Bisected theta series, least r-gaps in partitions, and polygonal numbers

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Abstract

The least r-gap, $g_r(\lambda)$, of a partition λ is the smallest part of λ appearing less than r times. In this article we introduce two new partition functions involving least r-gaps. We consider a bisection of a classical theta identity and prove new identities relating Euler's partition function $p(n)$, polygonal numbers, and the new partition functions. To prove the results we use an interplay of combinatorial and q-series methods.

We also give a combinatorial interpretation for

$$\sum_{n=0}^{\infty}(\pm 1)^{k(k+1)/2}p(n-r\cdot k(k+1)/2).$$

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1 Introduction

In [6], the second author considered a bisection of Euler's pentagonal number theorem

$$(q;q)_\infty = \sum_{k=0}^{\infty} (-1)^{\lceil k/2 \rceil} q^{G_k}$$

based on the parity of the k-th generalized pentagonal number

$$G_k = \frac{1}{2} \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{3k+1}{2} \right\rceil,$$

and obtained the following result:

$$\sum_{k=0}^{\infty} \frac{1+(-1)^{G_k}}{2} (-1)^{\lceil k/2 \rceil} q^{G_k} = (q^2, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}, q^{30}, q^{32}; q^{32})_\infty, \quad (1)$$

where

$$(a_1, a_2 \ldots, a_n; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_n;q)_\infty.$$

Because the infinite product

$$(a;q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k)$$

diverges when $a \neq 0$ and $|q| \geqslant 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$.

The following identity for Euler's partition function $p(n)$ was obtained in [6] as a combinatorial interpretation of (1):

$$\sum_{k=0}^{\infty} \frac{1+(-1)^{G_k}}{2} (-1)^{\lceil k/2 \rceil} p(n - G_k) = L(n), \quad (2)$$

where $L(n)$ is the number of partitions of n into parts not congruent to 0, 2, 12, 14, 16, 18, 20 or 30 mod 32. This identity is a bisection of Euler's well-known recurrence relation for the partition function $p(n)$:

$$\sum_{k=0}^{\infty} (-1)^{\lceil k/2 \rceil} p(n - G_k) = \delta_{0,n}, \quad (3)$$

where $\delta_{i,j}$ is the Kronecker delta function. For details on (3) see Andrews's book [1].

In this paper, motivated by these results, we consider a bisection of another classical theta identity [1, eq. 2.2.13]

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = \sum_{k=0}^{\infty} (-q)^{k(k+1)/2} \quad (4)$$

λ	5	4+1	3+2	3+1+1	2+2+1	2+1+1+1	1+1+1+1+1
$g_1(\lambda)$	1	2	1	2	3	3	2
$g_2(\lambda)$	1	1	1	2	1	2	2
$g_3(\lambda)$	1	1	1	1	1	2	2
$g_4(\lambda)$	1	1	1	1	1	1	2
$g_5(\lambda)$	1	1	1	1	1	1	2
$g_6(\lambda)$	1	1	1	1	1	1	1

Table 1: The partition functions g_r for $\lambda \vdash 5$

in order to derive new identities for Euler's partition function. These identities involve new partition functions which we define below.

For what follows, we denote by $g_r(l)$ the smallest part of the partition l appearing less than r times. The limit distribution of $g_r(l)$ has been studied in [9]. In the literature, $g_1(l)$ is referred to as the least gap of l. By analogy, we refer to $g_r(\lambda)$ as the *least r-gap* of λ. To make formulas more concise, we set $g_0(n) = \infty$. We denote by $S_r(n)$ the sum of the least r-gaps in all partitions of λ, i.e.,

$$S_r(n) = \sum_{l \vdash n} g_r(l).$$

Thus, $S_1(n)$ is the sum of the least gaps in all partitions of n. By Table 1, we see, for example, that

$$S_1(5) = 1 + 2 + 1 + 2 + 3 + 3 + 2 = 14$$

and

$$S_4(n) = 1 + 1 + 1 + 1 + 1 + 1 + 2 = 8.$$

When $r \geqslant 2$, for each partition l we have $g_r(l) \leqslant g_{r-1}(l)$. Let $G_r(n)$ be the number of partitions l of n satisfying $g_r(l) < g_{r-1}(l)$. It is clear that $G_1(n) = p(n)$ and $G_r(n) = 0$ for $r \geqslant n + 2$.

