,q,r))_{n \geq 0}$ is a Stieltjes moment sequence of polynomials.*

On the other hand, taking $a = q+r$, $b = q(p+r)$, $c = p+q+r$ and $e = q(p+r)$ in (8), we obtain the generating function of the first column $I(p,q,r)$ is

$$d(x,p,q,r) = \frac{2}{1 + (p-q-r)x + \sqrt{1 - 2(p+q+r)x + ((p+q+r)^2 - 4q(p+r))x^2}}.$$

Here is a list of the first few values of the sequence $(i_{n,0}(p,q,r))_{n\geq 0}$.

1

$q+r$

$(pq+q^2)+3qr+r^2$

$(p^2q+3pq^2+q^3)+\left(4pq+6q^2\right)r+6qr^2+r^3$

$(p^3q+6p^2q^2+6pq^3+q^4)+\left(5p^2q+20pq^2+10q^3\right)r+\left(10pq+20q^2\right)r^2+10qr^3+r^4$

Remark 17.

1. It follows from earlier results that $i_{n,0}(p,q,0)=\sum_{k=1}^{n}\frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1}q^kp^{n-k}$ from which it follows that $i_{n,0}(1,1,0)=C_n$ where $C_n=\frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

2. We can show that $i_{n,0}(p,q,1)=\sum_{k=1}^{n}\frac{1}{k+1}\binom{n+k}{k}\binom{n}{k}q^kp^{n-k}$ from which it follows that $(i_{n,0}(1,1,1))_{n\geq 0}$ is sequence [17, A006318] which is the sequence of large Schöoder numbers.

3. We can show that $(i_{n,0}(1,1,r))_{n\geq 0}$ is the triangle [17, A060693].

4. The sequence $(i_{n,0}(1,1,2))_{n\geq 0}$ starts out $1,3,12,57,300,1686,9912,\ldots$. This is sequence [17, A047891].

5. The sequence $(i_{n,0}(1,2,1))_{n\geq 0}$ starts out $1,3,13,67,381,2307,14598,\ldots$. This is sequence [17, A064062] which is the sequence of generalized Catalan numbers.

Next, in Examples 3.6-3.9, we will consider several sequences of polynomials in q that were studied by Zhu in [21]. In each case, Zhu showed that the sequence of polynomials is strongly q-log convex sequence. Chen et al. [8] proved that Narayana polynomials form a strongly q-log convex sequence. Chen et al. [9] also proved that Bell polynomials form a strongly q-log convex sequence. Wang and Zhu [20] further studied the polynomials of Zhu and showed that each sequence has the stronger property of being Stieltjes moment sequences of polynomials. Our results will show that there are natural (p,q)-analogues of these polynomials which are Stieltjes moment sequences of polynomials.

Example 3.6. Wang and Zhu [20] proved that the sequence of Narayana polynomials $(W_n(q))_{n\geq 0}$ of type B is a Stieltjes moment sequence of where $W_n(q)=\sum_{k=0}^{n}\binom{n}{k}^2q^k$. It follows from Theorem 5 that the sequence $(W_n(p,q))_{n\geq 0}$ where $W_n(p,q)=\sum_{k=0}^{n}\binom{n}{k}^2q^kp^{n-k}$ is a Stieltjes moment sequence of polynomials.

Proposition 18. *The sequence* $\left(\sum_{k=0}^{n}\binom{n}{k}^2p^kq^{n-k}\right)_{n\geq 0}$ *is a Stieltjes moment sequence of polynomials.*

Taking $a=p+q$, $b=2pq$, $c=p+q$ and $e=pq$ in (8), we obtain the generating function of the $e_{n,0}(p,q)$s is

$$d(x,p,q)=\frac{1}{\sqrt{1-2(p+q)x+(p-q)^2x^2}}.$$

Remark 19.

1. When we set $p=q=1$ in $(e_{n,0}(p,q))_{n\geq 0}$, we obtain the $1,2,6,20,70,252,\ldots$ which is sequence [17, A001850] which are central binomial coefficients $\binom{2n}{n}$. Thus we can view $(e_{n,0}(p,q))_{n\geq 0}$ as a (p,q)-analogue of the central binomial coefficients.