To our knowledge, the functions $S_r(n)$ and $G_r(n)$ have not been considered previously in the literature.

It is known [8, A022567] that the sum of the least gaps in all partitions of n can be expressed in terms of the Euler's partition function $p(n)$:

$$\sum_{k=0}^{\infty} p(n - T_k) = S_1(n), \tag{5}$$

where $T_n = n(n+1)/2$ is the n-th triangular number. Upon reflection, one expects that there might be infinite families of such identities where (5) is the first entry. As far as we know, the following identity has not been remarked before.

Theorem 1.1. *For $n \geqslant 0$ and $r \geqslant 1$,*

$$\sum_{k=0}^{\infty} p(n - rT_k) = S_r(n). \tag{6}$$

In section 2, we provide a combinatorial proof of Theorem 1.1. Then, theta identity (4) and Theorem 1.1 allow us to find the generating function for $S_r(n)$ and to prove the following result that also involves the partitions of n into parts not congruent to 0, r or $3r$ mod $4r$. We denote by $U_r(n)$ the number of these partitions.

Theorem 1.2. *For $n \geqslant 0$ and $r \geqslant 1$,*

(i) $\displaystyle\sum_{k=0}^{\infty} (p(n - rT_{4k}) + p(n - rT_{4k+3})) = \frac{S_r(n)}{2} + \frac{U_r(n)}{2};$

(ii) $\displaystyle\sum_{k=0}^{\infty} (p(n - rT_{4k+1}) + p(n - rT_{4k+2})) = \frac{S_r(n)}{2} - \frac{U_r(n)}{2}.$

By this theorem, we see that $S_r(n)$ and $U_r(n)$ have the same parity.

Corollary 1.3. *For $n \geqslant 0$ and $r \geqslant 1$, the sum of the least r-gaps in all partitions of n and the number of partitions of n into parts not congruent to 0, r or $3r$ mod $4r$ have the same parity.*

In addition, we have the following identity.

Corollary 1.4. *For $n \geqslant 0$ and $r \geqslant 1$,*

$$\sum_{k=0}^{\infty} (-1)^{T_k} p(n - rT_k) = U_r(n).$$

Replacing r by 1 in Corollary 1.4, we obtain another known identity (see [4, the proof of Theorem 2.3]).

Corollary 1.5. *For $n \geqslant 0$,*

$$\sum_{k=0}^{\infty} (-1)^{T_k} p(n - T_k) = \begin{cases} q\left(\frac{n}{2}\right), & \text{for } n \text{ even,} \\ 0, & \text{for } n \text{ odd,} \end{cases}$$

where $q(n)$ is the number of partitions of n into distinct parts.

It is shown in [7, Corollary 4.7] that $q(n)$ is odd if and only if n is a generalized pentagonal number. Thus, we deduce the following result related to the parity of $S_1(n)$.

Corollary 1.6. *For $n \geqslant 0$, the sum of the least gaps in all partitions of n is even except when n is twice a generalized pentagonal number.*

If $s \geqslant 3$ is the number of sides of a polygon, the nth s-polygonal number (or s-gonal number) is

$$P(s,n) = \frac{n^2(s-2) - n(s-4)}{2}.$$

If we allow $n \in \mathbb{Z}$, we obtain generalized s-gonal numbers. Note that, for $n > 0$, we have $P(3,-n) = P(3,n-1)$ and for all n we have $P(4,-n) = P(4,n)$. For $s \geqslant 5$ and $n > 0$, $P(s,-n)$ is not an ordinary s-gonal number. We remark that the n-th s-gonal number can be expressed in term of the triangular numbers T_n as follows:

$$P(s,n) = (s-3)T_{n-1} + T_n.$$

Beside Theorem 1.1, there is another infinite family of identities involving Euler's partition function $p(n)$ for which (5) is the special case $r = 1$.

Theorem 1.7. *For $n \geqslant 0$ and $r \geqslant 1$*

$$\sum_{k=0}^{\infty} p\left(n - P(r+2,-k)\right) = S_r(n) + G_r(n). \tag{7}$$

In this paper, we provide a purely combinatorial proof of this result and some applications involving partitions into even numbers of parts, partitions with nonnegative rank, and partitions with nonnegative crank.

2 Combinatorial proof of Theorem 1.1

Fix $r \geqslant 1$ and, for each $k \geqslant 0$ consider the fat staircase partition (written in exponential notation)

$$\delta_r(k) = (1^r, 2^r, \ldots, (k-1)^r, k^r).$$

This is the staircase partition with largest part k in which each part is repeated r times. Its size is equal to rT_k.

As before, fix $r \geqslant 1$ and also fix $n \geqslant 0$. For each $k \geqslant 0$ we create an injection from the set of partitions of $n - rT_k$ into the set of partitions of n

$$\varphi_{r,n,k} : \{\mu \vdash n - rT_k\} \hookrightarrow \{l \vdash n\}$$

where $\varphi_{r,n,k}(\mu)$ is the partition obtained from μ by inserting the parts of the staircase $\delta_r(k)$. Denote by $\mathcal{A}_{r,n,k}$ the image of $\{\mu \vdash n - rT_k\}$ under $\varphi_{r,n,k}$. Thus, $p(n - rT_k) = |\mathcal{A}_{r,n,k}|$ and $\mathcal{A}_{r,n,k}$ consists precisely of the partitions l of n satisfying $g_r(l) > k$.

Consider an arbitrary partition l of n with $g_r(l) = k$. Then $l \in \mathcal{A}_{r,n,i}$, $i = 0, 1, \ldots k-1$ and $l \notin \mathcal{A}_{r,n,j}$ with $j \geqslant k$. Therefore, each partition of n with $g_r(l) = k$ is counted by the left hand side of (6) exactly k times.

3 Proof of Theorem 1.2

We rewrite the identity (4) as

$$\sum_{k=0}^{\infty}(-q)^{T_k} = \frac{(q;q)_\infty}{(q^2;q^4)_\infty}. \tag{8}$$

Applying bisection on (8), we obtain:

$$\frac{1}{2}\sum_{k=0}^{\infty}\left(q^{T_k} \pm (-q)^{T_k}\right) = \frac{1}{2}\frac{(-q;-q)_\infty \pm (q;q)_\infty}{(q^2;q^4)_\infty}. \tag{9}$$

Multiplying both sides of (9) by the reciprocal of $(q;q)_\infty$, we give

$$\begin{aligned}
\frac{1}{2(q;q)_\infty}\sum_{k=0}^{\infty}\left(q^{T_k} \pm (-q)^{T_k}\right) &= \frac{(-q^2;q^2)_\infty}{2}\frac{(-q;-q)_\infty \pm (q;q)_\infty}{(q;q)_\infty} \\
&= \frac{(-q^2;q^2)_\infty}{2}\left(\frac{(-q;-q)_\infty}{(q;q)_\infty} \pm 1\right) \\
&= \frac{(-q;q)^2_\infty \pm (-q^2;q^2)_\infty}{2}.
\end{aligned}$$

By this identity, with q replaced by q^r, we obtain the relation

$$\frac{1}{2(q^r;q^r)_\infty}\sum_{k=0}^{\infty}\left(q^{rT_k} \pm (-q^r)^{T_k}\right) = \frac{(-q^r;q^r)^2_\infty \pm (-q^{2r};q^{2r})_\infty}{2},$$

that can be rewritten as

$$\begin{aligned}
&\frac{1}{2(q;q)_\infty}\sum_{k=0}^{\infty}\left(q^{rT_k} \pm (-q^r)^{T_k}\right) \\
&\quad = \frac{1}{2}\left(\frac{(-q^r;q^r)^2_\infty(q^r;q^r)_\infty}{(q;q)_\infty} \pm \frac{(-q^{2r};q^{2r})_\infty(q^r;q^r)_\infty}{(q;q)_\infty}\right) \\
&\quad = \frac{1}{2}\left(\frac{(-q^r;q^r)_\infty(q^r;q^r)_\infty}{(q;q)_\infty(q^r;q^{2r})_\infty} \pm \frac{(q^r,q^{2r};q^{2r})_\infty}{(q;q)_\infty(q^{2r};q^{4r})_\infty}\right) \\
&\quad = \frac{1}{2}\left(\frac{(q^{2r};q^{2r})_\infty}{(q;q)_\infty(q^r;q^{2r})_\infty} \pm \frac{(q^r,q^{3r},q^{4r};q^{4r})_\infty}{(q;q)_\infty}\right).
\end{aligned} \tag{10}$$