2. When we set $p = 2$ and $q = 1$ in $(e_{n,0}(p,q))_{n\geq 0}$, we obtain the $1, 3, 13, 63, 321, 1683, \ldots$ which is sequence [17, A000984] which are central Delannoy numbers. Thus we can view $(e_{n,0}(2p,q))_{n\geq 0}$ as a (p,q)-analogue of the central Delannoy numbers.

3. When we set $p = 2$ and $q = 2$ in $(e_{n,0}(p,q))_{n\geq 0}$, we obtain the $1, 4, 24, 160, 1120, 8064, \ldots$ which is sequence [17, A059304]. This n-th term of this sequence also counts the number of paths from (0,0) to (n,n) using steps (0,1) and two kinds of steps (1,0).

4. When we set $p = 2$ and $q = 4$ in $(e_{n,0}(p,q))_{n\geq 0}$, we obtain the $1, 6, 52, 504, 5136, \ldots$ which is sequence [17, A084773]. This sequence has a interpretation in terms of weighted Motzkin paths with a different set of weights than the ones that come out of our interpretation. Thus we can view $(e_{n,0}(2p,4q))_{n\geq 0}$ as a (p,q)-analogue of this sequence.

Example 3.7. Wang and Zhu [20] observed that the sequence $(\overline{N}_n(q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials where

$$\overline{N}_n(q) = \sum_{k=1}^{n} \frac{1}{n}\binom{n}{k-1}\binom{n}{k} q^k.$$

It follows from Theorem 5 that the sequence $(\overline{N}_n(p,q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials where

$$\overline{N}_n(p,q) = \sum_{k=1}^{n} \frac{1}{n}\binom{n}{k-1}\binom{n}{k} q^k p^{n-k}.$$

Proposition 20. *The sequence* $\left(\sum_{k=1}^{n} \frac{1}{n}\binom{n}{k-1}\binom{n}{k} p^{n-k} q^k\right)_{n\geq 0}$ *is a Stieltjes moment sequence of polynomials.*

Taking $a = q$, $b = pq$, $c = p + q$ and $e = pq$ in (8), we obtain the generating function of the $(f_{n,0}(q))$ is

$$d_F(x,p,q) = \sum_{n\geq 0} f_{n,0}(p,q)x^n = \frac{2}{1 + (p-q)x + \sqrt{1 - 2(p+q)x + (p-q)^2x^2}}.$$

Remark 21.

1. If we set $p = q = 1$ in the sequence of $(f_{n,0}(p,q))_{n\geq 0}$, we obtain the sequence of Catalan numbers $C_0, C_1, C_2, \ldots$. Thus we can view $(f_{n,0}(p,q))_{n\geq 0}$ as a (p,q)-analogue of the Catalan numbers.

2. If we set $p = 2$ and $q = 1$ in the sequence of $(f_{n,0}(p,q))_{n\geq 0}$, we obtain the sequence $1, 1, 3, 11, 45, 19, 903, 4279, \ldots$ which is the sequence of little Schröder numbers [17, A001003]. Thus we can view $(f_{n,0}(2p,q))_{n\geq 0}$ as a (p,q)-analogue of the little Schröder numbers.

3. If we set $p = 1$ and $q = 2$ in the sequence of $(f_{n,0}(p,q))_{n\geq 0}$, we obtain the sequence $1, 2, 6, 22, 90, 394, 1806, \ldots$ which is the sequence of large Schröder numbers [17, A0006318]. Thus we can view $(f_{n,0}(p,2q))_{n\geq 0}$ as a (p,q)-analogue of the large Schröder numbers.

4. If we set $p = 2$ and $q = 2$ in the sequence of $(f_{n,0}(p,q))_{n\geq 0}$, we obtain the sequence $1, 2, 8, 40, 224, 1344, 8448, 54912, \ldots$ which is the sequence [17, A151374]. This sequence counts the number of paths that start at (0,0) and stay in the first quadrant consisting of $2n$ steps $(1,1)$, $(-1,-1)$, and $(-1,0)$.

It follows that for any $a, b \geq 0$, the sequence $(f_{n,0}(a,b))_{n\geq 0}$ is a Stieltjes moment sequence. In particular, the Catalan numbers C_n, the little Schröder numbers S_n, and the large Schröder numbers r_n are all Stieltjes moment sequences.

Example 3.8. Wang and Zhu [20] proved that $(S_n(q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials where $S_n(q) = \sum_{k=1}^{n} S_{n,k} q^k$ and $S_{n,k}$ is Stirling number of the second kind which counts the number of set partitions of $\{1, 2, \ldots, n\}$ into k parts. It follows form Theorem 5 that $(S_n(p,q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials where $S_n(p,q) = \sum_{k=1}^{n} S_{n,k} q^k p^{n-k}$.

Proposition 22. *The sequence* $\left(\sum_{k=1}^{n} S_{n,k} p^{n-k} q^k\right)_{n\geq 0}$ *is a Stieltjes moment sequence of polynomials.*

It is straightforward to obtain the generating functions of the $g_{n,0}(p,q)$s. That is, Let $\mathbb{S}_k(x) = \sum_{n\geq k} S(n,k)x^n$. Then it is well-known that $\mathbb{S}_0(x) = 1$ and for $k \geq 1$,

$$\mathbb{S}_k(x) = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

It follows that

$$\sum_{n\geq 0} x^n \sum_{k=0}^{n} S(n,k) q^k = \sum_{k\geq 0} q^k \mathbb{S}_k(x) = 1 + \sum_{k\geq 1} \frac{q^k x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

Replacing x by xp and q by q/p in the generating function above gives

$$\sum_{n\geq 0} S_n(p,q) x^n = 1 + \sum_{k\geq 1} \frac{q^k x^k}{(1-px)(1-2px)\cdots(1-kpx)}.$$

Remark 23.
1. If we set $p = q = 1$ in the sequence of $(g_{n,0}(p,q))_{n\geq 0}$, we obtain the sequence of Bell numbers $B_0, B_1, B_2, \ldots$. If we set $p = q = k$ in the sequence of $(g_{n,0}(p,q))_{n\geq 0}$, we obtain the sequence $(k^n B_n)_{n\geq 0}$. In particular for all $k \geq 1$, the sequence $(k^n B_n)_{n\geq 0}$ is a Steiltjes moment sequence. Thus we can view $(g_{n,0}(p,q))_{n\geq 0}$ as a p,q-analogue of the Bell numbers.
2. If we set $p = 2$ and $q = 1$ in the sequence of $(g_{n,0}(p,q))_{n\geq 0}$, we obtain the sequence $1, 1, 3, 11, 49, 257, 1539, \ldots$ which is the sequence [17, A004211].
3. If we set $p = 1$ and $q = 2$ in the sequence of $(g_{n,0}(p,q))_{n\geq 0}$, we obtain the sequence $1, 2, 6, 22, 94, 454, 2430, \ldots$ which is the sequence [17, A001861].
4. If we set $p = 2$ and $q = 2$ in the sequence of $(g_{n,0}(p,q))_{n\geq 0}$, we obtain the sequence $1, 2, 8, 40, 224, 1344, 8448, 54912, \ldots$ which is the sequence [17, A055882].

Example 3.9. For any $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let $\mathrm{des}(\sigma) = |\{i : \sigma_i > \sigma_{i+1}\}|$ and $\mathrm{ris}(\sigma) = |\{i : \sigma_i < \sigma_{i+1}\}|$. Wang and Zhu [20] proved that $(E_{n+1}(q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials where for $n \geq 1$,

$$E_n(q) = \sum_{k=0}^{n-1} E_{n,k} q^k = \sum_{\sigma\in S_n} q^{\mathrm{des}(\sigma)}.$$

It follows from Theorem 5 that $(E_{n+1}(p,q))_{n\geq 0}$ is a Stieltjes moment sequence of polynomials where

$$E_n(p,q) = \sum_{k=0}^{n-1} E_{n,k} q^k p^{n-k} = \sum_{\sigma\in S_n} q^{\mathrm{des}(\sigma)} p^{\mathrm{ris}(\sigma)}.$$