Considering the generating function for $p(n)$, i.e.,

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}$$

and the theta identity (4), by Theorem 1.1 we deduce that

$$\sum_{k=0}^{\infty} S_r(k)q^k = \frac{(q^{2r};q^{2r})_\infty}{(q;q)_\infty(q^r;q^{2r})_\infty}.$$

On the other hand, we have

$$\sum_{k=0}^{\infty} U_r(k)q^k = \frac{(q^r, q^{3r}, q^{4r}; q^{4r})_\infty}{(q;q)_\infty}.$$

Taking into account the well-known Cauchy multiplication of two power series, we deduce our identities as combinatorial interpretations of (10).

4 Combinatorial proof of Theorem 1.7

The proof of Theorem 1.7 is analogous to the proof of Theorem 1.1.

For fixed $r \geqslant 1$ and, for each $k \geqslant 0$we denote by '

$$\delta_r(k) = (1^r, 2^r, \dots, (k-1)^r, k^{r-1})$$

the staircase partition in which the largest part is k and is repeated $r-1$ times and all other parts are repeated r times. Its size is equal to $P(r+2,-k)$.

As before, fix $r \geqslant 1$ and also fix $n \geqslant 0$. For each $k \geqslant 0$ we create an injection from the set of partitions of $n - P(r+2,-k)$ into the set of partitions of n

$$\varphi'_{r,n,k} : \{\mu \vdash n - P(r+2,-k)\} \hookrightarrow \{l \vdash n\}$$

where $\varphi'_{r,n,k}(\mu)$ is the partition obtained from μ by inserting the parts of the staircase $\delta'_r(k)$. If $\mathcal{A}'_{r,n,k}$ denotes the image of $\{\mu \vdash n - P(r+2,-k)\}$ under $\varphi_{r,n,k}$, we have that $p(n - P(r+2,-k)) = |\mathcal{A}'_{r,n,k}|$ and $\mathcal{A}'_{r,n,k}$ consists precisely of the partitions l of n satisfying $g_r(l) \geqslant k$ and $g_{r-1}(l) > k$.

If $l \vdash n$ has $g_r(l) = k$, then $l \in \mathcal{A}'_{r,n,i}$, $i = 0, 1, \dots k-1$. If $g_{r-1}(l) = k$, then $l \notin \mathcal{A}'_{r,n,j}$ with $j \geqslant k$. If $g_{r-1} > k$, then $l \in \mathcal{A}'_{r,n,k}$ but $l \notin \mathcal{A}'_{r,n,j}$ with $j > k$. Therefore, each partition of n with $g_r(l) = k$ is counted by the left hand side of (7) exactly k times if $g_r(l) = g_{r-1}(l)$ and exactly $k+1$ times if $g_r(l) < g_{r-1}(l)$.

5 Applications of Theorem 1.7

In this section we consider some special cases of Theorem 1.7 in order to discover and prove new identities involving Euler's partition function $p(n)$.

5.1 Partitions into even numbers of parts

Now we consider the following classical theta identity [1, eq. 2.2.12]

$$\frac{(q;q)_\infty}{(-q;q)_\infty} = 1 + 2\sum_{k=1}^{\infty} (-1)^k q^{k^2}. \tag{11}$$

Elementary techniques in the theory of partition [1] allow us to derive a known combinatorial interpretation of this identity, namely

$$p(n) + 2\sum_{j=k}^{n} (-1)^k p(n-k^2) = p_e(n) - p_o(n), \tag{12}$$

where $p_e(n)$ is the number of partitions of n into even number of parts and $p_o(n)$ is the number of partitions of n into odd number of parts. Moreover, it is known that

$$p_e(n) = p(n) + \sum_{k=1}^{n} (-1)^k p(n-k^2) \tag{13}$$

and

$$p_o(n) = -\sum_{k=1}^{n} (-1)^k p(n-k^2).$$

These relations can be considered a bisection of the identity (12). Combining identity (13) with the case $r = 2$ of Theorem 1.7, we derive the following result.