Proposition 24. *The sequence $\left(\sum_{\sigma\in S_n} p^{\mathrm{ris}(\sigma)} q^{\mathrm{des}(\sigma)}\right)_{n\geq 1}$ is a Stieltjes moment sequence of polynomials.*

It is easy to obtain a generating function for the $E_n(p,q)$s. That is, it is well-known that

$$1 + \sum_{n\geq 1} \frac{x^n}{n!} \sum_{\sigma\in S_n} q^{\mathrm{des}(\sigma)} = \frac{q-1}{q-e^{x(q-1)}}.$$

See [18]. Replacing x by px and q by q/p in this generating function we obtain that

$$1 + \sum_{n\geq 1} \frac{x^n}{n!} \sum_{\sigma\in S_n} q^{\mathrm{des}(\sigma)} p^{\mathrm{ris}(\sigma)+1} = \frac{q-p}{q-pe^{x(q-p)}}.$$

Thus

$$\sum_{n\geq 1} \frac{x^n}{n!} \sum_{\sigma\in S_n} q^{\mathrm{des}(\sigma)} p^{\mathrm{ris}(\sigma)} = \frac{e^{x(q-p)}-1}{q-pe^{x(q-p)}}.$$

References

[1] M. Aigner, Catalan-like numbers and determinants, J. Combin. Theory Ser. A 87 (1999) 33–51.

[2] M. Aigner, Catalan and other numbers: a recurrent theme, in: H. Crapo, D. Senato (Eds.), Algebraic Combinatorics and Computer Science, Springer, Berlin, 2001, 347–390.

[3] G. Bennett, Hausdorff means and moment sequences, Positivity, 15 (2011) 17–48.

[4] F. Brenti, Combinatorics and total positivity, J. Comb. Theory Ser. A 71 (1995) 175–218.

[5] F. Brenti, The applications of total positivity to combinatorics, and conversely, in: Total Positivity and Its Applications, Jaca, 1994, in: Math. Appl., vol. 359, Kluwer, Dordrecht, 1996, pp. 451–473.

[6] X. Chen, H. Liang and Y. Wang, Total positivity of Riordan arrays, European J. Combin. 46 (2015) 68–74.

[7] X. Chen, H. Liang and Y. Wang, Total positivity of recursive marices, Linear Algebra Appl. 471 (2015) 383–393.

[8] W.Y.C. Chen, L.X.W. Wang and A.L.B. Yang, Schur positivity and the q-log-convexity of the Narayana polynomials, J. Algebraic Combin. 32 (2010) 303–338.

[9] W.Y.C. Chen, L.X.W. Wang and A.L.B. Yang, Recurrence relations for strongly q-log-convex polynomials, Canad. Math. Bull. 54 (2011) 217–229.

[10] T.-X. He and R. Sprugnoli, Sequence characterization of Riordan arrays, Discrete Math. 309 (2009) 3962–3974.

[11] H. Liang, L. Mu and Y. Wang, Catalan-like numbers and Stieltjes moment sequences, Discrete Math. 339 (2016) 484–488.

[12] L.L. Liu and Y. Wang, On the log-convexity of combinatorial sequences, Adv. in. Appl. Math. 39 (2007) 453–476.

[13] H. Minc, Nonnegative Matrices, John Wiley & Sons, New York, 1988.

[14] A. Pinkus, Totally Positive Matrices, Cambridge University Press, Cambridge, 2010.

[15] L.W. Shapiro, S. Getu, W.-J. Woan and L.C. Woodson, The Riordan group, Discrete Appl. Math. 34 (1991) 229–239.

[16] J.A. Shohat and J.D. Tamarkin, The Problem of Moments, Amer. Math. Soc., New York, 1943.

[17] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, http://oeis.org.

[18] R.P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge University Press, Cambridge, 1999.

[19] Y. Wang and Z.-H. Zhang, Log-convexity of Aigner-Catalan-Riordan numbers, Linear Algebra Appl. 463 (2014) 45–55.

[20] Y. Wang and B.-X. Zhu, Log-convex and Stieltjes moment sequences, Advances in Applied Mathematics, vol. 81 (2016), 115-127.

[21] B.-X. Zhu, Log-convexity and strong q-log-convexity for some triangular arrays, Adv. in. Appl. Math. 50 (2013) 595–606.