Corollary 5.1. *For $n \geqslant 0$,*

(i) $\displaystyle\sum_{k=0}^{\infty} p\left(n-(2k)^2\right) = \frac{S_2(n)+G_2(n)+p_e(n)}{2}$;

(ii) $\displaystyle\sum_{k=0}^{\infty} p\left(n-(2k+1)^2\right) = \frac{S_2(n)+G_2(n)-p_e(n)}{2}$.

5.2 Partitions with nonnegative rank

In 1944, Dyson [5] defined the rank of a partition as the difference between its largest part and the number of its parts. Then he observed empirically that the partitions of $5n+4$ (respectively $7n+5$) form 5 (respectively 7) groups of equal size when sorted by their ranks modulo 5 (respectively 7). This interesting conjecture of Dyson was proved ten years later by Atkin and Swinnerton-Dyer [3]. In this section, we denote by $R(n)$ the number of partitions of n with nonnegative rank.

It is known [8, A064174] that the number of partitions of n with nonnegative rank can be expressed in terms of Euler's partition function as follows:

$$R(n) = \sum_{k=0}^{n} (-1)^k p\left(n-k(3k+1)/2\right). \tag{14}$$

Considering the case $r = 3$ of Theorem 1.7, we obtain the following result.

Corollary 5.2. *For $n \geqslant 0$,*

(i) $\displaystyle\sum_{k=0}^{\infty} p\left(n-k(6k+1)\right) = \frac{S_3(n)+G_3(n)+R(n)}{2}$;

(ii) $\displaystyle\sum_{k=0}^{\infty} p\left(n-(2k+1)(3k+2)\right) = \frac{S_3(n)+G_3(n)-R(n)}{2}$.

5.3 Partitions with nonnegative crank

Dyson [5] conjectured the existence of a crank function for partitions that would provide a combinatorial proof of Ramanujan's congruence modulo 11. Forty-four years later, Andrews and Garvan [2] successfully found such a function which yields a combinatorial explanation of Ramanujan congruences modulo 5, 7, and 11. For a partition λ, let $l(\lambda)$ denote the largest part of λ, $\omega(\lambda)$ denote the number of 1's in λ, and $\mu(\lambda)$ denote the number of parts of λ greater than $\omega(\lambda)$. The crank $c(\lambda)$ is defined by

$$c(\lambda) = \begin{cases} l(\lambda), & \text{for } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda), & \text{for } \omega(\lambda) > 0. \end{cases}$$

In this section, we denote by $C(n)$ the number of partitions of n with nonnegative crank.

We known [8, A064428] that the number of partitions of n with nonnegative crank can be expressed in terms of Euler's partition function $p(n)$:

$$C(n) = \sum_{k=0}^{\infty} (-1)^k p(n - T_k). \tag{15}$$

We have the following result related to the parity of $C(n)$.

Corollary 5.3. *For $n \geqslant 0$, the number of partitions of n with nonnegative crank is even except when n is twice a generalized pentagonal number.*

Proof. Considering the case $r = 1$ of Theorem 1.7 and the identity (15), we obtain

$$\sum_{k=0}^{\infty} p(n - T_{2k}) = \frac{C(n) + S_1(n)}{2}.$$

We see that the number of partitions of n with nonnegative crank and the sum of the least gaps in all partitions of n have the same parity. According to Corollary 1.6 the proof is finished. ☐

By the identity (15) and Corollary 1.4, we easily get two identities.

Corollary 5.4. *For $n \geqslant 0$,*

(i) $\displaystyle\sum_{k=0}^{\infty} (-1)^k p\left(n - T_{k+2\lfloor k/2 \rfloor}\right) = \frac{C(n) + U_1(n)}{2};$

(ii) $\displaystyle\sum_{k=0}^{\infty} (-1)^k p\left(n - T_{k+2\lfloor k/2 \rfloor+2}\right) = \frac{C(n) - U_1(n)}{2}.$

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